APPLICATION OF GRÖBNER BASES TO THE CUP-LENGTH OF ORIENTED GRASSMANN MANIFOLDS

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Abstract. For $n = 2^{m+1} - 4\,(m \geq 2)$, we determine the cup-length of $H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)$ by finding a Gröbner basis associated with a certain subring, where $\widetilde{G}_{n,3}$ is the oriented Grassmann manifold $SO(n+3)/SO(n) \times SO(3)$. As its applications, we provide not only a lower but also an upper bound for the LS-category of $\widetilde{G}_{n,3}$. We also study the immersion problem of $\widetilde{G}_{n,3}$.

1. Introduction

Let R be a commutative ring. The cup-length of R is defined by the greatest number n such that there exist $x_1, \ldots, x_n \in R \setminus R^\times$ with $x_1 \cdots x_n \neq 0$. We denote the cup-length of R by $\mathrm{cup}(R)$. In particular, for a space X and a commutative ring A, the cup-length of X with the coefficient A, is defined by $\mathrm{cup}(\tilde{H}^*(X; A))$. We denote it by $\mathrm{cup}_A(X)$. It is well-known that $\mathrm{cup}_A(X)$ is a lower bound for the LS-category of X.

The aim of this paper is to study $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$, where $\widetilde{G}_{n,k}$ is the oriented Grassmann manifold $SO(n+k)/SO(n) \times SO(k)$. Note that $\widetilde{G}_{n,k}$ is (nk)-dimensional. While the cohomology of $\widetilde{G}_{n,2}$ is well-known, that of $\widetilde{G}_{n,3}$ is in vague. However, Korbaš [Kor06] gave rough estimations for $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$ by considering the height of $w_2 \in H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)$, where w_2 is the second Stiefel-Whitney class.

The author studies $H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)$ by considering Gröbner bases associated with a certain subring of $H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)$. It seems that, in principle, the method of Gröbner bases works better in such complicated calculations than that of usual algebraic topology. The author employs a computer and carries a huge amount of calculations for finding the above Gröbner bases and then he dares to conjecture:

Conjecture 1.1.

$$\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3}) = \begin{cases} 2^{m+1} - 3 & \textit{when } 2^{m+1} - 4 \leq n \leq 2^{m+1} + 2^m - 6, \\ 2^{m+1} - 1 + k & \textit{when } n = 2^{m+1} + 2^m - 5 + k,\ 0 \leq k \leq 2, \\ 2^{m+1} + 2^m + \ldots & \textit{when } n = 2^{m+1} + 2^m + \cdots + 2^{j-1} - 2 + k, \\ \qquad +2^{j+1} + 2^{j-1} + k & \qquad 0 \leq k \leq 2^j - 1. \end{cases}$$

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When $n = 2^{m+1} - 4\,(m \geq 2)$, our method works very well and we obtain:

Theorem A. $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3}) = n + 1$ *when* $n = 2^{m+1} - 4\,(m \geq 2)$.

By a dimensional reason, we have

$$\mathrm{cat}(X) \leq \frac{3}{2}n, \tag{1}$$

where $\mathrm{cat}(X)$ denotes the LS-category of a space X normalized as $\mathrm{cat}(*) = 0$. Theorem A gives not only lower bounds for $\mathrm{cat}(\widetilde{G}_{n,3})$, but also refines the inequality (1). Actually we obtain:

Corollary . $n + 1 \leq \mathrm{cat}(\widetilde{G}_{n,3}) < \frac{3}{2}n$ *when* $n = 2^{m+1} - 4\,(m \geq 2)$. *In particular, we have* $\mathrm{cat}(\widetilde{G}_{4,3}) = 5$.

We will give applications of Theorem A for the immersion problem of $\widetilde{G}_{n,3}$. By the classical result of Whitney [Whi44], we know that $\widetilde{G}_{n,3}$ immerses into $\mathbb{R}^{6n-1}$. We will show:

Theorem B. *The oriented Grassmann manifold* $\widetilde{G}_{n,3}$ *immerses into* $\mathbb{R}^{6n-3}$ *but not into* $\mathbb{R}^{3n+8}$ *when* $n = 2^{m+1} - 4\,(m \geq 3)$ *and* $\widetilde{G}_{4,3}$ *immerse into* $\mathbb{R}^{21}$ *but not into* $\mathbb{R}^{17}$.

Remark : Walgenbach [Wal01] obtained better results on the non-immersion of $\widetilde{G}_{n,3}$: $\widetilde{G}_{n,3}$ does not immerses into $\mathbb{R}^{4n-2m+3}$. On the other hand, due to R. Cohen [Coh85], $\widetilde{G}_{n,3}$ is known to be immersed into $\mathbb{R}^{6n-m+1}$. Then Theorem B gives a better estimation when $m = 2, 3$.

The organization of this paper is as follows. In section 2, we consider the double covering map $p_n \colon \widetilde{G}_{n,3} \to G_{n,3}$, where $G_{n,3}$ is the unoriented Grassmann manifold $O(n+3)/O(n) \times O(3)$. We identify the subring $\mathbf{Im} p_n^*$ of $H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)$ with a certain algebra $\mathbb{Z}/2[\bar{w}_2, \bar{w}_3]/J_n$, where generators of J_n are given. In section 3, setting $n = 2^{m+1} - 4\,(m \geq 2)$, we will give an explicit description of generators of the ideal J_n by using the binary expansion. In section 4, we compute a Gröbner basis of J_n and obtain $\mathrm{cup}(\mathbf{Im} p_n^*)$. In section 5, we show $\mathrm{cup}(\mathbf{Im} p_n^*)$ determines $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$ and obtain it. As its applications, we give some estimations for $\mathrm{cat}(\widetilde{G}_{n,3})$ and study the immersion problem of $\widetilde{G}_{n,3}$.

2. Cohomology of $\widetilde{G}_{n,3}$

We consider the double covering

$$p_n \colon \widetilde{G}_{n,3} \to G_{n,3}. \tag{2}$$

It will be shown that $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$ can be determined by $\mathrm{cup}(\mathbf{Im} p_n^*)$. Then we shall investigate $\mathrm{cup}(\mathbf{Im} p_n^*)$.

The mod 2 cohomology of $BO(3)$ is given by

$$H^*(BO(3); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, w_3],$$

where w_i is the i-th universal Stiefel-Whitney class. It is well-known that the canonical map $i\colon G_{n,3} \to BO(3)$ induces an epimorphism $i^*\colon H^*(BO(3);\mathbb{Z}/2) \to H^*(G_{n,3};\mathbb{Z}/2)$. Hereafter we denote $i^*(w_i)$ by the same symbol w_i ambiguously.

One can easily see that the above double covering (2) induces the Wang sequence as:

$$\cdots \longrightarrow H^{q-1}(G_{n,3};\mathbb{Z}/2) \xrightarrow{\cdot w_1} H^q(G_{n,3};\mathbb{Z}/2) \xrightarrow{p_n^*} H^q(\widetilde{G}_{n,3};\mathbb{Z}/2) \longrightarrow \cdots .$$

Then we have

$$\mathbf{Im} p_n^* \cong \mathbb{Z}/2[w_1, w_2, w_3]\big/(w_1, \mathbf{Ker} i^*).$$

Let $\pi\colon \mathbb{Z}/2[w_1, w_2, w_3] \to \mathbb{Z}/2[w_2, w_3]$ be the abstract ring homomorphism defined by $\pi(w_1) = 0$, $\pi(w_2) = w_2$ and $\pi(w_3) = w_3$. Then it induces the isomorphism

$$\mathbf{Im} p_n^* \cong \mathbb{Z}/2[\bar{w}_2, \bar{w}_3]\big/J_n,$$

where $\pi(\mathbf{Ker} i^*) = J_n$ and we denote w_i in $H^*(\widetilde{G}_{n,3};\mathbb{Z}/2)$ by $\bar{w}_i$. Note that the commutative diagram

$$\begin{tikzcd} \widetilde{G}_{n,3} \arrow[r, "p_n"'] \arrow[d, "\tilde{\imath}"'] & G_{n,3} \arrow[d, "i"'] \\ BSO(3) \arrow[r, "p_\infty"'] & BO(3) \end{tikzcd}$$

yields that $\tilde{\imath}^*(w_i) = \bar{w}_i$ for $i = 2, 3$ and $p_\infty^*\colon H^*(BO(3);\mathbb{Z}/2) \to H^*(BSO(3);\mathbb{Z}/2)$ is expressed by $\pi\colon \mathbb{Z}/2[w_1, w_2, w_3] \to \mathbb{Z}/2[w_2, w_3]$.

Let us give explicit generators of J_n. Borel [Bor53] showed that $\mathbf{Ker} i^*$ is generated by the homogeneous components of degrees $n + 1$, $n + 2$ and $n + 3$ in

$$\frac{1}{1 + w_1 + w_2 + w_3}.$$

Then it follows that J_n is generated by the homogeneous components of degrees $n + 1$, $n + 2$ and $n + 3$ in

$$\frac{1}{1 + \bar{w}_2 + \bar{w}_3}.$$

Let N be the unique integer which satisfies $2^N < n \leq 2^{N+1}$. Since $\dim \widetilde{G}_{n,3} < 4n \leq 2^{N+3}$, we have

$$(1 + \bar{w}_2 + \bar{w}_3)^{2^{N+3}} = 1$$

in $H^*(\widetilde{G}_{n,3};\mathbb{Z}/2)$. Then it follows that

$$\frac{1}{1 + \bar{w}_2 + \bar{w}_3} = (1 + \bar{w}_2 + \bar{w}_3)^{2^{N+3}-1}$$

and hence J_n is generated by

$$g_r = \sum_{\frac{r}{3} \leq s \leq \frac{r}{2}} \binom{s}{3s - r} \bar{w}_2^{3s-r} \bar{w}_3^{r-2s} \tag{3}$$

for $r = n + 1, n + 2, n + 3$.

3. Investigating generators of J_n

In this section, we investigate generators g_{n+1}, g_{n+2} and g_{n+3} of J_n by exploiting the binary expansion.

Let us prepare notation for the binary expansion. To a non-negative integer x with $0 \le x < 2^k$, we assign a sequence

$$\epsilon_k(x) = (x_{k-1}, \dots, x_0) \in \{0, 1\}^k$$

such that

$$x = \sum_{i=0}^{k-1} x_i 2^i. \tag{4}$$

(4) is, of course, the binary expansion of x. We denote $1 - a$ by $\overline{a}$ with $a \in \{0, 1\}$. For example, we have

$$\epsilon_k(2^k - 1) = (1, \dots, 1)$$

and

$$\epsilon_k(2^k - 1 - x) = (\overline{x_{k-1}}, \dots, \overline{x_0})$$

for $\epsilon(x) = (x_{k-1}, \dots, x_0)$. We often denote $(x_k, \dots, x_0) \in \{0, 1\}^k$ by $\mathbf{x}_k$.

To calculate $\binom{s}{3s-r}$ modulo 2, we use the following well-known result from elementary number theory.

Lemma 3.1. *Let n and k be non-negative integers such that $k \le n \le 2^l - 1$ and $\epsilon_l(n) = (n_{l-1}, \dots, n_0)$, $\epsilon_l(k) = (k_{l-1}, \dots, k_0)$. Then we have $\binom{n}{k} \equiv 1 \pmod 2$ if and only if $k_i = 1$ implies $n_i = 1$ for each i.*

In the rest of this paper, we assume that

$$\boxed{n = 2^{m+1} - 4 \ (m \ge 2).}$$

Applying Lemma 3.1 to the coefficients of g_{n+1}, we have:

Proposition 3.2. *$\binom{s}{3s-(n+1)}$ is even for all integer s with $\frac{n+1}{3} \le s \le \frac{n+1}{2}$, that is $g_{n+1} = 0$.*

Proof. Let $\epsilon_m(s) = (s_{m-1}, \dots, s_0)$ for $\frac{n+1}{3} \le s \le \frac{n+1}{2}$ and let $\epsilon_{m+1}(n+1-2s) = (t_m, \dots, t_0)$. Since $s \le 2^m - 2$, there exists an integer i such that $s_i = 0$. Let i be the least integer satisfying $s_i = 0$, that is, $\epsilon_m(s) = (s_{m-1}, \dots, s_{i+1}, 0, 1, \dots, 1)$. Then it is easy to show that $t_i = 1$. Hence it follows from Lemma 3.1 that $\binom{s}{3s-(n+1)} = \binom{s}{n+1-2s} \equiv 0 \pmod 2$. □

Next we investigate g_{n+2}. Coefficients of g_{n+2} are well understood by considering their binary expansion as in the above case of g_{n+1}. Let

$$\mathcal{S}_k = \left\{ s \in \mathbb{Z} \,\middle|\, \frac{n(k)+2}{3} \le s \le \frac{n(k)+2}{2},\ \epsilon_k(s) = (s_{k-1}, \dots, s_0) \text{ satisfies that if } s_j = 0, \text{then } s_{j+1} = 1 \right\},$$

here $n(k) = 2^{k+1} - 4$. Note that $\frac{n(k)+2}{3} \le s \le \frac{n(k)+2}{2}$ implies that s_{k-1} is always equal to 1 for each $s \in \mathcal{S}_k$ with $\epsilon_k(s) = (s_{k-1}, \dots, s_0)$. There is a one-to-one correspondence between non-zero coefficients of g_{n+2} and $\mathcal{S}_m$ as:

Lemma 3.3.

$$\binom{s}{3s-(n+2)} \equiv 1 \pmod 2 \quad \textit{if and only if} \quad s \in \mathcal{S}_m.$$

Proof. Let $\epsilon_m(s) = (s_{m-1}, \dots, s_0)$ for $\frac{n(m)+2}{3} \le s \le \frac{n(m)+2}{2}$. Then we have

$$\epsilon_{m+1}(n+2-2s) = (\overline{s_{m-1}}, \dots, \overline{s_0}, 0)$$

and hence Lemma 3.3 follows from Lemma 3.1. □

It is convenient for calculations in section 4 to index coefficients of g_{n+2} by exponents of $\bar{w}_2$ in (3), that is, $3s-(n+2)$, not by $s \in \mathcal{S}_m$. Then we define a set $\mathcal{P}_k$ by

$$\mathcal{P}_k = \{\, p \in \mathbb{Z} |\, p = 3s - (n(k)+2),\ s \in \mathcal{S}_k \}\,.$$

$\mathcal{P}_m$ is expressed by the binary expansion as:

Proposition 3.4. *Let*

$$\Delta_k = \left\{(p_{k-1}, \dots, p_0) \in \{0,1\}^k \,|\textit{If } p_{l-1} = 1 \textit{ and } p_l = p_{l+1} = \cdots = p_{l+2t} = 0, \textit{ then } p_{l+2t+1} = 0\right\},$$

here we assume that $p_{-1} = 1$. Then we have

$$\mathcal{P}_m = \{\, p \in \mathbb{Z} |\, \epsilon_m(p) \in \Delta_m \}\,.$$

We list some properties of Δ_k which will be useful in the following discussion. The proof is straightforward.

Proposition 3.5. *The set Δ_k has the following properties.*

(a) $\mathbf{p}_k \in \Delta_k$ *implies* $(1, \mathbf{p}_k) \in \Delta_{k+1}$.
(b) $\mathbf{p}_k \in \Delta_k$ *implies* $(\mathbf{p}_k, 1) \in \Delta_{k+1}$.
(c) $\Delta_m = \{(1, \mathbf{p}_{m-1}) \in \{0,1\}^m |\, \mathbf{p}_{m-1} \in \Delta_{m-1}\} \sqcup \{(0,0,\mathbf{p}_{m-2}) \in \{0,1\}^m |\, \mathbf{p}_{m-2} \in \Delta_{m-2}\}$.

Proof of Proposition 3.4. Let $s \in \mathcal{S}_m$ with $\epsilon_m(s) = (s_{m-1}, \dots, s_0)$. If $s_{m-2} = 0$, then one has $s_{m-1} = 1$ and $s_{m-3} = 1$ by definition of $\mathcal{S}_m$. Then one can easily see that $(s_{m-2}, \dots, s_0) \in \mathcal{S}_{m-3}$. If $s_{m-2} = 1$, then one can see that $(s_{m-2}, \dots, s_0) \in \mathcal{S}_{m-1}$ as well. Hence one has obtained

$$\mathcal{S}_m = \left\{ s + 2^{m-1} \middle|\, s \in \mathcal{S}_{m-1} \right\} \sqcup \left\{ s + 2^{m-1} \middle|\, s \in \mathcal{S}_{m-2} \right\}. \tag{5}$$

We will show Proposition 3.4 by induction. We suppose that it is true for $m-1$ and $m-2$. Let $s \in \mathcal{S}_{m-1}$ and $p = 3(s+2^{m-1}) - (n+2)$. By the hypothesis of the induction, $\epsilon_{m-1}(3s-(n'+2)) = \mathbf{p}_{m-1} \in \Delta_{m-1}$, where $n' = 2^m - 4$. Since

$$p = 3(s+2^{m-1}) - (n+2) = 3s - 2^m + 2 + 2^{m-1} = 3s - (n'+2) + 2^{m-1},$$

we have

$$\epsilon_m(p) = (1, \mathbf{p}_{m-1}) \in \Delta_m.$$

Similarly, let $s \in \mathcal{S}_{m-2}$ and $p = 3(s+2^{m-1}) - (n+2)$. By the hypothesis of the induction, $\epsilon_{m-2}(3s-(n''+2)) = \mathbf{p}_{m-2} \in \Delta_{m-2}$, where $n'' = 2^{m-1} - 4$. Since

$$p = 3(s+2^{m-1}) - (n+2) = 3s - 2^{m-1} + 2 = 3s - (n''+2),$$

we have

$$\epsilon_m(p) = (0, 0, \mathbf{p}_{m-2}) \in \Delta_m.$$

Thus, by (5), we obtain

$$\mathcal{P}_m = \left\{ p + 2^{m-1} \middle| p \in \mathcal{P}_{m-1} \right\} \sqcup \mathcal{P}_{m-2}$$

and, by (c) of Proposition 3.5, we have established Proposition 3.4.

□

For the last of this section, we investigate g_{n+3}. Coefficients of g_{n+3} can be well understood by using the binary expansion as well as above. Let

$$\mathcal{S}'_k = \left\{ s' \in \mathbb{Z} \middle| \frac{n(k)+3}{3} \leq s' \leq \frac{n(k)+3}{2},\ \epsilon_k(s') = (s_{k-1}, \dots, s_1, 1) \text{ satisfies that if } s_j = 0, \text{then } s_{j+1} = 1 \right\}.$$

Quite similarly to Lemma 3.3, we can see:

Lemma 3.6.

$$\binom{s'}{3s' - (n+3)} \equiv 1 \pmod 2 \text{ *if and only if* } s' \in \mathcal{S}'_m.$$

We give an explicit description of the set

$$\mathcal{P}'_k = \left\{ p' \in \mathbb{Z} \middle| p' = 3s' - (n(k) + 3),\ s' \in \mathcal{S}'_k \right\}$$

as well. Define a map

$$\iota : \mathcal{S}_{m-1} \to \mathcal{S}'_m$$

by $\iota(s) = 2s + 1$. Then, obviously, it is bijective. Note that, for $s' = \iota(s)$,

$$3s' - (n + 3) = 3\iota(s) - 2^{m+1} + 1 = 6s - 2^{m+1} + 4 = 2(3s - (n' + 2))$$

where $n' = 2^m - 4$. Then we have $p \in \mathcal{P}_{m-1}$ if and only if $p' \in \mathcal{P}'_m$ such that $\epsilon_m(p') = (\mathbf{p}_{m-1}, 0)$ for $\epsilon_{m-1}(p) = \mathbf{p}_{m-1}$. Hence we have obtained:

Proposition 3.7. $\mathcal{P}'_m = \{ p \in \mathbb{Z} | \epsilon_m(p) = (\mathbf{p}_{m-1}, 0),\ \mathbf{p}_{m-1} \in \Delta_{m-1} \}$.

4. Gröbner basis and cup-length

In this section, by using the result of the previous section, we search for a Gröbner basis of J_n in order to determine cup($\mathbf{Im} p_n^*$).

4.1. **Gröbner bases.** We first recall the definition and some facts of Gröbner bases by restricting to our specific case. In order to clarify our discussion and to simplify notation, we shall make a convention of identifying a two variable polynomial ring with a certain set as follows. Let $\mathcal{X} = \{(p, q) \in \mathbb{Z}^2 | p \geq 0, q \geq 0\}$ and let $P[\mathcal{X}]$ denote the set of finite subset of $\mathcal{X}$. By assigning $F \in P[\mathcal{X}]$ to $\sum_{(p,q)\in F} \bar{w}_2^p \bar{w}_3^q$, we can identify $P[\mathcal{X}]$ with a polynomial ring $\mathbb{Z}/2[\bar{w}_2, \bar{w}_3]$ and we shall make this identification throughout this section. This identification translates the operations in $\mathbb{Z}/2[\bar{w}_2, \bar{w}_3]$ into $P[\mathcal{X}]$ as: For $F, G \in P[\mathcal{X}]$,

$$F + G = F \cup G \setminus F \cap G,$$

$$F \cdot G = \sum_{(p,q)\in F,\,(r,s)\in G} (p + r, q + s).$$

This translation of operations enables us to handle the following polynomial calculations easily.

The order of $\mathcal{X}$ is given by the usual lexicographic order. Namely, for $(p, q), (r, s) \in \mathcal{X}$,

$$(p, q) \geq (r, s) \text{ if and only if } p > r \text{ or } p = r, q \geq s.$$

By employing this order, we search for a Gröbner basis of the ideal $J_n \subset P[\mathcal{X}]$.

In order to define Gröbner bases, we prepare some notation and terminology. The leading term of a polynomial $F \in P[\mathcal{X}]$ is the monomial

$$\mathrm{LT}(F) = \max\{(p, q) \in F\}.$$

If there is a monomial $(p, q) \in \mathcal{X}$ such that $(p, q) \cdot \mathrm{LT}(G) \in F$, then the polynomial $F - (p, q) \cdot \mathrm{LT}(G)$ is called the remainder of F on division by G. We denote the remainder $R = F - (p, q) \cdot \mathrm{LT}(G)$ of F on division by G, by

$$F \xrightarrow{G_*} R.$$

Choose $F_1, \ldots, F_s \in P[\mathcal{X}]$ and give them an arbitrary order. Then it is known that there is an algorithm to provide the decomposition of $F \in P[\mathcal{X}]$ as

$$F = A_1F_1 + \cdots + A_sF_s + R$$

such that $A_1, \ldots, A_s \in P[\mathcal{X}]$ and R is a linear combination of monomials, none of which is divisible by each $\mathrm{LT}(F_1), \ldots, \mathrm{LT}(F_s)$. The above R is called the remainder of F on division by $\{F_1, \ldots, F_s\}$ as well. However, this decomposition depends on the choice of an order of $F_1, \ldots, F_s$ and $F \in (F_1, \ldots, F_s)$ does not imply the remainder $R = 0$. We can overcome this difficulty of remainders by choosing a Gröbner basis defined as:

Definition 4.1. *Let I be an ideal of $P[\mathcal{X}]$. A finite subset $G = \{G_1, \ldots, G_s\}$ is a Gröbner basis of I if*

$$(\{\mathrm{LT}(F) | F \in I\}) = (\mathrm{LT}(G_1), \ldots, \mathrm{LT}(G_s)).$$

Theorem 4.2. *Let I be an ideal of $P[\mathcal{X}]$ and let $\{G_1, \ldots, G_s\}$ be a Gröbner basis of I. Then the remainder of $F \in I$ on division by $\{G_1, \ldots, G_s\}$ is zero.*

Buchberger [CLO97] gave a criterion for a set of polynomials being a Gröbner basis of the ideal generated by it as follows. For $F, G \in P[\mathcal{X}]$, the least common multiple of F and G is the monomial

$$\mathrm{LCM}(F, G) = (\max\{p, r\}, \max\{q, s\}),$$

where $\mathrm{LT}(F) = (p, q)$ and $\mathrm{LT}(G) = (r, s)$. The S-polynomial of F and $G \in P[\mathcal{X}]$ is

$$S(F, G) = \frac{\mathrm{LCM}(F, G)}{\mathrm{LT}(F)} F + \frac{\mathrm{LCM}(F, G)}{\mathrm{LT}(G)} G.$$

Theorem 4.3 ([CLO97]). *The set of polynomials $\{G_1, \dots, G_s\} \subset P[\mathcal{X}]$ is a Gröbner basis of the ideal $(G_1, \dots, G_s)$ if and only if the remainder of $S(G_i, G_j)$ on division by $\{G_1, \dots, G_s\}$ is zero for each $i \neq j$.*

4.2. **Search for a Gröbner basis of J_n.** The author found the following polynomials experimentally by a computer calculation. For non-negative integers i, t with $t - 2(2^m - 2^i) \equiv 0 \pmod 3$, we define a polynomial $P(t, i)$ by

$$\begin{aligned} P(t,i) &= \left\{ \left(p, \tfrac{t-2p}{3}\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) = (\mathbf{p}_{m-i}, \overbrace{0, \dots, 0}^{i}),\ \mathbf{p}_{m-i} \in \Delta_{m-i} \right\}, \\ P_i &= P(2^i + n + 1, i). \end{aligned}$$

We shall prove that $\{P_0, \dots, P_m\}$ is a Gröbner basis of J_n.

In order to investigate P_i, we define the following sets which will be useful for expression. Let $\Delta(i, j, l)$ and $\bar{\Delta}(i, l)$ be

$$\Delta(i,j,l) = \left\{ (\mathbf{p}_{m-j}, \mathbf{p}_{j-i-l}, \overbrace{1, \dots, 1}^{l}, \overbrace{0, \dots, 0}^{i}) \in \{0,1\}^m \,\middle|\, (\mathbf{p}_{m-j}, \mathbf{p}_{j-i-l}) \in \Delta_{m-i-l},\ \mathbf{p}_{j-i-l} \neq (1, \dots, 1) \right\},$$

$$\bar{\Delta}(i,l) = \left\{ (\mathbf{p}_{m-i-l-2}, 0, 0, \overbrace{1, \dots, 1}^{l}, \overbrace{0, \dots, 0}^{i}) \in \{0,1\}^m \,\middle|\, \mathbf{p}_{m-i-l-2} \in \Delta_{m-i-l} \right\}.$$

It is easy to check:

Lemma 4.4.

$$\Delta(i, j, l) = \bar{\Delta}(i, l) \sqcup \Delta(i, j, l+1).$$

Let us begin investigating P_i. It is easy to verify that

$$\mathrm{LT}(P_i) = (2^m - 2^i, 2^i - 1). \tag{6}$$

Proposition 4.5. *We have $P_0, \dots, P_m \in J_n$. In particular $P_0 = g_{n+2}$, $P_1 = g_{n+3}$.*

Proof. By Proposition 3.4 and Proposition 3.7, one has $P_0 = g_{n+2}$ and $P_1 = g_{n+3}$. For $i < j$, it follows from (6) that

$$\begin{aligned} S(P_i, P_j) &= (0, 2^j - 2^i) \cdot P_i + (2^j - 2^i, 0) \cdot P_j \\ &= \left\{ \left(p, q_{i,j}\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) = (\mathbf{p}_{m-j}, \mathbf{p}_{j-i}, \overbrace{0, \dots, 0}^{i}),\ (\mathbf{p}_{m-j}, \mathbf{p}_{j-i}) \in \Delta_{m-i} \right\} \\ &\quad + \left\{ \left(p, q_{i,j}\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) = (\mathbf{p}_{m-j}, \overbrace{1, \dots, 1}^{j-i}, \overbrace{0, \dots, 0}^{i}),\ \mathbf{p}_{m-j} \in \Delta_{m-j} \right\} \\ &= \left\{ \left(p, q_{i,j}(p)\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) \in \Delta(i, j, 0) \right\}, \end{aligned}$$

where

$$q_{i,j}(p) = \frac{3 \cdot 2^j - 2 \cdot 2^i + n + 1 - 2p}{3}.$$

By the definition of Δ_k, one can easily see that $\Delta(i, 0, i+1) = \Delta_{m-i-2}$. Then it follows that $S(P_i, P_{i+1}) = P_{i+2}$ and hence we have established Proposition 4.5. □

We calculate the remainders of $S(P_i, P_j)$ on division by $\{P_0, \ldots, P_m\}$.

Lemma 4.6. *The remainder of* $Q_{i,j,l} = \left\{ \left(p, q_{i,j}(p)\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) \in \Delta(i, j, l) \right\}$ *on division by* P_{i+l+2} *is* $Q_{i,j,l+1}$.

Proof. Let $p(i,l)$ be $\epsilon_m(p(i,l)) = (\overbrace{1, \ldots, 1}^{m-i-l-2}, 0, 0, \overbrace{1, \ldots, 1}^{l}, \overbrace{0, \ldots, 0}^{i})$. Then it is easy to see

$$\mathrm{LT}(Q(i,j,l)) = \left(p(i,l), q_{i,j}(p(i,l))\right)$$

and it follows from (6) that

$$\mathrm{LT}(P_{i+l+2}) = \left(p(i+l, 0), q_{i+j+2, i+j+2}(p(i+l, 0))\right).$$

Hence we have

$$(2^{i+l} - 2^i, 2^j - 2^{i+l+1}) \cdot \mathrm{LT}(P_{i+l+2}) = Q(i,j,l).$$

On the other hand, one can easily check that

$$(2^{i+l} - 2^i, 2^j - 2^{i+l+1}) \cdot P_{i+l+2} = \left\{ \left(p, q_{i,j}(p)\right) \in \mathcal{X} \,\middle|\, \epsilon_m(p) \in \bar{\Delta}(i,l) \right\}$$

and then it follows from Lemma 4.4 that

$$Q(i,j,l) \xrightarrow{P_{i+l+2*}} Q(i,j,l) + (2^{i+l} - 2^i, 2^j - 2^{i+l+1}) \cdot P_{i+l+2} = Q(i,j,l+1).$$

□

Theorem 4.7. *The set* $\{P_0, \ldots, P_m\}$ *is a Gröbner basis of* J_n.

Proof. By Proposition 4.5, we have $J_n = (P_0, \ldots, P_m)$. As in the proof of Proposition 4.5, we have $S(P_i, P_j) = Q(i,j,0)$ and then it follows from Lemma 4.6 that, for $i < j$,

$$S(P_i, P_j) = Q(i,j,0) \xrightarrow{P_{i+2*}} Q(i,j,1) \xrightarrow{P_{i+3*}} \cdots \xrightarrow{P_{j*}} Q(i,j,j-i-1) \xrightarrow{P_{j+1*}} 0.$$

□

4.3. **Cup-length of $\mathbf{Im} p_n^*$.** In order to determine $\mathrm{cup}(\mathbf{Im} p_n^*)$, let us introduce new polynomials. For non-negative integers i, j, s with $s - 2^{m+1} + 2^{i+1} \equiv 0 \pmod 3$, we define a polynomial $\hat{P}(s, i, j)$ by

$$\hat{P}(s,i,j) = \left\{ \left(p, \tfrac{s-2p}{3}\right) \in \mathcal{X} \,\middle|\, \epsilon(p) \in \bar{\Delta}(i,j) \right\}.$$

Then we have

$$\text{(7)}\quad \mathrm{LT}(P_i) = P(2^i + n + 1, i) + P(2^i + n + 1, i+2) + \sum_{1 \le j \le m-i-2} \hat{P}(2^i + n + 1, i, j).$$

In order to investigate $\mathrm{cup}(\mathbf{Im} p_n^*)$, we shall calculate $\min\{p | (p, 0) \cdot \mathrm{LT}(P_i) \in J_n\}$ for each i as follows.

Lemma 4.8. *Let* $\alpha_i = \min\{\alpha | (\alpha, 0)\cdot P(t,i) \in J_n\}$ *for non-negative integers* i, t *with*

$$2^{i-2} + n + 1 \le t < 2^i + n + 1,\ t - 2(2^m - 2^i) \equiv 0 \pmod 3.$$

Then we have $\alpha_i = 2^m - 2^{i-1}$. *In particular,* α_i *is independent from* t *as above.*

Proof. Note that

$$(2^{i-1},0)\cdot P(t,i) = \left\{ \left(p, \tfrac{t+2^i-2p}{3}\right) \in X \,\middle|\, \epsilon_m(p) = (\mathbf{p}_{m-i}1, 0, \dots, 0) \in \{0,1\}^m,\ (\mathbf{p}_{m-i}, 1) \in \Delta_{m-i+1} \right\},$$

$$\left(0, \tfrac{t+2^{i-1}-n-1}{3}\right)\cdot P_{i-1} = \left\{ \left(p, \tfrac{t+2^i-2p}{3}\right) \in X \,\middle|\, \epsilon_m(p) = (\mathbf{p}_{m-i+1}, 0, \dots, 0) \in \{0,1\}^m,\ \mathbf{p}_{m-i+1} \in \Delta_{m-i+1} \right\}.$$

By Proposition 3.5, we have

$$(2^{i-1},0)\cdot P(t,i) + \left(0, \tfrac{t+2^{i-1}-n-1}{3}\right)\cdot P_{i-1} = P(t+2^i, i+1)$$

and hence

$$(2^{i-1},0)\cdot P(t,i) \xrightarrow{P_{i-1*}} P(t+2^i, i+1).$$

Then we obtain

$$\begin{array}{rcl}
(2^{i-1},0)\cdot P(t,i) & \xrightarrow{P_{i-1*}} & P(t+2^i, i+1), \\
(2^{i},0)\cdot P(t+2^i,i+1) & \xrightarrow{P_{i*}} & P(t+2^i+2^{i+1}, i+2), \\
& \vdots & \\
(2^{m-1},0)\cdot P(t+2^i+\cdots+2^{m-1},m) & \xrightarrow{P_{m-1*}} & 0
\end{array}$$

and this completes the proof of Lemma 4.8. □

Lemma 4.9. *Let* α_i *be as in Lemma 4.8. Then we have* $(\alpha_i, 0)\cdot \hat{P}(s,i,j) \in J_n$.

Proof. Quite similarly to the proof of Lemma 4.8, one has

$$(2^{j+i} + 2^i, 0)\cdot \hat{P}(s,i,j) + \left(0, \tfrac{s+2^{i+1}+1}{3}\right)\cdot P_{j+i+1} = P(s + 2^{j+i+1} + 2^{i+1}, j+i+3).$$

By Lemma 4.8, we have $2^{j+i} + 2^i + \alpha_{j+i+3} < \alpha_i$ and then Lemma 4.9 is accomplished. □

It follows from Lemma 4.8 and Lemma 4.9 that:

Proposition 4.10. *Let* α_i *be as in Lemma 4.8. Then we have* $\alpha_{i+1} = \min\{\alpha | (\alpha, 0)\cdot \mathrm{LT}(P_i) \in J_n\}$.

Corollary 4.11. *Let* χ_1 *be a fixed integer such that* $2^{m+1} - 2^{i+1} - 2^{i+2} \le \chi_1 < 2^{m+1} - 2^i - 2^{i+1}$ *and let* $\chi_2 = \max\{z | (\chi_1, z) \notin J_n\}$. *Then we have* $\chi_2 = 2^i - 1$.

Proof. Let $(p_i, q_i) = \mathrm{LT}(P_i)$. Then, by (6) and Lemma 4.8, we have

$$\cdots > p_{i-1} + \alpha_{i+1} > p_i + \alpha_{i+2} > p_{i+1} + \alpha_{i+3} \cdots,$$

$$\cdots < q_{i-1} < q_i < q_{i+1} < \cdots.$$

Hence, by Theorem 4.2 and Theorem 4.7, we have established that, for $p_{i+1} + \alpha_{i+3} = 2^{m+1} - 2^{i+1} - 2^{i+2} \le \chi_1 < 2^{m+1} - 2^i - 2^{i+1} = p_i + \alpha_{i+2}$, one has $\chi_2 = q_{i+1} - 1 = 2^{i+1} - 2$. □

From Corollary 4.11, $\chi_1+\chi_2$ takes the maximum when $(\chi_1,\chi_2)=(n,0)$ and it is n, of course, it is equal to $\mathrm{cup}(\mathbf{Im}p_n^*)$. Therefore we have obtained:

Corollary 4.12. $\mathrm{cup}(\mathbf{Im}p_n^*)=n$. *In particular,* $\bar{w}_2^n\neq 0$.

5. Cup-length of $\widetilde{G}_{n,3}$ and its applications

In this section, we determine $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$ and give its applications to immersion of $\widetilde{G}_{n,3}$ into a Euclidean space.

Proof of Theorem A. By Corollary 4.12, one has $\bar{w}_2^n\neq 0$. Then, by Poincaré duality, there exists $x\in H^n(\widetilde{G}_{n,3};\mathbb{Z}/2)$ such that $\bar{w}_2^n x\neq 0$ and hence we have $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})\geq n+1$.

Note that the canonical map $\widetilde{G}_{n,3}\to BSO(3)$ is an n-equivalence. Then it follows that $H^*(\widetilde{G}_{n,3};\mathbb{Z}/2)\cong\mathbf{Im}p_n^*$ in dimensions less than n. Now suppose that there exist $x_1,\dots,x_{n+2}\in\widetilde{H}^*(\widetilde{G}_{n,3};\mathbb{Z}/2)$ such that $x_1\cdots x_{n+2}\neq 0$. By a dimensional reason, one has $|x_i|<n$ for each i and then this contradicts to Corollary 4.12. Hence we have obtained Theorem A. □

Proof of Corollary. From Theorem A and the inequality $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})\leq\mathrm{cat}(\widetilde{G}_{n,3})$, it follows that $n+1\leq\mathrm{cat}(\widetilde{G}_{n,3})$.

Note that $\bar{w}_2\in H^2(\widetilde{G}_{n,3};\mathbb{Z}/2)$ is the fundamental class in the sense of James [Jam78]. By Corollary 4.12, we have $\bar{w}_2^{n+1}=0$ and then it follows from Proposition 5.3 in [Jam78] that $\mathrm{cat}(\widetilde{G}_{n,3})<\frac{3}{2}n$. Hence we have established Corollary. □

Let us consider the immersion of $\widetilde{G}_{n,3}$ into a Euclidean space as applications of Theorem A. Of course, as mentioned in section 1, we know, by the result of Whitney [Whi44], that $\widetilde{G}_{n,3}$ immerses into $\mathbb{R}^{6n-1}$. We shall give a slightly better estimation.

We denote the canonical vector bundle over $\widetilde{G}_{n,3}$ by γ and a stable normal bundle of $\widetilde{G}_{n,3}$ by ν. We abbreviate the classifying map $\widetilde{G}_{n,3}\to BSO(\infty)$ of ν by the same symbol ν. It is well-known that $T\widetilde{G}_{n,3}=\gamma\otimes\gamma^\perp$, then we have

$$\begin{aligned} T\widetilde{G}_{n,3}\oplus\gamma\otimes\gamma &= \gamma\otimes\gamma^\perp\oplus\gamma\otimes\gamma \\ &= \gamma\otimes(\gamma^\perp\oplus\gamma) \\ &= (n+3)\gamma. \end{aligned} \tag{8}$$

By Corollary 4.12, we have $(1+\bar{w}_2+\bar{w}_3)^{n+4}=1$. Using the formula for the Stiefel-Whitney class of a tensor product shows that $w(\gamma\otimes\gamma)=1+\bar{w}_2^2+\bar{w}_3^3$. Since $\nu\oplus T\widetilde{G}_{n,3}$ is trivial, we have

$$
\begin{aligned}
w(\nu) &= \frac{w(\gamma \otimes \gamma)}{w((n+3)\gamma)} \\
&= \frac{1 + \bar{w}_2^2 + \bar{w}_3^2}{(1 + \bar{w}_2 + \bar{w}_3)^{n+3}} \\
&= (1 + \bar{w}_2^2 + \bar{w}_3^2)(1 + \bar{w}_2 + \bar{w}_3) \\
&= 1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_2^2 + \bar{w}_2^3 + \bar{w}_3^2 + \bar{w}_2^2\bar{w}_3 + \bar{w}_2\bar{w}_3^2 + \bar{w}_3^3. \qquad (9)
\end{aligned}
$$

Then it immediately follows that $\widetilde{G}_{n,3}$ does not immerse into $\mathbb{R}^{3n+8}$ for $n = 2^{m+1} - 4\ m \geq 3$ and $\widetilde{G}_{4,3}$ does not immerse into $\mathbb{R}^{17}$.

Now let us consider the modified Postnikov tower of the fibration

$$BSO(3n-3) \to BSO(\infty)$$

following Gitler and Mahowald [GM66]. The A_2-free resolution of $H^*(SO(\infty)/SO(3n-3))$ in dimensions less than or equal to $3n$ is given as follows, where A_2 denotes the mod 2 Steenrod algebra.

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} H^*(SO(\infty)/SO(3n-3)) \to 0,$$

$$C_1 = \langle x_{3n-3}, x_{3n-1} \rangle,\ C_2 = \langle y_{3n-1} \rangle,$$
$$d_1(x_{3n-3}) = e_{3n-3},\ d_1(x_{3n-1}) = e_{3n-1},\ d_2(y_{3n-1}) = Sq^2 x_{3n-3},\ |x_i| = i,$$

where $\langle x \rangle$ and e_i denote the free A_2-module generated by x and a generator of

$$H^i(SO(\infty)/SO(3n-3)) \cong \mathbb{Z}/2$$

for $i = 3n-3, 3n-1$ respectively. Then the modified Postnikov tower of $BSO(3n-3) \to BSO(\infty)$ in dimensions less than or equal to $3n$ is given as:

$$
\begin{tikzcd}
BSO(3n-3) \arrow[d] & \\
E \arrow[r, "k_2"] \arrow[d] & K(\mathbb{Z}/2, 3n-1) \\
BSO(\infty) \arrow[r, "w_{3n-2} \times w_{3n}"] & K(\mathbb{Z}/2, 3n-2) \times K(\mathbb{Z}/2, 3n)
\end{tikzcd}
$$

It follows from (9) that $w_{3n-2}(\nu) = w_{3n}(\nu) = 0$ and then $\nu\colon \widetilde{G}_{n,3} \to BSO(\infty)$ lifts to $\tilde{\nu}\colon \widetilde{G}_{n,3} \to E$. By Poincaré duality, one has $H^{3n-1}(\widetilde{G}_{n,3}; \mathbb{Z}/2) = 0$. Then $\tilde{\nu}\colon \widetilde{G}_{n,3} \to E$ lifts to $\bar{\nu}\colon \widetilde{G}_{n,3} \to BSO(3n-3)$ and hence we can see from the result of Hirsch [Hir59] that $\widetilde{G}_{n,3}$ immerses into $\mathbb{R}^{6n-3}$. Then we have one obtains Theorem B.

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