SPECTRAL DENSITY FUNCTIONS OF GENERAL MODULES OVER FINITE VON NEUMANN ALGEBRAS AND THEIR APPLICATIONS

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ABSTRACT. We introduce a definition of weakly dilatational equivalence for density functions and study relations between density functions and their Laplace-Stieltjes transforms. Also we define spectral density functions of general modules over finite von Neumann algebras up to weakly dilatational equivalence and their Novikov-Shubin type invariants. By using these, we give some applications to random walks on discrete groups.

1. INTRODUCTION

In this paper we define spectral density functions of general modules over finite von Neumann algebras. Why do we define spectral density functions of general modules over finite von Neumann algebras? We have three motivations at least.

Novikov-Shubin invariants of finitely presented modules over finite von Neumann algebras are naturally defined by using spectral density functions of the modules. On the other hand Novikov-Shubin invariants (or capacities) of general modules over finite von Neumann algebras are defined in [10] by complicated arguments. Our first motivation is giving a conceptual definition of Novikov-Shubin invariants (or capacities) of general modules over finite von Neumann algebras by using spectral density functions of the modules.

For projective modules (more generally locally projective modules) over finite von Neumann algebras, taking their L^2-Betti numbers is faithful ([9, Chapter 6]). On the other hand for locally measurable modules over finite von Neumann algebras, taking their Novikov-Shubin invariants (or capacities) is not faithful without using a formal symbol ∞^+ different from ∞ (or 0^- different from 0). Our second motivation is giving faithful L^2-invariants for locally measurable modules.

Our final motivation is dealing with random walks on general discrete groups. In the applications to random walks, it is important to study relations between density functions, their Laplace-Stieltjes transforms and so on.

We remark that in [5] and so on they mainly dealt with comparing density functions with polynomial functions and Novikov-Shubin invariants. On the other hand we deal with more general functions and Novikov-Shubin type invariants (cf. [13]).

Also we introduce a definition of weakly dilatational equivalence for density functions, which is easier to deal with than dilatational equivalence.

Here we fix some conventions in this paper. Groups or discrete groups mean countable discrete groups. $\mathcal{A}$ is a finite von Neumann algebra and $\mathcal{A}$-modules mean right $\mathcal{A}$-modules. We call an $\mathcal{A}$-module M is locally something if any finitely generated submodule of M is something. For example M is locally non-projective if any finitely generated submodule of M is not projective.

Here we give explanation about each section.

In section 2 we introduce a definition of weakly dilatational equivalence for density functions, for their Laplace-Stieltjes transforms and for their discrete Laplace-Stieltjes transforms (Definition 2.2, Definition 2.4 and Definition 2.7). We remark that their Novikov-Shubin type invariants like Novikov-Shubin invariants and secondary Novikov-Shubin invariants are invariant under their weakly dilatational equivalence (Definition 2.1, Proposition 2.3 and Proposition 2.5). Also we

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study relations among (weakly) dilatational equivalence classes of density functions , (weakly) dilatational equivalence classes of their Laplace-Stieltjes transforms and (weakly) dilatational equivalence classes of their discrete Laplace-Stieltjes transforms. Indeed we give the following (more generally Theorem 2.18). Also see Theorem 4.7, too.

Theorem 1.1. Let $\mathcal{F}$ be a set of all density functions such that their Laplace-Stieltjes transforms are valued in $[0, \infty)$ and Θ be a set of all Laplace-Stieltjes transforms of density functions in $\mathcal{F}$. Then (weakly) dilatational equivalence classes of $\mathcal{F}$ and (weakly) dilatational equivalence classes of Θ are one-to-one correspondent. Moreover their Novikov-Shubin type invariants coincide.

Also we give a positive solution to a part of a technical question ([4, (1.15)]).

From section 3 we mainly deal with weakly dilatational equivalence because it is easier to deal with than dilatational equivalence. However surely we can deal with dilatational equivalence, too.

In section 3 we gather some examples of density functions from [9, Chapter 2] and so on, for example, we recall the definition of spectral density functions of finitely presented locally non-projective modules. Also we give a positive solution to a part of a technical question ([9, Remark 3.181]).

In section 4 we generalize the definition of density functions by using directed families of density functions. More generally we introduce double directed families of density functions. Also we define their weakly dilatational equivalence. Then we have the following (Lemma 4.3).

Lemma 1.2. A density function is regarded as a directed family of density functions and also a directed family of density function is regarded as a double directed family of density functions. Then their weakly dilatational equivalence are compatible.

In this context we have a type of Theorem 1.1, too (Theorem 4.7). Also we can define their Novikov-Shubin type invariants (or Novikov-Shubin type capacities) which are invariant under weakly dilatational equivalence.

In section 5 we gather some properties of modules over finite von Neumann algebras from [10], [17] and so on, for example, we recall the definition of cofinal measurable modules (in this paper we call them locally measurable modules).

In section 6 we define spectral density functions of general modules over finite von Neumann algebras by using tools in section 4 and section 5. We have the following (Theorem 6.6 and Definition 6.7).

Theorem 1.3. Spectral density functions of general modules over finite von Neumann algebras are well-defined.

Also we can define Novikov-Shubin type invariants (Novikov-Shubin type capacities) of general modules over finite von Neumann algebras. In particular our Novikov-Shubin capacities coincide capacities in [10] (Remark 6.9). Also we remark that for locally measurable modules taking their spectral density functions is faithful (Remark 6.10).

In section 7 we give some applications (Corollary 7.4, Corollary 7.5, Corollary 7.6, Corollary 7.8, Theorem 7.9, Theorem 7.10 and Theorem 7.11) to random walks on groups. We note that we can deal with not only finitely generated groups but also general discrete groups. In the case of finitely generated groups, Theorems are known in [14], however our some proofs are different from theirs and the author thinks our proofs are more conceptual.

In section 8 we refine a result about Novikov-Shubin invariants of groups in [16]. In particular we use a part of it in section 7.

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2. Density functions and their Laplace-Stieltjes transforms

First of all we prepare for and introduce some notations (cf. [9] for L^2-Betti numbers and Novikov-Shubin invariants and [13] for secondary Novikov-Shubin invariants).

Definition 2.1. Let $F : [0,\infty) \to [0,\infty)$ be a monotone non-decreasing function. Then we call $b^{(2)}(F) := F(0)$ the L^2-Betti number of F and we write $F^{\perp}(\lambda) = F(\lambda) - F(0)$. Also we write $F(\infty) := \lim_{\lambda\to\infty} F(\lambda)$. The Novikov-Shubin invariant of F is $\alpha(F) := \infty^+$ provided that there exists $\lambda > 0$ such that $F^{\perp}(\lambda) = 0$, and otherwise

$$\alpha(F) := \liminf_{\lambda\to 0+} \frac{\ln(F^{\perp}(\lambda))}{\ln(\lambda)}.$$

Also the secondary Novikov-Shubin invariant of F is $\beta(F) := \infty^+$ provided that there exists $\lambda > 0$ such that $F^{\perp}(\lambda) = 0$, and otherwise

$$\beta(F) := \liminf_{\lambda\to 0+} \frac{-\ln(-\ln(F^{\perp}(\lambda)))}{\ln(\lambda)}.$$

Let $\theta : (0,\infty) \to [0,\infty)$ be a monotone non-increasing function. Then we call $b^{(2)}(\theta) := \theta(\infty) := \lim_{t\to\infty} \theta(t)$ the L^2-Betti number of θ and we write $\theta^{\perp}(t) = \theta(t) - \theta(\infty)$. Also we write $\theta(0) := \lim_{t\to 0+} \theta(t)$. The Novikov-Shubin invariant of θ is $\alpha(\theta) := \infty^+$ provided that there exist $A, B \geq 1$ and $K > 0$ such that $\theta^{\perp}(t) \leq B\exp(-t/A)$ holds for all $t \in [K,\infty)$, and otherwise

$$\alpha(\theta) := \liminf_{t\to\infty} \frac{-\ln(\theta^{\perp}(t))}{\ln(t)}.$$

Also the secondary Novikov-Shubin invariant of θ is $\beta(\theta) := \infty^+$ provided that there exist $A, B \geq 1$ and $K > 0$ such that $\theta^{\perp}(t) \leq B\exp(-t/A)$ holds for all $t \in [K,\infty)$, and otherwise

$$\beta(\theta) := \liminf_{t\to\infty} \frac{\ln(-\ln(\theta^{\perp}(t)))}{\ln(t) - \ln(-\ln(\theta^{\perp}(t)))}.$$

Moreover we define that the lim sup version of the Novikov-Shubin invariant of F is $\bar{\alpha}(F) := \infty^+$ provided that there exists $\lambda > 0$ such that $F^{\perp}(\lambda) = 0$, and otherwise

$$\bar{\alpha}(F) := \limsup_{\lambda\to 0+} \frac{\ln(F^{\perp}(\lambda))}{\ln(\lambda)}.$$

Also $\bar{\beta}(F)$, $\bar{\alpha}(\theta)$ and $\bar{\beta}(\theta)$ are defined by the same way.

We recall the definition of dilatational equivalence and define weakly dilatational equivalence for monotone non-decreasing functions.

Definition 2.2. Let F_1 and F_2 be two monotone non-decreasing functions on $[0,\infty)$. We write $F_1 \preceq F_2$ if there exist $C \geq 1$ and $\epsilon > 0$ such that $F_1(\lambda) \leq F_2(C\lambda)$ holds for all $\lambda \in [0,\epsilon]$. We say that F_1 and F_2 are dilatationally equivalent (denoted by $F_1 \simeq F_2$) if $F_1 \preceq F_2$ and $F_2 \preceq F_1$. We write $F_1 \preceq_w F_2$ if there exists $D \geq 1$ such that $F_1^{\perp} \preceq DF_2^{\perp}$. We say that F_1 and F_2 are weakly dilatationally equivalent (denoted by $F_1 \simeq_w F_2$) if $F_1 \preceq_w F_2$ and $F_2 \preceq_w F_1$.

When F_1 and F_2 are bounded, $F_1 \preceq_w F_2$ if and only if there exist $C, D > 0$ such that $F_1^{\perp}(\lambda) \leq DF_2^{\perp}(C\lambda)$ holds for all $\lambda \in [0,\infty)$. The L^2-Betti number of F is invariant under dilatational equivalence. The dilatational equivalence between monotone non-decreasing functions induces certainly weakly dilatational equivalence. We note that weakly dilatational equivalence classes of monotone non-decreasing functions have no information about their L^2-Betti numbers.

The next proposition shows that Novikov-Shubin invariants and so on for monotone non-decreasing functions are invariant under weakly dilatational equivalence.

Proposition 2.3. Let $F : [0,\infty) \to [0,\infty)$ be a monotone non-decreasing function. When $\alpha(F)$, $\bar{\alpha}(F)$, $\beta(F)$ or $\bar{\beta}(F)$ is not ∞^+, then we have

$$\alpha(F) = \sup\left\{a | F^{\perp}(\lambda) \preceq_w \lambda^a\right\} = \sup\left\{a | \lim_{\lambda\to 0}(\lambda^{-a}F^{\perp}(\lambda)) = 0\right\}$$

$$\bar{\alpha}(F) = \inf\left\{a | \lambda^a \preceq_w F^{\perp}(\lambda)\right\} = \inf\left\{a | \lim_{\lambda\to 0}(\lambda^{-a}F^{\perp}(\lambda)) = \infty\right\}$$

$$\beta(F) = \sup\left\{b|F^{\perp}(\lambda) \preceq_w \exp(-\lambda^{-b})\right\} = \sup\left\{b|\lim_{\lambda\to 0}(\exp(\lambda^{-b})F^{\perp}(\lambda)) = 0\right\}$$

$$\bar{\beta}(F) = \inf\left\{b|\exp(-\lambda^{-b}) \preceq_w F^{\perp}(\lambda)\right\} = \inf\left\{b|\lim_{\lambda\to 0}(\exp(\lambda^{-b})F^{\perp}(\lambda)) = \infty\right\}$$

In particular these are invariant under weakly dilatational equivalence.

Proof. Let put $\alpha' = \sup\left\{a|F^{\perp}(\lambda) \preceq_w \lambda^a\right\}$ and $\alpha'' = \sup\left\{a|\lim_{\lambda\to 0}(\lambda^{-a}F^{\perp}(\lambda)) = 0\right\}$. Then since $\left\{a|F^{\perp}(\lambda) \preceq_w \lambda^a\right\} \supset \left\{a|\lim_{\lambda\to 0}(\lambda^{-a}F^{\perp}(\lambda)) = 0\right\}$ is clear, we get $\alpha' \geq \alpha''$. Also if we have $F^{\perp}(\lambda) \preceq_w \lambda^{\alpha'-\epsilon}$ for $\epsilon > 0$, then we observe $\lim_{\lambda\to 0}(\lambda^{-(\alpha'-2\epsilon)}F^{\perp}(\lambda)) = 0$. Hence we get $\alpha' \leq \alpha''$. Surely $\alpha(F) = \alpha'$ is proved by using the definition of '$\liminf$' and 'sup', too. Moreover the claims about $\bar{\alpha}(F)$, $\beta(F)$ and $\bar{\beta}(F)$ are shown by the same way. □

Also we can define other Novikov-Shubin type invariants which are invariant under weakly dilatational equivalence, but we do not deal in this paper (see Section 4, too).

Next we deal with monotone non-increasing functions.

Definition 2.4. Let θ_1 and θ_2 be two monotone non-increasing functions on $(0,\infty)$. We write $\theta_1 \preceq \theta_2$ if there exist $A, B, C \geq 1$ and $K > 0$ such that $\theta_1(t) \leq \theta_2(t/C) + B\exp(-t/A)$ holds for all $t \in [K,\infty)$. We say that θ_1 and θ_2 are dilatationally equivalent (denoted by $\theta_1 \simeq \theta_2$) if $\theta_1 \preceq \theta_2$ and $\theta_2 \preceq \theta_1$. We write $\theta_1 \preceq_w \theta_2$ if there exists $D \geq 1$ such that $\theta_1^{\perp} \preceq D\theta_2^{\perp}$. We say that θ_1 and θ_2 are weakly dilatationally equivalent (denoted by $\theta_1 \simeq_w \theta_2$) if $\theta_1 \preceq_w \theta_2$ and $\theta_2 \preceq_w \theta_1$. When a function is defined only on the discrete points, we extend it to the positive real axis by linear interpolation. We will use the same notation for the original function and its extension.

We note that our definition of dilatationally equivalent about monotone non-increasing functions are a little different from [5]. In our definition we add an error term $B\exp(-t/A)$. When θ_1 and θ_2 are bounded, $\theta_1 \preceq_w \theta_2$ if and only if there exist $A, B, C, D > 0$ such that $\theta_1^{\perp}(t) \leq D\theta_2^{\perp}(t/C) + A\exp(-t/B)$ holds for all $t \in [0,\infty)$. The L^2-Betti number of θ is invariant under dilatational equivalence. The dilatational equivalence between monotone non-increasing functions induces certainly weakly dilatational equivalence.

The next proposition shows that Novikov-Shubin invariants and so on for monotone non-increasing functions are invariant under weakly dilatational equivalence.

Proposition 2.5. Let $\theta : (0,\infty) \to [0,\infty)$ be a monotone non-increasing function. When $\alpha(\theta)$, $\bar{\alpha}(\theta)$, $\beta(\theta)$ or $\bar{\beta}(\theta)$ is not $\infty+$, then we have

$$\alpha(\theta) = \sup\left\{a|\theta^{\perp}(t) \preceq_w t^{-a}\right\} = \sup\left\{a|\lim_{t\to\infty} t^a(\theta^{\perp}(t)) = 0\right\}$$

$$\bar{\alpha}(\theta) = \inf\left\{a|t^{-a} \preceq_w \theta^{\perp}(t)\right\} = \inf\left\{a|\lim_{t\to\infty}(t^a\theta^{\perp}(t)) = \infty\right\}$$

$$\beta(\theta) = \sup\left\{\frac{d}{1-d}|\theta^{\perp}(t) \preceq_w \exp(-n^d)\right\} = \sup\left\{\frac{d}{1-d}|\lim_{t\to\infty}\exp(n^d)(\theta^{\perp}(t)) = 0\right\}$$

$$\bar{\beta}(\theta) = \inf\left\{\frac{d}{1-d}|\exp(-n^d) \preceq_w \theta^{\perp}(t)\right\} = \inf\left\{\frac{d}{1-d}|\lim_{t\to\infty}(\exp(n^d)\theta^{\perp}(t)) = \infty\right\}$$

In particular these are invariant under weakly dilatational equivalence.

Proof. When we put $\alpha' = \sup\left\{a|\theta^{\perp}(t) \preceq_w t^{-a}\right\}$, $\theta^{\perp}(t) \preceq_w t^{-(\alpha'-\epsilon)}$ for any $\epsilon > 0$. Then there are $D \geq 1, K > 0$ such that for any $t \geq K$ $\theta^{\perp}(t) \leq Dt^{-(\alpha'-\epsilon)}$ by the following lemma. Hence the claim about $\alpha(\theta)$ is shown by the same way in the proof of Proposition 2.3. Moreover the claims about $\bar{\alpha}(\theta)$, $\beta(\theta)$ and $\bar{\beta}(\theta)$ are shown by the same way. □

Lemma 2.6. Let θ_i for $i = 1, 2, 3$ be three monotone non-increasing functions on $(0, \infty)$ such that $\theta_i(\infty) = 0$ and satisfy that there exist $C, D \geq 1, K > 0$ such that for any $t \geq K$

$$\theta_1(t) \leq D\theta_2(t/C) + \theta_3(t).$$

If there exist $A, B \geq 1, L > 0$ such that for any $t \geq L$

$$\theta_3(t) \leq B\theta_2(t/A)$$

or there exist $A > 1, L > 0$ such that for any $t \geq L$

$$\theta_3(t) \leq \frac{1}{A}\theta_1(t),$$

then there exist $C, D \geq 1, K > 0$ such that for any $t \geq K$

$$\theta_1(t) \leq D\theta_2(t/C).$$

Proof. If there exist $C, D \geq 1, K > 0$ such that for any $t \geq K$

$$\theta_1(t) \leq D\theta_2(t/C) + \theta_3(t)$$

and there exist $A, B \geq 1, L > 0$ such that for any $t \geq L$

$$\theta_3(t) \leq B\theta_2(t/A),$$

then we have for any $t \geq K + L$

$$\theta_1(t) \leq D\theta_2(t/C) + \theta_3(t) \leq D\theta_2(t/C) + B\theta_2(t/A) \leq (B + D)\theta_2(t/(A + C)).$$

If there exist $C, D \geq 1, K > 0$ such that for any $t \geq K$

$$\theta_1(t) \leq D\theta_2(t/C) + \theta_3(t)$$

and there exist $A > 1, L > 0$ such that for any $t \geq L$

$$\theta_3(t) \leq \frac{1}{A}\theta_1(t),$$

then we have for any $t \geq K + L$

$$D\theta_2(t/C) \geq \theta_1(t) - \theta_3(t) \geq \frac{A-1}{A}\theta_1(t).$$

□

From now on, we can study density functions and their Laplace-Stieltjes transforms.

Definition 2.7. A function $F : [0, \infty) \to [0, \infty)$ is called a density function if it is monotone non-decreasing and right-continuous. We define its Laplace-Stieltjes transform by

$$\theta_F(t) := \int_{[0,\infty)} \exp(-t\lambda) dF(\lambda).$$

Also we define a discrete Laplace-Stieltjes transform of F which is a bounded density function on $[0, 1]$, that is, $F(\lambda) = F(1) < \infty$ for any $\lambda \geq 1$ by

$$q_F(t) := \int_{[0,1]} (1 - \lambda)^t dF(\lambda).$$

We give easily some properties for Laplace-Stieltjes transforms (cf. [9, Lemma 3.139.]).

Lemma 2.8. Let F be a density function. Then,
(1) If F is bounded, then θ_F is bounded.
(2) θ_F is valued in $[0, \infty)$ if and only if for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $F(\lambda) \leq C_\epsilon \exp(\epsilon\lambda)$ for any $\lambda \geq 0$. Then we have

$$\theta_F(t) = t \int_{[0,\infty)} \exp(-t\lambda) F(\lambda) d\lambda.$$

(3) When θ_F is valued in $[0,\infty)$, then we have for any $\lambda \in [0,\infty)$ and $t>0$

$$\exp(-t\lambda)F(\lambda) \le \theta_F(t) \le F(\lambda) + \int_{(\lambda,\infty)} \exp(-t\mu)dF(\mu).$$

In particular the L^2-Betti number of F and the L^2-Betti number of θ_F coincide and if $F \not\equiv 0$, then there exist $A>0$ and $K>0$ such that $\theta_F(t) \ge \exp(-At)$ holds for all $t \in [K,\infty)$.
(4) If F is a bounded density function on $[0,1]$, then we have for any $\lambda \in [0,1]$ and $t>0$

$$(1-\lambda)^t F(\lambda) \le q_F(t) \le F(\lambda) + (1-\lambda)^t F(1).$$

In particular the L^2-Betti number of F and the L^2-Betti number of q_F coincide and if there exists $\lambda \in (0,1)$ such that $F(\lambda) \ne 0$, there exist $A>0$ and $K>0$ such that $q_F(t) \ge \exp(-At)$ holds for all $t \in [K,\infty)$.

Proof. (1) is clear. (2) is well known.

When we fix $\lambda \in [0,\infty)$, then we have for any $\mu \in [0,\infty)$

$$\exp(-\lambda)\chi_{[0,\lambda]}(\mu) \le \exp(-\mu) \le \chi_{[0,\lambda]}(\mu) + \exp(-\mu)\chi_{(\lambda,\infty)}(\mu).$$

Hence the first part of (3) is clear and the latter part of (3) is shown by using Lebesgue's Theorem.

When we fix $\lambda \in [0,1]$, then we have for any $\mu \in [0,1]$

$$(1-\lambda)\chi_{[0,\lambda]}(\mu) \le 1-\lambda \le \chi_{[0,\lambda]}(\mu) + (1-\lambda)\chi_{(\lambda,1]}(\mu).$$

Hence the first part of (4) is clear and the latter part of (4) is shown by using Lebesgue's Theorem. □

When we deal with weakly dilatational equivalence about $\theta_F \not\equiv 0$ and $q_F \not\equiv 0$, then we can use the equivalent condition of Lemma 2.6, that is,

Lemma 2.9. Let F_i for $i=1,2$ be two density functions and $\theta_i := \theta_{F_i} \not\equiv 0$ be valued in $[0,\infty)$. Then $\theta_1 \preceq_w \theta_2$ if and only if there exist $C, D \ge 1$ such that $\theta_1^\perp(t) \le D\theta_2^\perp(t/C)$. Also when F_i for $i=1,2$ are two bounded density functions on $[0,1]$ such that $q_i := q_{F_i} \not\equiv 0$, then $q_1 \preceq_w q_2$ if and only if there exist $C, D \ge 1$ such that $q_1^\perp(t) \le Dq_2^\perp(t/C)$.

Proof. We put $\theta_3(t) = B\exp(t/A)$ in lemma 2.6. Then if we have $\theta_i := \theta_{F_i} \not\equiv 0$ resp. $q_i := q_{F_i} \not\equiv 0$, the assumption in lemma 2.6 is satisfied by the previous lemma (3) and (4). □

The next lemma shows that Novikov-Shubin invariants and so on are well-behaved under discrete Laplace-Stieltjes transforms.

Lemma 2.10. If F is a bounded density function on $[0,1]$, then we have $\alpha(F) = \alpha(q_F)$, $\bar{\alpha}(F) = \bar{\alpha}(q_F)$, $\beta(F) = \beta(q_F)$ and $\bar{\beta}(F) = \bar{\beta}(q_F)$.

Proof. We can assume that F is a bounded density function on $[0,1]$ with $F(0)=0$, $q = q_F(\infty) = 0$ and $q \not\equiv 0$. Then we have for any $\lambda \in [0,1]$ and $t>0$

$$(1-\lambda)^t F(\lambda) \le q_F(t) \le F(\lambda) + (1-\lambda)^t F(1).$$

When we define $a := \alpha(F) - \epsilon$ for any $\epsilon > 0$, then there exist $D \ge 1$ such that $F(\lambda) \le D\lambda^a$ for any sufficiently large $t>0$. Hence we get $q_F(t) \le D\lambda^a + (1-\lambda)^t F(1)$ by using $q_F(t) \le F(\lambda) + (1-\lambda)^t F(1)$. When we define $\lambda^a = \frac{1}{t^{a-\epsilon}}$, then we have $q_F(t) \le D\frac{1}{t^{a-\epsilon}} + (1-\lambda)^t F(1)$. Now since we have for any sufficiently large $t>0$ $(1-\lambda)^t F(1) \le D\frac{1}{t^{a-\epsilon}}$, we get $q_F(t) \le 2D\frac{1}{t^{a-\epsilon}}$. Hence we have $\alpha(F) \le \alpha(q)$. We can show $\bar{\alpha}(F) \ge \bar{\alpha}(q_F)$, $\beta(F) \le \beta(q_F)$ and $\bar{\beta}(F) \ge \bar{\beta}(q_F)$ by the same way in proving $\alpha(F) \le \alpha(q)$.

We can show $\alpha(F) \ge \alpha(q)$ by using $(1-\lambda)^t F(\lambda) \le q(t)$, $q(t) \le D\frac{1}{t^a}$ where $a := \alpha(q) - \epsilon$ and defining $\lambda^a = \frac{1}{t^a}$. We can show $\bar{\alpha}(F) \le \bar{\alpha}(q_F)$, $\beta(F) \ge \beta(q_F)$ and $\bar{\beta}(F) \le \bar{\beta}(q_F)$ by the same way in proving $\alpha(F) \ge \alpha(q)$. □

We define some sets of functions.

Definition 2.11. We denote by $\mathcal{F}$ the set of all density functions such that the Laplace-Stieltjes transforms are valued in $[0,\infty)$ and by Θ the set of the Laplace-Stieltjes transforms of all $F \in \mathcal{F}$. Moreover we denote by $\bar{\mathcal{F}}$ the set of all bounded density functions and by $\bar{\Theta}$ the set of the Laplace-Stieltjes transforms of all $F \in \bar{\mathcal{F}}$.

Moreover we denote by $\bar{\mathcal{F}}_{[0,1]}$ the set of all bounded density functions on $[0,1]$, by $\bar{\Theta}_{[0,1]}$ the set of the Laplace-Stieltjes transforms of all $F \in \bar{\mathcal{F}}_{[0,1]}$ and by $\mathcal{Q}$ the set of the discrete Laplace-Stieltjes transforms of all $F \in \bar{\mathcal{F}}_{[0,1]}$.

We denote by $\mathcal{F}^{\perp}$ the set of all density functions such that L^2-Betti numbers vanish and by $\Theta^{\perp}$ the set of the Laplace-Stieltjes transforms of all $\mathcal{F}^{\perp}$.

By the same way, we define $\bar{\mathcal{F}}^{\perp}$, $\bar{\Theta}^{\perp}$, $\bar{\mathcal{F}}^{\perp}_{[0,1]}$, $\bar{\Theta}^{\perp}_{[0,1]}$, and $\mathcal{Q}^{\perp}$.

We have certainly

$$\begin{aligned} \bar{\mathcal{F}}_{[0,1]} &\subset \bar{\mathcal{F}} \subset \mathcal{F}, \\ \bar{\mathcal{F}}^{\perp}_{[0,1]} &\subset \bar{\mathcal{F}}^{\perp} \subset \mathcal{F}^{\perp}, \\ \bar{\Theta}_{[0,1]} &\subset \bar{\Theta} \subset \Theta, \\ \bar{\Theta}^{\perp}_{[0,1]} &\subset \bar{\Theta}^{\perp} \subset \Theta^{\perp}, \end{aligned}$$

and these inclusions preserve the dilatational equivalence and the weakly dilatational equivalence.

Here we prepare for a convenient notation.

Definition 2.12. Let F be in $\mathcal{F}$. When $\lambda_0 > 0$ is fixed, we define $\bar{F}(\lambda) := F(\lambda)$ for any $\lambda \leq \lambda_0$ and $\bar{F}(\lambda) := F(\lambda_0)$ for any $\lambda > \lambda_0$, and denote by $\bar{\theta}$ the Laplace-Stieltjes transform of $\bar{F}$. Then $\bar{F}$ is in $\bar{\mathcal{F}}$ and $\bar{\theta}$ is in $\bar{\Theta}$. In particular when $1 \geq \lambda_0 > 0$, $\bar{F}$ is in $\bar{\mathcal{F}}_{[0,1]}$, $\bar{\theta}$ is in $\bar{\Theta}_{[0,1]}$ and $\bar{q} := q_{\bar{F}}$ is in $\mathcal{Q}$.

We will give the meaning of the above notation.

Lemma 2.13. Let F be in $\mathcal{F}$. Then for any $\lambda_0 > 0$, θ and $\bar{\theta}$ are dilatationally equivalent. Also if $\bar{F}$ is in $\bar{\mathcal{F}}_{[0,1]}$, then $\bar{\theta}$ and $\bar{q}$ are dilatationally equivalent.

Proof. We may assume that $F \not\equiv 0$. When we fix $\lambda_0 > 0$ such that $F(\lambda_0) \neq 0$, then we have

$$\begin{aligned} \theta(t) &:= \bar{\theta}(t) + \int_{[\lambda_0,\infty)} \exp(-t\lambda) d(F(\lambda) - F(\lambda_0)) \\ &= \bar{\theta}(t) + \exp(-t\lambda_0) \int_{[\lambda_0,\infty)} \exp(-t(\lambda - \lambda_0)) d(F(\lambda) - F(\lambda_0)) \\ &= \bar{\theta}(t) + \exp(-t\lambda_0)\theta_H(t), \end{aligned}$$

where $H(\mu) := F(\lambda) - F(\lambda_0)$ for $\mu = \lambda - \lambda_0$. Hence we get $\theta \preceq \bar{\theta}$ since $\theta_H(t) \leq 1$ for any sufficiently small $t > 0$. Hence we have $\theta \simeq \bar{\theta}$ since $\theta \succeq \bar{\theta}$ is clear. The latter part is proved by the same way. □

The following shows that Novikov-Shubin invariants and so on are well-behaved under Laplace-Stieltjes transforms.

Lemma 2.14. Let F be in $\mathcal{F}$ and $\bar{F}$ be in $\bar{\mathcal{F}}_{[0,1]}$. Then $\alpha(F) = \alpha(\theta_F) = \alpha(q_{\bar{F}})$, $\bar{\alpha}(F) = \bar{\alpha}(\theta_F) = \bar{\alpha}(q_{\bar{F}})$, $\beta(F) = \beta(\theta_F) = \beta(q_{\bar{F}})$ and $\bar{\beta}(F) = \bar{\beta}(\theta_F) = \bar{\beta}(q_{\bar{F}})$.

Proof. It is clear by Lemma 2.10 and Lemma 2.13. □

This is a positive solution to a question in [5, (1.15)].

Now we want to show that seven sets of dilatational equivalence classes $\bar{\mathcal{F}}_{[0,1]}/\simeq$, $\bar{\mathcal{F}}/\simeq$, $\mathcal{F}/\simeq$, $\bar{\Theta}_{[0,1]}/\simeq$, $\bar{\Theta}/\simeq$, $\Theta/\simeq$ and $\mathcal{Q}/\simeq$ are one-to-one correspondent and that seven sets of weakly dilatational equivalence classes $\bar{\mathcal{F}}_{[0,1]}/\simeq_w$, $\bar{\mathcal{F}}/\simeq_w$, $\mathcal{F}/\simeq_w$, $\bar{\Theta}_{[0,1]}/\simeq_w$, $\bar{\Theta}/\simeq_w$, $\Theta/\simeq_w$ and $\mathcal{Q}/\simeq_w$ are one-to-one correspondent. We need a few lemmas.

Lemma 2.15. Let F_1 and F_2 be in $\bar{\mathcal{F}}_{[0,1]}$. Then we have $F_1 \simeq F_2$ only if $\theta_1 \simeq \theta_2$, and $F_1 \simeq_w F_2$ only if $\theta_1 \simeq_w \theta_2$.

Proof. When $F_1 \preceq F_2$, we can assume that there exist $C \geq 1$ such that $F_1(\lambda) \leq F_2(C\lambda)$ for any $\lambda \geq 0$. Then $\theta_1 \preceq \theta_2$ is clear. The latter is proved by the same way. □

Lemma 2.16. Let F_1 and F_2 be in $\bar{\mathcal{F}}_{[0,1]}$. Then $F_1 \simeq F_2$ if $\theta_1 \simeq \theta_2$, and $F_1 \simeq_w F_2$ if $\theta_1 \simeq_w \theta_2$.

Proof. When $\theta_1 \preceq \theta_2$, there exist $A, B, C \geq 1$ such that $\theta_1(t) \leq \theta_2(t/C) + A\exp(-t/B)$ for any $t \geq 0$ since F_1 and F_2 are in $\bar{\mathcal{F}}_{[0,1]}$. Then we have

$$\int_{[0,\infty)} \exp(-t\lambda)dF_1(\lambda) \leq \int_{[0,\infty)} \exp(-t\lambda/C)dF_2(\lambda) + \int_{[0,\infty)} \exp(-t\lambda)dH(\lambda)$$
$$= \int_{[0,\infty)} \exp(-t\lambda)d(F_2(C\lambda) + H(\lambda))$$

for any $t \geq 0$ where $H(\lambda) := 0$ for any $\lambda < 1/B$ and $H(\lambda) := A$ for any $\lambda \geq 1/B$. We would like to show $F_1(\lambda) \leq F_0(\lambda)$ where we define $F_0(\lambda) := F_2(C\lambda) + H(\lambda)$. When we put $1 - x := \lambda$ and $G_i(x) := F_i(\lambda)$ for $i = 0, 1$, then we have

$$\int_{[0,1]} (1-x)^t dG_1(x) \leq \int_{[0,1]} (1-x)^t dG_0(x)$$

for any $t \geq 0$. Hence we have $G_1(x) \leq G_0(x)$ for any x by Weierstrass's approximation theorem. Thus we get $F_1(\lambda) \leq F_0(\lambda)$, moreover we get $F_1 \preceq F_2$. The latter is proved by the same way. □

Lemma 2.17. Let F_1 and F_2 be in $\bar{\mathcal{F}}_{[0,1]}$. Then $F_1 \simeq F_2$ if $q_1 \simeq q_2$, and $F_1 \simeq_w F_2$ if $q_1 \simeq_w q_2$.

Proof. This is proved by the same way in the previous lemma. □

Hence we have

Theorem 2.18. Seven sets of dilatational equivalence classes $\bar{\mathcal{F}}_{[0,1]}/\simeq$, $\bar{\mathcal{F}}/\simeq$, $\mathcal{F}/\simeq$, $\bar{\Theta}_{[0,1]}/\simeq$, $\bar{\Theta}/\simeq$, $\Theta/\simeq$ and $\mathcal{Q}/\simeq$ are one-to-one correspondent. Moreover this correspondence preserve $b^{(2)}$, α, $\bar{\alpha}$, β and $\bar{\beta}$.

Also seven sets of weakly dilatational equivalence classes $\bar{\mathcal{F}}_{[0,1]}/\simeq_w$, $\bar{\mathcal{F}}/\simeq_w$, $\mathcal{F}/\simeq_w$, $\bar{\Theta}_{[0,1]}/\simeq_w$, $\bar{\Theta}/\simeq_w$, $\Theta/\simeq_w$ and $\mathcal{Q}/\simeq_w$ are one-to-one correspondent. Moreover this correspondence preserve α, $\bar{\alpha}$, β and $\bar{\beta}$.

3. Examples of density functions

In this section, we recall some examples of density functions (see [9]).

From now on we mainly deal with α and β, but $\bar{\alpha}$ and $\bar{\beta}$ can be dealt with by the same way.

A von Neumann algebra $\mathcal{A}$ is called finite if $\mathcal{A}$ has a finite trace, that is, a finite, normal and faithful trace $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$. We will recall two examples of finite von Neumann algebras, that is, the group von Neumann algebra and the von Neumann crossed product.

Let G be a discrete group. The Hilbert space with basis G is denoted by $l^2(G)$. Then we have the left and right natural actions of the group algebra $\mathbb{C}G$ on $l^2(G)$ extending the left and right natural actions of G by linear. The bounded operators on $l^2(G)$ which are equivalent with respect to the right natural action of $\mathbb{C}G$ on $l^2(G)$ form a von Neumann algebra $\mathcal{N}(G) = B(l^2(G))^G$, called the group von Neumann algebra. Equivalently, $\mathcal{N}(G)$ can be defined as the weak closure of left natural action of $\mathbb{C}G$ in $B(l^2(G))$. The group von Neumann algebra $\mathcal{N}(G)$ with its standard trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$ given by $\mathrm{tr}_{\mathcal{N}(G)}(T) = \langle T(e), e\rangle_{l^2(G)}$. In particular, the trace of an element in the left natural action of $\mathbb{C}G$ is the coefficient of the unit element.

Let X be a standard Borel space equipped with a probability Borel measure μ. Then $L^\infty(X) = L^\infty(X;\mu)$ is a finite von Neumann algebra with the trace $\mathrm{tr}_{L^\infty(X)}(f) = \int_X f d\mu$. Assume X is additionally equipped with a μ-preserving left G-action. Then there is the von Neumann crossed product $\mathcal{N}(X \rtimes G)$ which contains the algebraic crossed product $L^\infty(X) \rtimes_{alg} G$ as a weakly dense subalgebra, where $L^\infty(X) \rtimes_{alg} G$ is a vector space generated by $f_g g$ ($f_g \in L^\infty(X)$, $g \in G$) with convolution product. Further, $\mathcal{N}(X \rtimes G)$ has a finite trace whose restriction to $L^\infty(X) \rtimes_{alg} G$ is given by $\mathrm{tr}_{\mathcal{N}(X\rtimes G)}(\sum_g f_g g) = \int_X f_e d\mu$.

For an n-dimensional square matrix $T \in M_{n,n}(\mathcal{A})$, we define

$$\mathrm{tr}_{\mathcal{A}}\big((T_{ij})_{1\leq i,j\leq n}\big) := \sum_{i=1}^{n} \mathrm{tr}_{\mathcal{A}}(T_{ii}).$$

The spectral density function of a operator $T \in M_{m,n}(\mathcal{A})$ is defined as $F(T)(\lambda) = \mathrm{tr}_{\mathcal{A}}\chi_{[0,\lambda^2]}(T^*T)$ by using spectral calculus. Here $\chi_{[0,\lambda^2]}$ is the characteristic function of the interval $[0,\lambda^2]$. Also we can write $F(T)(\lambda) = \mathrm{tr}_{\mathcal{A}} E_{\lambda^2}^{T^*T}$, where $\{E_\mu^{T^*T}\}_{\mu\in[0,\infty)}$ is the spectral family of the positive operator T^*T. Also we can write $F(T)(\lambda) = \mathrm{tr}_{\mathcal{A}} E_\lambda^{|T|}$. Surely we define $\theta(T) := \theta_{F(T)}$. Its L^2-Betti number $b^{(2)}(T)$ is defined as $F(T)(0)$, its Novikov-Shubin invariant $\alpha(T) \in [0,\infty] \cup \infty^+$ is defined as $\alpha(F(T))$ and its secondary Novikov-Shubin invariant $\beta(T) \in [0,\infty] \cup \infty^+$ is defined as $\beta(F(T))$.

In particular when $K \geq \|T\|$ is fixed, then $F(\frac{T}{K})$ is a bounded density function on $[0,1]$, and $F(T) \simeq_w F(\frac{T}{K})$. Thus we have $\alpha(T) = \alpha(q_{F(\frac{T}{K})})$ and $\beta(T) = \beta(q_{F(\frac{T}{K})})$. Surely we have $\bar{\alpha}(T) = \bar{\alpha}(q_{F(\frac{T}{K})})$ and $\bar{\beta}(T) = \bar{\beta}(q_{F(\frac{T}{K})})$. These are positive solutions for a part of a question in [9, Remark 3.181.].

From now on we will mainly deal with density functions up to weakly dilatational equivalence. In particular we ignore L^2-Betti numbers.

Definition 3.1. Let C_* be a finitely generated projective right $\mathcal{A}$-chain complex and c_p be its p-th boundary map. Also its p-th Laplacian is defined by $\Delta_p := c_p^* c_p + c_{p+1} c_{p+1}^*$. Then define its p-th spectral density function $F_p(C_*)$ as $F(c_p)$ up to its L^2-Betti number, and its p-th Novikov-Shubin invariant is $\alpha_p(C_*) := \alpha(F_p(C_*))$ and its p-th secondary Novikov-Shubin invariant is defined by the same way. Also $F_p^\Delta(C_*)$ is defined as $F(\Delta_p)$ up to its L^2-Betti number, and $\alpha_p^\Delta(C_*)$ is defined as $\alpha(F_p^\Delta(C_*))$ and $\beta_p^\Delta(C_*)$ is defined by the same way. Also $\theta_p(C_*)$ is defined as $\theta_{F_p(C_*)}$ up to its L^2-Betti number and $\theta_p^\Delta(C_*)$ is defined by the same way.

We remark that weakly dilatational equivalence classes of $F_p(C_*)$ and $F_p^\Delta(C_*)$ are homotopy invariants of C_*. In particular their Novikov-Shubin invariants and secondary Novikov-Shubin invariants are homotopy invariants of C_*.

Definition 3.2. If X be a free right G-CW-complex of finite type, then $C_* := C_*(X) \otimes_{\mathbb{Z}G} \mathcal{N}(G)$ is a finitely generated free right $\mathcal{N}(G)$-chain complex, where $C_*(X)$ is a cellular chain complex. Hence we define $F_p(X)$ as $F_p(C_*)$ up to weakly dilatational equivalence, $\alpha_p(X)$ as $\alpha_p(C_*)$ and so on.

Definition 3.3. Let M be a finitely presented locally non-projective module over $\mathcal{A}$ and fix its projective resolution P_* such that P_0 and P_1 are finitely generated. Then $F(M)$ is defined as $F(P_1 \to P_0)$ up to weakly dilatational equivalence. Surely $\alpha(M)$, $\beta(M)$, $\theta(M)$ and so on are defined as $\alpha(F(M))$, $\beta(F(M))$, $\theta_F(M)$ and so on.

Remark 3.4. M is a locally non-projective module over $\mathcal{A}$ if and only if $\dim_{\mathcal{A}} M = 0$ by definition of $\dim_{\mathcal{A}}$. We refer [9, Chapter 6.] about $\dim_{\mathcal{A}}$.

Lemma 3.5. Let L and L' be finitely presented locally non-projective modules. If L' is a submodule or a quotient module of L, then $F(L) \succeq_w F(L')$.

We refer [10, Lemma 2.3.] about its proof.

Remark 3.6. If C_* is a finitely generated projective right $\mathcal{A}$-chain complex and c_p is its p-th boundary map, then $F(H_p(C_*)) \simeq_w F_{p+1}(C_*)$.

Definition 3.7. If G has a finitely generated projective resolution P_* over $\mathbb{C}G$-module $\mathbb{C}$, then we define C_* as $P_* \otimes_{\mathbb{C}G} \mathcal{N}(G)$, $F_p(G)$ as $F_p(C_*)$ up to weakly dilatational equivalence and $F_p^\Delta(G)$ as

$F_p^{\Delta}(C_*)$ up to weakly dilatational equivalence. Also $\alpha_p(G)$ and $\beta_p(G)$ is defined by the same way. Also $\theta_p^{\Delta}(G)$ and $\theta_p(G)$ are defined up to weakly dilatational equivalence.

Definition 3.8. Let N be a free proper G cocompact Riemannian manifold without boundary. Let d_p be the p-th exterior differential and define the p-th Laplacian Δ_p^a as $d_p^* d_p + d_{p-1} d_{p-1}^*$, where d_p^* is the adjoint of d_p. Then $F_p^a(N)(\lambda)$ is defined as $\mathrm{tr}_{\mathcal{N}(G)}\chi_{(0,\lambda]}(d_p)$ up to its L^2-Betti number and $F_p^{\Delta^a}(\lambda)$ as $\mathrm{tr}_{\mathcal{N}(G)}\chi_{(0,\lambda]}(\Delta_p^a)$, up to its L^2-Betti number. Also $\alpha_p^a(N)$ is defined as $\alpha(F_{p-1}^a(N))$ and $\beta_p^a(N)$ is defined by the same way. Also $\theta_p^a(N)$ and $\theta_p^{\Delta^a}(N)$ are defined by the same way.

Remark 3.9. In above definition N can be regarded as a G-CW-complex. Thus we can define $F_p(N)$, too. Then $F_p(N) \simeq_w F_{p-1}^a(N)$, $\alpha_p(N) = \alpha_p^a(N)$ and so on (See [1] and [9, Chapter 2]).

4. A generalization of density functions

We generalize density functions by two steps. First step is the following.

Definition 4.1. A directed family of density functions F_J is defined as a family of density functions $\{F_j\}_{j\in J}$ such that $F_j \succeq_w F_{j'}$ if $j \le j'$, where J is a directed set. In particular if $F_j \in \mathcal{F}$ for any $j \in J$, then we call F_J a directed family in $\mathcal{F}$. $\mathcal{F}'$ is defined as the set of all directed families in $\mathcal{F}$.

Let F_J and $F_{J'}$ be directed families of density functions. $F_J \preceq_w F_{J'}$ if for any $F_{j'}$ there exists F_j such that $F_j \preceq_w F_{j'}$. F_J and $F_{J'}$ are weakly dilatationally equivalent denoted by $F_J \simeq_w F_{J'}$ if $F_J \preceq_w F_{J'}$ and $F_{J'} \preceq_w F_J$, where J and J' are directed sets.

Second step is the following.

Definition 4.2. A double directed family of density functions F_{J_I} is defined as a family of directed families of density functions $\{F_{J_i}\}_{i\in I}$ such that $F_{J_i} \preceq_w F_{J_{i'}}$ if $i \le i'$, where I and J_i are directed sets. In particular if $F_{J_i} \in \mathcal{F}'$ for any $i \in I$, then we call F_{J_I} a directed family in $\mathcal{F}'$. $\mathcal{F}$" is defined as the set of all directed families in $\mathcal{F}'$.

Let F_{J_I} and $F_{J_{I'}}$ be double directed families of density functions. $F_{J_I} \preceq_w F_{J_{I'}}$ if for any F_{J_i} there exists $F_{J_{i'}}$ such that $F_{J_i} \preceq_w F_{J_{i'}}$. F_{J_I} and $F_{J_{I'}}$ are weakly dilatationally equivalent denoted by $F_{J_I} \simeq_w F_{J_{I'}}$ if $F_{J_I} \preceq_w F_{J_{I'}}$ and $F_{J_{I'}} \preceq_w F_{J_I}$, where I, J_i, I' and $J_{i'}$ are directed sets.

A density function can be naturally regarded as a directed family of density functions and also a directed family of density functions can be naturally regarded as a double directed family of density functions. Then each weakly dilatational equivalence is compatible. More precisely we have the following.

Lemma 4.3. Let F and F' be density functions. These are weakly dilatationally equivalent as directed families of density functions if and only if these are weakly dilatationally equivalent as density functions.

Let F_J and $F_{J'}$ be directed families of density functions. These are weakly dilatationally equivalent as double directed families of density functions if and only if these are weakly dilatationally equivalent as directed families of density functions.

We regard real numbers $\mathbb{R}$ as a directed set by using a usual $\le$. Then we consider a directed family of density functions $F_{\mathbb{R}}$. Now we define Novikov-Shubin type capacities or the essentially inverse of Novikov-Shubin type invariants.

Definition 4.4. A Novikov-Shubin type capacity of F_{J_I} based on $F_{\mathbb{R}}$, denoted by $\mathrm{cap}_{F_{\mathbb{R}}}(F_{J_I})$, is defined by the following. If $F_{J_I} \simeq_w 0$, then $\mathrm{cap}_{F_{\mathbb{R}}}(F_{J_I}) = 0^-$. Otherwise

$$\mathrm{cap}_{F_{\mathbb{R}}}(F_{J_I}) = \inf\left\{\frac{1}{r} \mid F_{J_I} \preceq_w F_r\right\}.$$

Here we used a formal symbol 0^-.

We remark that Novikov-Shubin type capacity of F_{J_I} based on $F_{\mathbb{R}}$ is invariant under weakly dilatational equivalence.

We give two concrete examples of Novikov-Shubin type capacities.

Example 4.5. If $F_{\mathbb{R}} = \{F_r(\lambda) = \lambda^r\}$, then A Novikov-Shubin type capacity of F_{J_I} based on $F_{\mathbb{R}}$ should be called its capacity because it is the inverse of its Novikov-Shubin invariant if F_{J_I} is a density function (cf. [10]).

Example 4.6. If $F_{\mathbb{R}} = \{F_r(\lambda) = \exp(-\lambda^{-r})\}$, then Novikov-Shubin type capacity of F_{J_I} based on $F_{\mathbb{R}}$ is the essentially inverse of its secondary Novikov-Shubin invariant.

Surely we can define directed families, double directed families, their weakly dilatational equivalence and their Novikov-Shubin type capacities in Θ and so on instead of $\mathcal{F}$. Also we define Θ' as the set of all directed families in Θ, Θ" as the set of all double directed families in Θ and so on. We can repeat the arguments in Section 2 about directed families and double directed families. Then we can prove that the latter of Theorem 2.18 is true in the context of directed families and double directed families, too. Indeed we have the following which generalize Theorem 2.18.

Theorem 4.7. Seven sets of weakly dilatational equivalence classes $\bar{\mathcal{F}}_{[0,1]}"/\simeq_w$, $\bar{\mathcal{F}}"/\simeq_w$, $\mathcal{F}"/\simeq_w$, $\bar{\Theta}_{[0,1]}"/\simeq_w$, $\bar{\Theta}"/\simeq_w$, $\Theta"/\simeq_w$ and $\mathcal{Q}"/\simeq_w$ are one-to-one correspondent. Moreover these correspondence preserve Novikov-Shubin type capacities.

Also we have $\mathcal{F}/\simeq_w \subset \mathcal{F}'/\simeq_w \subset \mathcal{F}"/\simeq_w$, $\Theta"/\simeq_w \subset \Theta"/\simeq_w \subset \Theta"/\simeq_w$ and so on. Moreover these injections preserve Novikov-Shubin type capacities.

$\mathcal{F}/\simeq_w \subset \mathcal{F}'/\simeq_w \subset \mathcal{F}"/\simeq_w$ follows Lemma 4.3. This theorem claims that, for example, (double) directed families of density functions can be dealt with like usual density functions.

Remark 4.8. Also we can use $\preceq$ instead of $\preceq_w$ to define directed families and double directed families. In this definition, in particular we have the information of L^2-Betti numbers. Surely the former of Theorem 2.18 is true in this context. However we do not use this definition in this paper.

5. Some properties of modules over von Neumann algebras

In this section we deal with a finite von Neumann algebra $\mathcal{A}$ and right modules over $\mathcal{A}$. We call a right module over $\mathcal{A}$ simply a module. This algebra is semihereditary and has a dimension $\dim_{\mathcal{A}}$ ([8], [9, Chapter 6]). In particular it is known that a finitely presented module is locally finitely presented because A is semihereditary and also a locally non-projective module M if and only if $\dim_{\mathcal{A}} M = 0$ by definition of $\dim_{\mathcal{A}}$ ([8], [9, Chapter 6]). We gather some results in [10]. See [17] about Theorem 5.6.

We recall the definition of measurable modules.

Definition 5.1. Let M be a module. M is a measurable module if there exists a finitely presented locally non-projective module L and a surjective homomorphism $L \to M$.

In particular a measurable module is a finitely generated locally non-projective module. Also any measurable module is locally measurable because finitely presented modules are locally finitely presented.

Lemma 5.2. Let M be a measurable module. Given a finitely generated projective module P and a surjective homomorphism $p : P \to M$, then there exists a finitely generated submodule $K \subset \ker p$ such that P/K is a finitely presented locally non-projective module.

Proof. When M is a measurable module and any finitely presented locally non-projective module L, any surjective homomorphism $q : L \to M$, any finitely generated projective module P and a surjective homomorphism $p : P \to M$ are given, then there exists a homomorphism $r : P \to L$ such that $p = qr$ since P is projective. Thus we have $L \supset r(P)$. Then $r(P)$ is a finitely presented locally non-projective module and $q : r(P) \to M$ is a surjective homomorphism. ☐

In [10] they define cofinal measurable modules, however we do no use the name. We use the name of locally measurable instead of cofinal measurable because we trivially have the following.

Lemma 5.3. Any cofinal measurable module is locally measurable. Here M is a cofinal measurable module if there exist a directed system of measurable submodules $\{M_i\}_{i\in I}$ such that $M = \bigcup_{i\in I} M_i$.

In particular a locally measurable module is a locally non-projective module.

The following is clear by Definition 5.1 and Lemma 5.3

Lemma 5.4. M_1 is measurable and there exist a surjective homomorphism $M_1 \to M_2$, then M_2 is measurable, too. M_1 is locally measurable and there exist a surjective homomorphism $M_1 \to M_2$, then M_2 is locally measurable, too.

Let M be a module. If M_1 and M_2 are measurable submodules of M, then $M_1 + M_2$ is measurable, too. Thus the set of all measurable submodules $I(M)$ is regarded as a directed set. Thus we have the following definition.

Definition 5.5. Let M be a module. tM is a directed union by using a directed system of all of measurable submodules $\{M_i\}_{i\in I(M)}$ and $pM := M/tM$

In particular tM is locally measurable and pM is locally non-measurable. We call tM the locally measurable part of M.

The following is proved in [17].

Theorem 5.6. Let $\mathcal{A} \subset \mathcal{B}$ be a pair of finite von Neumann algebras with compatible traces. Then two short exact sequences
$0 \to t(M \otimes_{\mathcal{A}} \mathcal{B}) \to M \otimes_{\mathcal{A}} \mathcal{B} \to p(M \otimes_{\mathcal{A}} \mathcal{B}) \to 0$,
$0 \to (tM) \otimes_{\mathcal{A}} \mathcal{B} \to M \otimes_{\mathcal{A}} \mathcal{B} \to (pM) \otimes_{\mathcal{A}} \mathcal{B} \to 0$
are isomorphic.

6. Spectral density functions of general modules

In section 3 spectral density functions of finitely presented locally non-projective modules are defined up to weakly dilatational equivalence. In this section we define spectral density functions of locally measurable modules as double directed families of density functions up to weakly dilatational equivalence. Surely we can define weakly dilatationally equivalence classes of spectral density functions of general modules as weakly dilatational equivalence classes of spectral density functions of their locally measurable parts. We note that these generalized spectral density functions are not functions.

Firstly we deal with measurable modules.

Definition 6.1. Let M be a measurable module and $P \stackrel{p}{\to} M$ be a surjective homomorphism from a finitely generated projective module. Then $J(P \stackrel{p}{\to} M)$ is defined as a set of all of $P \stackrel{r_j}{\to} L_j \stackrel{q_j}{\to} M$, where L_j is a finitely presented locally non-projective module and r_j and q_j are surjective homomorphisms such that $p = q_j r_j$, in particular $\mathrm{Ker}(p) \subset \mathrm{Ker}(r_j)$.

Lemma 6.2. $J(P \stackrel{p}{\to} M)$ is a directed set. Hence we have $\mathrm{Ker}(p) = \bigcup_{J(P \stackrel{p}{\to} M)} \mathrm{Ker}(r_j)$, in other words, $\lim_{J(P \stackrel{p}{\to} M)} L_j = M$ in Definition 6.1.

Proof. Any $j', j" \in J(P \stackrel{p}{\to} M)$, we can take $P \stackrel{r_j}{\to} L_j \stackrel{q_j}{\to} M$ such that $\mathrm{Ker} p \supset \mathrm{Ker}(r_j) := \mathrm{Ker}(r_{j'}) + \mathrm{Ker}(r_{j"})$. □

Let M be a measurable module. Then we fix a surjective homomorphism from a finitely generated projective module $P \stackrel{p}{\to} M$, moreover a finite presentation for any L_j, where $P \stackrel{r_j}{\to} L_j \stackrel{q_j}{\to} M \in J(P \stackrel{p}{\to} M)$. Then we define a family of density functions $F_{J(P \stackrel{p}{\to} M)}$ as the family of spectral density functions $F(L_j)$, where we use a fixed finite presentation for L_j. Then $F_{J(P \stackrel{p}{\to} M)}$ is a directed family of density functions because $J(P \stackrel{p}{\to} M)$ is a directed set and Lemma 3.5

Theorem 6.3. Let M be a measurable module. Then a directed family of density functions $F_{J(P \xrightarrow{p} M)}$ depend only on M up to weakly dilatational equivalence. Moreover if J is a directed subset of $J(P \xrightarrow{p} M)$ such that $\lim_J L_j = M$, then $F_{J(P \xrightarrow{p} M)} \simeq_w F_J$.

Proof. For L_j, $F(L_j)$ is defined up to weakly dilatational equivalence. If we have two surjective homomorphisms from finitely generated projective modules $P \xrightarrow{p} M$ and $P' \xrightarrow{p'} M$, then we have two directed families of density functions $F_{J(P \xrightarrow{p} M)}$ and $F_{J'(P' \xrightarrow{p'} M)}$. For any $L_{j'}$, there exists $L_j \subset L_{j'}$ since the proof of Lemma 5.2. Thus $F(L_j) \preceq_w F(L_{j'})$ by Lemma 3.5. The latter part follows because $\mathrm{Ker}(r_j)$ is finitely generated and $\lim_J L_j = M$, in other words, $\bigcup_J \mathrm{Ker}(r_j) = \mathrm{Ker}(p)$. □

Definition 6.4. Let M be a measurable module. Its spectral density function $F(M)$ is defined as a directed family of density functions $F_{J(P \xrightarrow{p} M)}$ up to weakly dilatational equivalence, where $P \xrightarrow{p} M$ is a surjective homomorphism from a finitely generate projective module.

Lemma 6.5. Let M and M' be measurable modules. If M' is a submodule or a quotient module of M, then $F(M) \succeq_w F(M')$.

Proof. We fix a surjective homomorphism from a finitely generate projective module $P \xrightarrow{p} M$. If $M' \subset M$, then we can restrict $P \xrightarrow{p} M$ to get a surjective homomorphism from a finitely generate projective module $P' \xrightarrow{p'} M'$. Then any $P \xrightarrow{r_j} L_j \xrightarrow{q_j} M \in J(P \xrightarrow{p} M)$, we get a $P' \xrightarrow{r'_j} L'_j \xrightarrow{q'_j} M' \in J(P' \xrightarrow{p'} M')$ such that $L'_j := r_j(P')$ and $r'_j|_{P'}$. Then the set of such elements $J' \subset J(P' \xrightarrow{p'} M')$ is a directed subset such that $M' = \lim_{J'} L_{j'}$. Now $F(M) \succeq_w F(M')$ follows Lemma 3.5 and Theorem 6.3.

The case of quotient follows Lemma 3.5. □

Next we deal with locally measurable modules. Let M be a locally measurable module. Then $I(M)$ is defined as a directed set of all of measurable submodules $M_i \subset M$. Then for any measurable submodule M_i, we fix a surjective homomorphism from a finitely generated projective module $P_i \xrightarrow{p_i} M_i$, moreover a finite presentation for any L_{j_i}, where $P_i \xrightarrow{r_{j_i}} L_{j_i} \xrightarrow{q_{j_i}} M_i \in J(P_i \xrightarrow{p_i} M_i)$. Then we can define a directed family of density functions $F_{J_i(P_i \xrightarrow{p_i} M_i)}$. Thus we have a double directed family of density functions $F_{J(P_I \xrightarrow{p_I} M_I)} := \left\{ F_{J_i(P_i \xrightarrow{p_i} M_i)} \right\}_{i \in I(M)}$. We have the following because for any measurable submodule $M_{i'}$ and $M_{i''}$, $M_i := M_{i'} + M_{i''}$ is a measurable submodule, $F(M_{i'}) \preceq_w F(M_i)$ and $F(M_{i''}) \preceq_w F(M_i)$ by Lemma 6.5.

Theorem 6.6. Let M be a locally measurable module. Then a double directed family of density functions $F_{J(P_I \xrightarrow{p_I} M_I)}$ depends only on M up to weakly dilatational equivalence. Moreover if I is a directed subset of $I(M)$ such that $\bigcup_I M_i = M$, then $F_{J(P_I \xrightarrow{p_I} M_I)} \simeq_w F_{J_I}$.

Now we can define spectral density functions of general modules.

Definition 6.7. Let M be a locally measurable module. Its spectral density function $F(M)$ is defined as a double directed family of density functions $F_{J(P_I \xrightarrow{p_I} M_I)}$ up to weakly dilatational equivalence, where $P_i \xrightarrow{p_i} M_i$ is a surjective homomorphism from a finitely generated projective module.

If M is a general module, then its spectral density function $F(M)$ is defined as $F(tM)$ up to weakly dilatational equivalence.

We have the following by Lemma 6.5

Lemma 6.8. Let M and M' be locally measurable modules. If M' is a submodule or a quotient module of M, then $F(M) \succeq_w F(M')$.

Remark 6.9. Here we can define Novikov-Shubin capacities of general modules by using Definition 4.4 and Example 4.5. Then we can confirm that this Novikov-Shubin capacities coincide with the capacities in [10]. Moreover we can define secondary Novikov-Shubin capacities and so on.

Remark 6.10. $F(M)$ measures the size of the locally measurable part of a module M. In particular taking the weakly dilatational equivalence class of the spectral density function of a locally measurable module M is faithful, in other words, $F(M) \simeq_w 0$ if and only if $M = 0$. On the other hand, in order to make capacities (in this paper we call Novikov-Shubin capacities) be faithful, we have to introduce a formal symbol 0^- different from 0.

Here we consider a pair of von Neumann algebras with compatible traces $\mathcal{A} \subset \mathcal{B}$. Then $\otimes_{\mathcal{A}}\mathcal{B}$ is faithful flat functor from a category of $\mathcal{A}$-modules to a category of $\mathcal{B}$-modules ([9, Theorem 6.29]). Moreover this functor preserves locally measurable modules by Theorem 5.6. We have the following.

Proposition 6.11. Let M be a module over $\mathcal{A}$. Then $F(M) \simeq_w F(M \otimes_{\mathcal{A}} \mathcal{B})$.

Proof. It is enough to deal with the case that M is a finitely presented locally non-projective module. However this case follows the proof of [9, Theorem 2.55 (7)]. □

We get the following from the proof of [15, Theorem 4.20].

Proposition 6.12. Let M be a module over $\mathcal{A}$ and p be a full projection in $\mathcal{A}$. Then $F(M) \simeq_w F(M \otimes_{\mathcal{A}} \mathcal{A}p)$.

Here p is a full projection if p is a projection and $\mathcal{A} = \mathcal{A}p \otimes_{p\mathcal{A}p} p\mathcal{A}$. Then a category of $\mathcal{A}$-modules and a category of $p\mathcal{A}p$-modules are Morita equivalent by $\otimes_{\mathcal{A}}\mathcal{A}p$ and $\otimes_{p\mathcal{A}p}p\mathcal{A}$. These functors preserve projective, finitely generated, finitely presented, exact sequences and so on. Thus locally non-projective, measurable and locally measurable are preserved.

We define spectral density functions of $\mathcal{A}$-chain complexes and groups.

Definition 6.13. Let C_* be a right $\mathcal{A}$-chain complex. Then its p-th spectral density function $F_p(C_*)$ is defined as $F(H_{p-1}(C_*))$ up to weakly dilatational equivalence.

Let G be a discrete group. Then its p-th spectral density function $F_p(G)$ is defined as $F(H_{p-1}(G, \mathcal{N}(G)))$ up to weakly dilatational equivalence.

We remark that $F_1(G) = F(\mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G))$ because $\mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G) = H_0(G, \mathcal{N}(G))$.

7. Some applications to Random walks on discrete groups

In this section results in [14] about symmetric simple random walks on finitely generated groups are proved by the different way from theirs or generalized on general groups.

Let G be a finitely generated group and S be a finite set of generators of G. The Cayley graph $C_S(G)$ of (G, S) is the following connected one-dimensional free right G-CW-complex. (The Cayley graph is usually a connected one-dimensional free 'left' G-CW-complex, but there exist no essential difference.)

Its 0-skeleton is G. For each element $s \in S$ we attach free equivalent G-cells $G \times [-1, 1]$ by the attaching map $G \times \{-1, 1\} \to G$ which sends $(g, -1)$ to g and $(g, 1)$ to sg. We can identify the first boundary map of $C_*(C_S(G)) \otimes_{\mathbb{Z}G} \mathcal{N}(G)$

$$c_1 : C_1(C_S(G)) \otimes_{\mathbb{Z}G} \mathcal{N}(G) \to C_0(C_S(G)) \otimes_{\mathbb{Z}G} \mathcal{N}(G)$$

with

$$\bigoplus_{s \in S} l_{s-e} : \bigoplus_{s \in S} \mathcal{N}(G) \to \mathcal{N}(G),$$

where l is the natural left action of $\mathbb{C}G$ on $\mathcal{N}(G)$ and e is the unit element of G.

Lemma 7.1. Let G be a finitely generated group and let X be a connected free right G-CW-complex of finite type. Then for any finite set S of generators of G, we have $F_1(X) \simeq_w F_1(C_S(G))$. In particular $F_1(G)$ is defined as $F_1(C_S(G))$ up to weakly dilatational equivalence.

Proof. This is clear since it is known $F_1^{\perp}(X)$ and $F_1^{\perp}(C_S(G))$ are dilatationally equivalent ([9, Lemma 2.45, Theorem 2.55. (1)]). □

Remark 7.2. Let G be a finitely generated group with a finite set S of generators. Then we have $F_1(C_S(G))(\lambda) = F_0^{\Delta}(C_S(G))(\lambda^2)$. In particular $F_0^{\Delta}(G)$ is defined as $F_0^{\Delta}(C_S(G))$ up to weakly dilatational equivalence.

We can assume that S is symmetric, that is, $s \in S$ implies $s^{-1} \in S$ and S does not contain the unit element of G. We will recall symmetric simple random walk on $C_S(G)$. The probability distribution is

$$p_S : G \to [0,1],\ g \mapsto \begin{cases} |S|^{-1} & \text{if } g \in S, \\ 0 & \text{if } g \notin S. \end{cases}$$

Thus the transition probability operator is

$$P_S = \sum_{s \in S} \frac{1}{|S|} l_s : l^2(G) \to l^2(G),$$

in particular, we can confirm

$$P_S = id - \frac{1}{2|S|}(\bigoplus_{s \in S} l_{s-e})^*(\bigoplus_{s \in S} l_{s-e}) = id - \frac{1}{2|S|}\Delta_0.$$

Then for $n \in \mathbb{Z}_{\geq 0}$

$$p_S(n) := tr_{\mathcal{N}(G)} P_S^n$$

is the probability of return after n steps for the random walk on the Cayley graph. It is clear that $p_S(n)$ on $n \in 2\mathbb{Z}_{\geq 0}$ is a non-increasing function. we define $\phi_S(n) := p_S(2n)$ and we extend it to the positive real axis by linear interpolation. We will use the same notation for the original function and its extension.

A proof for the following can be found in [14].

Lemma 7.3. Let G be a finitely generated group with a finite symmetric set S of generators. Then we have $\phi_S \simeq_w \theta_0^{\Delta}(C_S(G))$.

Proof. We denote $p := p_S$, $F(\lambda) := F_0^{\Delta}(C_S(G))(2|S|\lambda)$, $\phi := \phi_S$ and $\theta := \theta_F$. Since $p(2n+1) = \int_{[0,2]}(1-\lambda)^{2n+1}dF(\lambda) \geq 0$, we have

$$0 \leq -\int_{[1,2]}(1-\lambda)^{2n+1}dF(\lambda) \leq \int_{[0,1]}(1-\lambda)^{2n+1}dF(\lambda) \leq$$
$$\leq \int_{[0,1]}(1-\lambda)^{2n}dF(\lambda) \leq \int_{[0,1]}\exp(-2n\lambda)dF(\lambda) \leq \theta(2n).$$

Hence we have

$$\phi(n+1) = p(2n+2) = \int_{[0,2]}(1-\lambda)^{2n+2}dF(\lambda)$$
$$= \int_{[0,1]}(1-\lambda)^{2n+2}dF(\lambda) + \int_{[1,2]}(1-\lambda)^{2n+2}dF(\lambda)$$
$$\leq \int_{[0,1]}(1-\lambda)^{2n+2}dF(\lambda) - \int_{[1,2]}(1-\lambda)^{2n+1}dF(\lambda) \leq \theta(2n+2) + \theta(2n) \leq 2\theta(2n).$$

Also we have

$$\theta(4n) = \int_{[0,2]}\exp(-4n\lambda)dF(\lambda) = \int_{[0,1/2]}\exp(-4n\lambda)dF(\lambda) + \int_{[1/2,2]}\exp(-4n\lambda)dF(\lambda)$$
$$\leq \int_{[0,1/2]}\exp(-4n\lambda)dF(\lambda) + \exp(-2n)\int_{[1/2,2]}dF(\lambda)$$
$$\leq \int_{[0,1/2]}\exp(-4n\lambda)dF(\lambda) + \exp(-2n)F(2) \leq \phi(n) + \exp(-2n)F(2).$$

□

We get the following by Theorem 2.18.

Corollary 7.4. In particular $\phi(G)$ is defined as ϕ_S up to weakly dilatational equivalence independent of S.

Compare the proof in [14].

By using the above corollaries, we can add the following (5), (6), (7) and so on for [9, Lemma 2.46].

Corollary 7.5. Let G be an infinite finitely generated group. The following are equivalent. (1) G is a virtually a-nilpotent group. (2) $\alpha_1(G) = a$, (3) G has polynomial growth of volume and the degree of the growth rate is a, (4) $\phi(G) \simeq_w t^{-a/2}$, (5) $F_1(G) \simeq_w \lambda^a$, (6) $F_0^{\Delta}(G) \simeq_w \lambda^{a/2}$, (7) $\theta_0^{\Delta}(G) \simeq_w t^{-a/2}$.

Here the group H is a-nilpotent if H is a nilpotent group with a lower central series $H = H_1 \supset H_2 \supset \cdots \supset H_s = \{e\}$ such that $a = \sum_{i=1}^{s-1} i d_i$, where d_i is the rank of H_i/H_{i+1}.

We note that we can use $\simeq$ instead of $\simeq_w$. See [13, Theorem 1.1] about secondary Novikov-Shubin invariants.

Also we get the following by Remark 3.9.

Corollary 7.6. Let G be a finitely generated group with a finite symmetric set S of generators and N be a free proper G cocompact Riemannian manifold without boundary. Then we have $\phi(G) \simeq_w \theta_0^{\Delta}(N)$.

Compare [14].

Lemma 7.7. Let G be a discrete group and $\{G_j\}_j \in J$ be a directed set of all of finitely generated subgroups of G. Then $F_1(G)$ is a directed family of density functions $\{F_1(G_j)\}_{j\in J}$ up to weakly dilatational equivalence.

Proof. Since Proposition 6.11 and $\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G_j) \otimes_{\mathcal{N}(G_j)} \mathcal{N}(G) = \mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G)$, $F(\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G_j)) \simeq_w F(\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G))$. Thus we will prove $F(\mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G)) \simeq_w \left\{F(\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G))\right\}_{j\in J}$. We have a natural surjective homomorphism $\mathcal{N}(G) \to \mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G)$. Moreover this factorize $\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G)$ for any $j \in J$. Thus $\mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G) = \lim_{j\in J} \mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G)$. Theorem 6.3 implies $F(\mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G)) \simeq_w \left\{F(\mathbb{C} \otimes_{\mathbb{C}G_j} \mathcal{N}(G))\right\}_{j\in J}$. ☐

In particular $F_0^{\Delta}(G)$ is defined as a directed family of density functions $\left\{F_0^{\Delta}(G_j)\right\}_{j\in J}$ up to weakly dilatational equivalence.

We get the following by Theorem 4.7.

Corollary 7.8. Let G be a discrete group and $\{G_j\}_j \in J$ be a directed set of all of finitely generated subgroups of G. Then $\phi(G)$ is defined as a directed family of Θ $\{\phi(G_j)\}_{j\in J}$ up to weakly dilatational equivalence.

Theorem 7.9. Let G and G' be discrete groups. If G' is a subgroup of G, then $\phi(G) \preceq_w \phi(G')$.

Proof. We have a natural surjective homomorphism $\mathbb{C} \otimes_{\mathbb{C}G'} \mathcal{N}(G) \to \mathbb{C} \otimes_{\mathbb{C}G} \mathcal{N}(G)$. Then we use Theorem 4.7 and Lemma 6.5. ☐

Theorem 7.10. Let G and G' be discrete groups. If G' is a quotient group of G, then $\phi(G) \preceq_w \phi(G')$.

Proof. We write this quotient map r. For any finitely generated subgroup $G'_{j'}$ of G', we take a finitely generated subgroup G_j of G such that $r(G_j) = G'_{j'}$. Moreover we take a finite symmetric set S_j of generators of G_j and write $S'_{j'} = r(S_j)$. Then $P^n_{S'_{j'}} = r(P^n_{S_j})$. Thus we get $p_{S_j}(n) \leq p_{S'_{j'}}(n)$. In particular $\phi(G_j) \preceq_w \phi(G'_{j'})$. Also we have $\phi(G) \preceq_w \phi(G_j)$. ☐

We get the following by Theorem 4.7 and Corollary 8.11.

Theorem 7.11. If G and G' are quasi-isometric groups, then $\phi(G) \simeq_w \phi(G')$.

See Definition 8.7 about the definition of quasi-isometric. If G and G' are finitely generated groups, then these three theorem are known in [14].

8. Spectral density functions of general discrete groups

In this section we give a stronger statement (Corollary 8.9 and Corollary 8.10) than [15, Theorem 4.24]. Hence this section depends on [15] and [16]. We recall the definition of weak orbit equivalence.

Definition 8.1. For $i = 1, 2$ standard actions $G_i \curvearrowright (X_i, \mu_i)$, that is, essentially free measure-preserving Borel actions of G_i on standard probability spaces (X_i, μ_i) are called weakly orbit equivalent, if there are Borel subsets $A_i \subset X_i$ with positive measure meeting almost every orbit and a Borel isomorphism $f : A_1 \to A_2$, which preserves the normalized measures on A_i and satisfies $f(G_1 x_1 \cap A_1) = G_2 f(x_1) \cap A_2$ for almost all $x_1 \in A_1$. The map f is called a weak orbit equivalence.

Remark 8.2. In the above definition two pairs of von Neumann algebras $L^\infty(A_1) \subset \chi_{A_1}\mathcal{N}(X_1 \rtimes G_1)\chi_{A_1}$ and $L^\infty(A_2) \subset \chi_{A_2}\mathcal{N}(X_2 \rtimes G_2)\chi_{A_2}$ are isomorphic preserving normalized traces.

We recall the definition of cocycles of a weak orbit equivalence.

Definition 8.3. Let $f : A_1 \to A_2$ be a weak orbit equivalence between the standard actions $G_i \curvearrowright X_i$ for $i = 1, 2$. The Borel mapping σ from the subset $\{(g_1, a_1); g_1 \in G_1, a_1 \in A_1 \cap g_1^{-1} A_1\} \subset G_1 \times A_1$ to G_2, determined up to null-sets by the condition $f(g_1 a_1) = \sigma(g_1, a_1) f(a_1)$ is called the cocycle of f. We say that σ is essentially bounded, if for each $g_1 \in G_1$ the restriction $\sigma|_{g_1 \times (A_1 \cap g_1^{-1} A_1)}$ is essentially bounded.

We recall the definition of bounded weak orbit equivalence.

Definition 8.4. The standard actions $G_i \curvearrowright X_i$ for $i = 1, 2$ are called boundedly weakly orbit equivalent if there is a weak orbit equivalence $f : A_1 \to A_2$ with the following additional properties:
(1) There are finite subsets $F_i \in G_i$, satisfying $X_i = F_i A_i$ up to null-sets.
(2) The cocycles of f and f^{-1} are essentially bounded.

Remark 8.5. In the above definition two pairs of algebras $L^\infty(A_1) \subset \chi_{A_1}(L^\infty(X_1) \rtimes_{alg} G_1)\chi_{A_1}$ and $L^\infty(A_2) \subset \chi_{A_2}(L^\infty(X_2) \rtimes_{alg} G_2)\chi_{A_2}$ are isomorphic. Also $L^\infty(X_i) \rtimes_{alg} G_i$ and $\chi_{A_i}(L^\infty(X_i) \rtimes_{alg} G_i)\chi_{A_i}$ is Morita equivalent because χ_{A_i} is a full projection in $L^\infty(X_i) \rtimes_{alg} G_i$.

We recall the definitions of measure equivalence and boundedly measure equivalence.

Definition 8.6. The countable groups G_i for $i = 1, 2$ are measure equivalent if there exist standard actions $G_i \curvearrowright X_i$ that are weakly orbit equivalent. Further, G_i are boundedly measure equivalent if there are boundedly weakly orbit equivalent standard actions $G_i \curvearrowright X_i$.

This definition is equivalent to usual one (see [15, Theorem 2.33, Remark 2.34.]).

We use the next definition of quasi-isometry.

Definition 8.7. Let G_1 and G_2 be discrete groups. They are quasi-isometric if they have their topological coupling Ω. Here Ω is their topological coupling if Ω is a locally compact space on which G_1 and G_2 act commuting continuously and each action is proper and cocompact.

If G_1 and G_2 are finitely generated, then this definition is equivalent to the usual definition of quasi-isometry about word metrics ([4], [15, Theorem 2.14]) Also boundedly measure equivalent imply quasi-isometric ([15, Lemma 2.25]).

Theorem 8.8. If G_1 and G_2 are boundedly measure equivalent groups, then

$$\begin{aligned} &H_p(G_1, \mathcal{N}(G_1)) \otimes_{\mathcal{N}(G_1)} \mathcal{N}(X_1 \rtimes G_1) \otimes_{\mathcal{N}(X_1 \rtimes G_1)} \mathcal{N}(X_1 \rtimes G_1)\chi_{A_1} \\ \cong\ &H_p(G_2, \mathcal{N}(G_2)) \otimes_{\mathcal{N}(G_2)} \mathcal{N}(X_2 \rtimes G_2) \otimes_{\mathcal{N}(X_2 \rtimes G_2)} \mathcal{N}(X_2 \rtimes G_2)\chi_{A_2}. \end{aligned}$$

Proof. We take a projective resolution P_*^i of $\mathbb{C}G_i$-module $\mathbb{C}$. Then $P_*^i \otimes_{\mathbb{C}G_i} L^\infty(X_i) \rtimes_{alg} G_i$ is a projective resolution of $L^\infty(X_i) \rtimes_{alg} G_i$-module $L^\infty(X_i)$. Moreover $P_*^i \otimes_{\mathbb{C}G_i} L^\infty(X_i) \rtimes_{alg} G_i \otimes_{L^\infty(X_i)\rtimes_{alg}G_i} L^\infty(X_i) \rtimes_{alg} G_i\chi_{A_i} = P_*^i \otimes_{\mathbb{C}G_i} L^\infty(X_i) \rtimes_{alg} G_i\chi_{A_i}$ is a projective resolution of $\chi_{A_i}L^\infty(X_i) \rtimes_{alg} G_i\chi_{A_i}$-module $L^\infty(A_i)$ since $\chi_{A_i}L^\infty(X_i) \rtimes_{alg} G_i\chi_{A_i}$ and $L^\infty(X_i) \rtimes_{alg} G_i$ are Morita equivalent (Remark 8.5). Hence $P_*^i \otimes_{\mathbb{C}G_i} \mathcal{N}(X_i \rtimes G_i)\chi_{A_i}$ for $i = 1, 2$ as $\chi_{A_i}\mathcal{N}(X_i \rtimes G_i)\chi_{A_i}$-chain complexes are homotopy equivalent by Remark 8.2. Hence $H_p(G_1, \mathcal{N}(X_1 \rtimes G_1)\chi_{A_1}) \cong H_p(G_2, \mathcal{N}(X_2 \rtimes G_2)\chi_{A_2})$. Moreover Hence $H_p(G_i, \mathcal{N}(X_i \rtimes G_i)\chi_{A_i}) \cong H_p(G_i, \mathcal{N}(X_i \rtimes G_i)) \otimes_{\mathcal{N}(X_i\rtimes G_i)} \mathcal{N}(X_i \rtimes G_i)\chi_{A_i} \cong H_p(G_i, \mathcal{N}(G_i)) \otimes_{\mathcal{N}(G_i)} \mathcal{N}(X_i \rtimes G_i) \otimes_{\mathcal{N}(X_i\rtimes G_i)} \mathcal{N}(X_i \rtimes G_i)\chi_{A_i}$ because $\mathcal{N}(G_i) \subset \mathcal{N}(X_i \rtimes G_i)$ are faithfully flat functor ([9, p.253]) and $\mathcal{N}(X_i \rtimes G_i)$ and $\chi_{A_i}\mathcal{N}(X_i \rtimes G_i)\chi_{A_i}$ are Morita equivalent. □

We have the following by Theorem 5.6, Proposition 6.11 and Proportion 6.12.

Corollary 8.9. If G_1 and G_2 are boundedly measure equivalent groups, then

$$\begin{aligned} &tH_p(G_1, \mathcal{N}(G_1)) \otimes_{\mathcal{N}(G_1)} \mathcal{N}(X_1 \rtimes G_1) \otimes_{\mathcal{N}(X_1\rtimes G_1)} \mathcal{N}(X_1 \rtimes G_1)\chi_{A_1} \\ \cong\ &tH_p(G_2, \mathcal{N}(G_2)) \otimes_{\mathcal{N}(G_2)} \mathcal{N}(X_2 \rtimes G_2) \otimes_{\mathcal{N}(X_2\rtimes G_2)} \mathcal{N}(X_2 \rtimes G_2)\chi_{A_2}. \end{aligned}$$

In particular $F_p(G_1)$ and $F_p(G_2)$ for all $p \geq 0$ are weakly dilatationally equivalent and also their Novikov-Shubin type capacities coincide.

If G_1 and G_2 are quasi-isometric amenable groups, then they are boundedly measure equivalent ([15, Theorem 2.38]). Hence we get the following.

Corollary 8.10. Let G_1 and G_2 be quasi-isometric amenable groups. Then $F_p(G_1) \simeq_w F_p(G_2)$ for any $p \geq 0$. In particular their Novikov-Shubin type capacities coincide.

We have the following because G is a non-amenable group if and only if $H_0(G, \mathcal{N}(G)) = 0$ ([9, p.448]).

Corollary 8.11. If G_1 and G_2 are quasi-isometric groups, then

$$\begin{aligned} &H_0(G_1, \mathcal{N}(G_1)) \otimes_{\mathcal{N}(G_1)} \mathcal{N}(X_1 \rtimes G_1) \otimes_{\mathcal{N}(X_1\rtimes G_1)} \mathcal{N}(X_1 \rtimes G_1)\chi_{A_1} \\ \cong\ &H_0(G_2, \mathcal{N}(G_2)) \otimes_{\mathcal{N}(G_2)} \mathcal{N}(X_2 \rtimes G_2) \otimes_{\mathcal{N}(X_2\rtimes G_2)} \mathcal{N}(X_2 \rtimes G_2)\chi_{A_2} \end{aligned}$$

and

$$\begin{aligned} &tH_0(G_1, \mathcal{N}(G_1)) \otimes_{\mathcal{N}(G_1)} \mathcal{N}(X_1 \rtimes G_1) \otimes_{\mathcal{N}(X_1\rtimes G_1)} \mathcal{N}(X_1 \rtimes G_1)\chi_{A_1} \\ \cong\ &tH_o(G_2, \mathcal{N}(G_2)) \otimes_{\mathcal{N}(G_2)} \mathcal{N}(X_2 \rtimes G_2) \otimes_{\mathcal{N}(X_2\rtimes G_2)} \mathcal{N}(X_2 \rtimes G_2)\chi_{A_2}. \end{aligned}$$

In particular $F_1(G_1)$ and $F_1(G_2)$ are weakly dilatationally equivalent and also their Novikov-Shubin type capacities coincide.

Gaboriau showed that $L^{(2)}$-Betti numbers for measure equivalent discrete groups proportionally coincide (see [2]). On the other hand $L^{(2)}$-Betti numbers of groups are not invariant under quasi-isometry. However whether the p-th $L^{(2)}$-Betti numbers of finite type groups vanish or not is invariant under quasi-isometry, where a group is finite type if its classifying space BG can be taken as a CW-complex of finite type. Moreover whether the p-th Novikov-Shubin invariants of finite type groups are ∞^+ or not is invariant under quasi-isometry (see [3, Section 11]). In particular whether a finite type group satisfies a topological version of the zero-in-the-spectrum conjecture ([12]) or not is quasi-isometric invariant property. Finally we give some questions.

Question 8.12.
(1) Is whether the p-th $L^{(2)}$-Betti numbers of general groups vanish or not invariant under quasi-isometry?
(2) Is whether the p-th Novikov-Shubin type capacities of general groups are 0^- or not invariant under quasi-isometry?
(3) Is whether a group satisfies an algebraic version of the zero-in-the-spectrum conjecture or not invariant under quasi-isometry?
(4) Are p-th spectral density functions of general groups invariant under quasi-isometry up to weakly dilatational equivalence?

Gromov indicates that Novikov-Shubin invariants of a certain class of groups may be invariant under quasi-isometry ([4, p.241]), but it is still open for finite type groups and so on.

References

[1] A. V. Efremov, Cell decompositions and the Novikov-Shubin invariants. (Russian) Uspekhi Mat. Nauk 46 (1991), no. 3(279), 189–190; translation in Russian Math. Surveys 46 (1991), no. 3, 219–220

[2] D. Gaboriau, Invariants l^2 de relations d'equivalence et de groupes. (French) [l^2-invariants of equivalence relations and groups] Publ. Math. Inst. Hautes Etudes Sci. No. 95 (2002), 93–150.

[3] S. M. Gersten, Bounded cohomology and combings of groups, preprint, Univ. of Utah, available at ftp://ftp.math.utah.edu/u/ma/gersten/bdd.dvi.Z.

[4] M. Gromov, Asymptotic invariants of infinite groups. Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[5] M. Gromov; M. A. Shubin, von Neumann spectra near zero. Geom. Funct. Anal. 1 (1991), no. 4, 375–404.

[6] J. Lott; W. Lück, L^2-topological invariants of 3-manifolds. Invent. Math. 120 (1995), no. 1, 15–60.

[7] W. Lück, L^2-torsion and 3-manifolds. Low-dimensional topology (Knoxville, TN, 1992), 75–107, Conf. Proc. Lecture Notes Geom. Topology, III, Internat. Press, Cambridge, MA, 1994.

[8] W. Lück, Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants. Math. Ann. 309 (1997), no. 2, 247–285.

[9] W. Lück, L^2-invariants: theory and applications to geometry and K-theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 44. Springer-Verlag, Berlin, 2002. xvi+595 pp.

[10] W. Lück; H. Reich; T. Schick, Novikov-Shubin invariants for arbitrary group actions and their positivity. Tel Aviv Topology Conference: Rothenberg Festschrift (1998), 159–176, Contemp. Math., 231, Amer. Math. Soc., Providence, RI, 1999.

[11] W. Lück; T. Schick, L^2-torsion of hyperbolic manifolds of finite volume. Geom. Funct. Anal. 9 (1999), no. 3, 518–567.

[12] S. Oguni, The group homology and an algebraic version of the zero-in-the-spectrum conjecture. to appear in J. Math. Kyoto Univ.

[13] S. Oguni, Secondary Novikov-Shubin invariants of groups and quasi-isometry. J. Math. Soc. Japan 59 (2007), no. 1, 223–237.

[14] C. Pittet; L. Saloff-Coste, On the stability of the behavior of random walks on groups. J. Geom. Anal. 10 (2000), no. 4, 713–737.

[15] R. Sauer, L^2-Invariants of Groups and Discrete Measured Groupoids. Dissertation, Universität Münster, 2003.

[16] R. Sauer, Homological invariants and quasi-isometry. Geom. Funct. Anal. 16 (2006), no. 2, 476–515.

[17] L. Vaš, Torsion theories for finite von Neumann algebras. Comm. Algebra 33 (2005), no. 3, 663–688.

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