SAMELSON PRODUCTS IN $Sp(2)$

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1. Introduction

We calculated certain generalized Samelson products of $Sp(2)$ and give two applications.

One is the classification of homotopy types of gauge groups. Let G be a compact Lie group, $\pi : P \to B$ a principal G-bundle over a finite complex B. We denote by $\mathcal{G}(P)$, the group of G-equivariant self maps covering the identity map of B. $\mathcal{G}(P)$ is called the (topological) gauge group of P. In [CS] M.Crabb and W.Sutherland prove as P ranges over all principal G-bundles over B, the number of homotopy types of $\mathcal{G}(P)$ is finite if B is connected and G is a compact connected Lie group. In some situations, exact number of homotopy types are calculated ([K], [HK2]).

In this paper we show the following:

Theorem 1.1. *Denote by ϵ_7' a generator of $\pi_7(Sp(2)) \cong \mathbb{Z}$ and by $\mathcal{G}_k$ the gauge group of principal $Sp(2)$ bundle over S^8 classified by $k\epsilon_7'$. Then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(140, k) = (140, k')$.*

The other application is on the homotopy commutativity of $Sp(2)$ localized at 3.

Theorem 1.2 (cf. [M])**.** *$Sp(2)_{(3)}$ is homotopy commutative.*

2. Notation

Here we give some notation and facts which we use throughout this note.

We use the same symbol c' for the inclusion $Sp(n) \hookrightarrow U(2n) \hookrightarrow U(2n+1)$, the complexifications $BSp(\infty) \to BU(\infty)$ and $BSp(n) \to BU(2n+1)$.

Let $W_n = U(\infty)/U(n), X_n = Sp(\infty)/Sp(n)$, and $\bar{c}' : X_n \to W_{2n+1}$. Then we have the following commutative diagram of fibration sequences

$$\begin{tikzcd}
Sp(\infty) \arrow[r, "p'"] \arrow[d, "c'"] & X_n \arrow[r, "i'"] \arrow[d, "\bar{c}'"] & BSp(n) \arrow[r] \arrow[d, "c'"] & BSp(\infty) \arrow[d, "c'"] \\
U(\infty) \arrow[r, "p"] & W_{2n+1} \arrow[r] & BU(2n+1) \arrow[r] & BU(\infty)
\end{tikzcd}$$

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Let σ be the cohomology suspension. Those facts listed below are well known:

$$\begin{aligned}
H^*(BU(\infty)) &= \mathbb{Z}[c_1, c_2, \ldots] \\
H^*(BSp(\infty)) &= \mathbb{Z}[q_1, q_2, \ldots] \\
H^*(W_{2n+1}) &= \bigwedge(x'_{4n+3}, x'_{4n+5}, \ldots) \\
H^*(X_n) &= \bigwedge(y'_{4n+3}, y'_{4n+7}, \ldots) \\
c'^*(c_{2j}) &= (-1)^j q_j, c'^*(c_{2j+1}) = 0 \\
p^*(x'_{4n+2j-1}) &= \sigma(c_{2n+j}) = x_{4n+2j-1} \\
p'^*(y'_{4n+4j-1}) &= \sigma(q_{n+j}) = y_{4n+4j-1} \\
\bar{c}'^*(x'_{4n+4j-1}) &= (-1)^{n+j} y'_{4n+4j-1}, \quad \bar{c}'^*(x'_{4n+4j-3}) = 0
\end{aligned}$$

Let $a_{4n+2j} = \sigma(x'_{4n+2j+1})$, $\quad b_{4n+4j-2} = \sigma(y'_{4n+4j-1})$ so that

$$\begin{aligned}
H^*(\Omega W_{2n+1}) &= \mathbb{Z}\{a_{4n+2}, \ldots, a_{8n+2}\}, \quad (* \leq 8n+2) \\
H^*(\Omega X_n) &= \mathbb{Z}\{b_{4n+2}, \ldots, b_{8n+3}\}, \quad (* \leq 8n+3).
\end{aligned}$$

We need the following Lemma which gives information on $\Omega p'$.

Lemma 2.1. *For a map $\alpha : \Sigma^2 X \to BSp(\infty)$, we have*

$$\left(\Omega \bar{c}' \circ \Omega p' \circ \sigma^2 \alpha\right)^* (a_{4n+4j-2}) = -(2n+2j-1)! \sigma^2(ch_{2n+2j}(c'(\alpha)))$$

Proof. Use the equality $(\Omega p)^* \sigma(x_{4n+4j-1}) = (2n+2j-1)! ch_{2n+2j}$ in [HK1]. $\square$

Precisely following the method in [HK1], we have the following Lemma [N].

Lemma 2.2. *There is a lift $\tilde{\gamma}'$ of the commutator map $\gamma' : Sp(n) \wedge Sp(n) \to Sp(n)$ such that $\delta' \circ \tilde{\gamma}' = \gamma'$ where $\delta' = \Omega i' : \Omega X_n \to Sp(n)$ and $\tilde{\gamma}'^*(b_{4n+4k-2}) = \sum_{i+j=n+k} y_{4i-1} \otimes y_{4j-1}$.*

Now we specialize to the case when $n = 2$.

Let $A = Sp(2)^{(7)} = S^3 \cup e^7$ and $\hat{\epsilon} : A \hookrightarrow Sp(2)$. Define two maps

$$\begin{aligned}
a : &\quad \Sigma A \subset \Sigma Sp(2) \xrightarrow{Ad(1)} BSp(2) \to BSp(\infty) \text{ and} \\
b : &\quad \Sigma A \xrightarrow{\pi} S^8 \xrightarrow{Ad(\epsilon'_7)} BSp(2) \to BSp(\infty), \text{ where } \pi \text{ is the projection.}
\end{aligned}$$

Then we have

$$ch(c'(a)) = \Sigma u_3 - \frac{1}{6} \Sigma u_7 \tag{1}$$

$$ch(c'(b)) = -2\Sigma u_7, \tag{2}$$

where $u_3 = \hat{\epsilon}^*(y_3)$ (resp. $u_7 = \hat{\epsilon}^*(y_7)$) is a generator of $H^3(A;\mathbb{Z}) \simeq \mathbb{Z}$ (resp. $H^7(A;\mathbb{Z}) \simeq \mathbb{Z}$).

Using the short exact sequence

$$0 = \tilde{KSp}^0(S^5) \to \tilde{KSp}^0(S^8) \to \tilde{KSp}^0(\Sigma A) \to \tilde{KSp}^0(S^4) \to \tilde{KSp}^0(S^7) = 0,$$

we have

Lemma 2.3. *$\tilde{KSp}^0(\Sigma A) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by a and b.*

3. The order of the Samelson product $\left\langle \epsilon_7', 1_{Sp(2)} \right\rangle \in [S^7 \wedge Sp(2), Sp(2)]$

First we consider the order of the Samelson product $\langle \epsilon_7', \hat{\epsilon} \rangle \in [S^7 \wedge A, Sp(2)]$.

Lemma 3.1. $[\Sigma^7 A, \Omega X_2] \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Recall that $\mathbf{dim}(\Sigma^7 A) = 14, \Omega X_2 = S^{10} \cup e^{14} \cup e^{18} \cdots$. Let F be the homotopy fiber of the inclusion $S^{11} \hookrightarrow X_2$. Then since

$$0 = \pi_{15}(S^{11}) \to \pi_{14}(\Omega X_2) \to \pi_{14}(F) \simeq \mathbb{Z}$$

is exact, we have $\pi_{14}(\Omega X_2) \simeq \mathbb{Z}$. Apply this to the following exact sequence

$$\mathbb{Z}/2 \simeq \pi_{11}(\Omega X_2) \to \pi_{14}(\Omega X_2) \to [\Sigma^7 A, \Omega X_2] \to \pi_{10}(Y) \simeq \mathbb{Z}.$$

□

Definition 3.2. *For $\alpha \in [\Sigma^7 A, \Omega X_2]$, define $\lambda(\alpha) = (\lambda_1(\alpha), \lambda_2(\alpha)) \in \mathbb{Z} \oplus \mathbb{Z}$, where $(\Omega \bar{c'} \circ \alpha)^*(a_{10}) = \lambda_1(\alpha)\Sigma^7 u_3$ and $(\Omega \bar{c'} \circ \alpha)^*(a_{14}) = \lambda_2(\alpha)\Sigma^7 u_7$.*

Lemma 3.3. *$\lambda : [\Sigma^7 A, \Omega X_2] \to \mathbb{Z} \oplus \mathbb{Z}$ is a homomorphism and monic.*

Proof. The map $\xi = y_{11}' \times y_{15}' : X_2 \to K(\mathbb{Z}, 11) \times K(\mathbb{Z}, 15)$ induces a 18 equivalence $\xi_{(0)} : (X_2)_{(0)} \to K(\mathbb{Q}, 11) \times K(\mathbb{Q}, 15)$. Since $\mathbf{dim}\Sigma^7 A = 14$,

$$(\Omega\xi_{(0)})_* : [\Sigma^7 A, (\Omega X_2)_{(0)}] \to H^{10}(\Sigma A; \mathbb{Q}) \oplus H^{14}(\Sigma A; \mathbb{Q})$$

is an isomorphism. By the commutative diagram

$$\begin{tikzcd}
 & {[\Sigma^7 A, \Omega X_2]} \arrow[r, "(\Omega\xi)_*"] \arrow[d] & H^{10}(\Sigma A) \oplus H^{14}(\Sigma A) \arrow[d] \\
{[\Sigma^7 A, \Omega X_2]_{(0)}} \arrow[r, equal] & {[\Sigma^7 A, (\Omega X_2)_{(0)}]} \arrow[r, "(\Omega\xi_{(0)})_*"'] & H^{10}(\Sigma A; \mathbb{Q}) \oplus H^{14}(\Sigma A; \mathbb{Q}),
\end{tikzcd}$$

we have the lemma since $[\Sigma^7 A, \Omega X_2]$ is free and $\lambda = (\Omega\xi)_*$. □

Let $D \in \tilde{KSp}^0(S^8) \simeq \mathbb{Z}$ be a generator. Then we have $ch(c'(D)) = v_8$, where v_8 is a generator of $H^8(S^8; \mathbb{Z})$.

Consider the following diagram

$$\begin{tikzcd}
 & & \Omega Sp(\infty) \arrow[d, "\Omega p'"] & \\
 & & \Omega X_2 \arrow[r, "\Omega \bar{c'}"] \arrow[d, "\delta"] & \Omega W_5 \\
S^7 \wedge A \arrow[r, "\epsilon_7' \wedge \hat{\epsilon}"] & Sp(2) \wedge Sp(2). \arrow[ur, "\tilde{\gamma'}"] \arrow[r, "\gamma'"] & Sp(2) &
\end{tikzcd}$$

Lemma 3.4. *$\gamma' \circ (\epsilon_7' \wedge \hat{\epsilon})$ has order 140.*

Proof. Put $\gamma_1 = \tilde{\gamma'} \circ (\epsilon_7' \wedge \hat{\epsilon})$, $\alpha_1 = (\Omega p')_*(\sigma^2(D \hat{\otimes} a))$ and $\beta_1 = (\Omega p')_*(\sigma^2(D \hat{\otimes} b))$. Recall that

$$\begin{aligned}
\bar{c'}^*(a_{10}) &= -b_{10}, \bar{c'}^*(a_{14}) = b_{14} \\
\bar{\gamma'}^*(b_{10}) &= y_3 \otimes y_7 + y_7 \otimes y_3 \\
\bar{\gamma'}^*(b_{14}) &= y_3 \otimes y_{11} + y_7 \otimes y_7 + y_{11} \otimes y_3 \\
(\epsilon_7')^*(y_7) &= 12 v_7.
\end{aligned}$$

Hence we have $\lambda(\gamma_1) = (-12, 12)$.

By Lemma 2.1, we have $\lambda(\alpha_1) = (-5!, \frac{7!}{6}), \lambda(\beta_1) = (0, 2 \cdot 7!)$. Since λ is monic and $\lambda(140\gamma_1 - 14\alpha_1 + \beta_1) = 0$ we get $140\gamma_1 = 14\alpha_1 - \beta_1$.

Consider the following exact sequence

$$0 \to \mathbf{Im}(\Omega p')_* \to [\Sigma^7 A, \Omega X_2] \xrightarrow{\delta} [\Sigma^7 A, Sp(2)].$$

This shows that $140\gamma' \circ (\epsilon'_7 \wedge \hat{\epsilon}) = 140\delta \circ \gamma_1 = 14\delta \circ \alpha_1 - \delta \circ \beta_1 = 0$. □

Proposition 3.5. *The order of the Samelson product* $\langle \epsilon'_7, 1_{Sp(2)} \rangle$ *is 140.*

Proof. Since the attaching map of the top cell of $Sp(2)$ become trivial after double suspension, there exists a map $i : S^{17} \to \Sigma^7 Sp(2)$ such that $S^{17} \vee \Sigma^7 A \xrightarrow{i \vee \Sigma^7 \hat{\epsilon}} \Sigma^7 Sp(2)$ is a homotopy equivalence. Hence we only have to show that $140\gamma_2 = 0$, where $\gamma_2 = \gamma' \circ (\epsilon'_7 \wedge 1_{Sp(2)}) \circ i : S^{17} \to Sp(2)$.

Let $\epsilon_7 \in \pi_7(SU(4)) \simeq \mathbb{Z}$ be a generator. Since $c'_*(\epsilon'_7) = 2\epsilon_7$, we have the following commutative diagram

$$\begin{tikzcd}
 & Sp(2) \wedge Sp(2) \arrow[r, "\gamma'"] \arrow[d, "c' \wedge c'"] & Sp(2) \arrow[d, "c'"] \\
S^7 \wedge Sp(2) \arrow[ur, "\epsilon'_7 \wedge 1"] \arrow[r, "2\epsilon_7 \wedge c'"] & SU(4) \wedge SU(4) \arrow[r, "\gamma"] & SU(4)
\end{tikzcd},$$

where $\gamma : SU(4) \wedge SU(4) \to SU(4)$ is the commutator.

Consider the map of fibrations

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[dd] & Sp(2) \arrow[dr, "p'"] \arrow[dd, "c'"] & \\
 & & S^7 \\
SU(3) \arrow[r] & SU(4) \arrow[ur, "p"] &
\end{tikzcd}$$

By the above diagram, we have $p' \circ \gamma_2 = p \circ c' \circ \gamma_2 = 2p \circ \gamma \circ (\epsilon_7 \wedge c')$. Since

$$p'_* : \mathbb{Z}/8 \oplus \mathbb{Z}/5 \simeq \pi_{17}(Sp(2)) \to \pi_{17}(S^7) \simeq \mathbb{Z}/24 \oplus \mathbb{Z}/2$$

induces an injection on 2-primary part by Mimura-Toda [MT], we have $20\gamma_2 = 0$. □

Proof of Theorem 1.1. By [AB], the classifying space $B\mathcal{G}(P)$ of the gauge group of a principal G-bundle P over a finite complex B, is homotopy equivalent to $\mathbf{Map}_P(B, BG)$, the connected component of maps from B to BG containing the classifying map of P. Consider the fibre sequence arose from the evaluation fibration

$$\mathcal{G}_k \to Sp(2) \xrightarrow{\alpha_k} \mathbf{Map}^*_{k\epsilon'_7}(S^8, BSp(2)) \to \mathbf{Map}_{k\epsilon'_7}(S^8, BSp(2)) \xrightarrow{e_k} BSp(2).$$

By Lang [L] $\mathbf{Map}^*_{k\epsilon'_7}(S^8, BSp(2))$ is homotopy equivalent to $\mathbf{Map}^*_0(S^8, BSp(2))$ and α_k can be identified with $\langle 1_{Sp(2)}, k\epsilon_7 \rangle = k \langle 1_{Sp(2)}, \epsilon_7 \rangle$ in

$$[Sp(2), \mathbf{Map}^*_0(S^8, BSp(2))] \cong [\Sigma^8 BSp(2), BSp(2)] \cong [\Sigma^7 Sp(2), Sp(2)],$$

where ϵ_7 is the adjoint of ϵ'_7 and $\langle , \rangle$ denotes the Samelson product.

Previous Proposition and the method in [HK2] completes the proof. □

4. The order of the Samelson product $\langle 1_{Sp(2)}, 1_{Sp(2)}\rangle$

In [M] McGibbon shows that $Sp(2)_{(3)}$ is homotopy commutative. Here we give another proof of this fact.

Denote the mod 3 reduction of y'_{4j+3} $(j \geq 2)$ by the same symbol. Then we have $H^*(X_2; \mathbb{Z}/3) = \bigwedge(y'_{11}, y'_{15}, y'_{19}, \ldots)$ and $\mathcal{P}^1 y'_{11} = \pm y'_{15}$. Let E be the homotopy fiber of $\rho\beta\mathcal{P}^1 u_{11} : K(\mathbb{Z}_{(3)}, 11) \to K(\mathbb{Z}_{(3)}, 16)$, where ρ is mod 3 reduction and u_{11} is a generator of $H^{11}(K(\mathbb{Z}_{(3)}, 11), \mathbb{Z}/3)$. Since $\mathcal{P}^1 y'_{11} = \pm y'_{15}$ and $\beta\mathcal{P}^1 y'_{11} = 0$, the map $y'_{11} : (\Omega X_2)_{(3)} \to K(\mathbb{Z}_{(3)}, 11)$ lifts to a 17 equivalence $f : (\Omega X_2)_{(3)} \to E$. Since $\mathbf{dim}(A \wedge A) = 14$,

$$(\Omega f)_* : [A \wedge A, (\Omega X_2)_{(3)}] \to [A \wedge A, \Omega E]$$

is an isomorphism of groups. Consider the following exact sequence:

$$H^9(A\wedge A; \mathbb{Z}_{(3)}) \to H^{14}(A\wedge A; \mathbb{Z}_{(3)}) \to [A\wedge A, \Omega E] \to H^{10}(A\wedge A, \mathbb{Z}_{(3)}) \to H^{15}(A\wedge A; \mathbb{Z}_{(3)}).$$

Since $H^k(A \wedge A; \mathbb{Z}_{(3)}) = \begin{cases} 0 & k = 9, 15 \\ \mathbb{Z}_{(3)} & k = 10, 14, \end{cases}$ we have $[A \wedge A, \Omega E] \simeq \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(3)}$.

Define $\tilde{\lambda} : [A \wedge A, (\Omega X_2)_{(3)}] \to (\mathbb{Z}_{(3)})^3$ by $\tilde{\lambda}(\alpha) = (\tilde{\lambda}_1(\alpha), \tilde{\lambda}'_1(\alpha), \tilde{\lambda}_2(\alpha))$ where $\alpha^*(\Omega c')^*(a_{10}) = \tilde{\lambda}_1(\alpha) u_3 \otimes u_7 + \tilde{\lambda}'_1(\alpha) u_7 \otimes u_3$ and $\alpha^*(\Omega c')^*(a_{14}) = \tilde{\lambda}_2(\alpha) u_7 \otimes u_7$ for $\alpha \in [A \wedge A, (\Omega X_2)_{(3)}]$. Since $\tilde{\lambda}_{(0)} : [A \wedge A, (\Omega X_2)_{(3)}] \to (\mathbb{Q})^3$ is an isomorphism (see section 3), $\tilde{\lambda}$ is monic.

It is not hard to show $c' : \tilde{KSp}(\Sigma^2 A \wedge A)_{(3)} \to \tilde{K}(\Sigma^2 A \wedge A)_{(3)}$ is an isomorphism. Therefore we may consider $a\hat{\otimes}a, a\hat{\otimes}b + b\hat{\otimes}a \in \tilde{KSp}(\Sigma^2 A \wedge A)_{(3)}$.

Put $\alpha_1 = \frac{6}{5!}(\Omega p')_*(\sigma^2(a\hat{\otimes}a))$ and $\alpha_2 = \frac{9}{2\cdot 6!}(\Omega p')_*(\sigma^2(a\hat{\otimes}b + b\hat{\otimes}a))$.Then $\alpha_1, \alpha_2 \in [A \wedge A, \Omega X_2]_{(3)}$. Using equalities $ch(c'(a)) = \Sigma u_3 - \frac{1}{6}\Sigma u_7$ and $ch(c'(b)) = -2\Sigma u_7$, we can easily show

$$\tilde{\lambda}(\alpha_1) = (1, 1, -7), \tilde{\lambda}(\alpha_2) = (3, 3, -42).$$

By the same method as in the proof of Lemma 3.4, we have

$$\tilde{\lambda}(\tilde{\gamma}' \circ (\hat{\epsilon} \wedge \hat{\epsilon})) = (-1, -1, 1).$$

Since $\tilde{\lambda}(\tilde{\gamma}' \circ (\hat{\epsilon} \wedge \hat{\epsilon}) + \alpha_1 + \frac{2}{7}(3\alpha_1 - \alpha_2)) = 0$ and $\tilde{\lambda}$ is monic, we have

Lemma 4.1. *$\gamma' \circ (\hat{\epsilon} \wedge \hat{\epsilon}) = 0$ in $[A \wedge A, Sp(2)]_{(3)}$.*

Proof of Theorem 1.2 . Since $\Sigma A \hookrightarrow \Sigma Sp(2)$ has a retraction. Therefore we have to show that the generalized Whitehead product $\Sigma A \wedge \Sigma A \to BSp(2)$ vanish. Taking the adjoint, previous Lemma completes the proof. □

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