GAP OF DEPTHS OF LEAVES ON CODIMENSION ONE FOLIATIONS

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Abstract. In this paper, we introduce a new invariant called gap of codimension one foliations. Roughly speaking, the gap of a foliation $\mathcal{F}$ is the maximal value of $\mathrm{depth}(L') - \mathrm{depth}(L)$, where L and L' are leaves of $\mathcal{F}$ such that $L \subset \overline{L'} \setminus L'$, and that there does not exist a leaf L'' with $L \subset \overline{L''} \setminus L''$, and with $L'' \subset \overline{L'} \setminus L'$. Let $\Sigma^{(n)}(K, 0)$ be the n-fold cyclic covering space of $S^3(K, 0)$, where $S^3(K, 0)$ is the manifold obtained from S^3 by performing 0-surgery along the 0-twisted double K of a non-cable knot. By using gap, we give an estimation of minimal value of the depths of the codimension one C^0 foliations with C^∞ leaves on $\Sigma^{(n)}(K, 0)$, which are transversely oriented and taut, and each of which has exactly one depth 0 leaf corresponding to a generator of $H_2(S^3(K, 0))$.

1. Introduction

In 1980's D.Gabai developed the theory of codimension one foliations on three manifolds. He gave a powerful method which is called sutured manifold theory, for constructing taut foliations on three dimensional manifolds [4]. Particularly in [5], he showed that for any knot K in S^3, there exists a codimension one, transversely oriented, taut C^0 foliation $\mathcal{F}$ of finite depth on the knot exterior $E(K)$ such that $\mathcal{F}|_{\partial E(K)}$ is a foliation by circles. (As a consequence of this theorem, Property R Conjecture, which was one of the most important problems in knot theory, follows immediately.) Inspired by this result, Cantwell-Conlon introduced invariants of 3-manifolds by making use of depths of foliations. More precisely, for a given knot K, they considered the minimal value of the depths of the transversely oriented, taut C^r foliations on the exterior $E(K)$ of K which is transverse to $\partial E(K)$, and called it C^r depth of K ([2]). From this viewpoint (, that is, consider the minimal value of the depths of certain foliations on a given manifold), it is natural to ask: Question "Does this value changes or not if we pose qualitative conditions on the foliations?"

In [7], the author studied the question in the case when we pose the condition that each of the foliation has exactly one depth 0 leaf, and showed the following. Let K be a 0-twisted double of a non-cable knot, $S^3(K, 0)$ the manifold obtained from S^3 by performing 0-surgery along K, and $\Sigma^{(n)}(K, 0)$ the n-fold cyclic covering space of $S^3(K, 0)$. Let $\mathcal{F}$ be a codimension one, transversely oriented, and taut C^0 foliation on $\Sigma^{(n)}(K, 0)$ with exactly one depth 0 leaf representing a generator of $H_2(S^3(K, 0))$. Then we have: $\mathrm{depth}(\mathcal{F}) \geq 1 + [\frac{n}{2}]$. Here we note the following. Let k be the depth of a foliation on $E(K)$ given by Gabai's construction in [5]. It is easy to see that for each n, $\Sigma^{(n)}(K, 0)$ admits a codimension one, transversely oriented, and taut C^0 foliation of depth k with at least n depth 0 leaves. This shows that the condition (each of the foliation has exactly one depth 0 leaf) is essential. The purpose of this paper is, as a sequel to these researches, to propose more delicate qualitative conditions.

In fact, we introduce a quantity called "gap" of foliation to deal with behaviors of depths of leaves of foliations of finite depth. The naive idea of the concept "gap" is as follows. We know by the definition of depth of leaves (see Section 2.2) that each depth $k(\geq 1)$ leaf of $\mathcal{F}$ is adjacent to a depth $k-1$ leaf. Note that even if k does not represent the maximal depth in $\mathcal{F}$, it is not necessary the case that there exists a depth $k+1$ leaf which is adjacent to the leaf. One possibility of this phenomenon is that k is loccaly maximal and the other is that there is a leaf with depth greater than $k+1$ which is adjacent to the depth k leaf. We phrase the latter situation "There is a gap between the depths of the leaves." More precisely, for a leaf L, we consider the minimal value of the differences between the depth of L and the depths of leaves which are adjacent to L with depth greater than that of L. Then the gap of the foliation is the maximum of such values among the leaves of the foliation. For the formal definition of gap, see Section 2.3. By using this invariant, we give an estimation of depth of foliations on the above manifold $\Sigma^{(n)}(K, 0)$.

Theorem 1.1. *Suppose K is a 0-twisted double of a non-cable knot. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut, C^0 foliation with C^∞ leaves on $\Sigma^{(n)}(K,0)$ with exactly one depth 0 leaf representing $[\alpha]$, where α is corresponding to a generator of $H_2(S^3(K,0)) \cong \mathbb{Z}$. Suppose $\hat{G}(\hat{\mathcal{F}})$ is a tree. Then for each n, we have:*

$$depth(\mathcal{F}) \geq \frac{n + gap(\widetilde{\mathcal{F}})}{2}.$$

For the notations $\hat{G}(\hat{\mathcal{F}})$ and $\mathrm{gap}(\widetilde{\mathcal{F}})$, see Section 2.3, and for the definition of 0-twisted double and cable knot, see Section 4.

2. Preliminaries

2.1. **Codimension one foliations.** Let M be a Riemannian manifold of dimension n. In this subsection, we suppose that M is compact and orientable.

Definition 2.1. A *codimension q (or dimension $n-q$) $C^r(0 \leq r \leq \infty)$ foliation* on M is a C^r atlas $\mathcal{F}$ on M with the following properties.

1. If $(U, \varphi) \in \mathcal{F}$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$ where U_1 (U_2 resp.) is an open disk in $\mathbb{R}^{n-q}$ ($\mathbb{R}^q$ resp.).
2. If (U, φ) and $(V, \psi) \in \mathcal{F}$ are such that $U \cap V \neq \emptyset$, then the change of coordinates map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is of the form $\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y))$.

The charts $(U, \varphi) \in \mathcal{F}$ will be called *foliation charts.* We call the pair $(M, \mathcal{F})$ a *foliated manifold.*

Definition 2.2. Let (U, φ) be a foliation chart. The sets of the form $\varphi^{-1}(U_1 \times \{c\})$, $c \in U_2$ are called *plaques of U*, or else *plaques of $\mathcal{F}$*.

For the basic terminologies concerning foliations (leaf, holonomy, etc.), see [1]. In the remainder of this section, let $\mathcal{F}$ be a codimension one $C^r(0 \leq r \leq \infty)$ foliation on M.

Definition 2.3. We say that a leaf of $\mathcal{F}$ is *proper* if its topology as a manifold coincides with the topology induced from that of M. A foliation $\mathcal{F}$ is called *proper* if every leaf of $\mathcal{F}$ is proper.

Definition 2.4. We say that $\mathcal{F}$ is *taut* if for any leaf L of $\mathcal{F}$, there is a properly embedded (possibly, closed) transverse curve which meets L.

Definition 2.5. Let $\mathcal{F}$ be a C^r foliation. We say that $\mathcal{F}$ is a C^r foliation with C^∞ leaves on M if each leaf is a C^∞ immersed manifold.

By using a partition of unity argument, we can show that any codimension one, transversely oriented foliation with C^∞ leaves has a one dimensional C^∞ foliation which is transverse to $\mathcal{F}$. In the remainder of this section, we fix a one dimensional C^∞ foliation $\mathcal{F}^\perp$ which is transverse to $\mathcal{F}$.

Notation 2.6. A subset U of M is called *saturated* if U is a union of leaves of $\mathcal{F}$. Let U be an open saturated set, and $\iota : U \to M$ the inclusion. Then $\hat{U}$ denotes the path-metric completion of U by the Riemannian metric induced from M, and $\hat{\iota} : \hat{U} \to M$ denotes the extended isometric immersion. Let $\hat{\mathcal{F}} = \hat{\iota}^{-1}(\mathcal{F})$ and $\hat{\mathcal{F}}^\perp = \hat{\iota}^{-1}(\mathcal{F}^\perp)$, the induced foliations on $\hat{U}$.

In the remainder of this section, we suppose $\mathcal{F}$ is transversely oriented. The unit tangent bundle $q : \widetilde{M} \to M$ of $\mathcal{F}^\perp$ is a C^∞ double covering of M. Since $\mathcal{F}$ is transversely oriented, for each leaf L of $\mathcal{F}$, $q^{-1}(L)$ consists of two components. Each component of $q^{-1}(L)$ is called a *side* of L.

Definition 2.7. A side $\widetilde{L}$ of $q(\widetilde{L}) = L$ is *proper* if there are a transverse curve $\tau : [0,1] \to M$ starting from L in the direction of $\widetilde{L}$ and $\varepsilon(> 0)$ such that $\tau(t) \notin L$ for $0 < t < \varepsilon$.

Remark 2.8. Any side of a proper leaf is proper.

Definition 2.9. Let $\widetilde{L}$ be a proper side of L. The leaf L has *unbounded holonomy* on the side $\widetilde{L}$ if there are a transverse curve $\gamma : [0,1] \to M$ starting from L in the direction of $\widetilde{L}$ and a sequence $h_1, h_2, \dots$ of holonomy pseudogroup elements with domain containing $\mathrm{im}(\gamma)$ such that

$$h_i(\mathrm{im}(\gamma)) = \gamma([0,\varepsilon_i]),\ \varepsilon_i \searrow 0.$$

The leaf L is *semistable* on the side $\widetilde{L}$ if there is a sequence $e_1, e_2, \dots$ of C^∞ immersions of $\widetilde{L}\times[0,1]$ (with its manifold structure) into M such that $e_i(x,0) = q(x)$ for all x and i, $e_{i*}(\frac{\partial}{\partial t})|_{t=0}$ points in the direction $\widetilde{L}$, $e_{i*}(\frac{\partial}{\partial t})$ is always tangent to $\mathcal{F}^\perp$, each $e_i(\widetilde{L}\times\{1\})$ is a leaf of $\mathcal{F}$, and $\bigcap_i e_i(\widetilde{L}\times[0,1]) = L$.

In [3], Dippolito showed the following.

Theorem 2.10. (Semistability Theorem [3]) *Let $\mathcal{F}$ be a codimension one foliation with C^∞ leaves on a closed manifold. If $\widetilde{L}$ is a proper side of a leaf L of $\mathcal{F}$, then L either is semistable or has unbounded holonomy on the side $\widetilde{L}$.*

2.2. **Depth of foliations.**

Definition 2.11. A leaf L of $\mathcal{F}$ is at *depth 0* if it is compact. Inductively, when leaves of at depth less than k are defined, L is at *depth* $k \geq 1$ if $\overline{L} \setminus L$ consists of leaves at depth strictly less than k, and at least one of which is at depth $k-1$.

If L is at depth k, we use the notation depth$(L) = k$, and call L a *depth k leaf.* The foliation $\mathcal{F}$ is of depth $k < \infty$ if every leaf of $\mathcal{F}$ is at depth at most k and k is the least integer for which this is true. If $\mathcal{F}$ is of depth k, we use the notation depth$(\mathcal{F}) = k$. If there is no integer $k < \infty$ which satisfies the above condition, the foliation $\mathcal{F}$ is of infinite depth.

In the remainder of this section, we suppose $\mathcal{F}$ is of finite depth. By Definition 2.11, we have the following.

Facts 2.12. 1. Let L, L' be leaves of $\mathcal{F}$. If $\overline{L} \setminus L \subset \overline{L'} \setminus L'$, then we have depth$(L) \leq$ depth(L').

2. For any leaf L of $\mathcal{F}$, there exists a depth 0 leaf of $\mathcal{F}$ in $\overline{L}$.

It is known :

Fact 2.13. Each leaf of any finite depth foliation is proper.

Lemma 2.14. *Let L be a leaf of $\mathcal{F}$. Suppose L has unbounded holonomy on the side $\widetilde{L}$ and let γ be as in Definition 2.9. Then for any leaf L' of $\mathcal{F}$ such that $L' \cap \gamma \neq \emptyset$, we have depth$(L) <$ depth(L').*

Proof. Let $h_1, h_2, \dots$ be as in Definition 2.9. Fix a point $x_0 \in L' \cap \gamma$. Let $x_i = h_i(x_0)$ $(i = 1, 2, \dots)$. Then $x_i \in L'$, and x_i converges to the point $\gamma(0) \in L$. This shows that $L \subset \overline{L'}$. Since $L \neq L'$, this implies depth$(L) <$ depth(L'). □

Lemma 2.15. *Let L, L' be leaves of $\mathcal{F}$. Suppose L is semistable on the side $\widetilde{L}$ and let $e_1, e_2, \dots$ be as in Definition 2.9. Suppose there exists i such that $e_i(\widetilde{L} \times [0,1]) \supset L'$. Then we have depth$(L) \leq$ depth(L').*

Proof. If L is compact, then obviously the lemma holds. Suppose L is noncompact. Then $\overline{L} \setminus L \neq \emptyset$. Let L^* be a leaf contained in $\overline{L} \setminus L$. Fix a point x^* in L^*. Let P be a plaque of $\mathcal{F}^\perp$ through x^*. Let P' be the closure of a component of $P \setminus x^*$ such that $x^* \in \overline{P' \cap L}$. Then we can take points $x_1, x_2, \dots$ in $P' \cap L$ such that x_i monotonously converges to x^*. Let $\widetilde{x}_j$ be the points in $\widetilde{L}$ such that $e_i(\widetilde{x}_j \times \{0\}) = x_j$. Let $P_j = e_i(\widetilde{x}_j \times [0,1])$. Then $P_2, P_3, \dots$ are mutually disjoint arcs embedded in P'. Since $L' \subset e_i(\widetilde{L} \times [0,1])$, $L' \cap P_j \neq \emptyset$ $(j = 2, 3, \dots)$. Fix a point $x'_j \in L' \cap P_j$. Then $\{x'_j\}_{j=2,3,\dots}$ converges to x^*. Hence $L^* \subset \overline{L'}$. Since $L^* \neq L'$, this implies that $L^* \subset \overline{L'} \setminus L'$. Hence $\overline{L} \setminus L \subset \overline{L'} \setminus L'$. By Facts 2.12, we have depth$(L) \leq$ depth(L'). □

Let $\{L_i^{(d)}\}$ be a set of depth d leaves of $\mathcal{F}$, U a component of $M \setminus \overline{\cup L_i^{(d)}}$, and F a component of $\partial \hat{U}$. Let L be the leaf $\hat{\imath}(F)$ of $\mathcal{F}$ and $x \in L$. Let P be a plaque of $\mathcal{F}^\perp$ through x. Since $x \in L = \hat{\imath}(F) \subset \hat{\imath}(\partial \hat{U})$ and U is open, $(U_\varepsilon(x) \cap P) \cap U \neq \emptyset$ for any $\varepsilon > 0$, where $U_\varepsilon(x)$ denotes the ε-neighborhood of x. Let P_1, P_2 be the closures of the components of $P \setminus x$. We may suppose $(U_\varepsilon(x) \cap P_2) \cap U \neq \emptyset$ for any $\varepsilon > 0$.

Lemma 2.16. *Under the above notations, there exists a subarc P'_2 in P_2 such that $x \in \partial P'_2$ and $P'_2 \subset \hat{\imath}(\hat{U})$.*

Proof. Assume $x \in \overline{P_2 \cap (\cup L_i^{(d)})}$. For a point z in P_2, let $L(z)$ be the leaf of $\mathcal{F}$ through z. For a constant $c > 0$, let $N_c^{L(z)}(z)$ be a c-neighborhood of z in $L(z)$. Then, we can take $c > 0$ such that for each point $z \in P_2$, $N_c^{L(z)}(z)$ is homeomorphic to an $n-1$ dimensional disk D^{n-1}. Let $N_c = \bigcup_{z\in P_2} N_c^{L(z)}(z)$, and $N_{\frac{c}{2}} = \bigcup_{z\in P_2} N_{\frac{c}{2}}^{L(z)}(z)$. Let y be a point in $\partial\hat{U}$ such that $\hat{\imath}(y) = x$. Since $y \in \partial\hat{U}$, there exists a Cauchy sequence $\{y_i\} \subset \hat{U} \setminus \partial\hat{U}$ such that y_i converges to y. By retaking subscripts if necessary, we may suppose that $\hat{\imath}(y_i) \in N_{\frac{c}{2}}$ for each i. Moreover by taking a subsequence if necessary, we may suppose that if $i \neq j$, then $\hat{\imath}(y_i)$ and $\hat{\imath}(y_j)$ are contained in different components of $N_{\frac{c}{2}} \setminus \cup L_i^{(d)}$. Thus we have $d(y_i, y_j) > \frac{c}{2}$ where d is a path metric in U. This contradicts that $\{y_i\}$ is a Cauchy sequence. □

Lemma 2.17. *Under the above notations, we have depth*$(L) \leq d$.

Proof. If there exists a subarc P' of P_1 with $x \in \partial P'$ such that P' does not intersect $\cup L_i^{(d)}$, since $P_2' \subset \hat{\imath}(\hat{U})$ (Lemma 2.16), this implies $L \in \{L_i^{(d)}\}$, hence depth$(L) = d$.

Suppose for any subarc P' of P_1 with $x \in \partial P'$, there exists $L' \in \{L_i^{(d)}\}$ such that $L' \cap P' \neq \emptyset$. Then the situation is divided into the following two cases.

Case 1 L is semistable on the side $\widetilde{L}$ which contains P_1.

Let e_1, e_2, ... be as in Definition 2.9. Take a subarc P'' of P_1 with $x \in \partial P''$ and $P'' \subset \mathrm{im}(e_1)$. Let L'' be an element of $\{L_i^{(d)}\}$ such that $L'' \cap P'' \neq \emptyset$. By Lemma 2.15, we have depth$(L) \leq$ depth$(L'') = d$.

Case 2 L has unbounded holonomy on the side $\widetilde{L}$ which contains P_1.

Take a subarc γ'' of P_1 with $x \in \partial\gamma''$ and satisfies the condition of γ in Definition 2.9. Let L'' be an element of $\{L_i^{(d)}\}$ such that $L'' \cap \gamma'' \neq \emptyset$. By Lemma 2.14, we have depth$(L) <$ depth$(L') = d$. □

Lemma 2.18. *Suppose there exists a pair of components of $\partial\hat{U}$ representing the same leaf L of $\mathcal{F}$. Then L is an element of $\{L_i^{(d)}\}$.*

Proof. Note that P_1 also satisfies the condition of Lemma 2.16, i.e., there exists a subarc P_1' in P_1 such that $x \in \partial P_1'$ and $P_1' \subset \hat{\imath}(\hat{U})$. Then $P' = P_1' \cup P_2'$ is a plaque of x such that $P' \cup L = x$. This obviously implies $L \in \{L_i^{(d)}\}$. □

2.3. **Gap of foliations.**

Definition 2.19. For leaves L_1 and L_2 of $\mathcal{F}$, we say that L_1 is *equivalent* to L_2 if $L_1 = L_2$ or there exisits an embedding $\phi : L_1 \times [0,1] \longrightarrow M$ such that the image of $L_1 \times \{0\}$ ($L_1 \times \{1\}$ resp.) coincides with L_1 (L_2 resp.), and the image of $\{x\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$ for each $x \in L_1$. Moreover, if $\widetilde{L}$ is the side of L such that $\phi_*(\frac{\partial}{\partial t})|_{t=0}/\|\phi_*(\frac{\partial}{\partial t})|_{t=0}\|$ is contained in $\widetilde{L}$, then we say that L is equivalent to L' *through the side* $\widetilde{L}$.

Lemma 2.20. *Let L be a leaf of $\mathcal{F}$. Suppose that L is semistable on the proper side $\widetilde{L}$ and let $e_1, e_2, \ldots$ be as in Definition 2.9. For each i, let L' be a leaf in $e_i(\widetilde{L} \times [0,1])$ such that depth$(L) =$ depth(L'). Then, L' is equivalent to L through the side $\widetilde{L}$.*

Proof. Let P be a leaf of $\mathcal{F}^\perp |_{e_i(\widetilde{L}\times[0,1])}$. If $L' \cap P$ contains more than one point, then by considering the holonomy along L, we can show that $L' \cap P$ contains infinitely many points, say x_i. Let x_∞ be an accumulating point of $\{x_i\}$. If $x_\infty \in L$, then $\overline{L'} \setminus L' \supset L$, this contradicts that depth$(L) =$ depth(L'). Suppose $x_\infty \notin L$. Let L_∞ be the leaf which contains x_∞. By Lemma 2.15, we have depth$(L) \leq$ depth(L_∞), this implies depth$(L) <$ depth(L'), a contradiction. Hence $L' \cap P$ contains exactly one point. Since we can choose P as any leaf of $\mathcal{F}^\perp \mid_{e_i(\widetilde{L} \times [0,1])}$, this implies that L' is equivalent to L through the side $\tilde{L}$. $\square$

Suppose further M is closed. Let $\widetilde{\mathcal{F}}$ be a codimension one, transversely oriented C^r foliation of finite depth with C^∞ leaves on M which satisfies the following properties.

Properties 2.21. 1. *The number of equivalence classes of the leaves of $\widetilde{\mathcal{F}}$ is finite;*

2. *Let L_1, L_2 be leaves of $\widetilde{\mathcal{F}}$ such that L_1 is equivalent to L_2 through the side $\widetilde{L}_1$. Let $\phi : L \times I \to M$ be an embedding giving the equivalence relation between L_1 and L_2 through the side $\widetilde{L}_1$, then $\mathcal{F} \mid_{\phi(L_1 \times [0,1])}$ is a product foliation with each leaf is homeomorphic to L_1.*

Let $[L_i^0]$ be the equivalence classes of the depth 0 leaves of $\widetilde{\mathcal{F}}$. Let $\hat{M}$ be the union of the path-metric completions of the components of $M \setminus (\bigcup_i L_i^0)$. Let $\hat{\mathcal{F}}$ be the foliation on $\hat{M}$ induced from $\widetilde{\mathcal{F}}$. By the definition of depth, we immediately have the following.

Lemma 2.22. *Under the above notations, we have depth$(\hat{\mathcal{F}}) =$ depth$(\widetilde{\mathcal{F}})$.*

Let $[L_j]$ be the equivalence classes of the leaves of $\hat{\mathcal{F}}$.

Definition 2.23. The graph of $\hat{\mathcal{F}}$ denoted by $\hat{G}(\hat{\mathcal{F}}) = \{V, E\}$ is the directed graph with the vertex set $V = \{v_j\}$ and the edge set $E = \{e_{k\ell}\}$ such that each v_j corresponds to the equivalence class $[L_j]$ of the leaves of $\hat{\mathcal{F}}$ and there is an edge $e_{k\ell}$ from v_k to v_ℓ if there exists a leaf L'_k (L'_ℓ resp.) representing v_k (v_ℓ resp.) such that $L'_\ell \subset \overline{L'_k} \setminus L'_k$, and there does not exist a leaf L such that $L \subset \overline{L'_k} \setminus L'_k$ and $L'_\ell \subset \overline{L} \setminus L$.

By the construction, the foliated manifold $(M, \widetilde{\mathcal{F}})$ is recovered from $(\hat{M}, \hat{\mathcal{F}})$ by identifying pairs of depth 0 leaves L_i^{0+} and L_i^{0-}, each corresponding to L_i^0. Then we define the graph of $\widetilde{\mathcal{F}}$ as follows.

Definition 2.24. The graph of $\widetilde{\mathcal{F}}$ denoted by $G(\widetilde{\mathcal{F}})$ is the graph obtained from $\hat{G}(\hat{\mathcal{F}})$ by identifying pairs of vertices corresponding to L_i^{0+} and L_i^{0-} for each depth 0 leaf L_i^0 of $\tilde{\mathcal{F}}$.

By the definition, we immediately have the following.

Lemma 2.25. *The following three conditions are equivalent to each other.*

1. *$\hat{G}(\hat{\mathcal{F}})$ is the graph consisting of exactly one vertex.*
2. *$\widetilde{\mathcal{F}}$ is a foliation given by a fiber bundle structure over S^1.*
3. *There exists a leaf L_i^0 of $\widetilde{\mathcal{F}}$ such that L_i^{0+} and L_i^{0-} corresponding to the same vertex of $\hat{G}(\hat{\mathcal{F}})$.*

Definition 2.26. Let v be a vertex of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. We say that v is at *depth k* if v represents a leaf at depth k. If v is at depth k, we use the notation depth$(v) = k$, and call v a *depth k vertex*.

Definition 2.27. Let e be an edge of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. Let v be the initial point and v' the terminal point of e. Then, we define the *length* of e as follows:

$$\text{length}(e) = \text{depth}(v) - \text{depth}(v').$$

Remark 2.28. Let v be a vertex of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. If $\text{depth}(v) \neq 0$, then by the definition of the depth, we see that there exists a directed path $\Gamma = e_1 \cup \cdots \cup e_n$ from v to a depth 0 vertex such that $\text{length}(e_i) = 1$ $(i = 1, \ldots, n)$

Definition 2.29. We define the *gap* of the foliation $\widetilde{\mathcal{F}}$ as follows:

$$\text{gap}(\widetilde{\mathcal{F}}) = \begin{cases} 0 & \text{if } G(\widetilde{\mathcal{F}}) \text{ has no edges,} \\ \max_{e:\text{edges of } G(\widetilde{\mathcal{F}})}\{\text{length}(e)\} & \text{if } G(\widetilde{\mathcal{F}}) \text{ has an edge.} \end{cases}$$

3. Modifying foliations

Let M be a closed, connected, oriented n dimensional manifold and $\mathcal{F}$ a codimension one, transversely oriented C^r foliation of finite depth with C^∞ leaves on M. We further suppose $\text{depth}(\mathcal{F}) \neq 0$. In this section, we show that for any foliation as above, we can modify $\mathcal{F}$ to obtain a foliation satisfying Properties 2.21. Let $\mathcal{F}^\perp$ be a one dimensional C^∞ foliation on M which is transverse to $\mathcal{F}$.

Definition 3.1. An $(\mathcal{F}, \mathcal{F}^\perp)$ *coordinate atlas* is a locally finite collection of C^r embeddings $\varphi_i : D^{n-1} \times [0,1] \to M$ such that the interior of the images cover M, and the restriction of φ_i to each $D^{n-1} \times \{t\}$ (to each $\{x\} \times [0,1]$ resp.) is a $C^\infty(C^r$ resp.) embedding into a leaf of $\mathcal{F}(\mathcal{F}^\perp$ resp.).

Since M is compact, we can take an $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ of $m(< \infty)$ components.

3.1. **First step of Modification.** In this subsection, we describe a procedure for modifying $\mathcal{F}$ by using depth 0 leaves of $\mathcal{F}$.

Lemma 3.2. *Under the equivalence relation of Definition 2.19, the number of the equivalence classes represented by the depth 0 leaves is at most* $2m+$*rank* $H_1(M, \mathbb{R})-1$.

Proof. Let $\{L_j^{(0)}\}$ be representatives of the equivalence classes of the depth 0 leaves of $\mathcal{F}$. We assume that $\{L_j^{(0)}\}$ has $2m + \text{rank } \mathrm{H}_1(M, \mathbb{R})$ elements. By slightly modifying the $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ if necessary, we may suppose that $(\cup L_j^{(0)}) \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that if we take any subset of $\{L_j^{(0)}\}$ consisting of at least $\text{rank } \mathrm{H}_1(M, \mathbb{R}) + 1$ elements, then the union of them separates M. Hence the number of the components of $M \setminus \cup L_j^{(0)}$ is at least $2m+1$. Hence we can find U, a component of $M \setminus \cup L_j^{(0)}$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that U is a saturated set. Hence we use notations in Notation 2.6. For any point x in $\partial\hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^\perp$ which meets x. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some i and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints x and y, say. Since $\mathcal{F}$ is transversely oriented, x and y are contained in different components of $\partial\hat{U}$. If $\hat{\imath}(x)$ and $\hat{\imath}(y)$ are contained in the same leaf of $\mathcal{F}$, this implies that $\{L_j^{(0)}\}$ consists of one element, contradicting the assumption that $\{L_j^{(0)}\}$ has $2m + \text{rank } \mathrm{H}_1(M, \mathbb{R})$ elements. Thus $\hat{\imath}(x)$ and $\hat{\imath}(y)$ are contained in different leaves, say F_x and F_y of $\mathcal{F}$.

Obviously, we can take an embedding $\phi : F_x \times [0,1] \to M$ which gives equivalence relation between F_x and F_y such that $\phi(F_x \times (0,1)) = U$. This contradicts the assumption that each pair of elements of $\{L_j^{(0)}\}$ is not mutually equivalent. □

For an equivalence class $[L]$ represented by a depth 0 leaf L, $\cup L_\alpha$ denotes the union of the leaves of $\mathcal{F}$ representing $[L]$.

Claim Under the above notations, $\cup L_\alpha$ is closed.

Proof of Claim. Let $\{x_i\}_{i=1,2,\ldots}$ be a Cauchy sequence in M such that each x_i is contained in $\cup L_\alpha$ and converges to x_∞. We show that $x_\infty \in \cup L_\alpha$. Let L_∞ be the leaf of $\mathcal{F}$ which contains x_∞. Let P be a plaque of $\mathcal{F}^\perp$ through x_∞. We may suppose each x_i is contained in $P+$ or P_-, say P_+. Let P_+, P_- be the components of $P \setminus x_\infty$. Then, by retaking x_i if necessary, we may suppose that each x_i is contained in P_+. Let L_i be the leaf of $\mathcal{F}$ which contains x_i. Since L_i is compact, L_i intersects P finitely many times. Thus we may suppose that x_i is the nearest to x_∞ among all the points of $L_i \cap P$. Suppose L_∞ has unbounded holonomy. Let $\gamma, h_1, h_2, \ldots$ be as in Definition 2.9. Since x_i converges to x_∞, we may suppose $x_n \in \mathrm{im}(\gamma)$, for $n \gg 0$. Fix such n. Take δ_n such that $\gamma([0,\delta_n])$ is the subarc of $\mathrm{im}(\gamma)$ with endpoints x_∞, x_n. Since x_n is the nearest to x_∞, $h_i(\mathrm{im}(\gamma)) \not\subset \gamma([0,\delta_n])$ for any i, a contradiction. Hence L_∞ does not have unbounded holonomy on the side which contains x_i. By Theorem 2.10, L_∞ is semistable on the side which contains x_i. Let e_i be as in Definition 2.9. For each L_i, there exists j such that $e_j(\widetilde{L}_\infty \times [0,1]) \supset L_i$. Since $\mathrm{depth}(L_i) = 0$, L_∞ is equivalent to L_i (Lemma 2.20). Thus x_∞ is contained in $\cup L_\alpha$. □

Now, we describe how to modify $\mathcal{F}$ near L to obtain a new finite depth foliation $\mathcal{F}^1$. The situation is divided into the following two cases.

Case 1 There exist more than one leaves of $\mathcal{F}$ representing $[L]$.

Case 2 There exists exactly one leaf of $\mathcal{F}$ representing $[L]$.

In Case 1, let $\{\phi_\beta\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$. Let $\mathcal{U} = \cup \phi_\beta(L \times [0,1])$. The situation is divided into the following two subcases.

Case 1.1 $\mathcal{U} \neq M$.

In this case, we first show the following claims (Claims 1∼3).

Claim 1 $\mathcal{U}$ is closed.

Proof of Claim 1. Let $\{x_i\}_{i=1,2,\ldots}$ be a Cauchy sequence in $\mathcal{U}$ which converges to x_∞. Assume $x_\infty \notin \mathcal{U}$. This implies that for each β, $x_\infty \notin \phi_\beta(L \times [0,1])$. For each i, we fix an embedding ϕ_i giving an equivalence relation such that $x_i \in \phi_i(L \times [0,1])$. Let L_∞ be the leaf of $\mathcal{F}$ which contains x_∞. Let $L_i = \phi_i(L \times \{1\})$ such that $L_i \neq L$. Let P be the plaque of $\mathcal{F}^\perp$ through x_∞. We may suppose each x_i is contained in P. Let P_+, P_- be the components of $P \setminus x_\infty$. By taking subsequence if necessary, we may suppose all of the x_i's are contained in P_+ or P_-, say P_+. Let y_i be the point of $L_i \cap P_+$ which is the nearest to x_∞. Then, by the same argument as in the proof of Claim given soon after the proof of Lemma 3.2 with regarding the above y_i as x_i, L_∞ is equivalent to L, hence $x_\infty \in \mathcal{U}$, a contradiction. □

Let τ be a leaf of $\mathcal{F}^\perp|_{\mathcal{U}}$ which meets a component of $\partial\mathcal{U}$, say L_0.

Claim 2 The leaf τ is an arc properly embedded in $\mathcal{U}$.

Proof of Claim 2. Assume not, i.e., τ meets $\partial\mathcal{U}$ in one point, say x_0. Let $\{x_i\}_{i=1,2,\ldots}$ be a sequence of points on τ such that $d_\tau(x_0, x_i) > i$, where d_τ is the path metric on τ induced from M. By the above Claim 1, $\mathcal{U}$ is compact. Hence there exists an accumulating point of $\cup x_i$. By taking a subsequence of $\{x_i\}_{i=1,2,\ldots}$ if necessary, we may suppose that x_i converges to x_∞. Since $\mathcal{U}$ is closed, $x_\infty \in \mathcal{U}$. Note that $x_\infty \notin \partial\mathcal{U}$ because if $x_\infty \in \partial\mathcal{U}$, then a plaque of $\mathcal{F}^\perp$ through x_n for $n >> 0$ intersects $\partial\mathcal{U}$, contradicting the fact that $\tau \cap \partial\mathcal{U} = x_0$. We have the following two cases.

Case A $x_\infty \in L$.

In this case, since $x_\infty \notin \partial\mathcal{U}$, we have $L_0 \neq L$. Let ϕ_0 be an embedding which gives equivalence relation between L and L_0. Since x_i converges to x_∞, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_0(L \times [0,1])$ is a union of infinitely many leaves of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$ contained in τ, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Case B $x_\infty \notin L$.

In this case, there exists $\phi_\infty \in \{\phi_\beta\}$ such that $x_\infty \in \phi_\infty(L \times [0,1])$. Let $L_\infty = \phi_\infty(L \times \{1\})$. On the other hand, since $\mathcal{U}$ is closed, $L_0 \subset \mathcal{U}$. This implies that L_0 is equivalent to L. Suppose $L_0 = L$. Since $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many leaves of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$ those are contained in τ, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Suppose $L_0 \neq L$. Let ϕ_0 be an embedding which gives equivalence relation between L and L_0. The situation is divided into the following two cases.

Case B.1 $\phi_0(L \times [0,1]) \cap \phi_\infty(L \times [0,1]) = \phi_\infty(L \times [0,1])$.

In this case, since the length of each fiber of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$ is finite, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many leaves of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$. Hence $\tau \cap \phi_0(L \times [0,1])$ is also a union of infinitely many subarcs of τ which are properly embedded in $\phi_0(L \times [0,1])$, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Case B.2 $\phi_0(L \times [0,1]) \cap \phi_\infty(L \times [0,1]) = L$.

In this case, by applying the argument as in Case B.1, we can show that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many leaves of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$. Since each leaf of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$ is adjacent to a leaf of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$, $\tau \cap \phi_0(L \times [0,1])$ is also a union of infinitely many leaves of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$. This contradicts the assumption that τ meets $\partial\mathcal{U}$ in one point. □

Claim 3 The boundary of $\mathcal{U}$ consists of two components.

Proof of Claim 3. Since $\mathcal{F}$ is transversely oriented, we see by Claim 2 of Case 1.1 that $\partial\mathcal{U}$ consists of at least two components. Suppose $L \subset \partial\mathcal{U}$. Let L'' be another component of $\partial\mathcal{U}$. Since $\mathcal{U}$ is closed, $L'' \subset \mathcal{U}$. Hence L'' is equivalent to L. Let ϕ'' be an embedding which gives equivalence relation between L and L''. Then, it is obvious that $\mathcal{U} = \phi''(L \times [0,1])$, hence $\partial\mathcal{U} = L \cup L''$. Suppose $L \subset \operatorname{int}\mathcal{U}$. Let L_1, L_2 be different components of $\partial\mathcal{U}$. Let ϕ_1 (ϕ_2 resp.) be an embedding which gives equivalence relation between L and L_1 (L_2 resp.). Then it is obvious that $\mathcal{U} = \phi_1(L \times [0,1]) \cup \phi_2(L \times [0,1])$. Hence $\partial\mathcal{U} = L_1 \cup L_2$. This completes the proof of the claim. □

Let $\partial\mathcal{U} = L_\infty \cup L_{-\infty}$. Obviously, L_∞ is equivalent to $L_{-\infty}$, i.e., there exists $\phi^* : L_\infty \times [0,1] \to M$ such that $\phi^*(L_\infty \times [0,1]) = \mathcal{U}$. Now, we modify $\mathcal{F}$ by replacing $\mathcal{F}|_\mathcal{U}$ with the image of the product foliation on $L_\infty \times [0,1]$. The modification near the depth 0 leaf L is completed.

Case 1.2 $\mathcal{U} = M$.

Claim 1 For each side of L, there is a leaf L' to which L is equivalent through the side.

Proof of Claim 1. Fix a side L. Let P be a plaque of $\mathcal{F}^{\perp}$ through $x \in L$. Let P' be the component of $P \setminus x$ corresponding to the side. Let x_i be a sequence of points in P_+ which converges to x. Since $\mathcal{U} = M$, for each i, there exists a leaf L_i to which L is equivalent via embedding $\phi_i : L \times [0,1] \to M$ such that $\phi_i(L \times [0,1]) \ni x_i$. If there exists i such that $\phi_i(L \times [0,1])$ contains a subarc of P' with endpoints x and x_i, then the leaf $\phi_i(L \times \{1\})$ is equivalent to L through the side corresponding to P'. Suppose for each i, $\phi_i(L \times [0,1])$ does not contain a subarc of P' with endpoints x and x_i. Let $L_i = \phi_i(L \times \{1\})$. Then L_i is a depth 0 leaf such that $L_i \cap P$ contains a point y_i such that $y_i = x_i$ or y_i is nearer to x than x_i in P'. By applying the argument as in the proof of Claim given soon after the proof of Lemma 3.2, we see that L is semistable on the side corresponding to P'. Hence by Lemma 2.20, there exists a leaf L' to which L is equivalent through the side corresponding to P', and this completes the proof of the claim. $\square$

Let $\{\phi_\beta^{\pm}\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$ through the side $\widetilde{L}_{\pm}$. Let $\mathcal{U}^{\pm} = \cup \phi_\beta^{\pm}(L \times [0,1])$.

Remark 3.3. The above Claim 1 implies that $\mathcal{U}^+ \neq \emptyset$ and $\mathcal{U}^- \neq \emptyset$.

Claim 2 Both $\mathcal{U}^+$ and $\mathcal{U}^-$ are closed.

Proof of Claim 2. Since the situation is symmetric, we give the proof for $\mathcal{U}^+$. If $\mathcal{U}^+ = M$, then the claim clearly holds. Suppose $\mathcal{U}^+ \neq M$. We can apply the same argument as in the proof of Claim 1 of Case 1.1. to show that $\mathcal{U}^+$ is closed. $\square$

Remark 3.4. $\mathcal{U}^+$ and $\mathcal{U}^-$ coincide with M or homeomorphic to $L \times [0,1]$.

Claim 3 There exists a point y in M such that $y \notin L$ and $y \in \mathcal{U}^+ \cap \mathcal{U}^-$.

Proof of Claim 3. If $\mathcal{U}^+ = M$, the claim clearly holds. Suppose $\mathcal{U}^+ \neq M$. Let x be a point in L. Let τ_+ be the leaf of $\mathcal{F}^{\perp} |_{\mathcal{U}^+}$ which meets x. Note that $\mathcal{U}^+ \neq \emptyset$ (Remark 3.3). By Claim 2 of Case 1.1, τ_+ is an arc properly embedded in $\mathcal{U}^+$. Let $y = \partial\tau_+ \setminus x$ and L^+ the leaf containing y. Since $\mathcal{U}^+ \cup \mathcal{U}^- = M$ and M is closed, for each point y in L^+, every neighborhood of y contains a point in $\mathcal{U}^-$. Since $\mathcal{U}^-$ is closed, this shows $y \in \mathcal{U}^-$. $\square$

Claim 4 There exists a leaf L_* to which L is equivalent through both sides of L.

Proof of Claim 4. By Claim 3, we can take a point y in $\mathcal{U}^+ \cap \mathcal{U}^-$. Let $\phi^{\pm}$ be an embedding from $L \times [0,1]$ to M which gives equivalence relation through the side $\widetilde{L}_{\pm}$ such that $\phi^{\pm}(L \times [0,1])$ contains y. Let $L^{\pm} = \phi^{\pm}(L \times \{1\})$. If $L^+ = L^-$, then L is equivalent to $L^+ = L^-$ through both sides of L. Suppose $L^+ \neq L^-$. In this case, $L^+ \subset \mathrm{int}\phi^-(L \times [0,1])$. Since L^+ is transverse to $\mathcal{F}^{\perp}$, L is equivalent to L^+ through the side $\widetilde{L}^-$. Thus L^+ satisfies the condition of L_*. $\square$

By joining the embeddings giving equivalence relation between L and L_* in the above Claim 4, we see that there is an immersion $\phi' : L \times [0,1] \to M$ such that the image of $\{x\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^{\perp}$, $\phi'(L \times \{0\}) = \phi'(L \times \{1\}) = L$. Hence M admits a fiber bundle structure over S^1 with each fiber homeomorphic to

L and transverse to $\mathcal{F}^\perp$, and L is a fiber. In this case, we let $\mathcal{F}^1$ be the foliation given by this bundle structure, i.e., each leaf of $\mathcal{F}^1$ is a fiber of the fibration.

In Case 2 (the case that there exists exactly one leaf of $\mathcal{F}$ representing $[L]$), let $U = M \setminus L$. Then $\partial\hat{U} = L_+ \cup L_-$, where L_+ (L_- resp.) is homeomorprhic to L. The situation is divided into the following two subcases.

Case 2.1 There exists a homeomorphism $h : L \times [0,1] \to \hat{U}$ such that the image of each $x \times [0,1]$ is a leaf of $\hat{\mathcal{F}}^\perp$.

In this case M admits a fiber bundle structure over S^1 with each fiber homeomorphic to L and transverse to $\mathcal{F}^\perp$, and L is a fiber. Then, we let $\mathcal{F}^1$ be the foliation given by this bundle structure,

Case 2.2 There does not exist a homeomorphism from $L \times [0,1]$ to $\hat{U}$ as in Case 2.1.

In this case $\mathcal{F}$ is unchanged by the modification.

In Cases 1.1 and 2.2, we further modify the foliation by using another equivalence class of the depth 0 leaves. By Lemma 3.2, this terminates in finitely many steps. Let $\mathcal{F}^1$ be the foliation which is obtained by repeating the procedure for all equivalence classes of the depth 0 leaves. Namely, $\mathcal{F}^1$ satisfies the following.

Property 3.5. Let L_1, L_2 be depth 0 leaves of $\mathcal{F}^1$. Suppose that L_1 is equivalent to L_2 via an embedding $\phi : L_1 \times [0,1] \to M$, then $\mathcal{F}^1|_{\phi(L_1\times[0,1])}$ is a product foliation.

Note that this modification does not change the transverse foliation $\mathcal{F}^\perp$, i.e., $\mathcal{F}^{1\perp} = \mathcal{F}^\perp$.

3.2. **Second step of Modification.** Suppose $\mathcal{F}^1$ has a depth 1 leaf. In this subsection, we describe a procedure for modifying $\mathcal{F}^1$ obtained in Section 3.1 by using depth 1 leaves of $\mathcal{F}^1$.

Lemma 3.6. *For the modified foliation $\mathcal{F}^1$, the number of the equivalence classes represented by the depth 1 leaves is finite.*

Proof. Let $\{L_k^{(1)}\}$ be a set of depth 1 leaves of $\mathcal{F}^1$ such that each pair of elements is not mutually equivalent. We assume that $\{L_k^{(1)}\}$ has infinitely many elements. By Theorem 2.10, Lemma 2.14 and Lemma 2.20, we can show that each leaf of $\{L_k^{(1)}\}$ is isolated in $\cup L_k^{(1)}$. By slightly modifying the $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ if necessary, we may suppose that $\overline{(\cup L_k^{(1)})} \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Hence we can find U, a component of $M \setminus \overline{\cup L_k^{(1)}}$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. For any point x in $\partial\hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^{1\perp} (= \hat{\mathcal{F}}^\perp)$ which meets x. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some i and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints x and y, say. Since $\mathcal{F}$ is transversely oriented, x and y are contained in different components of $\partial\hat{U}$. Let F_x (F_y resp.) be the leaf of $\mathcal{F}^1$ which meets $\hat{\imath}(x)$ ($\hat{\imath}(y)$ resp.). Then we immediately have the following.

Claim 1 $\hat{U}$ is homeomorphic to $F_x \times [0,1]$, where each $\{p\} \times [0,1]$ is correspondig to a leaf of $\hat{\mathcal{F}}^\perp$.

Moreover we have the following.

Claim 2 If $F_x \neq F_y$, then $\mathcal{F}^1|_U$ is a product foliation with each leaf is at depth 0.

Proof of Claim 2. By Lemma 2.17, F_x (F_y resp.) is either depth 0 or depth 1. If F_x or F_y is a depth 1 leaf, then obviously F_x is equivalent to F_y, this contradicts the assumption that each pair of elements of $\{L_k^{(1)}\}$ is not mutually equivalent. Hence F_x, F_y are depth 0 leaves. By Property 3.5, $\mathcal{F}^1|_U$ is a product foliation with each leaf is at depth 0. □

Suppose $F_x = F_y$. Let $\widetilde{U} = U \cup F_x$.

Claim 3 Under the above conditions, we have the following:

1. F_x is a depth 1 leaf;
2. $\partial\widetilde{U} = \overline{F_x} \setminus F_x$; and
3. F_x is the only element of $\{L_k^{(1)}\}$ which meets $\widetilde{U}$.

Remark 3.7. By 1 and 2 of the above Claim 3, we see that $\partial\widetilde{U}$ consists of depth 0 leaves.

Proof of Claim 3. By Lemma 2.18, we see that 1 of the claim holds. We show that $\partial\widetilde{U} \supset \overline{F_x} \setminus F_x$. Note that since F_x is a depth 1 leaf, $\overline{F_x} \setminus F_x$ is a union of depth 0 leaves. Let L_0 be a leaf contained in $\overline{F_x} \setminus F_x$. Since $U \cong F_x \times (0,1)$(the above Claim 1), F_x is noncompact, and each leaf of $\hat{\mathcal{F}}^1$ is transverse to $\hat{\mathcal{F}}^\perp$, every leaf of $\mathcal{F}^1|_U$ is noncompact. Since L_0 is compact, this shows that $L_0 \cap U = \emptyset$. Hence $L_0 \cap \widetilde{U} = \emptyset$. On the other hand, since $L_0 \subset \overline{F_x}$, we have $L_0 \subset \overline{\widetilde{U}}$. These imply $L_0 \subset \partial\widetilde{U}$, thus $\partial\widetilde{U} \supset \overline{F_x} \setminus F_x$. Then we show that $\partial\widetilde{U} \subset \overline{F_x} \setminus F_x$. Let a be a point in $\partial\widetilde{U}$ and N_a a neighborhood of a. Then there exist points b_1 and b_2 of N_a such that $b_1 \in \widetilde{U}$ and $b_2 \notin \widetilde{U}$. Suppose $b_1 \notin F_x$. Take an arc $\overline{b_1 b_2}$ in N_a connecting b_1 and b_2. Then there is a point $b \in \overline{b_1 b_2}$ such that $b \in F_x$. This shows that for any neiborhood N_a, there is a point of F_x in N_a, which implies $a \in \overline{F_x}$. Note that $\widetilde{U}$ is homeomorphic to a manifold obtained from $F_x \times [0,1]$ by identifying $F_x \times \{0\}$ and $F_x \times \{1\}$ with F_x corresponding to $F_x \times \{0\}$ $(= F_x \times \{1\})$. This implies $F_x \subset \operatorname{int} \widetilde{U}$. This shows that $a \notin F_x$. These show $\partial\widetilde{U} \subset \overline{F_x} \setminus F_x$, and 2 of the claim holds. We see by Claim 1 that $\partial\hat{U}$ consists of two components. Since these are identified in M, it is clear that 3 holds. □

We know that the number of the equivalence classes represented by the depth 0 leaves of $\mathcal{F}$ is finite (Lemma 3.2). Since the modification does not change the number of the equivalence classes represented by the depth 0 leaves, the number of the equivalence classes represented by the depth 0 leaves of $\mathcal{F}^1$ is also finite. This fact together with the above Claim 2, 3 of the above Claim 3 and Remark 3.7 imply that there exist at most finitely many components of $M \setminus \overline{\cup L_k^{(1)}}$ which are disjoint from $\bigcup_{i=1}^m \varphi(D^{n-1} \times \partial[0,1])$. Since there exist at most $2m$ components of $M \setminus \overline{\cup L_k^{(1)}}$ which intersect $\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])$, this shows that $\{L_k^{(1)}\}$ consists of finitely many elements, a contradiction. □

Now, we modify the foliation $\mathcal{F}^1$. Since the number of the equivalence classes represented by the depth 0 leaves is finite (Lemma 3.2) and $\mathcal{F}^1$ satisfies Property 3.5, $M \setminus \cup$(depth 0 leaves) consists of finite number of components, say $U_1, U_2, \ldots, U_k$.

Note that there is a depth 1 leaf in each U_i. Let $L(\subset U_1)$ be a depth 1 leaf. The situation is divided into the following two cases.

Case 1 There exist more than one leaves of $\mathcal{F}^1$ representing $[L]$.

Case 2 There exists exactly one leaf of $\mathcal{F}^1$ representing $[L]$.

In Case 1, let $\{\phi_\beta\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$. Let $\mathcal{U}^{(1)} = \cup\phi_\beta(L \times [0,1])$. The situation is divided into the following two subcases.

Case 1.1 $\mathcal{U}^{(1)} = U_1$.

In this case, by applying the argument as in the proof of Claim 4 of Case 1.2 in Section 3.1, we can show that there exists a depth 1 leaf $L' \subset U$ such that for each side of L, L is equivalent to L' through the side. This implies that $U_1 \setminus L \cong L \times (0,1)$ with each $x \times (0,1)$ is contained in a leaf of $\mathcal{F}^\perp$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L \times [0,1]$. Note that in this case, the modification on U_1 is completed.

Case 1.2 $\mathcal{U}^{(1)} \neq U_1$.

In this case, by applying the argument as in the proof of Claim 1 of Case 1.1 in Section 3.1, we can show that $\mathcal{U}^{(1)}$ is closed, which implies that $\mathcal{U}^{(1)} \cong L \times [0,1]$ with each $x \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L \times [0,1]$.

Case 2 is divided into the following two subcases.

Case 2.1 There exists a homeomorphism $h : L \times (0,1) \to U_1 \setminus L$ such that the image of each $x \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$.

In this case, we replace $\mathcal{F}^1|_{U_1 \setminus L}$ by the image of the product foliation on $L \times [0,1]$. Note that in this case, the modification on U_1 is completed.

Case 2.2 There does not exist a homeomorphism h as in Case 2.1.

In this case $\mathcal{F}^1$ is unchanged by the modification.

In Cases 1.2 and 2.2, we further modify the foliation for another equivalence class of a depth 1 leaf in U_1. Then repeat the procedure to modify the foliation restricted to U_1. Since the number of equivalent classes of depth 1 leaf is finite (Lemma 3.6), this terminates in finitely many steps. Then the desired foliation $\mathcal{F}^2$ is obtained by applying the procedure for all U_i's. Since $\{U_i\}$ consists of finitely many components, this terminates in finitely many steps.

Note that this modification does not change the transverse foliation $\mathcal{F}^\perp$, i.e., $\mathcal{F}^{2\perp} = \mathcal{F}^\perp$.

By the construction, we see that $\mathcal{F}^2$ satisfies the following.

Property 3.8. Let L_1, L_2 be depth 0 or depth 1 leaves of $\mathcal{F}^2$. Suppose that L_1 is equivalent to L_2 and $L_1 \neq L_2$. If $\phi : L_1 \times [0,1] \to M$ is an embedding giving an equivalence relation, then $\mathcal{F}^2|_{\phi(L_1 \times [0,1])}$ is a product foliation.

3.3. **Completion of modification.** It is easy to see that the above arguments work for depth 2 leaves of $\mathcal{F}^2$, to obtain a modified foliation $\mathcal{F}^3$, and so on, that is, we can gradually modify the foliation $\mathcal{F}$ by using depth 0 leaves, depth 1 leaves, $\ldots$. Since depth$(\mathcal{F}) < \infty$, this terminates in finitely many steps to obtain a finite depth foliation, say $\widetilde{\mathcal{F}}$.

Note that once $\mathcal{F}^\perp$ is fixed, then $\widetilde{\mathcal{F}}$ is uniquely determined. Moreover, by the construction, it is easy to show the following.

Facts 3.9. 1. The number of the equivalence classes of the leaves of $\widetilde{\mathcal{F}}$ is finite.
2. Let L_1, L_2 be leaves of $\widetilde{\mathcal{F}}$. Suppose that L_1 is equivalent to L_2, via an embedding $\phi : L_1 \times [0,1] \to M$. Then $\widetilde{\mathcal{F}}|_{\phi(L_1\times[0,1])}$ is a product foliation.
3. $\mathrm{depth}(\widetilde{\mathcal{F}}) \leq \mathrm{depth}(\mathcal{F})$.

Here, we note that 1,2 of Facts 3.9 is exactly Properties 2.21.

4. Theorem 1.1

In this section, we give a proof of Theorem 1.1. Firstly, we give some definitions of terminologies that appears in the theorem.

Let K' be a knot in S^3, and $V = D^2 \times S^1$ an unknotted solid torus in S^3. Let L be a link in V such that L is not contained in any 3-ball in V, and $h : V \to N(K')$ a homeomorphism. Then the link $h(L)$ is called a *satellite* for K', and K' is called a *companion* for $h(L)$. Let ℓ (m resp.) be a longitude (a meridian resp.) of V. If L is a knot in ∂V representing a homology class $p[m]+q[\ell](\in H_1(\partial V,\mathbb{Z}))$ with $|q| \geq 2$ then $h(L)$ is called a *cable* of K'. We say that a knot K is a *cable knot* if there exists a knot K'' such that K is a cable of K''. A knot which is not a cable knot is called a *non-cable knot*. Let C be the knot in V as in Figure 4.1 and ℓ' (m' resp.) ($\subset \partial N(K')$) a longitude (a meridian resp.) of $N(K')$. For an integer q, let $h_q : V \to N(K')$ be a homeomorphism with $(h_q)_*([m]) = [m'], (h_q)_*([\ell]) = [\ell'] + q[m']$. Then, we call the satellite $h_q(C)$ a *q-twisted double* of K'. Let S be the genus one surface in V, as in Figure 4.2. Clearly $h_q(S)$ is a Seifert surface for $h_q(C)$. We often use the notation S_q for denoting this Seifert surface. Note that if K' is a non-trivial knot, then for any $q \in \mathbb{Z}$, $h_q(C)$ is a non-trivial knot [8, IV.10]. Since S_q is of genus one, this implies that if K' is a non-trivial knot, then S_q is a minimal genus Seifert surface for $h_q(C)$. We call S_q a *standard Seifert surface* for $h_q(C)$.

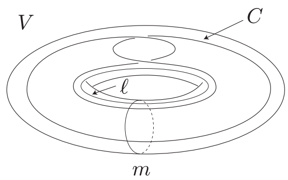

Figure 4.1

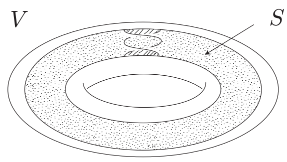

Figure 4.2

Proof of Theorem 1.1 Recall that $S^3(K,0)$ is the manifold obtained from S^3 by performing 0-surgery along K. Let T be a surface in $S^3(K,0)$ obtained from S_q by capping off its boundary with a meridian disk of the solid torus attached to $E(K)$. Let M_S be the manifold obtained from $S^3(K,0)$ by cutting along T. Recall that $\Sigma^{(n)}(K,0)$ is the n-fold cyclic covering space of $S^3(K,0)$. Note that $\Sigma^{(n)}(K,0)$

admits a decomposition $\Sigma^{(n)}(K,0) = M_1 \cup \cdots \cup M_n$ where each M_i $(i = 1, \ldots, n)$ is homeomorphic to M_S and $M_1, \ldots, M_n$ are arrayed cyclically i.e., $M_i \cap M_{i+1} = \partial M_i \cap \partial M_{i+1}$ consists of a torus, say T_i, which is a lift of T (if $n > 2$) (subscript is taken in mod n) or $M_1 \cap M_2 = \partial M_1 = \partial M_2$ $(= T_1 \cup T_2$, say)(if $n = 2$). Let $\mathcal{F}$ be a codimension one, transversely oriented, taut C^0 foliation of finite depth, with C^∞ leaves on $\Sigma^{(n)}(K,0)$ with exactly one depth 0 leaf $\hat{T}$ representing α, where α is corresponding to a generator of $H_1(S^3(K,0)) \cong \mathbb{Z}$. Let $\widetilde{\mathcal{F}}$ be the foliation obtained by modifying $\mathcal{F}$ as in Section 3, hence $\widetilde{\mathcal{F}}$ satisfies Properties 2.21. Let S be a standard Seifert surface for K. Since $\hat{T}$ is the compact leaf of the taut foliation $\check{\mathcal{F}}$, $\hat{T}$ is taut, i.e., incompressible and norm minimizing. Note that $[\hat{T}] = \alpha = \pm[T_i]$ $(1 \le i \le n)$ and each T_i is a torus. Hence $\hat{T}$ is a torus or a 2-sphere. Assume that $\hat{T}$ is a 2-sphere. Theorem 3 of [9] implies that $\Sigma^{(n)}(K,0) \cong S^2 \times S^1$. By Theorem 7 of [6] , $\Sigma^{(n)}(K,0) \cong S^2 \times S^1$ implies that $S^3(K,0) \cong S^2 \times S^1$. However since K is a non-trivial knot, Corollary 8.3 (and its remark) [5] implies that $S^3(K,0) \ncong S^2 \times S^1$. Hence $\check{T}$ is an incompressible torus.

Then, by applying the argument as in the proof of Claim in Section 4 of [7], we may assume that the compact leaf of $\widetilde{\mathcal{F}}$ is isotopic to T_n. Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K,0)$ by cutting along T_n. Then, $M^{(n)}$ clearly corresponds to $\hat{M}$ which appears in the paragraph preceding Lemma 2.22 in Section 2.3. Then, we abuse notation T_n for denoting the component of $\partial M^{(n)}$ such that $T_n \subset M_n$ and T_0 denotes the other component of $\partial M^{(n)}$. Let $\hat{\mathcal{F}}$ be the foliation on $M^{(n)}$ induced from $\widetilde{\mathcal{F}}$. Let $\hat{G}(\hat{\mathcal{F}})$ and $G(\widetilde{\mathcal{F}})$ be as in Definitions 2.23 and 2.24. Let $g = \mathrm{gap}(\widetilde{\mathcal{F}})$. Since $M^{(n)}$ is not homeomorphic to (torus) $\times [0,1]$, we see that $\widetilde{\mathcal{F}}$ is not a foliation given by a surface bundle structure over S^1. Thus $G(\widetilde{\mathcal{F}})$ has an edge, hence $g \ge 1$. By the construction of $\widetilde{\mathcal{F}}$ described in Section 3, we see that $\widetilde{\mathcal{F}}$ contains exactly one depth 0 leaf. These imply that $\hat{G}(\hat{\mathcal{F}})$ contains exactly two vertices at depth 0.

Lemma 4.1. *$\hat{G}(\hat{\mathcal{F}})$ is connected.*

Proof. Assume that $\hat{G}(\hat{\mathcal{F}})$ is not connected. Since the union of the compact leaves of $\hat{\mathcal{F}}$ is $T_0 \cup T_n$, there are exactly two vertices at depth 0. By Remark 2.28, any component of $\hat{G}(\hat{\mathcal{F}})$ must have a depth 0 vertex. Hence $\hat{G}(\hat{\mathcal{F}})$ consists of two components, say G_1 and G_2. Let $\{u_j\}$ ($\{v_k\}$ resp.) be the vertices of G_1 (G_2 resp.). Let U_j (V_k resp.) be the union of the leaves representing u_j (v_k resp.). Then we show that $\bigcup_j U_j$ is closed. Let $\{x_i\}_{i=1,2,\ldots}$ be a Cauchy sequence in $\hat{M}$ such that each x_i is contained in $\bigcup_j U_j$ and converges to x_∞. We show that $x_\infty \in \bigcup_j U_j$. Let L_∞ be the leaf of $\hat{\mathcal{F}}$ which contains x_∞. Let P be a plaque of $\hat{\mathcal{F}}^\perp$ through x_∞. We may suppose each x_i is contained in P. Let P_+, P_- be the components of $P \setminus x_\infty$. Then, by retaking x_i if necessary, we may suppose that each x_i is contained in P_+. If there is a leaf L of $\hat{\mathcal{F}}$ which contains infinitely many x_i, this implies that $\overline{L} \setminus L \supset L_\infty$. Thus there exists a path in G_1 connecting some u_j and the vertex representing L_∞. Hence we have $x_\infty \in \bigcup_j U_j$. Suppose there does not exist such L. Let L_i be the leaf of $\hat{\mathcal{F}}$ which contains x_i. Since L_i intersects P finitely many times, we may suppose that x_i is the nearest to x_∞ among all the points of $L_i \cap P$. By applying the arguments as in the proof of Claim given soon after the proof of Lemma 3.2, we can show that L_∞ is equivalent to L_i. This implies that $x_\infty \in \bigcup_j U_j$. Hence $\bigcup_j U_j$ is closed. By applying the arguments as above, we can show that $\bigcup_k V_k$ is also closed. Note that $M^{(n)} = (\bigcup_j U_j) \cup (\bigcup_k V_k)$ and that $(\bigcup_j U_j) \cap (\bigcup_k V_k) = \emptyset$, contradicting the fact that $M^{(n)}$ is connected. $\square$

We say that a graph Γ is a *tree* if Γ is connected and Γ does not contain a cycle.

Lemma 4.2. *The following two conditions are equivalent to each other.*

1. *$\hat{G}(\hat{\mathcal{F}})$ is a tree.*
2. *The number of the cycles of $G(\widetilde{\mathcal{F}})$ is one.*

Proof. We first show that 1 implies 2. Suppose 1 holds. Recall that the number of the depth 0 vertices of $\hat{G}(\hat{\mathcal{F}})$ is two, and $G(\widetilde{\mathcal{F}})$ is obtained from $\hat{G}(\hat{\mathcal{F}})$ by identifying them. Since $\hat{G}(\hat{\mathcal{F}})$ is a tree, $\hat{G}(\hat{\mathcal{F}})$ does not have a cycle and there is a unique path in $\hat{G}(\hat{\mathcal{F}})$ connecting the depth 0 vertices, the path becomes a cycle in $G(\widetilde{\mathcal{F}})$ and this is the only cycle in $G(\widetilde{\mathcal{F}})$.

Suppose 2 holds. Since $\hat{G}(\hat{\mathcal{F}})$ is connected (Lemma 4.1), we only need to prove that $\hat{G}(\hat{\mathcal{F}})$ does not have a cycle. Assume that $\hat{G}(\hat{\mathcal{F}})$ has a cycle. By applying the argument as above, a path in $\hat{G}(\hat{\mathcal{F}})$ connecting the depth 0 vertices become a cycle in $G(\widetilde{\mathcal{F}})$ and since the operation obtaining $G(\widetilde{\mathcal{F}})$ does not remove a cycle, this implies that the number of the cycles of $G(\widetilde{\mathcal{F}})$ is two, a contradiction. $\square$

In the following, we suppose that $\hat{G}(\hat{\mathcal{F}})$ is a tree. Let Γ be the path connecting the depth 0 vertices of $\hat{G}(\hat{\mathcal{F}})$. Then, clearly $\mathrm{gap}(\hat{\mathcal{F}}) = g$. Suppose $g = 1$. Note that Theorem in [7] implies that $\mathrm{depth}(\mathcal{F}) \geq 1 + \left[\frac{n}{2}\right] \geq \frac{1+n}{2}$. Thus Theorem 1.1 holds.

Hence in the remainder of this proof, we suppose $g > 1$.

Lemma 4.3. *There exists exactly one edge, say e, of $\hat{G}(\hat{\mathcal{F}})$ with $\mathrm{length}(e) > 1$. (Hence we have $\mathrm{length}(e) = g$.)*

Proof. Let e be an edge of $\hat{G}(\hat{\mathcal{F}})$ such that $\mathrm{length}(e) > 1$. Let v, v' be the endpoints of e such that $\mathrm{depth}(v) < \mathrm{depth}(v')$. Let Γ_1 (Γ_2 resp.) be a directed path from v (v' resp.) to a vertex at depth 0 such that each edge of Γ_1 (Γ_2 resp.) has length one (Remark 2.28). Here we regard the vertex v as Γ_1 if $\mathrm{depth}(v) = 0$. Then, since $\hat{G}(\hat{\mathcal{F}})$ is a tree and the number of the vertices at depth 0 is two, it is clear that $\Gamma = \Gamma_1 \cup e \cup \Gamma_2$. Take any edge e' of $\hat{G}(\hat{\mathcal{F}})$ with $\mathrm{length}(e') > 1$. By applying the argument as above, we can show that there exist directed paths Γ_1', Γ_2' from the endpoints of e' to the depth 0 vertices, each edge of which has length one. Moreover we have $\Gamma = \Gamma_1' \cup e' \cup \Gamma_2'$. Since each edge of Γ_1, Γ_2, Γ_1', Γ_2' has length one, this shows that $e' = e$. Hence e is the only edge of length greater than one, thus we have $\mathrm{length}(e) = g$. $\square$

Let v, v', Γ_1, Γ_2 be as in the proof of Lemma 4.3. Since the situation is symmetric, we may suppose Γ_1 (Γ_2 resp.) contains the vertex representing T_0 (T_n resp.). Let m be the number of edges of Γ_1. Since $\mathrm{gap}(\hat{\mathcal{F}}) = g$, the number of edges of Γ_2 is $m+g$. Rename the vertices in $\Gamma_1 \cup \Gamma_2$ by $v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m+g+1}$ so that v_i $(0 \leq i \leq 2m+g+1)$ are on Γ in this order, and that $v_0 = [T_0]$, $v_{2m+g+1} = [T_n]$.

Let L_k be a leaf representing v_k. Let $\mathbb{T} = \bigcup_{i=1}^{n-1} T_i$. Let $\mathring{T}_0, \mathring{T}_n$ be tori in $\mathrm{int} M^{(n)}$ such that $\mathring{T}_0$ ($\mathring{T}_n$ resp.) is parallel to T_0 (T_n resp.), and $\mathring{T}_0 \cap \mathbb{T} = \emptyset$ ($\mathring{T}_n \cap \mathbb{T} = \emptyset$ resp.). Since $\mathcal{F}$ is taut, we can show that by deforming $\mathbb{T}$ by ambient isotopy in $M^{(n)}$, we may suppose that $\mathbb{T}$ is transverse to $\mathcal{F}$. Let $M^{(n)\prime}$ be the closure of the component of $M^{(n)} \setminus (\mathring{T}_0 \cup \mathring{T}_n)$ which does not meet $\partial M^{(n)}$. Let M_i' be the closure of the component of $M^{(n)\prime} \setminus \mathbb{T}$ corresponding to M_i. Note that for $i \neq 1, n$, we have $M_i' = M_i$.

Claim depth$(\mathcal{F}) \geq m + g$.

Proof of Claim. It is clear that depth$(\hat{\mathcal{F}}) \geq \max\{\text{depth}(v_i)\}$. Note that v_{m+1} corresponds to v' in the proof of Lemma 4.3. Hence $\max\{\text{depth}(v_i)\} = \text{depth}(v_{m+1})$. Note that $\text{depth}(v_{m+1}) = \Sigma_{\varepsilon:\text{edges of } \Gamma_1} \text{length}(\varepsilon) + \text{length}(e)$ (see Figure 4.3). Since the length of each edge of Γ_1 is one, this implies that $\text{depth}(v_{m+1}) =$

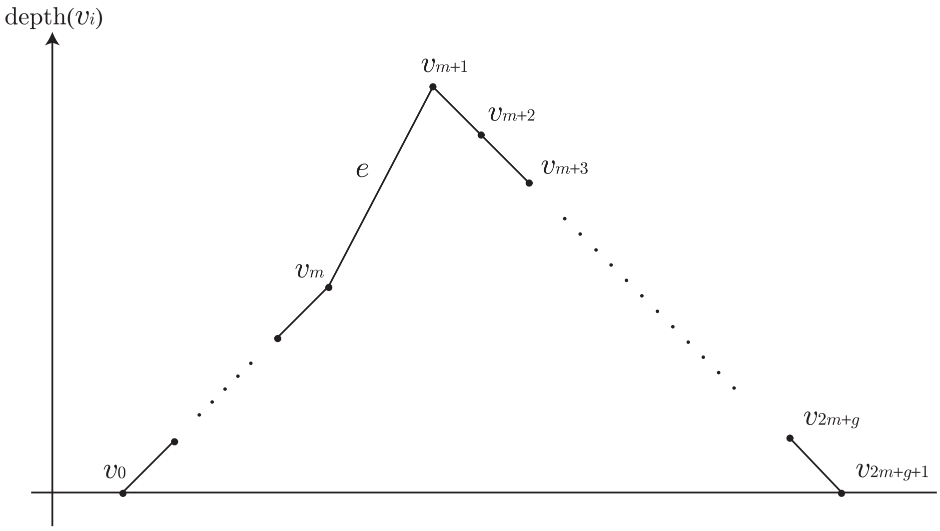

Figure 4.3

(number of edges of Γ_1) $+ g = m + g$. Hence we have depth$(\hat{\mathcal{F}}) \geq m + g$. By Lemma 2.22 and 3 of Facts 3.9, we see that depth$(\mathcal{F}) \geq$ depth$(\hat{\mathcal{F}}) \geq m + g$. $\square$

Now we estimate the value $m + g$. If $m \geq n$, we have $m + g \geq n + g > \frac{n+g}{2}$. By the above Claim, this shows that Theorem 1.1 holds. Hence in the remainder of this section, we suppose $m < n$.

Lemma 4.4. *There is an ambient isotopy f_t $(0 \leq t \leq 1)$ of $M^{(n)}$ whose support is contained in $\bigcup_{i=1}^{m+1} M_i$ satisfying the following two conditions:*

1. *$f_1(\mathbb{T})$ is transverse to $\hat{\mathcal{F}}$;*
2. *for k $(1 \leq k \leq m)$, $L_k \subset \bigcup_{i=1}^{k} \widetilde{M}_i$, where $\widetilde{M}_i$ is the closure of the component of $M^{(n)} \setminus f_1(\mathbb{T})$ corresponding to M_i.*

Proof. We consider for $k = 1$. Since the union of the compact leaves of $\mathcal{F}$ is $T_0 \cup T_n$, by applying the argument as in the proof of Assertion(i) in the proof of Lemma 3.3 of [7], we see that $L_1 \cap M^{(n)\prime} = \emptyset$ or $L_1 \cap M^{(n)\prime}$ is compact. Suppose $L_1 \cap M_2 \neq \emptyset$. Then, by applying the argument as in the proof of Lemma 4.2 of [7], we can show that there is an ambient isotopy f_t^1 $(0 \leq t \leq 1)$ whose support is contained in $M_1 \cup M_2$ such that $L_1 \subset \check{M}_1$, where $\check{M}_i$ is the closure of the component of $M^{(n)} \setminus f_1^1(\mathbb{T})$ corresponding to M_i, and that $f_1^1(\mathbb{T})$ is transverse to $\hat{\mathcal{F}}$. Suppose $L_1 \cap M_2 = \emptyset$. Then we let $f_t^1 = \text{id}_{M^{(n)}}$ $(0 \leq t \leq 1)$. Then, we consider for $k = 2$.

Suppose $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1} \neq \emptyset$. If $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1}$ is noncompact, then there exists a depth 1 leaf L'_1 such that $L'_1 \subset \overline{L_2}$ and $L'_1 \cap \overline{M^{(n)} \setminus \check{M}_1} \neq \emptyset$. Now, since $L_1 \subset \check{M}_1$, $L'_1 \neq L_1$. Let v'_1 be the vertex representing L'_1. We claim that $v'_1 \neq v_1$. In fact, if $v'_1 = v_1$, then there is an embedding $\phi : L_1 \times [0,1] \to M^{(n)}$ giving equivalence relation between L_1 and L'_1. Note that $\hat{\mathcal{F}}|_{\phi(L_1 \times [0,1])}$ is a product foliation, and $L_1 \cap T_1 = \emptyset$. These imply that there is a point x in $T_1 \cap \phi(L_1 \times [0,1])$ such that $\hat{\mathcal{F}}$ and T_1 are not transverse at x, a contradiction. By Remark 2.28, there exists a directed path Γ'_1 from v'_1 to v_0 or v_{2m+g+1}. This contradicts the assumption that $\hat{G}(\hat{\mathcal{F}})$ is a tree. Hence $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1}$ is compact. Suppose $L_2 \cap \check{M}_3 \neq \emptyset$. Then by applying the argument as in the proof of Lemma 3.10 of [7], we can show that there is an ambient isotopy f_t^2 whose support is contained in $\check{M}_2 \cup \check{M}_3$ such that $L_2 \subset \check{\check{M}}_1 \cup \check{\check{M}}_2$, where $\check{\check{M}}_i$ is the closure of the component of $M^{(n)} \setminus f_1^2(f_1^1(\mathbb{T}))$ corresponding to M_i, and $f_1^2(f_1^1(\mathbb{T}))$ is transverse to $\hat{\mathcal{F}}$. Suppose $L_2 \cap \check{M}_3 = \emptyset$. Then we let $f_t^2 = \mathrm{id}_{M^{(n)}}$ $(0 \leq t \leq 1)$. By applying the argument as above, we can obtain a sequence of ambient isotopies $f_t^1, f_t^2, \ldots, f_t^{m-1}, f_t^m$. Then, the desired ambient isotopy f_t is obtained by applying $f_t^1, f_t^2, \ldots, f_t^m$ successively in this order (with reparametrizing the parameter t). $\square$

In the following, we abuse notation $\mathbb{T}$ for denoting $f_1(\mathbb{T})$ for simplicity, hence for k $(1 \leq k \leq m)$, $L_k \subset \bigcup_{i=1}^k M_i$ holds. For k $(1 \leq k \leq m)$, let j_k be the integer which satisfies $L_k \cap M_{j_k} \neq \emptyset$ and $L_k \cap M_{j_k+1} = \emptyset$. We extend the definition of j_k by putting $j_0 = 0$. Since $L_k \subset \bigcup_{i=1}^k M_i$, we immediately have the following.

Lemma 4.5. *For k $(1 \leq k \leq m)$, we have $j_k \leq k$.*

Suppose $j_m \geq n-m-g+1$. By applying Lemma 4.5 for the case $k = m$, we have $j_m \leq m$. These inequalities imply $m \geq n-m-g+1$, hence $m+g \geq \frac{n+g+1}{2} > \frac{n+g}{2}$. This together with the claim in this section shows that Theorem 1.1 holds. Hence in the remainder of this section, we may suppose $j_m < n-m-g+1$. Note that Γ_2 contains $m+g+1$ vertices, $v_{m+1}, v_{m+2}, \ldots, v_{2m+g+1}$. By applying the argument as in the proof of Lemma 4.4 to leaves corresponding to the $m+g-1$ vertices $v_{2m+g}, v_{2m+g-1}, \ldots, v_{m+2}$, we can obtain the following lemma. (Note that $L_{2m+g+1-k'}(M_{n+1-k'}$ resp.) in Lemma 4.6 corresponds to $L_{k'}$ ($M_{k'}$ resp.) in Lemma 4.4.)

Lemma 4.6. *There is an ambient isotopy f'_t $(0 \leq t \leq 1)$ of $M^{(n)}$ whose support is contained in $\bigcup_{i=1}^{m+g} M_{n+1-i}$ satisfying the following two conditions:*

1. *$f'_1(\mathbb{T})$is transverse to $\hat{\mathcal{F}}$;*
2. *for k' $(1 \leq k' \leq m+g-1)$, $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} \widetilde{\widetilde{M}}_{n+1-i}$, where $\widetilde{\widetilde{M}}_{n+1-i}$ is the closure of the component of $M^{(n)} \setminus f'_1(\mathbb{T})$ corresponding to M_{n+1-i}.*

Note that since $j_m < n-m-g+1 = n+1-(m+g)$, f'_t does not change $\bigcup_{i=0}^m L_i$. In the following, we abuse notation $\mathbb{T}$ for denoting $f'_1(\mathbb{T})$ for simplicity, i.e., for k' $(1 \leq k' \leq m+g-1)$, $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i}$. For k' $(1 \leq k' \leq m+g-1)$, let $j'_{k'}$ be the integer which satisfies $L_{2m+g+1-k'} \cap M_{n-j'_{k'}+1} \neq \emptyset$, and $L_{2m+g+1-k'} \cap M_{n-j'_{k'}} = \emptyset$. Since $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i}$, we immediately have the following.

Lemma 4.7. *For $1 \leq k' \leq m+g-1$, we have $j'_{k'} \leq k'$.*

Then, we have the following.

Lemma 4.8. $n - j'_{m+g-1} \leq j_m + 1$.

Proof. Assume that $n - j'_{m+g-1} \geq j_m + 2$. By the definition of j_k, we see that $(M_{j_m+1} \cup M_{j_m+2}) \cap L_m = \emptyset$. Note that if $k' = m+g-1$, then $L_{2m+g+1-k'} = L_{m+2}$. Since $L_{2m+g+1-k'} \cap M_{n-j'_{k'}} = \emptyset$, this implies $L_{m+2} \cap M_{n-j'_{m+g-1}} = \emptyset$. Hence the assumption $n - j'_{m+g-1} \geq j_m + 2$ implies $(M_{j_m+1} \cup M_{j_m+2}) \cap L_{m+2} = \emptyset$. Since L_{m+1} approaches both L_m and L_{m+2}, L_{m+1} intersects both M_{j_m+1} and M_{j_m+2}. By applying the argument as in the proof of Lemma 3.10 of [7], we can show that this implies a contradiction. □

By Lemma 4.5, we see that $j_m \leq m$ and by Lemma 4.7, we see that $j'_{m+g-1} \leq m + g - 1$. These together with Lemma 4.8 imply that

$$n - m - g + 1 \leq m + 1.$$

Thus we obtain

$$m + g \geq \frac{n+g}{2}.$$

This together with the claim of this section shows that Theorem 1.1 holds.

This completes the proof of Theorem 1.1. □

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