# A CHARACTERIZATION OF SYMMETRIC CONES BY AN ORDER-REVERSING PROPERTY OF THE PSEUDOINVERSE MAPS

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Abstract. When a homogeneous convex cone is given, a natural partial order is introduced in the ambient vector space. We shall show that a homogeneous convex cone is a symmetric cone if and only if Vinberg's $*$-map and its inverse reverse the order. Actually our theorem is formulated in terms of the family of pseudoinverse maps including the $*$-map, and states that the above order-reversing property is typical of the $*$-map of a symmetric cone which coincides with the inverse map of the Jordan algebra associated with the symmetric cone.

## 1. Introduction

Let $\Omega$ be an open convex cone in a finite-dimensional real vector space $V$ which is regular, that is, $\overline{\Omega} \cap (-\overline{\Omega}) = \{0\}$, where $\overline{\Omega}$ stands for the closure of $\Omega$. For $x, y \in V$, we write $x \succeq_\Omega y$ if $x - y \in \overline{\Omega}$. Clearly, this defines a partial order in $V$. In the special case that $\Omega$ is the one-dimensional symmetric cone $\mathbb{R}_{>0}$, the order $\succeq_\Omega$ is the usual one and is reversed by taking inverse numbers in $\mathbb{R}_{>0}$. In general, let a symmetric cone $\Omega \subset V$ be given. Then $V$ has a structure of the Jordan algebra associated with $\Omega$. In this case, the Jordan algebra inverse map is an involution on $\Omega$ and reverses the order $\succeq_\Omega$. This order-reversing property helps our geometric understanding of the Jordan algebra inverse map. In this paper, we shall investigate this order-reversing property in a more general setting and give a characterization of symmetric cones.

One of natural ways to generalize a symmetric cone is to consider a *homogeneous convex cone*, that is, a regular open convex cone $\Omega \subset V$ on which its linear automorphism group

$$G(\Omega) := \{g \in GL(V) \mid g\Omega = \Omega\}$$

acts transitively. For a non-symmetric homogeneous convex cone, no algebraic structure of $V$ associated with $\Omega$ is known where a natural inverse map arises. However, it is known that the Jordan algebra inverse map associated with a symmetric cone can be generalized to an analytic map on a homogeneous convex cone which is called *Vinberg's $*$-map* [17].

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Let us recall its definition briefly to present our main theorem. Let $\Omega \subset V$ be a homogeneous convex cone. We denote by $\Omega^*$ the dual cone of $\Omega$:

$$\Omega^* := \{f \in V^* \mid \forall x \in \overline{\Omega} \setminus \{0\}, \langle x, f \rangle > 0\},$$

which is also a homogeneous convex cone. Let $\phi : \Omega \to \mathbb{R}_{>0}$ be the *characteristic function* of $\Omega$ defined by

$$\phi(x) := \int_{\Omega^*} e^{-\langle x,f \rangle}\, df \qquad (x \in \Omega),$$

where $df$ stands for a Euclidean measure on $V^*$. Vinberg's $*$-map $\Omega \ni x \mapsto x^* \in V^*$ is introduced by

$$\langle v, x^* \rangle = -D_v \log \phi(x) \qquad (v \in V, x \in \Omega), \tag{1.1}$$

where $D_v f(w) := \frac{d}{dt} f(w+tv)\big|_{t=0}$ $(v \in V, w \in \Omega)$ for functions $f$ on $\Omega$. This $*$-map is known to be a bijection from $\Omega$ onto $\Omega^*$. When $\Omega$ is a symmetric cone, the $*$-map coincides with the Jordan algebra inverse map under a suitable identification of $V^*$ with $V$, so that the $*$-map reverses the order (see Section 3). First we state a simple version of our main theorem:

**Theorem 1.1.** *Let $\Omega$ be a homogeneous convex cone. Then the following conditions are equivalent*:

(A) *The cone $\Omega$ is a symmetric cone.*
(B) *For $x, y \in \Omega$, we have $x \succeq_\Omega y$ if and only if $y^* \succeq_{\Omega^*} x^*$.*
(C) *For $x, y \in \Omega$, the pair $(x, y)$ is comparable if and only if $(x^*, y^*)$ is comparable.*

Actually our theorem is stronger and is formulated in terms of the *family of pseudoinverse maps* introduced in [10] which contains Vinberg's $*$-map as a special member. This family is defined as follows. By [17, Chapter I, Theorem 1], we know that there exists a maximal connected split solvable subgroup $H \subset G(\Omega)$ acting simply transitively on $\Omega$. Let $\Delta : \Omega \to \mathbb{R}_{>0}$ be any $H$-relatively invariant function. The *pseudoinverse map* $\mathcal{I}_\Delta : \Omega \to V^*$ is introduced by

$$\langle v, \mathcal{I}_\Delta(x) \rangle = -D_v \log \Delta(x) \qquad (x \in \Omega, v \in V). \tag{1.2}$$

When $\mathcal{I}_\Delta(\Omega) \subset \Omega^*$, we say that $\Delta$ is *admissible*. In fact, $\Delta$ is admissible if and only if $\mathcal{I}_\Delta$ is a bijection from $\Omega$ onto $\Omega^*$. The characteristic function $\phi$ is one of such functions on $\Omega$. Another example comes from the Bergman (resp. Szegö) kernel of a homogeneous Siegel domain whose base cone is $\Omega$, and the corresponding pseudoinverse map appears in [4] and [9] (resp. [11]). While the family of pseudoinverse maps defined above is an interesting object in itself, it is also used to define the family of Cayley transforms for a homogeneous Siegel domain. For the study in this direction, see [12], [9], [10], [11] and [7].

Now our main theorem in its precise form is stated as follows.

**Theorem 1.2.** *Let $\Omega$ be an irreducible homogeneous convex cone and $\Delta : \Omega \to \mathbb{R}_{>0}$ an admissible $H$-relatively invariant function. Then the following conditions are equivalent*:

(A) *The cone* $\Omega$ *is a symmetric cone and* $\Delta(x) = \phi(x)^p$ $(x \in \Omega)$ *for some* $p > 0$ *up to a constant multiple.*
(B) *For* $x, y \in \Omega$, *one has* $x \succeq_\Omega y$ *if and only if* $\mathcal{I}_\Delta(y) \succeq_{\Omega^*} \mathcal{I}_\Delta(x)$.
(C) *For* $x, y \in \Omega$, *the pair* $(x, y)$ *is comparable if and only if* $(\mathcal{I}_\Delta(x), \mathcal{I}_\Delta(y))$ *is comparable.*

Concerning the order-reversing property, we would like to mention Güler's work [6] which deals with pseudoinverse maps associated with certain polynomials called *hyperbolic homogeneous polynomials*. In the paper, it is shown that for every homogeneous convex cone $\Omega \subset V$, there exists a hyperbolic homogeneous polynomial $p(x)$ on $V$ such that $\Omega$ is one of the connected components of the set $\{p(x) \neq 0\}$. As in (1.2), a map $\widetilde{\mathcal{I}}_{p^{-1}} : \Omega \to \Omega^*$ is introduced by

$$\langle v, \widetilde{\mathcal{I}}_{p^{-1}}(x) \rangle = -D_v \log p(x)^{-1} \qquad (x \in \Omega, v \in V).$$

While our pseudoinverse maps are associated with relatively invariant functions on $\Omega$ and Theorem 1.2 is a *characterization* of symmetric cones by the order-reversing property, it is interesting to note that [6, Corollary 6.1] states that, for $x, y \in \Omega$, $x \succeq_\Omega y$ always implies $\widetilde{\mathcal{I}}_{p^{-1}}(y) \succeq_{\Omega^*} \widetilde{\mathcal{I}}_{p^{-1}}(x)$. It should be noted that when $\Omega$ is a symmetric cone, the inverse of the characteristic function is a hyperbolic homogeneous polynomial defining $\Omega$. However, if $\Omega$ is non-symmetric, the characteristic function is not necessarily (a negative power of) a polynomial, as we will see in Section 6. Additionally, in [6], $F(x) := \log p(x)^{-1}$ $(x \in \Omega)$ is called a hyperbolic barrier function for $\Omega$, and the following characterization is given: if the Legendre transform of $F(x)$ is also a hyperbolic barrier function for $\Omega^*$, then $\Omega$ is symmetric.

Our previous paper [8] is also related to the pseudoinverse maps, where we characterized symmetric cones by the condition that the analytic continuation of $\mathcal{I}_\Delta$ to the complexification $V_\mathbb{C}$ maps the tube domain $V + i\Omega$ onto $V^* + i\Omega^*$. Here are some of characterizations of symmetric cones: a characterization by a constancy of the dimensions of a certain eigenspace decomposition of $V$ [18, Proposition 3] which we quote in the present paper as Proposition 4.8, a characterization by the condition that the Riemannian curvature tensor for a standard Riemannian metric of $\Omega$ is parallel ([14], [15]), and some Jordan-theoretic characterizations ([3, Theorem 4.7], [16]).

We organize this paper as follows. Section 2 is preliminary. In Section 2.1 we review the theory of a non-commutative left-symmetric algebra called a *clan* associated with a homogeneous convex cone. Then, in Section 2.2, we introduce the family of pseudoinverse maps and its parametrization for the convenience of computation. We describe the parameter of Vinberg's $*$-map in Section 2.3.

In Section 3 we deal with the case of symmetric cones $\Omega \subset V$. First we present how to introduce into $V$ the structure of Jordan algebra associated with $\Omega$. Then, using the Hua identity, we verify that the Jordan algebra inverse maps and the $*$-maps are order-reversing.

After we collect basic formulas and some criterions in Section 4, we prove Theorem 1.2 in Section 5. It is easy to see that (B) is equivalent to (C) and that (A) implies (B). In Section 5.2, we prove that (B) implies (A). The proof is divided into several steps of computations. First we show that the assumption (B) restricts the parameter of the pseudoinverse map. Continuing the computation, finally we obtain (A) by Vinberg's criterion, Proposition 4.8.

In Section 6 we show that the $*$-map associated with the *dual Vinberg cone* which is one of the lowest-dimensional non-symmetric cones is not order-reversing by giving a pair $x, y \in \Omega$ with $x \succeq_\Omega y$ and $y^* \not\preceq_{\Omega^*} x^*$.

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## 2. Preliminaries

Let $\Omega$ be a homogeneous convex cone in a finite-dimensional real vector space $V$. If the dual cone $\Omega^*$ coincides with $\Omega$ under the identification of $V^*$ with $V$ by means of a positive definite inner product $\langle\cdot|\cdot\rangle$ on $V$, then $\Omega$ is said to be *self-dual* relative to $\langle\cdot|\cdot\rangle$. As usual, we call a self-dual homogeneous convex cone a *symmetric cone*. Symmetric cones form a special subclass in the class of homogeneous convex cones.

2.1. **Clan associated with a homogeneous convex cone.** Let $\Omega \subset V$ be a homogeneous convex cone. Then, $V$ has a structure of a non-commutative left-symmetric algebra called a *clan* as follows. Our basic reference is [17]. We know by [17, Chapter I, Theorem 1] that there exists a maximal connected split solvable subgroup $H \subset G(\Omega)$ which acts simply transitively on $\Omega$. Moreover, by [17, Chapter I, Propositions 8, 9], $H$ acts simply transitively also on $\Omega^*$ by the contragradient action. We take any $E \in \Omega$ and fix it. Then the orbit map $H \ni h \mapsto hE \in \Omega$ is a diffeomorphism. Differentiating it at the unit element of $H$, we obtain the linear isomorphism $\mathfrak{h} := \mathrm{Lie}(H) \ni T \mapsto TE \in V$, where we identify the tangent space of $\Omega$ at $E$ with $V$. We denote its inverse map by $V \ni x \mapsto L_x \in \mathfrak{h}$, that is, $L_xE = x$ $(x \in V)$. We introduce a product $(x, y) \mapsto x \triangle y$ on $V$ by $x \triangle y := L_x y$. Then the (non-associative) algebra $(V, \triangle)$ has the following properties:

- (C1) $[L_x, L_y] = L_{(x\triangle y - y\triangle x)}$,
- (C2) the bilinear form $(x, y) \mapsto \mathrm{Tr}\, L_{x\triangle y}$ defines a positive definite inner product on $V$,
- (C3) for every $x \in V$, the linear operator $L_x$ has only real eigenvalues.

Moreover, it follows that $E$ is the unit element. In general, we call a non-associative algebra with the properties (C1) to (C3) a clan after Vinberg. Thus we have constructed a clan with the unit element $E$ associated with $\Omega$. It is known that any homogeneous convex cone arises from the associated clan with the unit element and that there is a one-to-one correspondence up to isomorphisms between the class of homogeneous convex cones and that of clans with the unit element.

The clan $V$ has a direct sum decomposition called a *normal decomposition*. Namely, there exist a positive integer $r$, and primitive idempotents $E_1, \dots, E_r$ such that $V$ decomposes into

$$V = \sum_{1\le j\le k\le r} V_{kj}, \tag{2.1}$$

where, for each pair $(k,j)$ of integers with $1 \le j \le k \le r$, we have put

$$V_{kj} := \Big\{ x \in V \,;\, \forall c = \sum_{m=1}^{r} c_m E_m, c \triangle x = \tfrac{1}{2}(c_j + c_k)x, x \triangle c = c_j x \Big\}.$$

In this case, it follows that $E = E_1 + \cdots + E_r$ and $V_{ii} = \mathbb{R}E_i$ $(i = 1, \ldots, r)$. We obtain the following multiplication table:

$$\begin{aligned} &\bullet\ V_{lk} \triangle V_{kj} \subset V_{lj}, \\ &\bullet\ \text{if } k \neq i, j, \text{ then } V_{lk} \triangle V_{ij} = 0, \\ &\bullet\ V_{lk} \triangle V_{mk} \subset V_{lm} \text{ or } V_{ml} \text{ according to } l \ge m \text{ or } m \ge l. \end{aligned} \tag{2.2}$$

2.2. **Pseudoinverse maps.** First we shall parametrize $H$-relatively invariant functions on $\Omega$. We put $\mathfrak{a} := \{L_x \in \mathfrak{h} \mid x \in \sum_{m=1}^{r} \mathbb{R}E_m\}$ and $\mathfrak{n} := \{L_x \in \mathfrak{h} \mid x \in \sum_{1 \le j < k \le r} V_{kj}\}$. Then $\mathfrak{a}$ (resp. $\mathfrak{n}$) is an abelian (resp. nilpotent) subalgebra of $\mathfrak{h}$. Moreover we have $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}$, so that $H = A \ltimes N$, where $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we define a one-dimensional representation $\chi_{\mathbf{s}}$ of $A$ by

$$\chi_{\mathbf{s}}\Big(\exp\Big(\sum t_m L_{E_m}\Big)\Big) := \exp\Big(\sum s_m t_m\Big) \qquad (t_m \in \mathbb{R}). \tag{2.3}$$

Since $H = A \ltimes N$, we can extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of $H$ by defining $\chi_{\mathbf{s}}|_N \equiv 1$. Using the diffeomorphism $H \ni h \mapsto hE \in \Omega$, we transfer $\chi_{\mathbf{s}}$ to a function $\Delta_{\mathbf{s}}$ on $\Omega$: $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$. It is clear that

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \qquad (h \in H, x \in \Omega), \tag{2.4}$$

that is, $\Delta_{\mathbf{s}}$ is $H$-relatively invariant. In particular, putting $h := \exp(\log \lambda\, L_E)$ for any $\lambda > 0$, we see that

$$\Delta_{\mathbf{s}}(\lambda x) = \lambda^{|\mathbf{s}|}\Delta_{\mathbf{s}}(x) \qquad (x \in \Omega), \tag{2.5}$$

where $|\mathbf{s}| := s_1 + \cdots + s_r$. Every $H$-relatively invariant function on $\Omega$ arises as a constant multiple of $\Delta_{\mathbf{s}}$ for some $\mathbf{s} \in \mathbb{R}^r$.

For $x \in \Omega$, we define the *pseudoinverse* $\mathcal{I}_{\mathbf{s}}(x) \in V^*$ by

$$\langle v, \mathcal{I}_{\mathbf{s}}(x) \rangle = -D_v \log \Delta_{-\mathbf{s}}(x) \qquad (v \in V). \tag{2.6}$$

We call $\mathcal{I}_{\mathbf{s}} : \Omega \to V^*$ the *pseudoinverse map*. It is easy to show by (2.4) that

$$\mathcal{I}_{\mathbf{s}}(hx) = h \cdot \mathcal{I}_{\mathbf{s}}(x) \qquad (x \in \Omega, h \in H), \tag{2.7}$$

where the action in the right-hand side is the contragradient action of $H$ on $V^*$: for $f \in V^*$ and $h \in H$, $\langle v, h \cdot f \rangle := \langle h^{-1}v, f \rangle$ $(v \in V)$. Moreover we see the following. We define $E_i^* \in V^*$ $(i = 1, \ldots, r)$ by

$$\Big\langle \sum_{m=1}^{r} \lambda_m E_m + \sum_{1 \le j < k \le r} X_{kj}, E_i^* \Big\rangle := \lambda_i \qquad (\lambda_m \in \mathbb{R}, X_{kj} \in V_{kj})$$

and set $E_{\mathbf{s}}^* := \sum s_i E_i^*$. Then it follows from [10, Lemma 3.10 (ii)] that

$$\mathcal{I}_{\mathbf{s}}(E) = E_{\mathbf{s}}^*. \tag{2.8}$$

Here we refer the reader to [2, Section 2] for the translation of the normal $j$-algebra language into our language of clan. In addition, we know easily by (2.5) that

$$\langle x, \mathcal{I}_{\mathbf{s}}(x)\rangle = |\mathbf{s}| \qquad (x \in \Omega). \tag{2.9}$$

For every $\mathbf{s} \in \mathbb{R}^r$, we introduce a symmetric bilinear form $\langle\cdot|\cdot\rangle_{\mathbf{s}}$ on $V$ by

$$\langle x|y\rangle_{\mathbf{s}} = D_x D_y \log \Delta_{-\mathbf{s}}(E) \qquad (x, y \in V).$$

It is easy to see by the proof of [10, Lemma 3.11] that

$$\langle x|y\rangle_{\mathbf{s}} = \langle x \triangle y, E^*_{\mathbf{s}}\rangle \qquad (x, y \in V). \tag{2.10}$$

We say that a parameter $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$ is *positive*, if $s_1, \dots, s_r > 0$. Actually, it follows that

$$E^*_{\mathbf{s}} \in \Omega^* \text{ if and only if } \mathbf{s} \text{ is positive}, \tag{2.11}$$

as we shall see later at the end of Section 2.3. This together with [10, Lemma 3.3], (2.7), (2.8) and the fact that $H$ acts simply transitively on $\Omega$ and $\Omega^*$ tells us the following lemma.

**Lemma 2.1.** *The following conditions are equivalent*:

(i) $\mathbf{s}$ *is positive.*
(ii) $E^*_{\mathbf{s}} \in \Omega^*$.
(iii) *The $H$-relatively invariant function $\Delta_{-\mathbf{s}}$ is admissible, that is, $\mathcal{I}_{\mathbf{s}}(\Omega) \subset \Omega^*$.*
(iv) $\mathcal{I}_{\mathbf{s}}$ *is a bijection from $\Omega$ onto $\Omega^*$.*
(v) *The bilinear form $\langle\cdot|\cdot\rangle_{\mathbf{s}}$ defines a positive definite inner product on $V$.*

In this paper, we consider only the pseudoinverse maps $\mathcal{I}_{\mathbf{s}}$ with positive parameters $\mathbf{s}$. Let $\mathbf{s} \in \mathbb{R}^r$ be positive. In this case, we can give explicitly $\mathcal{I}_{\mathbf{s}}^{-1}$ in the following way. We introduce a function $\Delta^*_{\mathbf{s}}$ on $\Omega^*$ by

$$\Delta^*_{\mathbf{s}}(h \cdot E^*_{\mathbf{s}}) := \chi_{\mathbf{s}}(h) \qquad (h \in H).$$

Then we have clearly

$$\Delta^*_{\mathbf{s}}(h \cdot f) = \chi_{\mathbf{s}}(h)\Delta^*_{\mathbf{s}}(f) \qquad (h \in H, f \in \Omega^*). \tag{2.12}$$

Let us define the *dual pseudoinverse map* $\mathcal{I}^*_{\mathbf{s}} : \Omega^* \to V$ by

$$\langle \mathcal{I}^*_{\mathbf{s}}(\xi), f\rangle = -D_f \log \Delta^*_{\mathbf{s}}(\xi) \qquad (\xi \in \Omega^*, f \in V^*).$$

By (2.12), one has $\mathcal{I}^*_{\mathbf{s}}(h \cdot \xi) = h\mathcal{I}^*_{\mathbf{s}}(\xi)$ $(h \in H, \xi \in \Omega^*)$. Moreover, it follows from [10, Lemma 3.13] that $\mathcal{I}^*_{\mathbf{s}}(E^*_{\mathbf{s}}) = E$. Hence we know by (2.7) and (2.8) that $\mathcal{I}_{\mathbf{s}}^{-1} = \mathcal{I}^*_{\mathbf{s}}$.

2.3. **Vinberg's $*$-map as a pseudoinverse map.** By [5, Proposition I.3.1], one has

$$\phi(gx) = (\operatorname{Det} g)^{-1}\phi(x) \qquad (g \in G(\Omega), x \in \Omega), \tag{2.13}$$

which clearly implies that $\phi$ is $H$-relatively invariant. Hence the $*$-map is a member of the family of pseudoinverse maps. Moreover, by [17, Chapter I, Proposition 5], the bilinear form

$$\langle x|y\rangle_{\phi} := D_x D_y \log \phi(E) \qquad (x, y \in V)$$

defines a positive definite inner product on $V$. We introduce a parameter $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{R}^r$ by

$$d_i := \operatorname{Tr} L_{E_i} \qquad (i = 1, \dots, r).$$

Then we know by (2.1) that

$$d_i = 1 + \tfrac{1}{2} \sum_{\alpha > i} n_{\alpha i} + \tfrac{1}{2} \sum_{\alpha < i} n_{i\alpha} \qquad (i = 1, \dots, r), \tag{2.14}$$

where we have put

$$n_{kj} := \dim V_{kj} \qquad (1 \leq j < k \leq r). \tag{2.15}$$

Clearly, $\mathbf{d}$ is positive. It follows from (2.3) that $\chi_{\mathbf{d}}(a) = \operatorname{Det} a$ $(a \in A)$. Since $H = A \ltimes N$ and $N$ is nilpotent, we have $\chi_{\mathbf{d}}(h) = \operatorname{Det} h$ $(h \in H)$. Hence we obtain

$$\Delta_{\mathbf{d}}(hx) = \operatorname{Det} h \, \Delta_{\mathbf{d}}(x) \qquad (h \in H, x \in \Omega).$$

This together with (2.13) gives

$$\phi(x) = \Delta_{-\mathbf{d}}(x) \quad (x \in \Omega) \tag{2.16}$$

up to a positive constant multiple. Therefore it holds that

$$\langle \cdot | \cdot \rangle_\phi = \langle \cdot | \cdot \rangle_{\mathbf{d}}, \qquad x^* = \mathcal{I}_{\mathbf{d}}(x) \qquad (x \in \Omega). \tag{2.17}$$

*Proof of* (2.11). Since $\sum e^{t_m} E_m = \exp(\sum t_m L_{E_m}) E \in \Omega$ $(t_m \in \mathbb{R})$, we have

$$\sum \lambda_m E_m \in \overline{\Omega} \qquad (\lambda_m \geq 0). \tag{2.18}$$

In particular, $E_m \in \overline{\Omega}$ $(m = 1, \dots, r)$. Hence, if $E^*_{\mathbf{s}} \in \Omega^*$, then $s_m = \langle E_m, E^*_{\mathbf{s}} \rangle > 0$ $(m = 1, \dots, r)$. Conversely, let $\mathbf{s}$ be positive. Then we see easily that $E^*_{\mathbf{s}} = \exp(-\sum \log(s_m/d_m) L_{E_m}) \cdot E^*_{\mathbf{d}}$. Hence we have $E^*_{\mathbf{s}} \in \Omega^*$, because $E^*_{\mathbf{d}} = E^* \in \Omega^*$ by [17, Chapter I, Proposition 7]. $\square$

## 3. The case of symmetric cones

In this section we assume that the homogeneous convex cone $\Omega \subset V$ is a symmetric cone.

### 3.1. Coincidence of Vinberg's $*$-map and the Jordan algebra inverse map.

Since $\Omega$ is a symmetric cone, $V$ has a structure of a Jordan algebra associated with $\Omega$ as follows. First, it follows from [13, Chapter I, §8, Theorem 8.5] and [13, Chapter I, §8, Exercise 5] that $\Omega$ is self-dual with respect to $\langle \cdot | \cdot \rangle_\phi$. We transfer the image of Vinberg's $*$-map by means of the inner product $\langle \cdot | \cdot \rangle_\phi$ and denote this map by $\Omega \ni x \mapsto x^\phi \in \Omega$:

$$\langle x^\phi | y \rangle_\phi = -D_y \log \phi(x) \qquad (x \in \Omega, y \in V). \tag{3.1}$$

We see easily that $E^\phi = E$.

We introduce a commutative product $(x, y) \mapsto x \circ y$ on $V$ by

$$\langle x \circ y | z \rangle_\phi = -\tfrac{1}{2} D_x D_y D_z \log \phi(E) \qquad (x, y, z \in V).$$

In view of $E^\phi = E$, we see by [5, Theorem III.3.1] and [5, Chapter III, Exercise 5] that $V$ with the product $\circ$ is a Jordan algebra (we note that the third equality in [5, Chapter III, Exercise 5] has an error and its right-hand side should be $-\frac{1}{2}D_uD_vD_w \log \phi(e)$). This means that in addition to the commutativity of the product, one has

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y) \qquad (x, y \in V).$$

Moreover, $E$ is the unit element of the Jordan algebra $V$, and one has

$$\langle x|y\rangle_\phi = \operatorname{Tr} L(xy) \qquad (x, y \in V),$$

where, for $v, w \in V$, we define $L(v)w := v \circ w$. Therefore it follows from [5, Proposition III.4.2 (i)] and [5, Proposition III.4.3] that for every invertible $x \in V$,

$$x^\phi = x^{-1}, \tag{3.2}$$

where $x^{-1}$ denotes the Jordan algebra inverse of $x$. In particular we know that the Jordan algebra inverse map $x \mapsto x^{-1}$ is an involution on $\Omega$.

In addition, the structure of Jordan algebra is related to that of clan as follows: for all $x \in V$, one has

$$L(x) = \tfrac{1}{2}(L_x + {}^{t_\phi}L_x),$$

where ${}^{t_\phi}L_x$ is the transpose of $L_x$ relative to the inner product $\langle\cdot|\cdot\rangle_\phi$. See [9, Lemma 4.1] for the proof.

3.2. **Order-reversing property of the Jordan algebra inverse maps.** For $x \in V$, we define a linear operator $P(x)$ on $V$ by

$$P(x) := 2L(x)^2 - L(x^2).$$

It follows from [5, Proposition III.2.2] that for every invertible $x \in V$, $P(x)$ belongs to $G(\Omega)$. We quote the following identity called Hua's identity (see [5, Chapter II, Exercise 5]).

**Lemma 3.1.** *For $a \in \Omega$, one has*

$$(a + P(a)b^{-1})^{-1} + (a + b)^{-1} = a^{-1},$$

*when $b, a + b, a + P(a)b^{-1}$ are invertible.*

**Proposition 3.2.** *For every $x, y \in \Omega$, one has $x \succeq_\Omega y$ if and only if $y^{-1} \succeq_\Omega x^{-1}$.*

*Proof.* Since the Jordan algebra inverse map is an involution on $\Omega$, it suffices to prove that $x \succeq_\Omega y$ implies $y^{-1} \succeq_\Omega x^{-1}$. First we assume $z := x - y \in \Omega$. Since $y \in \Omega$, we know that $y$ is invertible, so that $P(y) \in G(\Omega)$. This together with $z^{-1} \in \Omega$ gives $P(y)z^{-1} \in \Omega$. Hence $y + P(y)z^{-1}$ belongs to $\Omega$ and is invertible. In Lemma 3.1 we set $a := y$ and $b := z$. Then we know that

$$y^{-1} - x^{-1} = (y + P(y)z^{-1})^{-1},$$

which implies $y^{-1} - x^{-1} \in \Omega$.

Next we suppose $x \succeq_\Omega y$, that is, $z := x - y \in \overline{\Omega}$. Let $\{z_m\}$ be a sequence in $\Omega$ which converges to $z$. We put $x_m := z_m + y$. Then $\{x_m\}$ is a sequence in $\Omega$

converging to $x$. Since $x_m, y$ and $x_m - y = z_m$ belong to $\Omega$, we know by the above argument that $y^{-1} - x_m^{-1} \in \Omega$. Moreover the sequence $\{y^{-1} - x_m^{-1}\}$ converges to $y^{-1} - x^{-1}$, because the inverse map is continuous in $\Omega$. Thus we have $y^{-1} - x^{-1} \in \overline{\Omega}$, that is, $y^{-1} \succeq_\Omega x^{-1}$. $\square$

By (3.2), we have a conclusion similar to Proposition 3.2 for the original $*$-map $\Omega \xrightarrow{\sim} \Omega^*$, that is, for every $x, y \in \Omega$, one has $x \succeq_\Omega y$ if and only if $y^* \succeq_{\Omega^*} x^*$.

## 4. Basic formulas and some criterions

In this section, we suppose that a parameter $\mathbf{s} \in \mathbb{R}^r$ is positive, and we identify $V^*$ with $V$ by means of the positive definite inner product $\langle\cdot|\cdot\rangle_{\mathbf{s}}$ to simplify the notation. Under this identification, we denote by $\Omega^{\mathbf{s}}$ the dual cone of $\Omega$:

$$\Omega^{\mathbf{s}} := \{x \in V \mid \forall y \in \overline{\Omega} \setminus \{0\}, \langle x|y\rangle_{\mathbf{s}} > 0\}.$$

The Lie group $H$ acts on $\Omega^{\mathbf{s}}$ simply transitively by the action $x \mapsto {}^{\mathbf{s}}h^{-1}x$ $(h \in H, x \in \Omega^{\mathbf{s}})$, where ${}^{\mathbf{s}}h$ stands for the transpose of $h$ relative to $\langle\cdot|\cdot\rangle_{\mathbf{s}}$. It follows from (2.10) and (2.2) that the subspaces $V_{kj}$ $(1 \le j \le k \le r)$ are orthogonal to each other with respect to $\langle\cdot|\cdot\rangle_{\mathbf{s}}$. Moreover, for $v_{kj}, w_{kj} \in V_{kj}$ $(1 \le j < k \le r)$, we see easily by (2.10) and (2.2) that

$$v_{kj} \triangle w_{kj} = s_k^{-1}\langle v_{kj}|w_{kj}\rangle_{\mathbf{s}} E_k. \tag{4.1}$$

In addition, for all $x = \sum x_m E_m + \sum_{\beta>\alpha} x_{\beta\alpha} \in V$ $(x_m \in \mathbb{R}, x_{\beta\alpha} \in V_{\beta\alpha})$, we have

$$\langle x, E_i^* \rangle = x_i = \langle x|s_i^{-1}E_i\rangle_{\mathbf{s}} \qquad (i = 1, \ldots, r). \tag{4.2}$$

Hence,

$$E_{\mathbf{s}}^* \in \Omega^* \text{ is identified with } E \in \Omega^{\mathbf{s}}. \tag{4.3}$$

From now on, we always assume that the integers $j, k, l$ satisfy $1 \le j < k < l \le r$. We quote the following Propositions 4.1 and 4.2 to compute the actions of $H$ on $\Omega$ and $\Omega^{\mathbf{s}}$. For $w_{kj} \in V_{kj}, w_{lj} \in V_{lj}$, we set

$$S_{lk} := \tfrac{1}{2}(w_{kj} \triangle w_{lj} + w_{lj} \triangle w_{kj}) \in V_{lk}.$$

**Proposition 4.1** ([8, Proposition 4.2]). *Let $t_j, t_k, t_l \in \mathbb{R}$ and $w_{kj} \in V_{kj}, w_{lj} \in V_{lj}$ and $w_{lk} \in V_{lk}$. Then one has*

$$\begin{aligned}
&\exp\left(L_{w_{lj}} + L_{w_{kj}}\right)\exp\left(L_{w_{lk}}\right)\exp\left(t_j L_{E_j} + t_k L_{E_k} + t_l L_{E_l}\right) E \\
&= \sum_{m \ne j,k,l} E_m + e^{t_j} E_j + \left(e^{t_k} + e^{t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2\right) E_k \\
&\quad + \left(e^{t_l} + e^{t_k}(2s_l)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 + e^{t_j}(2s_l)^{-1}\|w_{lj}\|_{\mathbf{s}}^2\right) E_l \\
&\quad + e^{t_j} w_{lj} + e^{t_j} w_{kj} + \left(e^{t_j} S_{lk} + e^{t_k} w_{lk}\right).
\end{aligned}$$

**Proposition 4.2** ([8, Proposition 4.6]). *Let $t_j, t_k, t_l \in \mathbb{R}$ and $w_{kj} \in V_{kj}, w_{lj} \in V_{lj}$ and $w_{lk} \in V_{lk}$. Then one has*

$$\begin{aligned}
&{}^{\mathbf{s}}\!\big(\exp(L_{w_{lj}} + L_{w_{kj}}) \exp(L_{w_{lk}}) \exp\left(t_j L_{E_j} + t_k L_{E_k} + t_l L_{E_l}\right)\big)^{-1} E \\
&\quad = \sum_{m \neq j,k,l} E_m + \Big(e^{-t_j} + \left(e^{-t_k} + e^{-t_l}(2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2\right)(2s_j)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 \\
&\qquad\qquad + e^{-t_l}(2s_j)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 - e^{-t_l} s_j^{-1}\langle S_{lk}|w_{lk}\rangle_{\mathbf{s}}\Big) E_j \\
&\qquad + \left(e^{-t_k} + e^{-t_l}(2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2\right) E_k + e^{-t_l} E_l \\
&\qquad + \left(e^{-t_l}\, {}^{\mathbf{s}}L_{w_{lj}} w_{lk} - (e^{-t_k} + e^{-t_l}(2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2) w_{kj}\right) \\
&\qquad + e^{-t_l}\left({}^{\mathbf{s}}L_{w_{kj}} w_{lk} - w_{lj}\right) - e^{-t_l} w_{lk}.
\end{aligned}$$

We present some formulas to prove Proposition 4.6.

**Lemma 4.3.** *For every $x = \sum x_m E_m + \sum_{\beta>\alpha} x_{\beta\alpha}$ $(x_m \in \mathbb{R}, x_{\beta\alpha} \in V_{\beta\alpha})$, $w_{lj} \in V_{lj}$ and $w_{lk} \in V_{lk}$, one has*

$$\begin{aligned}
&\exp(L_{w_{lj}} + L_{w_{lk}})x = x + (x_j w_{lj} + w_{lk}\triangle x_{kj}) + (x_k w_{lk} + w_{lj}\triangle x_{kj}) \\
&\quad + \sum_{\alpha<j} w_{lj}\triangle x_{j\alpha} + \sum_{\beta>j,\beta\neq k} w_{lj}\triangle x_{\beta j} + \sum_{\alpha<k,\alpha\neq j} w_{lk}\triangle x_{k\alpha} + \sum_{\beta>k} w_{lk}\triangle x_{\beta k} \\
&\quad + (2s_l)^{-1}(x_j\|w_{lj}\|_{\mathbf{s}}^2 + x_k\|w_{lk}\|_{\mathbf{s}}^2 + \langle w_{lk}|w_{lj}\triangle x_{kj}\rangle_{\mathbf{s}} + \langle w_{lj}|w_{lk}\triangle x_{kj}\rangle_{\mathbf{s}})E_l.
\end{aligned}$$

*Proof.* First we know by (2.2) that

$$\begin{aligned}
L_{w_{lj}}x &= w_{lj}\triangle(x_j E_j + x_{kj} + \sum_{\alpha<j} x_{j\alpha} + \sum_{\beta>j,\beta\neq k} x_{\beta j}) \\
&= x_j w_{lj} + w_{lj}\triangle x_{kj} + \sum_{\alpha<j} w_{lj}\triangle x_{j\alpha} + \sum_{\beta>j,\beta\neq k} w_{lj}\triangle x_{\beta j}.
\end{aligned}$$

By a similar argument for $L_{w_{lk}}x$, we have

$$\begin{aligned}
(L_{w_{lj}} + L_{w_{lk}})x &= x_j w_{lj} + x_k w_{lk} + w_{lj}\triangle x_{kj} + w_{lk}\triangle x_{kj} \\
&\quad + \sum_{\alpha<j} w_{lj}\triangle x_{j\alpha} + \sum_{\beta>j,\beta\neq k} w_{lj}\triangle x_{\beta j} + \sum_{\alpha<k,\alpha\neq j} w_{lk}\triangle x_{k\alpha} + \sum_{\beta>k} w_{lk}\triangle x_{\beta k}.
\end{aligned}$$

Further we see by (2.2) that

$$(L_{w_{lj}} + L_{w_{lk}})(\sum_{\alpha<j} w_{lj}\triangle x_{j\alpha} + \sum_{\beta>j,\beta\neq k} w_{lj}\triangle x_{\beta j} + \sum_{\alpha<k,\alpha\neq j} w_{lk}\triangle x_{k\alpha} + \sum_{\beta>k} w_{lk}\triangle x_{\beta k}) = 0.$$

Hence it follows from (2.2) and (4.1) that

$$\begin{aligned}
(L_{w_{lj}} + L_{w_{lk}})^2 x &= x_j w_{lj}\triangle w_{lj} + x_k w_{lk}\triangle w_{lk} + w_{lk}\triangle(w_{lj}\triangle x_{kj}) + w_{lj}\triangle(w_{lk}\triangle x_{kj}) \\
&= s_l^{-1}(x_j\|w_{lj}\|_{\mathbf{s}}^2 + x_k\|w_{lk}\|_{\mathbf{s}}^2 + \langle w_{lk}|w_{lj}\triangle x_{kj}\rangle_{\mathbf{s}} + \langle w_{lj}|w_{lk}\triangle x_{kj}\rangle_{\mathbf{s}})E_l.
\end{aligned}$$

Clearly one has $(L_{w_{lj}} + L_{w_{lk}})^3 x = 0$. Now the proof is complete. $\square$

**Lemma 4.4.** *Let $a_m \in \mathbb{R}$ $(m = 1, \dots, r), v_{kj} \in V_{kj}, v_{lj}, w_{lj} \in V_{lj}$ and $v_{lk}, w_{lk} \in V_{lk}$. Then one has*

$$\begin{aligned}
&{}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})(\sum a_m E_m + v_{kj} + v_{lj} + v_{lk}) \\
&\quad = \sum_{m \neq j,k} a_m E_m + (a_j + a_l(2s_j)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 + s_j^{-1}\langle v_{lj}|w_{lj}\rangle_{\mathbf{s}})E_j \\
&\qquad + (a_k + a_l(2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 + s_k^{-1}\langle v_{lk}|w_{lk}\rangle_{\mathbf{s}})E_k \\
&\qquad + (v_{kj} + a_l {}^{\mathbf{s}}L_{w_{lj}} w_{lk} + {}^{\mathbf{s}}L_{w_{lk}} v_{lj} + {}^{\mathbf{s}}L_{w_{lj}} v_{lk}) + (v_{lj} + a_l w_{lj}) + (v_{lk} + a_l w_{lk}).
\end{aligned}$$

*Proof.* Let us take any $x = \sum x_m E_m + \sum_{\beta > \alpha} x_{\beta\alpha} \in V$ $(x_m \in \mathbb{R}, x_{\beta\alpha} \in V_{\beta\alpha})$. Since the subspaces $V_{\beta\alpha}$ $(1 \leq \alpha \leq \beta \leq r)$ are orthogonal to each other with respect to $\langle\cdot|\cdot\rangle_{\mathbf{s}}$, it follows from Lemma 4.3 and (2.2) that

$$\begin{aligned}
&\langle {}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})(\sum a_m E_m)|x\rangle_{\mathbf{s}} \\
&\quad = \langle \sum a_m E_m | x + w_{lj} \triangle x_{lj} + w_{lk} \triangle x_{lk} \\
&\qquad + (2s_l)^{-1}(x_j\|w_{lj}\|_{\mathbf{s}}^2 + x_k\|w_{lk}\|_{\mathbf{s}}^2 + \langle w_{lk}|w_{lj} \triangle x_{kj}\rangle_{\mathbf{s}} + \langle w_{lj}|w_{lk} \triangle x_{kj}\rangle_{\mathbf{s}})E_l\rangle_{\mathbf{s}}.
\end{aligned}$$

We see by (4.1) and (2.10) that the last term equals

$$\begin{aligned}
&\langle \sum a_m E_m | x\rangle_{\mathbf{s}} + a_l(\langle w_{lj}|x_{lj}\rangle_{\mathbf{s}} + \langle w_{lk}|x_{lk}\rangle_{\mathbf{s}} \\
&\qquad + \tfrac{1}{2}(x_j\|w_{lj}\|_{\mathbf{s}}^2 + x_k\|w_{lk}\|_{\mathbf{s}}^2 + \langle {}^{\mathbf{s}}L_{w_{lj}} w_{lk}|x_{kj}\rangle_{\mathbf{s}} + \langle {}^{\mathbf{s}}L_{w_{lk}} w_{lj}|x_{kj}\rangle_{\mathbf{s}})).
\end{aligned} \tag{4.4}$$

It follows from [8, Lemma 7.7] and [8, Lemma 4.4] that ${}^{\mathbf{s}}L_{w_{lj}} w_{lk} = {}^{\mathbf{s}}L_{w_{lk}} w_{lj} \in V_{kj}$. Hence we know by (4.2) that (4.4) is equal to

$$\langle \sum a_m E_m + a_l(w_{lj} + w_{lk} + (2s_j)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 E_j + (2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 E_k + {}^{\mathbf{s}}L_{w_{lj}} w_{lk})|x\rangle_{\mathbf{s}}.$$

This implies

$$\begin{aligned}
&{}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})(\sum a_m E_m) \\
&\quad = \sum a_m E_m + a_l(w_{lj} + w_{lk} + (2s_j)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 E_j + (2s_k)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 E_k + {}^{\mathbf{s}}L_{w_{lj}} w_{lk}).
\end{aligned}$$

By a similar argument we have

$$\begin{aligned}
{}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})v_{kj} &= v_{kj}, \\
{}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})v_{lj} &= v_{lj} + s_j^{-1}\langle v_{lj}|w_{lj}\rangle_{\mathbf{s}} E_j + {}^{\mathbf{s}}L_{w_{lk}} v_{lj}, \\
{}^{\mathbf{s}}\exp(L_{w_{lj}} + L_{w_{lk}})v_{lk} &= v_{lk} + s_k^{-1}\langle v_{lk}|w_{lk}\rangle_{\mathbf{s}} E_k + {}^{\mathbf{s}}L_{w_{lj}} v_{lk}.
\end{aligned}$$

Summing up these results, we obtain the assertion. □

4.1. **Some criterions.** In this section we improve [7, Lemma 5.9] and [8, Lemma 7.6]. Before proceeding, we note that for any $x = \sum x_m E_m + \sum_{\beta > \alpha} x_{\beta\alpha} \in \Omega$ $(x_m \in \mathbb{R}, x_{\beta\alpha} \in V_{\beta\alpha})$, one has $x_m > 0$ $(m = 1, \dots, r)$. In fact, we know by (2.11) that $E_m^* \in \overline{\Omega^*} \setminus \{0\}$, so that $x_m = \langle x, E_m^*\rangle > 0$ because $\Omega = (\Omega^*)^*$. In view of (2.18), we have a similar conclusion for the elements of $\Omega^{\mathbf{s}}$.

**Proposition 4.5.** *Let $a_m \in \mathbb{R}$ $(m = 1, \dots, r), v_{kj} \in V_{kj}, v_{lj} \in V_{lj}$ and $v_{lk} \in V_{lk}$. We set $U_{lk} := \frac{1}{2}(v_{kj} \triangle v_{lj} + v_{lj} \triangle v_{kj}) \in V_{lk}$. Then we have $\sum a_m E_m + v_{kj} + v_{lj} + v_{lk} \in \Omega$ if and only if*

(i) $a_m > 0$ $(m = 1, \dots, r)$,
(ii) $a_j a_k - (2s_k)^{-1} \|v_{kj}\|_{\mathbf{s}}^2 > 0,\ a_j a_l - (2s_l)^{-1} \|v_{lj}\|_{\mathbf{s}}^2 > 0$,
(iii) $(a_j a_k - (2s_k)^{-1} \|v_{kj}\|_{\mathbf{s}}^2)(a_j a_l - (2s_l)^{-1} \|v_{lj}\|_{\mathbf{s}}^2) - (2s_l)^{-1} \|a_j v_{lk} - U_{lk}\|_{\mathbf{s}}^2 > 0$.

*Proof.* For simplicity, we set $v_1 := \sum a_m E_m + v_{kj} + v_{lj} + v_{lk}$. Let us assume that $v_1 \in \Omega$. Then we have $a_m > 0$ $(m = 1, \dots, r)$. In [8, Lemma 4.1], we set $w_{lj} := -a_j^{-1} v_{lj}, w_{kj} := -a_j^{-1} v_{kj}$ and $x := v_1$. Then, it follows from (4.1) that

$$\begin{aligned} v_2 :=& \exp(L_{w_{lj}} + L_{w_{kj}}) v_1 \\ =& \sum_{m \neq k,l} a_m E_m + (a_k - a_j^{-1}(2s_k)^{-1} \|v_{kj}\|_{\mathbf{s}}^2) E_k + (a_l - a_j^{-1}(2s_l)^{-1} \|v_{lj}\|_{\mathbf{s}}^2) E_l \\ &+ v_{lk} - a_j^{-1} U_{lk}. \end{aligned}$$

Since $\exp(L_{w_{lj}} + L_{w_{kj}}) \in H$, one has $v_2 \in \Omega$. Therefore we obtain (ii) and (iii) by [8, Lemma 7.5].

Conversely we assume that (i), (ii) and (iii) hold. Then we have $v_2 \in \Omega$, so that $v_1 = (\exp(L_{w_{lj}} + L_{w_{kj}}))^{-1} v_2 \in \Omega$. □

**Proposition 4.6.** *Let $a_m \in \mathbb{R}$ $(m = 1, \dots, r), v_{kj} \in V_{kj}, v_{lj} \in V_{lj}$ and $v_{lk} \in V_{lk}$. Then we have $\sum a_m E_m + v_{kj} + v_{lj} + v_{lk} \in \Omega^{\mathbf{s}}$ if and only if*

(i) $a_m > 0$ $(m = 1, \dots, r)$,
(ii) $a_j a_l - (2s_j)^{-1} \|v_{lj}\|_{\mathbf{s}}^2 > 0,\ a_k a_l - (2s_k)^{-1} \|v_{lk}\|_{\mathbf{s}}^2 > 0$,
(iii) $(a_j a_l - (2s_j)^{-1} \|v_{lj}\|_{\mathbf{s}}^2)(a_k a_l - (2s_k)^{-1} \|v_{lk}\|_{\mathbf{s}}^2) - (2s_j)^{-1} \|a_l v_{kj} - {}^{\mathbf{s}}L_{v_{lk}} v_{lj}\|_{\mathbf{s}}^2 > 0$.

*Proof.* For simplicity, we set $v_1 := \sum a_m E_m + v_{kj} + v_{lj} + v_{lk}$. We suppose that $v_1 \in \Omega^{\mathbf{s}}$. Then one has $a_m > 0$ $(m = 1, \dots, r)$. In Lemma 4.4, we set $w_{lj} := -a_l^{-1} v_{lj}, w_{lk} := -a_l^{-1} v_{lk}$. Then we obtain by (4.1) and [8, Lemma 7.7] that

$$\begin{aligned} v_2 :=& {}^{\mathbf{s}}(\exp L_{w_{lj}} + L_{w_{lk}}) v_1 \\ =& \sum_{m \neq j,k} a_m E_m + (a_j - a_l^{-1}(2s_j)^{-1} \|v_{lj}\|_{\mathbf{s}}^2) E_j + (a_k - a_l^{-1}(2s_k)^{-1} \|v_{lk}\|_{\mathbf{s}}^2) E_k \\ &+ (v_{kj} - a_l^{-1} {}^{\mathbf{s}}L_{v_{lk}} v_{lj}). \end{aligned}$$

Since $\exp(L_{w_{lj}} + L_{w_{lk}}) \in H$ and $\Omega^{\mathbf{s}} = \{{}^{\mathbf{s}}hE \mid h \in H\}$, one has $v_2 \in \Omega^{\mathbf{s}}$. Hence (ii) and (iii) follow from [8, Lemma 7.6].

Conversely we assume that (i), (ii) and (iii) hold. Then one has $v_2 \in \Omega^{\mathbf{s}}$, so that $v_1 = {}^{\mathbf{s}}(\exp L_{w_{lj}} + L_{w_{lk}})^{-1} v_2 \in \Omega^{\mathbf{s}}$. □

Also we use the following criterions. Recall the definition (2.15) of the numbers $n_{kj}$ $(1 \leq j < k \leq r)$.

**Proposition 4.7** ([1, Theorem 4]). *The homogeneous convex cone $\Omega$ is irreducible if and only if for each pair $(j, k)$ of integers with $1 \leq j < k \leq r$, there exists a series $j_0, \dots, j_m$ of distinct positive integers such that $j_0 = k, j_m = j$ and $n_{j_{\lambda-1} j_\lambda} \neq 0$ for $\lambda = 1, \dots, m$, where if $j_{\lambda-1} < j_\lambda$, then one puts $n_{j_{\lambda-1} j_\lambda} := n_{j_\lambda j_{\lambda-1}}$.*

**Proposition 4.8** ([18, Proposition 3]). *Let us suppose that the homogeneous convex cone $\Omega$ is irreducible. Then $\Omega$ is a symmetric cone if and only if the numbers $n_{kj}$ $(1 \le j < k \le r)$ are independent of $j, k$.*

## 5. Proof of Theorem 1.2

As in Theorem 1.2, we assume that $\Omega \subset V$ is an irreducible homogeneous convex cone and $\Delta : \Omega \to \mathbb{R}_{>0}$ an admissible $H$-relatively invariant function. We know by Lemma 2.1, (1.2) and (2.6) that $\Delta = \Delta_{-\mathbf{s}}$ with a positive $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ up to a positive constant multiple, so that $\mathcal{I}_\Delta = \mathcal{I}_{\mathbf{s}}$.

5.1. **Proof of the equivalence of (B) and (C).** It is clear that (B) implies (C). To show that (C) implies (B), we prove the following fact:

**Lemma 5.1.** *Let $x, y \in \Omega$. If $x \succeq_\Omega y$ and $\mathcal{I}_{\mathbf{s}}(x) \succeq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(y)$, then one has $x = y$.*

*Proof.* We put $v := x - y \in \overline{\Omega}$. It follows from $\mathcal{I}_{\mathbf{s}}(x) \in \Omega^*$ that $\langle v, \mathcal{I}_{\mathbf{s}}(x)\rangle \ge 0$. This together with (2.9) gives

$$\langle y, \mathcal{I}_{\mathbf{s}}(x) - \mathcal{I}_{\mathbf{s}}(y)\rangle = \langle y, \mathcal{I}_{\mathbf{s}}(x)\rangle - |s| \le \langle x, \mathcal{I}_{\mathbf{s}}(x)\rangle - |s| = 0.$$

Since $y \in \Omega = (\Omega^*)^*$, one has $\langle y, f\rangle > 0$ for any $f \in \overline{\Omega^*} \setminus \{0\}$. In view of $\mathcal{I}_{\mathbf{s}}(x) - \mathcal{I}_{\mathbf{s}}(y) \in \overline{\Omega^*}$, we know that $\mathcal{I}_{\mathbf{s}}(x) = \mathcal{I}_{\mathbf{s}}(y)$, which implies $x = y$. □

The above lemma tells us that if $x \succeq_\Omega y$ and $\mathcal{I}_{\mathbf{s}}(y) \not\succeq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(x)$, then $\mathcal{I}_{\mathbf{s}}(x) \not\succeq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(y)$. Let us suppose that (C) holds. If $x \succeq_\Omega y$, then the pair $(\mathcal{I}_{\mathbf{s}}(x), \mathcal{I}_{\mathbf{s}}(y))$ is comparable, so that we must have $\mathcal{I}_{\mathbf{s}}(y) \succeq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(x)$. By a similar argument for $\mathcal{I}_{\mathbf{s}}^* = \mathcal{I}_{\mathbf{s}}^{-1}$, it follows that $\mathcal{I}_{\mathbf{s}}(y) \succeq_{\Omega^*} \mathcal{I}_{\mathbf{s}}(x)$ implies $x \succeq_\Omega y$. Now we have (B).

5.2. **Proof of the equivalence of (A) and (B).** First we suppose that (A) holds. Then we see by (1.1) and (1.2) that $\mathcal{I}_{\mathbf{s}}(x) = px^*$ $(x \in \Omega)$. Therefore we know by Section 3 that (B) holds.

Now we suppose that (B) holds. As in Section 4, we identify $V^*$ with $V$ by means of the positive definite inner product $\langle \cdot | \cdot \rangle_{\mathbf{s}}$, and denote the pseudoinverse map by the same symbol $\mathcal{I}_{\mathbf{s}} : \Omega \to \Omega^{\mathbf{s}}$. Then we see by (2.7), (2.8) and (4.3) that

$$\mathcal{I}_{\mathbf{s}}(hE) = {}^{\mathbf{s}}h^{-1}E \qquad (h \in H). \tag{5.1}$$

Also we assume that the integers $j, k, l$ always satisfy $1 \le j < k < l \le r$.

5.2.1. *First step.* We shall show that $s_1 = \cdots = s_r$.

**Lemma 5.2.** *If $n_{kj} \neq 0$, then one has $s_j \ge s_k$.*

*Proof.* We suppose $n_{kj} \neq 0$. Let us take any non-zero $v_{kj} \in V_{kj}$ and $\xi_j, \xi_k > 0$ satisfying

$$\xi_j \xi_k - (2s_k)^{-1} \|v_{kj}\|_{\mathbf{s}}^2 = 0. \tag{5.2}$$

For simplicity we set $v := \xi_j E_j + \xi_k E_k + v_{kj}$. Then we see that $v \in \overline{\Omega}$. In fact, we know by Proposition 4.5 that $v_\varepsilon := \sum_{m \neq j,k} \varepsilon E_m + (\xi_j + \varepsilon)E_j + (\xi_k + \varepsilon)E_k + v_{kj} \in \Omega$ for any $\varepsilon > 0$, and $v_\varepsilon$ converges to $\xi_j E_j + \xi_k E_k + v_{kj}$ when $\varepsilon$ approaches 0.

Thus we have $E+v \succeq_\Omega E$, so that $E \succeq_{\Omega^{\mathbf{s}}} \mathcal{I}_{\mathbf{s}}(E+v)$ by the assumption. In Proposition 4.1 we set

$$\begin{gathered} t_j := \log(1+\xi_j), \qquad t_k := \log(1+\xi_k-(1+\xi_j)^{-1}(2s_k)^{-1}\|v_{kj}\|^2_{\mathbf{s}}), \\ t_l := 0, \qquad w_{kj} := (1+\xi_j)^{-1}v_{kj}, \qquad w_{lj} = w_{lk} = 0, \end{gathered} \tag{5.3}$$

where we note that $t_k$ is a real number by (5.2). Then, the right-hand side of the formula in Proposition 4.1 becomes $E+v$. Hence we see by (5.1) and Proposition 4.2 that

$$\mathcal{I}_{\mathbf{s}}(E+v) = \sum_{m\neq j,k} E_m + (e^{-t_j}+e^{-t_k}(2s_j)^{-1}\|w_{kj}\|^2_{\mathbf{s}})E_j + e^{-t_k}E_k - e^{-t_k}w_{kj}.$$

It follows from $E \succeq_{\Omega^{\mathbf{s}}} \mathcal{I}_{\mathbf{s}}(E+v)$ that

$$(1-(e^{-t_j}+e^{-t_k}(2s_j)^{-1}\|w_{kj}\|^2_{\mathbf{s}}))E_j + (1-e^{-t_k})E_k + e^{-t_k}w_{kj} \in \overline{\Omega^{\mathbf{s}}}.$$

Hence we know by Proposition 4.6 that

$$(e^{-t_j}+e^{-t_k}(2s_j)^{-1}\|w_{kj}\|^2_{\mathbf{s}}-1)(e^{-t_k}-1) - e^{-2t_k}(2s_j)^{-1}\|w_{kj}\|^2_{\mathbf{s}} \geq 0.$$

After some simplification, we obtain

$$(e^{-t_j}-1)(e^{-t_k}-1) - e^{-t_k}(2s_j)^{-1}\|w_{kj}\|^2_{\mathbf{s}} \geq 0.$$

We multiply both sides by $e^{2t_j}e^{t_k}$. Then we have by (5.3) that

$$\xi_j((1+\xi_j)\xi_k - (2s_k)^{-1}\|v_{kj}\|^2_{\mathbf{s}}) - (2s_j)^{-1}\|v_{kj}\|^2_{\mathbf{s}} \geq 0.$$

This together with (5.2) gives

$$(s_j-s_k)(2s_js_k)^{-1}\|v_{kj}\|^2_{\mathbf{s}} \geq 0.$$

This implies $s_j \geq s_k$, because $v_{kj} \neq 0$. □

**Lemma 5.3.** *If $n_{kj} \neq 0$, then one has $s_j \leq s_k$.*

*Proof.* The proof is similar to that of Lemma 5.2. We suppose $n_{kj} \neq 0$. Let us take any non-zero $v_{kj} \in V_{kj}$ and $\xi_j, \xi_k > 0$ satisfying

$$\xi_j\xi_k - (2s_j)^{-1}\|v_{kj}\|^2_{\mathbf{s}} = 0. \tag{5.4}$$

Discussing as in the proof of Lemma 5.2, we see that $v := \xi_jE_j + \xi_kE_k + v_{kj} \in \overline{\Omega^{\mathbf{s}}}$. Hence it follows that $E+v \succeq_{\Omega^{\mathbf{s}}} E$, so that $E \succeq_\Omega \mathcal{I}^*_{\mathbf{s}}(E+v)$ by the assumption. In Proposition 4.2 we set

$$\begin{gathered} t_j := -\log(1+\xi_j-(1+\xi_k)^{-1}(2s_j)^{-1}\|v_{kj}\|^2_{\mathbf{s}}), \qquad t_k := -\log(1+\xi_k), \\ t_l := 0, \qquad w_{kj} := -(1+\xi_k)^{-1}v_{kj}, \qquad w_{lj} = w_{lk} = 0, \end{gathered} \tag{5.5}$$

where we have $t_j \in \mathbb{R}$ by (5.4). Then, the right-hand side of the formula in Proposition 4.2 becomes $E+v$. It follows from (5.1) and Proposition 4.1 that

$$\mathcal{I}^*_{\mathbf{s}}(E+v) = \sum_{m\neq j,k} E_m + e^{t_j}E_j + (e^{t_k}+e^{t_j}(2s_k)^{-1}\|w_{kj}\|^2_{\mathbf{s}})E_k + e^{t_j}w_{kj}.$$

Since $E \succeq_\Omega \mathcal{I}^*_{\mathbf{s}}(E+v)$, we obtain

$$(1-e^{t_j})E_j + (1-(e^{t_k}+e^{t_j}(2s_k)^{-1}\|w_{kj}\|^2_{\mathbf{s}}))E_k - e^{t_j}w_{kj} \in \overline{\Omega}.$$

We know by Proposition 4.5 that
$$(e^{t_j}-1)(e^{t_k}+e^{t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2-1)-e^{2t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2\geq 0.$$
After some simplification, we obtain
$$(e^{t_j}-1)(e^{t_k}-1)-e^{t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2\geq 0.$$
We multiply both sides by $e^{-t_j}e^{-2t_k}$. Then we have by (5.5) that
$$((1+\xi_k)\xi_j-(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2)\xi_k-(2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2\geq 0.$$
It follows from (5.4) that
$$(s_k-s_j)(2s_js_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2\geq 0,$$
which implies $s_k\geq s_j$, because $v_{kj}\neq 0$. □

Lemmas 5.2 and 5.3 yield

**Proposition 5.4.** *If $n_{kj}\neq 0$, then one has $s_j=s_k$.*

This together with Proposition 4.7 tells us that $s_1=\cdots=s_r$. In fact, let $j,k$ be any integers with $1\leq j<k\leq r$. Let $j_0,\dots,j_m$ be a series appearing in Proposition 4.7. Then it follows from Proposition 5.4 that $s_{j_{\lambda-1}}=s_{j_\lambda}$ for $\lambda=1,\dots,m$, so that one has $s_j=s_k$.

5.2.2. *Second step.* We set $s:=s_1=\cdots=s_r$. The purpose of this section is to show that if $n_{lk}\neq 0$, then $n_{lj}\leq n_{kj}$. For $v_{lk}\in V_{lk}$, we consider the linear map
$$V_{lj}\ni v_{lj}\mapsto {}^{\mathbf{s}}L_{v_{lk}}v_{lj}\in V_{kj},$$
where we know indeed by [8, Lemma 4.4] and [8, Lemma 7.7] that ${}^{\mathbf{s}}L_{v_{lk}}v_{lj}\in V_{kj}$.

**Lemma 5.5.** *Let us suppose $n_{lk}\neq 0$. Then, for every non-zero $v_{lk}\in V_{lk}$, the linear map $V_{lj}\ni v_{lj}\mapsto {}^{\mathbf{s}}L_{v_{lk}}v_{lj}\in V_{kj}$ is injective. Hence one has $n_{lj}\leq n_{kj}$.*

*Proof.* We assume that ${}^{\mathbf{s}}L_{v_{lk}}v_{lj}=0$. We shall show that $v_{lj}=0$. Contrary we suppose that $v_{lj}\neq 0$. Then, there exist $\xi_j,\xi_k,\xi_l>0$ such that
$$\xi_j\xi_l-(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2=0,\qquad \xi_k\xi_l-(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2=0. \tag{5.6}$$
Discussing as in the proof of Lemma 5.2, we see by Proposition 4.6 and the assumption ${}^{\mathbf{s}}L_{v_{lk}}v_{lj}=0$ that
$$v:=\xi_jE_j+\xi_kE_k+\xi_lE_l+v_{lj}+v_{lk}\in\overline{\Omega^{\mathbf{s}}}.$$
Hence we have $E+v\succeq_{\Omega^{\mathbf{s}}}E$, so that $E\succeq_\Omega\mathcal{I}_{\mathbf{s}}^*(E+v)$. In Proposition 4.2 we set
$$\begin{gathered}t_j:=-\log(1+\xi_j-(1+\xi_l)^{-1}(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2),\\ t_k:=-\log(1+\xi_k-(1+\xi_l)^{-1}(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2),\qquad t_l:=-\log(1+\xi_l),\\ w_{kj}:=0,\qquad w_{lj}:=-(1+\xi_l)^{-1}v_{lj},\qquad w_{lk}:=-(1+\xi_l)^{-1}v_{lk},\end{gathered}\tag{5.7}$$

where $t_j, t_k$ are real numbers by (5.6). Then, the right-hand side of the formula in Proposition 4.2 becomes $E + v$, because ${}^{\mathbf{s}}L_{v_{lk}} v_{lj} = 0$. We obtain by (5.1) and Proposition 4.1 that

$$\mathcal{I}_{\mathbf{s}}^*(E+v) = \sum_{m \neq j,k,l} E_m + e^{t_j} E_j + e^{t_k} E_k \\ + (e^{t_l} + e^{t_k}(2s)^{-1} \|w_{lk}\|_{\mathbf{s}}^2 + e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2) E_l + e^{t_j} w_{lj} + e^{t_k} w_{lk}.$$

Since $E \succeq_\Omega \mathcal{I}_{\mathbf{s}}^*(E+v)$, one has

$$(1-e^{t_j})E_j + (1-e^{t_k})E_k + (1-(e^{t_l} + e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 + e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2))E_l \\ - e^{t_j} w_{lj} - e^{t_k} w_{lk} \in \overline{\Omega}.$$

Hence it follows from Proposition 4.5 (iii) that

$$(1-e^{t_j})(1-e^{t_k})\{(1-e^{t_j})(1-(e^{t_l} + e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 + e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2)) \\ - e^{2t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2\} - (1-e^{t_j})^2 e^{2t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 \geq 0.$$

We divide both sides by $(1-e^{t_j})$, where we note that $1 - e^{t_j} > 0$ by (5.7) and (5.6). After some simplification we obtain

$$(1-e^{t_j})(1-e^{t_k})(1-e^{t_l}) \\ - (1-e^{t_j})e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 - (1-e^{t_k})e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 \geq 0.$$

Multiplying both sides by $e^{-t_j}e^{-t_k}e^{-3t_l}$, we have by (5.7) that

$$e^{-t_l}(e^{-t_j}-1)e^{-t_l}(e^{-t_k}-1)(e^{-t_l}-1) - e^{-t_l}(e^{-t_j}-1)(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 \\ - e^{-t_l}(e^{-t_k}-1)(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 \geq 0.$$

Here we see by (5.7) and (5.6) that

$$e^{-t_l} - 1 = \xi_l, \qquad e^{-t_l}(e^{-t_j}-1) = \xi_j, \qquad e^{-t_l}(e^{-t_k}-1) = \xi_k.$$

Then it follows that

$$\xi_j\xi_k\xi_l - \xi_j(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 - \xi_k(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 \geq 0.$$

This together with (5.6) gives $-\xi_j\xi_k\xi_l \geq 0$, which is a contradiction. Therefore we have $v_{lj} \neq 0$. □

5.2.3. *Third step.*

**Lemma 5.6.** *We suppose $n_{kj} \neq 0$. Then, for every non-zero $v_{kj} \neq 0$, the linear map*

$$V_{lj} \ni v_{lj} \mapsto U_{lk} := \tfrac{1}{2}(v_{lj} \triangle v_{kj} + v_{kj} \triangle v_{lj}) \in V_{lk}$$

*is injective. Hence we have $n_{lj} \leq n_{lk}$.*

*Proof.* Let us assume that $U_{lk} = 0$ for some $v_{lj} \in V_{lj}$. We shall show that $v_{lj} = 0$. Contrary we suppose $v_{lj} \neq 0$.

We can take $\xi_j, \xi_k, \xi_l > 0$ satisfying

$$\xi_j\xi_k - (2s)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 = 0, \qquad \xi_j\xi_l - (2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 = 0. \tag{5.8}$$

Then we see by Proposition 4.5 and the assumption $U_{lk} = 0$ that

$$v := \xi_j E_j + \xi_k E_k + \xi_l E_l + v_{lj} + v_{kj} \in \overline{\Omega}.$$

Hence we have $E + v \succeq_\Omega E$, so that $E \succeq_{\Omega^{\mathbf{s}}} \mathcal{I}_{\mathbf{s}}(E+v)$. In Proposition 4.1 we set

$$\begin{gathered} t_j := \log(1+\xi_j), \qquad t_k := \log(1+\xi_k - (1+\xi_j)^{-1}(2s)^{-1}\|v_{kj}\|_{\mathbf{s}}^2), \\ t_l := \log(1+\xi_l - (1+\xi_j)^{-1}(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2), \\ w_{kj} := (1+\xi_j)^{-1}v_{kj}, \qquad w_{lj} := (1+\xi_j)^{-1}v_{lj}, \qquad w_{lk} := 0, \end{gathered} \tag{5.9}$$

where $t_k, t_l$ are real numbers by (5.8). Then, since $U_{lk} = 0$, the right-hand side of the formula in Proposition 4.1 becomes $E+v$. Hence we know by (5.1) and Proposition 4.2 that

$$\begin{aligned} \mathcal{I}_{\mathbf{s}}(E+v) = \sum_{m\neq j,k,l} E_m &+ (e^{-t_j} + e^{-t_k}(2s)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 + e^{-t_l}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2)E_j \\ &+ e^{-t_k}E_k + e^{-t_l}E_l - e^{-t_k}w_{kj} - e^{-t_l}w_{lj}. \end{aligned}$$

Since $E \succeq_{\Omega^{\mathbf{s}}} \mathcal{I}_{\mathbf{s}}(E+v)$, one has

$$\begin{aligned} &(1 - (e^{-t_j} + e^{-t_k}(2s)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 + e^{-t_l}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2))E_j \\ &\qquad + (1-e^{-t_k})E_k + (1-e^{-t_l})E_l + e^{-t_k}w_{kj} + e^{-t_l}w_{lj} \in \overline{\Omega^{\mathbf{s}}}. \end{aligned}$$

It follows from Proposition 4.6 (iii) that

$$\begin{aligned} \{(1 - (e^{-t_j} + e^{-t_k}(2s)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 + e^{-t_l}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2))(1-e^{-t_l}) - e^{-2t_l}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2\} \\ (1-e^{-t_k})(1-e^{-t_l}) - (1-e^{-t_l})^2 e^{-2t_k}(2s)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 \geq 0. \end{aligned}$$

We divide both sides by $(1-e^{-t_l})$, where we note that $1 - e^{-t_l} > 0$ by (5.9) and (5.8). After some simplification we have

$$\begin{aligned} &(1-e^{-t_j})(1-e^{-t_k})(1-e^{-t_l}) \\ &\qquad - (1-e^{-t_l})e^{-t_k}(2s)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 - (1-e^{-t_k})e^{-t_l}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 \geq 0. \end{aligned}$$

Multiplying both sides by $e^{3t_j}e^{t_k}e^{t_l}$, we obtain by (5.9) that

$$\begin{aligned} &(e^{t_j}-1)e^{t_j}(e^{t_k}-1)e^{t_j}(e^{t_l}-1) \\ &\qquad - e^{t_j}(e^{t_l}-1)(2s)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 - e^{t_j}(e^{t_k}-1)(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 \geq 0. \end{aligned}$$

It follows from (5.9) and (5.8) that

$$e^{t_j} - 1 = \xi_j, \qquad e^{t_j}(e^{t_k}-1) = \xi_k, \qquad e^{t_j}(e^{t_l}-1) = \xi_l.$$

Hence we have

$$\xi_j\xi_k\xi_l - \xi_l(2s)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 - \xi_k(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 \geq 0.$$

Therefore it holds by (5.8) that $-\xi_j\xi_k\xi_l \geq 0$, which is a contradiction. Now we have $v_{lj} = 0$, which we had to show. □

5.2.4. *Last step.* Now that we have Lemmas 5.5 and 5.6, we can prove that the numbers $n_{kj}$ $(1 \le j < k \le r)$ are independent of $j, k$ as in [11, Lemma 5.15] and [11, Proposition 5.16]. Here we give a sketch of the proof. First, by Lemmas 5.5 and 5.6 we know the following.

**Lemma 5.7.** *Fix integers $j, k, l$ with $j < k < l$. If at least two of $n_{kj}, n_{lj}, n_{lk}$ are non-zero, then they are all equal.*

A discussion using Lemma 5.7 and Proposition 4.7 gives that $n_{kj} \neq 0$ for all $j < k$. Then we see easily by Lemma 5.7 that the numbers $n_{kj}$ $(1 \le j < k \le r)$ are all equal. Therefore Proposition 4.8 tells us that the irreducible homogeneous convex cone $\Omega$ is a symmetric cone. Moreover, since $s_1 = \cdots = s_r$ and $d_1 = \cdots = d_r$ by (2.14), we have $\mathbf{s} = p\mathbf{d}$ for some $p > 0$. Hence it follows from (2.16) that $\Delta(x) = \Delta_{-\mathbf{s}}(x) = \phi(x)^p$ $(x \in \Omega)$ up to a positive constant multiple. Now (A) of Theorem 1.2 holds.

## 6. Example of non-order-reversing Vinberg's $*$-map

In this section, we shall show that Vinberg's $*$-map associated with the dual Vinberg cone is not order-reversing. We note that the $*$-map associated with the Vinberg cone is not either, though we do not prove it here. See [9, Section 5] for an explicit computation of the $*$-map associated with the Vinberg cone.

Let $V$ be the real vector space defined by

$$V := \left\{ v = \begin{pmatrix} v_1 & 0 & v_2 \\ 0 & v_3 & v_4 \\ v_2 & v_4 & v_5 \end{pmatrix} ; v_i \in \mathbb{R} \right\}.$$

The dual Vinberg cone is given by

$$\Omega := \{ v \in V \mid v \text{ is positive definite} \},$$

which is one of the lowest-dimensional non-symmetric cones.

Let us define a Lie group $H$ by

$$H := \left\{ h = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_3 & 0 \\ h_2 & h_4 & h_5 \end{pmatrix} \in GL(3, \mathbb{R}); h_1, h_3, h_5 > 0 \right\}.$$

We see easily that $H$ is a split solvable Lie group acting on $V$ by $v \mapsto \rho(h)v := hv{}^{\mathrm{t}}h$ $(h \in H, v \in V)$. It is clear that $\rho(h) \in G(\Omega)$. In addition, $H$ acts on $\Omega$ simply transitively. We take the unit matrix in $V$ as the base point $E$. Then the product of the clan induced by the action of $H$ is described as

$$v \triangle w = \check{v} w + w \hat{v} \qquad (v, w \in V),$$

where

$$\check{v} := \begin{pmatrix} v_1/2 & 0 & 0 \\ 0 & v_3/2 & 0 \\ v_2 & v_4 & v_5/2 \end{pmatrix}, \qquad \hat{v} := \begin{pmatrix} v_1/2 & 0 & v_2 \\ 0 & v_3/2 & v_4 \\ 0 & 0 & v_5/2 \end{pmatrix}.$$

The normal decomposition is given by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$V_{21} = \{0\}, \qquad V_{31} = \left\{ \begin{pmatrix} 0 & 0 & v_2 \\ 0 & 0 & 0 \\ v_2 & 0 & 0 \end{pmatrix} \right\}, \qquad V_{32} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v_4 \\ 0 & v_4 & 0 \end{pmatrix} \right\}.$$

Hence it follows from (2.14) that $d_1 = 3/2, d_2 = 3/2, d_3 = 2$. Thus we know by (2.10) and (2.16) that

$$\langle v|w\rangle_\phi = \tfrac{3}{2}v_1w_1 + 4v_2w_2 + \tfrac{3}{2}v_3w_3 + 4v_4w_4 + 2v_5w_5 \qquad (v, w \in V).$$

Let us compute the $*$-map explicitly. For $x \in \Omega$, let $h(x)$ be the element of $H$ such that $\rho(h(x))E = x$. Then we see that $h(x)$ is given by $h(x)_i := \alpha_i$ $(i = 1, \dots, 5)$, where we have set

$$\alpha_1 := \sqrt{x_1}, \qquad \alpha_3 := \sqrt{x_3}, \qquad \alpha_5 := \sqrt{x_5 - x_2^2/x_1 - x_4^2/x_3},$$
$$\alpha_2 := x_2/\sqrt{x_1}, \qquad \alpha_4 := x_4/\sqrt{x_3}.$$

Since $x^\phi = (\rho(h(x))E)^\phi = {}^{t_\phi}\rho(h(x))^{-1}E$ by (5.1), a straightforward computation yields

$$x^\phi = \begin{pmatrix} \alpha_1^{-2} + \frac{4}{3}(\alpha_2/(\alpha_1\alpha_5))^2 & 0 & -\alpha_2/(\alpha_1\alpha_5^2) \\ 0 & \alpha_3^{-2} + \frac{4}{3}(\alpha_4/(\alpha_3\alpha_5))^2 & -\alpha_4/(\alpha_3\alpha_5^2) \\ -\alpha_2/(\alpha_1\alpha_5^2) & -\alpha_4/(\alpha_3\alpha_5^2) & \alpha_5^{-2} \end{pmatrix} \qquad (x \in \Omega). \tag{6.1}$$

Additionally, we know by (2.13) that for $x \in \Omega$,

$$\phi(x) = \alpha_1^{-3}\alpha_3^{-3}\alpha_5^{-4}\phi(E) = x_1^{1/2}x_3^{1/2}(x_1x_3x_5 - x_2^2x_3 - x_4^2x_1)^{-2}\phi(E).$$

We shall give a pair $x, y \in \Omega$ such that $x \succeq_\Omega y$ and $y^\phi \not\succeq_{\Omega^\phi} x^\phi$. We set $y := E$. It is clear that

$$v_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \in \overline{\Omega},$$

so that we have $x := y + v_1 \succeq_\Omega y$. We know by (6.1) that

$$v_2 := y^\phi - x^\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{18} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

However, it holds that

$$v_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{pmatrix} \in \overline{\Omega}$$

and $\langle v_2|v_3\rangle_\phi = -1$, which implies $v_2 \notin \overline{\Omega^\phi}$. Thus we obtain $y^\phi \not\succeq_{\Omega^\phi} x^\phi$.

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