Auslander-Reiten Theory and noncommutative projective schemes

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February 27, 2007

Abstract

We prove that the category of representations of the N-Kronecker quiver and that of coherent sheaves on the noncommutative projective scheme of $R = k\langle X_1, \dots, X_N\rangle/(\sum_{i=1}^N X_i^2)$ are derived equivalent. The quadratic relation $\sum_{i=1}^N X_i^2$ naturally arises from Auslander-Reiten Theory.

0 Introduction

In noncommutative projective algebraic geometry one thinks of $\mathrm{qgr}\, R$ as the category of coherent sheaves on noncommutative projective scheme $\mathrm{proj}\, R$ of graded ring R ([AZ],[SvB],[Po]).

The main result of this paper is following.

Theorem 0.1 (Theorem 2.12)**.** *Let k be an algebraically closed field, let $A = k\overrightarrow{\Omega}_N$ be the path algebra of the N-Kronecker quiver $\overrightarrow{\Omega}_N$ (see Figure 1) and let $R = k\langle X_1, \dots, X_N\rangle/(\sum_{i=1}^N X_i^2)$.*

For $N \geq 2$ we have the following equivalence of derived categories :

$$D^b(\mathsf{mod}\text{-}A) \cong D^b(\mathrm{qgr}(R)).$$

$$0 \bullet \ \underset{x_N}{\overset{x_1}{\underset{\vdots}{\rightrightarrows}}} \ \bullet 1$$

Figure 1: the N-Kronecker quiver $\overrightarrow{\Omega}_N$

It is well known that there is a finite dimensional algebra B such that the category $\mathsf{mod}\text{-}B$ of finite B-modules is derived equivalent to the category $\mathrm{coh}\, \mathbb{P}^n_k$ of coherent sheaves on $\mathbb{P}^N_k$. Although the projective space $\mathbb{P}^n_k$ is a basic object in algebraic geometry, the corresponding finite dimensional algebra B is not basic. The main theorem states that in noncommutative projective algebraic geometry there is an object $\operatorname{qgr} R$ corresponding to $k\overrightarrow{\Omega}_N$ which is one of the simplest non-trivial finite dimensional algebra.

In [KR], Kontsevich and Rosenberg constructed the category Spaces_k of "noncommutative spaces" and showed that the functor which is represented by projective space $\mathbb{P}_k^n$ in (commutative) algebraic geometry is represented by an object $N\mathbb{P}_k^n$ of Spaces_k. They prove that $\operatorname{coh} N\mathbb{P}_k^n$ is derived equivalent to $\mathsf{mod}\text{-}k\overrightarrow{\Omega}_N$. Therefore it is natural to consider whether the graded ring $k\langle X_1, \dots, X_N\rangle/(\sum_{i=1}^N X_i^2)$ may be the coordinate ring of $N\mathbb{P}_k^n$,i.e., $\operatorname{coh} N\mathbb{P}_k^n$ is equivalent to $\operatorname{qgr} k\langle X_1, \dots, X_N\rangle/(\sum_{i=1}^N X_i^2)$.

To prove Theorem0.1 we use the result on the derived Picard group by J. Miyachi and A. Yekutieli. They computed the derived Picard groups of finite dimensional path algebras of quivers by using Happel's derived version of Auslander-Reiten Theory from which the quadratic relation $\sum_{i=1}^N X_i^2$ naturally arises.

The organization of this paper is as follows. In Section 1 we introduce some definitions and results. In Section 2 we prove Theorem 0.1 (Theorem 2.12).

Acknowledgment I am gratefull to Takuro Abe for his encouragement. I am also gratefull to Atushi Moriwaki for his hospitality and kindness.

1 Preliminaries

In this section we introduce some definitions and results. One is about noncommutative projective schemes and the other is about derived Picard groups.

1.1 Noncommutative Projective Schemes

This subsection is a summary of the paper [Po] by A. Polishchuk. We only treat $\mathbb{N}$-graded algebras although $\mathbb{Z}$-algebras which is the more general notion are treated in [Po].

Let k be a field and let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ be a connected graded coherent ring. $\operatorname{Gr} R$ (resp. $\operatorname{gr} R$) denote the category of graded right R-modules (resp. finitely generated graded right R-modules). $\operatorname{Tor} R$ (resp. $\operatorname{tor} R$) denote the full subcategory of torsion modules (resp. finite k-dimensional modules). Note that $\operatorname{Tor} R$ and $\operatorname{tor} R$ are dense subcategories of $\operatorname{Gr} R$ and $\operatorname{gr} R$ respectively, hence the quotient categories $\operatorname{QGr} R = \operatorname{Gr} R/\operatorname{Tor} R$ and $\operatorname{qgr} R = \operatorname{gr} R/\operatorname{tor} R$ are abelian categories.

The degree shift operator $(1) : \operatorname{coh} R \longrightarrow \operatorname{coh} R$ induces the autoquivalence (1) on $\operatorname{qgr} R$. We denote by $\overline{R}$ the image in $\operatorname{qgr} R$ of the regular module R_R. The (coherent) *noncommutative projective scheme* $\operatorname{proj} R$ associated to R is the triple $(\operatorname{qgr} R, \overline{R}, (1))$. The autequivalence (1) is called the *canonical polization* on $\operatorname{proj} R$.

Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple consisting of a k-linear abelian category $\mathcal{C}$, an object $\mathcal{O}$ and an autequivalence s on $\mathcal{C}$. For $\mathcal{F} \in \mathcal{C}$, we define

$$\Gamma_*(\mathcal{F}) = \oplus_{n\geq 0} \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}[n]),$$

where $\mathcal{F}(n) = s^n\mathcal{F}$, and we set

$$R = \Gamma_*(\mathcal{C}, \mathcal{O}, s) = \Gamma_*(\mathcal{O}).$$

Multiplication is defined as follows:
If $x \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(l))$, $b \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(m))$ and $a \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(n))$ then

$$x \cdot a = s^m(x) \circ a \quad \text{and} \quad a \cdot b = s^n(a) \circ b$$

With this low of composition, $\Gamma_*(\mathcal{F})$ become a graded right module over the graded algebra R over k.

Definition 1.1 ([AZ] Section 4.2, [Po] Section 2). *Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple as above. Then s is called ample if the following conditions hold:*

(1) For every object $\mathcal{F} \in \mathcal{C}$, there are positive integers $l_1, dots, l_p$ and an epimorphism $\oplus_{i=1}^{p}\mathcal{O}(-l_i) \longrightarrow \mathcal{F}$.

(2) For every epimorphism $f : \mathcal{F} \longrightarrow \mathcal{G}$, there exists an integer n_0 such that for every $n \geq n_0$ the induced map $\operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(n)) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{G}(n))$ is surjective.

Let $\pi : \operatorname{Gr} R \longrightarrow \operatorname{QGr} R$ be the quotient functor. Set $\overline{\Gamma_*} = \pi \circ \Gamma_*$.

Theorem 1.2 ([Po] Theorem 2.4). *Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple as above. If s is ample, then the graded ring $R = \Gamma_*(\mathcal{C}, \mathcal{O}, s)$ is coherent, $\Gamma_*(\mathcal{F})$ is finitely generated R-module for $\mathcal{F} \in \mathcal{C}$ and the functor $\overline{\Gamma_*} : \mathcal{C} \longrightarrow \operatorname{qgr} R$ is equivalence of triples, i.e., $\overline{\Gamma_*} : \mathcal{C} \longrightarrow \operatorname{qgr} R$ is equivalence of categories, $\overline{\Gamma_*}(\mathcal{O}) \cong \overline{R}$ and $\overline{\Gamma_*} \circ s = (1) \circ \overline{\Gamma_*}$.*

It is well known that there is an eqivalence of triangulated categories $D^b(\operatorname{coh} \mathbb{P}^1) \cong D^b(\mathsf{mod}\text{-}k\overrightarrow{\Omega}_2)$ (See [Be]). Therefore by Theorem 0.1 $\operatorname{coh} \mathbb{P}^1$ and $\operatorname{qgr}(k\langle X_1, X_2\rangle/(X_1^2 + X_2^2))$ are derived equivalent. However, it turns out that these two categories are equivalent.

Example 1.1 ([SvB] Section 3). *Let $\sigma : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, $[x_1 : x_2] \mapsto [x_2 : -x_1]$ be the automorphism of $\mathbb{P}^1 = \operatorname{proj} k[X_1, X_2]$ and set $s = \sigma_*(- \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1})$. Then s is ample on the triple $(\operatorname{coh}(\mathbb{P}^1), \mathcal{O}_{\mathbb{P}^1}(1), s)$ and*

$$\Gamma_*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}, s) \cong k\langle X_1, X_2\rangle/(X_1^2 + X_2^2).$$

(Traditionally the graded ring $\Gamma_*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}, s)$ is denoted by $B(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \sigma)$ and is called the twisted homogeneous coordinate ring associated to the pair $(\mathcal{O}_{\mathbb{P}^1}(1), s)$.)

By Theorem1.2 there is an equivalence

$$\operatorname{coh} \mathbb{P}^1 \xrightarrow{\overline{\Gamma_*}} \operatorname{qgr}(k\langle X_1, X_2\rangle/(X_1^2 + X_2^2)).$$

1.2 Derived Picard Groups of Finite Dimensional Hereditary Algebras

This subsection is a summary of [MY] by J. Miyachi and A. Yekutieli.

First note that we work with right modules, hence the defintion of Reidtmann quiver differs from that of [Ha] and [MY]. (They work with left modules.)

Let A be a finite dimensional k-algebra. A complex $T \in D^b(\mathsf{Mod}\text{-}A \otimes_k A^{\mathrm{op}})$ is called a *two-sided tilting complex* if there exists another complex $T^\vee \in D^b(\mathsf{Mod}\text{-}A \otimes_k A^{\mathrm{op}})$ such that $T \otimes^{\mathbf{L}}_A T^\vee \cong T^\vee \otimes^{\mathbf{L}}_A T \cong A$. The *derived Picard group* of A (relative to k) is

$$\mathrm{DPic}_k(A) := \{\text{two-sided tilting complexes}\}/\text{isomorphisms}$$

with the identity element A, product $(T_1, T_2) \mapsto T_1 \otimes^L_A T_2$ and inverse $T \mapsto T^\vee := \mathbb{R}\mathrm{Hom}_A(T, A)$.

A tilting complex induces an equivalence of triangulated categories,

$$- \otimes^L_A T : D^b(\mathsf{Mod}\text{-}A) \longrightarrow D^b(\mathsf{Mod}\text{-}A).$$

See [Y],[MY] for more details.

Let $\vec{\Delta} = (\vec{\Delta}_0, \vec{\Delta}_1)$ be a finite quiver. The Reidtmann quiver $\vec{\mathbb{Z}}\vec{\Delta}$ of $\vec{\Delta}$ is defined as follows : The set of vertices $(\vec{\mathbb{Z}}\vec{\Delta})_0$ is given by $\mathbb{Z} \times \vec{\Delta}_0$, given an arrow $a : x \to y$ in $\vec{\Delta}$, there are the arrows $(n, a) : (n, y) \to (n, x)$ and $(n, a)^* : (n, x) \to (n + 1, y)$. The translation τ and the polization μ on $\vec{\mathbb{Z}}\vec{\Delta}$ are defined by $\tau(n, x) = (n - 1, x)$ and $\mu(n, a) = (n - 1, a)^*$, $\mu((n, a)^*) = (n, a)$. For an arrow $\alpha : \xi \to \eta$ in $\vec{\mathbb{Z}}\vec{\Delta}$, we have $\mu(\alpha) : \tau(\eta) \to \xi$.

The *path category* $k\langle\vec{\mathbb{Z}}\vec{\Delta}\rangle$ (in the sence of [MY]) is a category whose set of objects is $(\vec{\mathbb{Z}}\vec{\Delta})_0$, morphisms are generated by identites and the arrows and the only relations arise from incomposability of paths.

Let η be a vertex of $\vec{\mathbb{Z}}\vec{\Delta}$, let $\xi_1, \cdots, \xi_p$ be complete representative of the set $\{\xi \mid \text{there is an arrow}\, \alpha : \xi \to \eta\}$ and let $\{\alpha_{ij}\}_{j=1}^{d_i}$ be the set of arrows from ξ_i to η. The *mesh* ending at η is the subquiver of $\vec{\mathbb{Z}}\vec{\Delta}$ with vertex $\{\eta, \mu(\eta), \xi_1, \ldots, \xi_p\}$ and arrows $\{\alpha_{ij}, \mu(\alpha_{ij}) \mid i = 1, \ldots, p,\ j = 1, \ldots, d_i\}$. The *mesh ideal* in the path category $k\langle\vec{\mathbb{Z}}\vec{\Delta}\rangle$ is the ideal generated by the elements

$$\sum_{i=1}^{p}\sum_{j=1}^{d_i} \alpha_{ij} \circ \mu(\alpha_{ij}) \in \mathrm{Hom}_{k\langle\vec{\mathbb{Z}}\vec{\Delta}\rangle}(\mu(\eta), \eta).$$

The *mesh category* $k\langle\vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle$ is defined as the quotient category of the path category modulo mesh ideal.

Let $A = k\vec{\Delta}$ be the path algebra of $\vec{\Delta}$ and let $\mathsf{mod}\text{-}A$ denote the category of finite right A-modules. The k-dual $A^* = \mathrm{Hom}_k(A, k)$ of A is a two-sided tilting complex. We write $\tau_A \in \mathrm{DPic}_k(A)$ for the element represented by $A^*[1]$. Let us agree that τ_A also denotes the autequivalence $- \otimes^{\mathbf{L}}_A A^*$ of $D^b(\mathsf{mod}\text{-}A)$.

Let P_x be the indecomposable projective right A-module corresponding to a vertex $x \in \vec{\Delta}_0$. Define $\mathsf{B} \subset D^b(\mathsf{mod}\text{-}A)$ to be the full subcategory with objects $\{\tau_A^n P_x \mid x \in \vec{\Delta}_0, n \in \mathbb{Z}\}$.

Theorem 1.3 ([MY] Theorem 2.6). *There is a k-linear equivalence*

$$G : k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m \rangle \longrightarrow \mathsf{B}, \quad G(n,x) = \tau_A^{-n} P_x.$$

The equivalence G sends the mesh ending at $\eta = (n,y)$ to the exact triangle called *Auslander-Reiten triangle*

$$\tau_A(G(\eta)) = G(\tau(\eta)) \overset{\oplus G(\beta_{ij})}{\longrightarrow} \bigoplus_{i=1}^{p} \bigoplus_{j=1}^{d_i} G(\xi_i) \overset{\oplus G(\alpha_{ij})}{\longrightarrow} G(\eta) \overset{[1]}{\longrightarrow} \tag{1}$$

where $\beta_{ij} = \mu(\alpha_{ij})$.

By the definition of the path category the sets of arrows $\{\alpha_{ij}\}_{j=1}^{d_i}$ and $\{\beta_{ij}\}_{j=1}^{d_i}$ are basis for $\mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\xi_i, \eta)$ and $\mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\tau(\eta), \xi_i)$ respectively.

Fix isomorphisms

$$\mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\xi_i, \eta) \cong \mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\tau(\eta), \xi_i)^*$$
$$\alpha_{ij} \mapsto \beta_{ij}^*$$

where $\{\beta_{ij}^*\}_{j=1}^{d_i}$ is the dual basis of $\{\beta_{ij}\}_{i=1}^{d_i}$.

Set $V_i = \mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\xi_i, \eta) \cong \mathrm{Hom}_{k\langle \vec{\mathbb{Z}}\vec{\Delta}, I_m\rangle}(\tau(\eta), \xi_i)^*$. Then there are canonical morphisms

$$\varphi_i : V_i \otimes_k G(\xi_i) \longrightarrow G(\eta), \quad \psi_i : \tau_A(G(\eta)) \longrightarrow V_i \otimes_k G(\xi_i). \tag{2}$$

Then the Auslander-Reiten triangle (1) has the following form:

$$\tau_A(G(\eta)) \overset{\oplus\psi_i}{\longrightarrow} \bigoplus_{i=1}^{p} V_i \otimes_k G(\xi_i) \overset{\oplus\varphi_i}{\longrightarrow} G(\eta) \overset{[1]}{\longrightarrow}. \tag{3}$$

Let $\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0)$ be the permutation group of the vertex set $(\vec{\mathbb{Z}}\vec{\Delta})_0$ and let $\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d)^{\langle\tau\rangle}$ be the subgroup of permutations which preserves arrow-multiplicties and commute with τ , namely

$$\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d)^{\langle\tau\rangle} = \{\pi \in \mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})_0 \mid d(x,y) = d(\pi(x), \pi(y)) \,\text{for all}\, x, y \in (\vec{\mathbb{Z}}\vec{\Delta})_0, \,\text{and}\, \pi\tau = \tau\pi\}$$

where $d(x,y)$ denotes the arrow-multiplicity from x to y.

Theorem 1.4 ([MY] Theorem 3.8.). *Let $\vec{\Delta}$ be a finite quiver without oriented cycles and let $A = k\vec{\Delta}$ be the path algebra of $\vec{\Delta}$ over an algebraically closed field k. If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k A \cong (\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d) \ltimes \mathrm{Out}_k^0(A)) \times \mathbb{Z}$$

where $\mathrm{Out}_k^0(A)$ denotes the identity component of the group of outer automorphisms $\mathrm{Out}_k(A)$.

Remark 1.1. *In* [MY], *the case of the finite representation type is also computed.*

2 Proof of Theorem 0.1

From now on we assume that k is an algebraically closed field and $N \geq 2$. We denote by R_n the degree n component of the graded ring $R = \oplus_{n\geq 0} R_n = k\langle X_1, \dots, X_N\rangle / (\sum_{i=1}^{N} X_i^2)$.

The following Lemma is easy to show.

Lemma 2.1. *For each $n \geq 1$, there is a following short exact sequences of k-vector spaces.*

$$0 \longrightarrow R_{n-1} \xrightarrow{\Psi} R_1 \otimes_k R_n \xrightarrow{\Phi} R_{n+1} \longrightarrow 0, \tag{4}$$

where the second and third morphisms are defined as follows

$$\begin{aligned} \Phi : & R_1 \otimes_k R_n \ni f \otimes_k r \mapsto fr \in R_{n+1} \\ \Psi : & R_{n-1} \ni r_{n-1} \mapsto \sum_{i=1}^{N} X_i \otimes_k X_i r_{n-1} \in R_1 \otimes_k R_n. \end{aligned}$$

.

Let A be the path algebra of the N-Kronecker quiver $\vec{\Omega}_N$ shown in Figure 1. We denote by P_0, P_1 indecomposable projective right A-modules associated to vertices $0, 1$ of $\vec{\Omega}_N$, and S_0, S_1 simple A modules associated to vertices $0, 1$ of $\vec{\Omega}_N$. Then $S_1 = P_1$ and P_0 is the projective cover of S_0.

The Reidetmann quiver $\vec{\mathbb{Z}}\vec{\Omega}_N$ of the N-Kronecker quiver $\vec{\Omega}_N$ is shown in Figure 2 below. (In Figure 2 vertices $(-n, 1)$ and $(-n, 0)$ are replaced by $\tau_A^n P_1$ and $\tau_A^n P_0$ respectively.)

$$\begin{tikzcd}
\tau^{n+1}P_1 \arrow[dr, shift left, "x_1"] \arrow[dr, shift right, "x_N"', "\vdots" description] & & \tau^n P_1 \arrow[dr, shift left, "x_1"] \arrow[dr, shift right, "x_N"', "\vdots" description] & & \tau^{(n-1)}P_1 \\
& \tau^{n+1}P_0 \arrow[ur, shift left, "x_1"] \arrow[ur, shift right, "x_N"', "\vdots" description] & & \tau^n P_0 \arrow[ur, shift left, "x_1"] \arrow[ur, shift right, "x_N"', "\vdots" description] &
\end{tikzcd}$$

Figure 2: Reidetmann quiver $\vec{\mathbb{Z}}\vec{\Omega}_N$

It is easy to see that

$$\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Omega}_N)_0; d) \cong \mathbb{Z}$$

and there is a genertor ρ such that $\rho(0, 1) = (0, 0), \rho(0, 0) = (1, 0)$. This ρ satisfies the relation $\rho^{-2} = \tau$. By Theorem1.4 there exists the two-sided tilting complex ρ_A such that $\rho_A^{-2} \cong \tau_A$ and $\rho_A P_1 \cong P_0$.

From now on we write $\rho = \rho_A$ and $\tau = \tau_A$.

For $M^{\cdot} \in D^b(\mathsf{mod}\text{-}A)$ we use the following notation

$$\rho^n M^{\cdot} = M^{\cdot} \otimes^{\mathbf{L}}_A \overbrace{\rho \otimes^{\mathbf{L}}_A \cdots \otimes^{\mathbf{L}}_A \rho}^{n}.$$

Since $P_0 \cong \rho P_1$ and $\rho^2 \cong \tau^{-1}$ it follows that $\tau^{-}nP_1 \cong \rho^{2n}P_1$, $\tau^n P_0 \cong \rho^{2n+1}$. Hence the Reidetmann quiver $\overrightarrow{\mathbb{Z}}\overrightarrow{\Omega}_N$ has forms shown in Figure 3.

$$\begin{tikzcd}
P_1 \arrow[dr, shift left=2, "x_1"] \arrow[dr, phantom, "\vdots"] \arrow[dr, shift right=2, "x_N"'] & & \rho^2 P_1 \arrow[dr, shift left=2, "x_1"] \arrow[dr, phantom, "\vdots"] \arrow[dr, shift right=2, "x_N"'] & & \rho^4 P_1 \\
& \rho P_1 \arrow[ur, shift left=2, "x_1"] \arrow[ur, phantom, "\vdots"] \arrow[ur, shift right=2, "x_N"'] & & \rho^3 P_1 \arrow[ur, shift left=2, "x_1"] \arrow[ur, phantom, "\vdots"] \arrow[ur, shift right=2, "x_N"'] &
\end{tikzcd}$$

Figure 3: Reidetmann quiver $\overrightarrow{\mathbb{Z}}\overrightarrow{\Omega}_N$

Set $V = \mathrm{Hom}_A(P_1, P_0)$. The set of arrows $x_1, \ldots, x_N$ is a basis for V. The graded vector space $\oplus_{\geq n} \mathrm{Hom}(P_1, \rho^n P_1)$ has graded algebra structure as in the same way of Section 1.1.

By the mesh relation and Theorem1.3 we have the following:

Proposition 2.2. *There is an isomrphism of graded algebras* $\oplus_{\geq n} \mathrm{Hom}(P_1, \rho^n P_1) \longrightarrow R$ *which sends* $x_i \in V$ *to* $X_i \in R_1$ *for* $i = 1, \ldots, N$.

Fix isomorphisms $V \cong \mathrm{Hom}_{D^b(\mathsf{mod}\text{-}A)}(\rho^n P_1, \rho^{n+1} P_1)$ for $n \geq 0$ and identify with V and its dual V^* by sending $x_i \mapsto x_i^*$. Then the canonical morphisms φ_i and ψ_i in (2) of Section 1.2 have following forms:

$$\varphi : V \otimes_k \rho^n P_1 \longrightarrow \rho^{n+1} P_1,\; v \otimes p_n \mapsto v(p_n),$$

$$\psi : \rho^{n-1} P_1 \longrightarrow V \otimes \rho^n P_1,\; p_{n-1} \mapsto \sum_{i=1}^{N} x_i \otimes x_i(p_{n-1}).$$

The Auslander-Reiten triangle (3) has following form:

$$(\mathcal{S}_n) \qquad \rho^{n-1} P_1 \simeq k \otimes_k \rho^{n-1} P_1 \xrightarrow{\psi} V \otimes_k \rho^n P_1 \xrightarrow{\varphi} \rho^{n+1} P_1 \xrightarrow{[1]} .$$

Proposition 2.3. $\rho^n = \rho \otimes^{\mathbf{L}}_A \cdots \otimes^{\mathbf{L}}_A \rho$ *is a pure module for* $n \geq 0$, *(i.e.,*$H^i(\rho^n) = 0$ *for all* $i \neq 0$*).*

Proof. Since $A \cong P_1 \oplus \rho P_1$, it suffices to show that $\rho^{n+1} P_1$ is pure for $n \geq 1$. By induction we may assume that $\rho^{n-1} P_1$ and $\rho^n P_1$ are pure. Then the long cohomology sequence of $(\mathcal{S}_n)$ implies that $\mathrm{H}^i(\rho^{n+1} P_1) = 0$ for $i \neq -1, 0$.

If $i = -1$, then since $A = P_1 \oplus \rho P_1$ and P_1 is a projective module, we have only to show that $\mathrm{Ext}^{-1}_A(P_1, \rho^{n+1} P_1) = \mathrm{Hom}_A(P_1, \mathrm{H}^{-1}(\rho^{n+1} P_1)) = 0$.

Observe that taking $\mathrm{Hom}(P_1, -)$ of exact triangles $(\mathcal{S}_n)$ yields exact sequences (4) of Lemma 2.1 under isomorphismes $R_n \cong \mathrm{Hom}(P_0, \rho^n P_0)$ of proposition 2.2. By induction hypothes $\mathrm{Ext}^{-1}_A(P_0, \rho^n P_0) = 0$, hence the injectivity of $R_{n+1} \to V \otimes R_n$ implies $\mathrm{Ext}^{-1}_A(P_0, \rho^{n+1} P_0) = 0$. $\square$

Corollary 2.4.

(1) $\tau^{n+1}[1]$ *are pure modules for all* $n \geq 0$.

(2) $\rho^{-n}S_0$ *are pure modules for all* $n \geq 0$.

Proof. (1) By Proposition 2.3 $\tau^{-n} \cong \rho^{2n}$ is pure for $n \geq 0$. Hence $\tau^{n+1}[1] \cong \tau^n \otimes A^* \cong (\tau^{-n})^*$ is pure for $n \geq 0$.

(2) By the direct calculation we have $S_1 \cong P_1 \otimes^{\mathbf{L}}_A A^* \cong (\rho^{-1}[1])P_0$. Hence $\rho^{-2n}S_1 \cong \tau^{n+1}[1]P_1$ and $\rho^{-(2n+1)} \cong \tau^{n+1}[1]P_0$. $\square$

Remark 2.1. *In this paper we assume* $N \geq 2$. *When* $N = 1$ *it is known that* $\tau^3 \cong A[-2]$ *(*[MY] *Theorem 4.1). Hence above corollary fails.*

Reversing Serre vanishing theorem we define as the following.

Definition 2.5. *The fullsubcategory* $D^{\rho,\geq 0}$ *(resp.* $D^{\rho,\leq 0}$*) of* $D^b(\mathsf{mod}\text{-}A)$ *consists of objects* $M^{\cdot}$ *which satisfies*

$$\begin{aligned} &\mathbb{R}\mathrm{Hom}^{\cdot}(P_1, \rho^n M^{\cdot}) \in D^{\geq 0}(k - vect) \quad \textit{for } n \gg 0 \\ (\textit{resp.} \quad &\mathbb{R}\mathrm{Hom}^{\cdot}(P_1, \rho^n M^{\cdot}) \in D^{\leq 0}(k - vect) \quad \textit{for } n \gg 0) \end{aligned}$$

Remark 2.2. *Since* $A \cong P_1 \oplus \rho P_1$, $M^{\cdot} \in D^{\rho,\geq 0}$ *(resp.*$D^{\rho,\leq 0}$*) if and only if* $\rho^n M^{\cdot} \in D^{\geq 0}(\mathsf{mod}\text{-}A)$ *(resp.*$D^{\leq 0}(\mathsf{mod}\text{-}A)$*) for* $n \gg 0$

Proposition 2.6. *The pair of fullsubcategories* $D^{\rho} = (D^{\rho,\geq 0}, D^{\rho,\leq 0})$ *is a t-structure in* $D^b(\mathsf{mod}\text{-}A)$.

Proof. Since the other two axioms of t-structure are obvious by above Remark 2.2, it suffices to show that for every complex $M^{\cdot} \in D^b(\mathsf{mod}\text{-}A)$ there is an exact triangle

$$A^{\cdot} \longrightarrow M^{\cdot} \longrightarrow B^{\cdot} \xrightarrow{[1]}$$

such that $A^{\cdot} \in D^{\rho,\leq 0}$ and $B^{\cdot} \in D^{\rho,\geq 1} = D^{\rho,\geq 0}[-1]$.

By [Ha] Lemma I.5.2 an indecomposable object of $D^b(\mathsf{mod}\text{-}A)$ is of the form $M[-i]$, where $M \in \mathsf{mod}\text{-}A$ and $i \in \mathbb{Z}$. By Lemma 2.3, $\rho^n M[-i] \in D^{[i-1,i]}(\mathsf{mod}\text{-}A)$. Hence $M[-i] \in D^{\rho,\geq 1}$ for $i \geq 2$ and $M[-i] \in D^{\rho,\leq 0}$ for $i \leq 0$.

The case when $i = -1$ is reduced to the following Lemma. $\square$

Lemma 2.7. *For every* $M \in \mathsf{mod}\text{-}A$ *there exists a submodule* M' *such that* $\mathrm{Hom}(P_1, \rho^n M') = 0$ *for* $n \gg 0$ *and* $\mathrm{Ext}^{-1}_A(P_1, \rho^n M'') = 0$ *for* $n \gg n_0$, *where we set* $M'' = M/M'$.

Proof. Define $\mathcal{T}_n = \{N \in \mathsf{mod}\text{-}A \mid N \text{ is generated by } \rho^n S_0\}$ and $\mathcal{F}_n = \{N \in \mathsf{mod}\text{-}A \mid Hom_A(\rho^{-n}S_0, N) = 0\}$. Since $\mathrm{Ext}^1_A(\rho^{-n}S_0, \rho^{-n}S_0) \cong \mathrm{Ext}^1_A(P_1, P_1) = 0$ and A is hereditary, by [Ha] Lemma III.4.2, the pair $(\mathcal{T}_n, \mathcal{F}_n)$ is a torsion theory on the abelian category $\mathsf{mod}\text{-}A$. So, if we set $t_n(M), f_n(M)$ to be the image and cokernel of the canonical morphism $\mathrm{Hom}(\rho^{-n}S_1, M) \otimes_k M \to M$, then $t_n(M) \in$

$\mathcal{T}_n$, $f_n(M) \in \mathcal{F}_n$. Since the canonical morphism $V \otimes_k \rho^{-(n+1)} S_1 \to \rho^{-n} S_0$ is surjective. Hence $t_n(M) \subset t_{n+1}(M)$. Since $\dim_k M < \infty$, there is an integer n_0 such that $t_n(M) = t_{n_0}(M)$ for all $n \geq n_0$. Then by definition $f_n(M) = f_{n_0}(M)$ for all $n \geq n_0$. Set $M' = t_{n_0}(M), M'' = f_{n_0}(M)$.

It is easy to see that $\mathrm{Hom}(P_0, \rho^n N) = 0$ for $N \in \mathcal{T}_n$ and $\mathrm{Ext}_A^{-1}(P_0, \rho^n N) \cong \mathrm{Hom}_A(\rho^{-n} S_1, N) = 0$ for $N \in \mathcal{F}_n$. Therefore $\mathrm{Hom}(P_0, \rho^n M') = 0$ for $n \geq n_0$ and $\mathrm{Ext}_A^{-1}(P_0, \rho^n M'') = 0$ for $n \geq n_0$ □

Remark 2.3. *Set* $\mathcal{T} = \{N \in \mathsf{mod}\text{-}A \mid \mathrm{Hom}(P_1, \rho^n N) = 0 \, for \, n \gg 0\}$ *and* $\mathcal{F} = \{N \in \mathsf{mod}\text{-}A \mid \mathrm{Ext}^{-1}(P_1, \rho^n N) = 0 \, for \, n \gg 0\}$. *Then by the Lemma 2.7 it is easy to see that* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair on* $\mathsf{mod}\text{-}A$.

We can define a t-structure on $D^b(\mathsf{mod}\text{-}A)$ *from this torsin pair by*

$$\begin{aligned} D'^{\geq 0} &:= \{M^{\cdot} \in D^b(\mathsf{mod}\text{-}A)^{\geq 0} \mid H^0(M^{\cdot}) \in \mathcal{F}\} \\ D'^{\leq 0} &:= \{M^{\cdot} \in D^b(\mathsf{mod}\text{-}A)^{\leq 1} \mid H^1(M^{\cdot}) \in \mathcal{T}\}. \end{aligned}$$

(See [HRS] *Proposition I.2.1). However, this is not a new t-structure. It is easy to see that* $(D'^{\geq 0}, D'^{\leq 0}) = (D^{\rho, \geq 0}, D^{\rho, \leq 0})$.

Let $\mathcal{H}^\rho$ be the heart of the t-structure D^ρ. By Proposition 2.3, $P_1 \in \mathcal{H}^\rho$. Now le us consider the triple $(\mathcal{H}^\rho, P_1, \rho)$.

Proposition 2.8. ρ *is ample on the triple* $(\mathcal{H}^\rho, P_1, \rho)$.

Proof. We check the conditions (1) and (2) of Definiton 1.1.

First note that the cokernel of the morphism $f : M^{\cdot} \longrightarrow N^{\cdot}$ in the abelian category $\mathcal{H}^\rho$ is $\tau^\rho_{\geq 0}(\mathrm{Cone}(f))$, where $\tau^\rho_{\geq 0} : D^b(\mathsf{mod}\text{-}A) \longrightarrow D^{\rho, \geq 0}$ is the truncation functor (See [GM] IV.4). So f is surjective in $\mathcal{H}^\rho$ if and only if $\mathrm{Cone}(f) \in D^{\rho, \leq -1}$.

(1) It is easy to see that for every $M \in \mathsf{mod}\text{-}A$ there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(P_1, M) \otimes_k P_1 \longrightarrow M \longrightarrow \mathrm{Hom}_A(P_0, M) \otimes_k S_0 \longrightarrow 0$$

which is functorial in M. So for every $M^{\cdot} \in D^b(A\text{-mod})$ and for every n there is an exact triangle

$$\mathbb{R}\mathrm{Hom}(P_1, \rho^n M^{\cdot}) \otimes_k \rho^{-n} P_1 \longrightarrow M^{\cdot} \longrightarrow \mathbb{R}\mathrm{Hom}(P_1, \rho^{n-1} M^{\cdot}) \otimes_k \rho^{-n} S_0 \xrightarrow{[1]}$$

Since $\mathrm{Hom}(P_1, \rho^{m-n} S_0) \cong \mathrm{Hom}(P_1, \rho^{m-n-1} P_1[1]) = 0$ for $m \geq n+1$, hence $\rho^{-n} S_0 \in D^{\rho, \leq -1}$.

Now assume that $M^{\cdot} \in \mathcal{H}^\rho$ and take an intger n such that the complex $\mathbb{R}\mathrm{Hom}(P_1, \rho^{n-1} M^{\cdot})$ is pure. Then $\mathbb{R}\mathrm{Hom}(P_1, \rho^{n-1} M^{\cdot}) \otimes_k \rho^{-n} S_0 \in D^{\rho, \leq -1}$.

(2) Let

$$M^{\cdot} \xrightarrow{f} N^{\cdot} \longrightarrow L^{\cdot} \xrightarrow{[1]}$$

be a triangle such that $M^{\cdot}, N^{\cdot} \in \mathcal{H}^\rho$ and $L^{\cdot} \in D^{\rho \leq -1}$.

Take an intger n such that $\mathrm{Hom}(P_1, \rho^n L^{\cdot}) = 0$. Then the induced morphism $\mathrm{Hom}(P_1, \rho^n M^{\cdot}) \longrightarrow \mathrm{Hom}(P_1, \rho^n N^{\cdot})$ is surjective. □

Combining Proposition 2.2 and Proposition 2.8 and applying Theorem 1.2 we obtain :

Proposition 2.9. *The triple $(\mathcal{H}^\rho, P_1, \rho)$ is equivalent to* $\operatorname{proj} R$ *as triples.*

The functor

$$\Gamma_* : \mathsf{mod}\text{-}A \longrightarrow \operatorname{Gr} R, \quad \Gamma_*(M) := \oplus_{i \geq 0} \operatorname{Hom}(P_1, M \otimes_A \rho^n)$$

is right exact. Let $\mathbf{L}\Gamma_*$ be the left derived functor of Γ_*. A finite projective A-module P is of the form $P \cong P_1^{\oplus n} \oplus P_0^{\oplus m}$ for some $n, m \in \mathbb{Z}_{\geq 0}$. Then $\Gamma_*(P) \cong R^{\oplus n} \oplus R(1)^{\oplus m}$ is a finite R-module. Hence for $M^\cdot \in D^b(\mathsf{mod}\text{-}A)$, $\mathbf{L}\Gamma_*(M^\cdot)$ has coherent cohomologies.

Thus there is an exact functor

$$\mathbf{L}\Gamma_* : D^b(\mathsf{mod}\text{-}A) \longrightarrow D^b_{\operatorname{gr} R}(\operatorname{Gr} R).$$

Since the quotient functor $\pi : \operatorname{Gr} R \longrightarrow \operatorname{QGr} R$ is exact by [Pop] Theorem 4.3.8, this functor can be extended to the exact functor $\pi : D^b_{\operatorname{gr} R}(\operatorname{Gr} R) \longrightarrow D^b_{\operatorname{qgr} R}(\operatorname{QGr} R)$. Define $\mathbf{L}\overline{\Gamma_*} := \pi \circ \mathbf{L}\Gamma_*$.

Lemma 2.10. *The set $\{\overline{R}, \overline{R(1)}\}$ generates $D^b(\operatorname{qgr} R)$, i.e., the minimal triangulated full subcategory of $D^b(\operatorname{qgr} R)$ containing $\overline{R}$ and $\overline{R(1)}$ is $D^b(\operatorname{qgr} R)$ itself.*

Proof. The canonical functor $c : D^b(\operatorname{qgr} R) \longrightarrow D^b_{\operatorname{qgr} R}(\operatorname{QGr} R)$ is an eqivalence by [BvB] Lemma 4.3.3.

There is a following commutative diagram :

(5)
$$\begin{tikzcd}
D^b(\mathsf{mod}\text{-}A) \arrow[rr, "\mathbf{L}\overline{\Gamma_*}"] & & D^b_{\operatorname{qgr} R}(\operatorname{QGr} R) \\
\mathcal{H}^\rho \arrow[u, hook, "i_A"] \arrow[r, "\overline{\Gamma_*}"] & \operatorname{qgr} R \arrow[r, hook, "i_R"] & D^b(\operatorname{qgr} R) \arrow[u, "c"']
\end{tikzcd}$$

where i_A, i_R are inclusions.

It is clear that $\operatorname{qgr} R$ generates $D^b(\operatorname{qgr} R)$. Thus $\mathbf{L}\overline{\Gamma_*}$ is essentially surjective. Since the set $\{P_1, P_0\}$ generates $D^b(\mathsf{mod}\text{-}A)$, $\overline{R} = \mathbf{L}\overline{\Gamma_*}(P_1)$ and $\overline{R(1)} = \mathbf{L}\overline{\Gamma_*}(P_0)$ generate $D^b(\operatorname{qgr} R)$. $\square$

Lemma 2.11. *The set $\{\overline{R}, \overline{R(1)}\}$ is a strongly exceptional sequence in $D^b(\operatorname{qgr} R)$, i.e., $\operatorname{Hom}_{D^b(\operatorname{qgr} R)}(\overline{R}, \overline{R(1)}[i]) = 0$ for every $i \neq 0$ and $\operatorname{Hom}_{D^b(\operatorname{qgr} R)}(\overline{R(1)}, \overline{R}[i]) = 0$ for every i.*

Proof. By Appendix of [Be2] there exists an exact functor $F : D^b(\mathcal{H}^\rho) \longrightarrow D^b(\mathsf{mod}\text{-}A)$ and following commutative diagram

$$\begin{tikzcd}
D^b(\mathcal{H}^\rho) \arrow[r, "F"] & D^b(\mathsf{mod}\text{-}A) \\
\mathcal{H}^\rho \arrow[u, hook, "i_H"] \arrow[ur, hook, "i_A"'] &
\end{tikzcd},$$

where i_H, i_A are inclusions.

The restriction $\mathbf{L}\overline{\Gamma_*} \circ F|_{\mathcal{H}^\rho}$ to $\mathcal{H}^\rho$ is equal to $c \circ i_R \circ \overline{\Gamma_*}$ with the notation in the diagram (5). Hence $\mathbf{L}\overline{\Gamma_*} \circ F$ is an equivalence. Therefore, the morhpism of Hom-sets

$$\mathbf{L}\overline{\Gamma_*}_{(M^\cdot, N^\cdot[n])} : \mathrm{Hom}_{D^b(\mathsf{mod}\text{-}A)}(M^\cdot, N^\cdot[n]) \longrightarrow \mathrm{Hom}_{D^b_{cohprojR}(QGrR)}(\mathbf{L}\overline{\Gamma_*}(M^\cdot), \mathbf{L}\overline{\Gamma_*}(N^\cdot)[n])$$

is surjective for $M^\cdot, N^\cdot \in D^b(\mathsf{mod}\text{-}A)$ and $n \in \mathbb{Z}$. Now it is easy to prove the statement because P_1, P_0 is a strongly exceptional sequence in $D^b(\mathsf{mod}\text{-}A)$ and $\overline{R} = \mathbf{L}\overline{\Gamma_*}(P_1), \overline{R(1)} = \mathbf{L}\overline{\Gamma_*}(P_0)$. □

Since $\mathrm{QGr}\, R$ has enough injectives by [Pop] Theorem 4.5.2, Theorem 6.2 of [Bo] completes the proof of Theorem 0.1.

Theorem 2.12 (Theorem 0.1). *Let k be an algebraically closed field, let $A = k\overrightarrow{\Omega}_N$ be the path algebra of the N-Kronecker quiver $\overrightarrow{\Omega}_N$ and let $R = k\langle X_1, \ldots, X_N\rangle/(\sum_{i=1}^N X_i^2)$. Set $T := \overline{R} \oplus \overline{R(1)}$. For $N \geq 2$, the functor*

$$\mathbb{R}\mathrm{Hom}_{D^b(\mathrm{qgr}\, R)}(T, -) : D^b(\mathrm{qgr}(R)) \longrightarrow D^b(\mathsf{mod}\text{-}A)$$

is an equivalence of triangulated categories.

Remark 2.4. *It is known that the functor $- \otimes^{\mathbf{L}}_A T$ is a quasi-inverse to the functor $\mathbb{R}\mathrm{Hom}(T, -)$.*

It is easy to see that there is a following commutaive diagram:

$$\begin{tikzcd} D^b(\mathsf{mod}\text{-}A) \arrow[r, "\mathbf{L}\overline{\Gamma_*}"] \arrow[dr, "- \otimes^{\mathbf{L}}_A T"'] & D^b_{\mathrm{qgr}\, R}(\mathrm{QGr}\, R) \\ & D^b(\mathrm{qgr}\, R) \arrow[u, "c"'] \end{tikzcd} .$$

Thus the functors $\mathbf{L}\overline{\Gamma_*}$ and $- \otimes^{\mathbf{L}}_A T$ are essentially isomorphic. In paticular $\mathbf{L}\overline{\Gamma_*}$ is an equivalence.

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