HOMOTOPY NILPOTENCY IN LOCALIZED GROUPS

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Abstract. Let G be a compact, simply connected Lie group. A prime p is called regular for G if G has the homotopy type of the product of odd spheres at p-local. When p is regular for G, the localization of G at p, $G_{(p)}$, is known to be homotopy nilpotent. We determine the homotopy nilpotency class of $G_{(p)}$ when p is regular for G.

1. Introduction

Let p be a prime. We denote the localization at p by $-_{(p)}$ throughout. Each space is assumed to have the homotopy type of a CW-complex. Quite often, we identify maps with their homotopy classes ambiguously.

Let us first recall words and facts on finite H-spaces. Let X be a connected H-space with $\dim H_*(X;\mathbf{Q}) < \infty$. By the Hopf theorem, one has

$$X_{(0)} \simeq S^{2n_1-1}_{(0)} \times \cdots \times S^{2n_l-1}_{(0)},$$

where $-_{(0)}$ means the rationalization. In this case, we say that X is of type $(n_1, \ldots, n_l)$. The types of compact, connected, simple Lie groups are listed in the following table.

A_l	$(2,3,\ldots,l+1)$	G_2	$(2,6)$
B_l	$(2,4,\ldots,2l)$	F_4	$(2,6,8,12)$
C_l	$(2,4,\ldots,2l)$	E_6	$(2,5,6,8,9,12)$
D_l	$(2,4,\ldots,2l-2,l)$	E_7	$(2,6,8,10,12,14,18)$
		E_8	$(2,8,12,14,18,20,24,30)$

Let G be a compact, connected Lie group of type $(n_1, \ldots, n_l)$ with $n_1 \le \cdots \le n_l$. Serre [16] defined that a prime p is regular for G if there is a homotopy equivalence

$$G_{(p)} \simeq S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_l-1}_{(p)}. \tag{1.1}$$

It is shown that p is regular for G if and only if $p \ge n_l$ when G is simple. Kumpel [9] generalized Serre's result above as follows. Let X be a p-local, simply connected finite H-space of type $(n_1, \ldots, n_l)$ with $n_1 \le \cdots \le n_l$. Kumpel [9] showed that if $p \ge n_l - n_1 + 2$,

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then p is regular for X in the sense of Serre. In particular, if p is regular for X, the mod p cohomology of X is given by

$$H^*(X;\mathbf{Z}/p)=\Lambda(x_1,\dots,x_l),\ |x_i|=2n_i-1.$$

Moreover, it is known that if X has a classifying space BX, then

$$H^*(BX;\mathbf{Z}/p)=\mathbf{Z}/p[y_1,\dots,y_l],\ |y_i|=2n_i$$

and generators y_i of $H^*(X;\mathbf{Z}/p)$ can be chosen as

$$y_i=\sigma(x_i), \tag{1.2}$$

where σ denotes the cohomology suspension.

For simplicity, we make a convention that each loop space is assumed to be *strictly associative* in the standard way.

We consider the group structure of a loop space X in a homotopy theoretical point of view. Here, we avoid the complexity of considering general homotopy associative H-spaces. Regarding the group structure of X, the commutator map

$$\gamma : X\times X\to X,\ (x,y)\mapsto xyx^{-1}y^{-1}$$

is obviously important. We say that X is homotopy commutative if γ is null-homotopic, that is, X is an abelian group up to homotopy.

The homotopy commutativity of H-spaces has been extensively studied. In particular, regarding finite H-spaces, Hubbuck [5] got the celebrated result that a connected, homotopy commutative, finite H-space is equivalent to a torus. Meanwhile, McGibbon [11] studied the homotopy commutativity of localized Lie groups, which are infinite H-spaces, and proved:

Theorem 1.1 (McGibbon [11]). *Let G be a compact, simply connected, simple Lie group of type $(n_1,\dots,n_l)$ with $n_1\le\dots\le n_l$.*

(1) *If $p>2n_l$, then $G_{(p)}$ is homotopy commutative.*
(2) *If $p<2n_l$, then $G_{(p)}$ is not homotopy commutative except for the cases that $(G,p)=(Sp(2),3)$, $(G_2,5)$.*

There are several generalizations of the notion of the homotopy commutativity. One generalization is the higher commutativity which gives levels between the homotopy commutativity and the strict commutativity. This notion is first formulated by Sugawara [17] and later refined by Williams [19]. Saumell [15] generalized McGibbon's result above along this direction.

On the other hand, one way of generalizing the notion of the homotopy commutativity of a loop space X is the homotopy nilpotency which measures how non-commutative X is. The precise definition of the homotopy nilpotency is as follows. Let X be a connected loop space and let $\gamma_k : X^{k+1}\to X$ denote the k-iterated commutator map

$$\gamma_k=\gamma\circ(1\times\gamma)\circ\dots\circ(1\times\dots\times1\times\gamma) : X^{k+1}\to X,$$

where X^{k+1} denotes the direct product of $(k+1)$-copies of X. We say that X is homotopy nilpotent if there exists a positive integer N such that γ_N is null-homotopic, that is, X is a nilpotent group up to homotopy. The homotopy nilpotency class of a homotopy nilpotent loop space X is defined as the least number n such that γ_n is null-homotopic, and denoted by $\mathbf{nil}(X)$, which, of course, corresponds to the class of a nilpotent group. In particular, X is homotopy commutative if and only if $\mathbf{nil}(X) = 1$. The reader may refer to [21] for general facts on the homotopy nilpotency.

The homotopy nilpotency of H-spaces has been extensively studied as well as the homotopy commutativity. In particular, Hopkins [4] made a big progress by giving (co)homological criteria for homotopy associative finite H-spaces to be homotopy nilpotent. For example, he showed that if a homotopy associative H-space has no torsion in the integral homology, then it is homotopy nilpotent. Later, Rao [14] showed that the converse of the above criterion is true in the case of $Spin(n)$ and $SO(n)$. Eventually, Yagita [20] proved that, when G is a compact, simply connected Lie group, $G_{(p)}$ is homotopy nilpotent if and only if it has no torsion in the integral homology. Although many results on the homotopy nilpotency are obtained as above, the homotopy nilpotency classes have not been determined in almost all cases.

The aim of this paper is to generalize McGibbon's result above along the concept of the homotopy nilpotency. Precisely, we determine the homotopy nilpotency class of compact, simply connected, simple Lie groups localized at regular primes as follows. Of course, from this result, we can see the homotopy nilpotency classes of compact, simply connected Lie groups localized at regular primes which are not necessarily simple.

Theorem 1.2. *Let G be a compact, simply connected, simple Lie group of type $(n_1, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$. If p is regular, then $G_{(p)}$ is homotopy nilpotent with:*

(1) *If $\frac{3}{2}n_l < p < 2n_l$, then $\mathbf{nil}(G_{(p)}) = 2$.*
(2) *If $n_l \leq p \leq \frac{3}{2}n_l$, then $\mathbf{nil}(G_{(p)}) = 3$ except for the cases that $(G, p) = (F_4, 17)$, $(E_6, 17)$, $(E_8, 41)$, $(E_8, 43)$ or $\mathrm{rank} G = 1$ with $p = 2$.*
(3) *In the above exceptional cases, $\mathbf{nil}(G_{(p)}) = 2$*

The organization of this paper is as follows. In §2, we consider the homotopy nilpotency of p-local finite loop space X when p is regular for X. We decompose the above iterated commutator map γ_n into smaller pieces which can be detected by iterated Samelson products in $\pi_*(X)$. Moreover, we see that such Samelson products can be handled with the data of homotopy groups of spheres. Then we prove some of Theorem 1.2 in a more general setting. As a consequence, the proof of Theorem 1.2 is reduced to find non-trivial iterated Samelson products on a case-by-case analysis.

In §3, we deal with the case of classical groups by use of the result of Bott [1].

In §4 and §5, we consider the cases of E_7 with $p = 23$ and E_8 with $p = 37$ respectively. The main idea to search for non-trivial Samelson products is due to Kono and Ōshima

[10], which detects non-zero Samelson products by the primary operation $\wp^1$. Then we will make some cohomology calculations.

In §6, we deal with the remaining cases by referring Hamanaka and Kono [6] and by an easy dimension counting.

2. Generalized Samelson products

We begin this section with an elementary commutator calculus. Let H be a group generated by $x_1, \ldots, x_n$ and let $[a,b]$ denote the commutator of $a, b \in H$, that is, $[a,b] = aba^{-1}b^{-1}$. We define a subset Z_k of H inductively by

$$Z_0 = \{x_i^\nu | 1 \leq i \leq n, \nu = \pm 1\},\ Z_k = \{[a,b] | a \in Z_0, b \in Z_{k-1}\}.$$

Denote the subgroup of H generated by $\bigcup_{i=k}^\infty Z_i$ by $\bar{Z}_k$. Using the formulae

$$[x, yz] = [x,y][y,[x,z]][x,z],\ [xy, z] = [x,[y,z]][y,z][x,z],$$

one can see the following by induction on k and on the word lengths of a_1 in $[a_1, [\cdots[a_k, a_{k+1}]\cdots]]$, where $a_1, \ldots, a_{k+1} \in H$.

Lemma 2.1. *Define the subgroup H_k of H by $H_0 = H$ and $H_i = [H, H_{i-1}]$, the group generated by $\{[a,b] | a \in H, b \in H_{i-1}\}$. Then we have*

$$H_k = \bar{Z}_k.$$

Let us recall the definition of generalized Samelson products. Let X be a loop space. The generalized Samelson product of maps $\alpha : A \to X$ and $\beta : B \to X$ is defined as the composition of maps

$$A \wedge B \xrightarrow{\alpha \wedge \beta} X \wedge X \xrightarrow{\bar{\gamma}} X$$

and denoted by $\langle \alpha, \beta \rangle$, where $\bar{\gamma}$ is the reduced commutator map of X. Then it is a usual Samelson product in $\pi_*(X)$ if both A and B are spheres. Let $\{\alpha, \beta\}$ denote the composition

$$A \times B \xrightarrow{\alpha \times \beta} X \times X \xrightarrow{\gamma} X,$$

where $\gamma : X \times X \to X$ is the unreduced commutator map of X as in the previous section. Note that $\pi^* : [A_1, \wedge \cdots \wedge A_k, X] \to [A_1 \times \cdots \times A_k, X]$ is monic, where $\pi : A_1 \times \cdots \times A_k \to A_1 \wedge \cdots \wedge A_k$ is the projection. Actually, it is an isomorphism onto a direct summand. Then, for $\alpha_i : A_i \to X$ $(i = 1, \ldots, k)$, we have $\langle \alpha_1, \langle \cdots \langle \alpha_{k-1}, \alpha_k \rangle \cdots \rangle \rangle = 0$ if and only if $\{\alpha_1, \{\cdots \{\alpha_{k-1}, \alpha_k\} \cdots\}\} = 0$. Here we mean by $f = 0$ that a map f is null homotopic and we shall make use of this notation unless any confusion occurs.

By definition, X is homotopy nilpotent of class $\mathbf{nil}(X) < k$ if and only if the k-iterated commutator $\{1_X, \{\cdots \{1_X, 1_X\} \cdots\}\} = 0$. Then we shall consider this map.

Let X be a p-local, simply connected finite loop space of type $(n_1, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$. We denote a generator of a free part of $\pi_{2n_i-1}(X)$, that is, $\mathbf{Z}_{(p)}$ in $\pi_{2n_i-1}(X)$, corresponding to the entry n_i in the type of X by ϵ_i. Define a map

$$\iota : S^{2n_1-1} \times \cdots \times S^{2n_l-1} \to X$$

by

$$\iota(x_1,\ldots,x_l)=\epsilon_1(x_1)\cdots\epsilon_l(x_l).$$

Suppose that p is regular for X. Then the homotopy equivalence (1.1) is given by the map $\iota_{(p)}:S^{2n_1-1}_{(p)}\times\cdots\times S^{2n_l-1}_{(p)}\to X$. Hence it follows that

$$1_X(x)=\epsilon_1'(x)\cdots\epsilon_l'(x) \tag{2.1}$$

for $x\in X$, where $\epsilon_i'=(\epsilon_i)_{(p)}\circ\pi_i\circ\iota_{(p)}^{-1}$ for the projection $\pi_i:S^{2n_1-1}_{(p)}\times\cdots\times S^{2n_l-1}_{(p)}\to S^{2n_i-1}_{(p)}$.

Now we decompose the iterated commutator $\{1_X,\{1_X,\{\cdots\{1_X,1_X\}\cdots\}\}\}$. Let us consider the group $[X^n,X]$, where the group structure of $[X^n,X]$ is given by the pointwise multiplication. Let $\rho_j:X^n\to X$ denote the j-th projection. We define a subset $\mathcal{Z}^n$ of $[X^n,X]$ by

$$\mathcal{Z}^n=\{(\epsilon_i'\circ\rho_j)^\nu|1\le i\le l,1\le j\le n,\nu=\pm1\}.$$

We consider the subgroup $\mathcal{H}^n$ of $[X^n,X]$ generated by $\mathcal{Z}^n$. Let $\bar{\mathcal{Z}}^n_k$ denote the subgroup of $[X^n,X]$ corresponding to $\bar{Z}_k$ in Lemma 2.1 putting $Z_0=\mathcal{Z}^n$

We denote the commutator of $[X^n,X]$ by $[\,\cdot\,,\,\cdot\,]_n$. For $\alpha_i\in[X^{k+1},X]$ $(i=1,\ldots,k+1)$, one can see that the k-iterated commutator $[\alpha_1,[\cdots[\alpha_k,\alpha_{k+1}]_{k+1}\cdots]_{k+1}]_{k+1}$ in $[X^{k+1},X]$ is the composition

$$X^{k+1}\xrightarrow{\Delta}X^{(k+1)^2}\xrightarrow{\alpha_1\times\cdots\times\alpha_{k+1}}X^{k+1}\xrightarrow{\{1_X,\{\cdots\{1_X,1_X\}\cdots\}\}}X. \tag{2.2}$$

Then it follows from the formula

$$(\rho_1\times\cdots\times\rho_{k+1})\circ\Delta=1_{X^{k+1}} \tag{2.3}$$

that the k-iterated commutator can be written down as

$$\{1_X,\{\cdots\{1_X,1_X\}\cdots\}\}=[\rho_1,[\cdots[\rho_k,\rho_{k+1}]_{k+1}\cdots]_{k+1}]_{k+1}$$

(cf. Lemma 2.6.1 in [21]). Hence, for (2.3) and (2.1), we can apply Lemma 2.1 to the group $\mathcal{H}^{k+1}$ and obtain:

Proposition 2.1. *Let X be a p-local, simply connected finite loop space. If p is regular for X, then the k-iterated commutator $\{1_X,\{\cdots\{1_X,1_X\}\cdots\}\}$ belongs to $\bar{\mathcal{Z}}^{k+1}_k$.*

Corollary 2.1. *Let X and p be as in Proposition 2.1. Then $\mathbf{nil}(X)<k$ if and only if $\langle\epsilon_{i_1},\langle\cdots\langle\epsilon_{i_k},\epsilon_{i_{k+1}}\rangle\cdots\rangle\rangle=0$ for each $1\le i_1,\ldots,i_{k+1}\le l$.*

Proof. From Proposition 2.1 and (2.2), one can see that the k-iterated commutator

$$\{1_X,\{\cdots\{1_X,1_X\}\cdots\}\}=0$$

if and only if

$$[(\epsilon_{i_1}'\circ\rho_{j_1})^{\nu_1},[\cdots[(\epsilon_{i_k}'\circ\rho_{j_k})^{\nu_k},(\epsilon_{i_{k+1}}'\circ\rho_{j_{k+1}})^{\nu_{k+1}}]_{k+1}\cdots]_{k+1}]_{k+1}=0$$

for each $1\le i_1,\ldots,i_{k+1},j_1,\ldots,j_{k+1}\le l$ and $\nu_i=\pm1$. Moreover, (2.2) and (2.3) yield that the above holds if and only if

$$\{(\epsilon_{i_1}')^{\nu_1},\{\cdots\{(\epsilon_{i_k}')^{\nu_k},(\epsilon_{i_{k+1}}')^{\nu_{k+1}}\}\cdots\}\}=0$$

for each $1 \leq i_1, \ldots, i_{k+1} \leq l$ and $\nu_i = \pm 1$. For the above observation on the generalized Samelson products and commutators, this is equivalent to that

$$\langle (\epsilon'_{i_1})^{\nu_1}, \langle \cdots \langle (\epsilon'_{i_k})^{\nu_k}, (\epsilon'_{i_{k+1}})^{\nu_{k+1}} \rangle \cdots \rangle\rangle = 0$$

for each $1 \leq i_1, \ldots, i_{k+1} \leq l$ and $\nu_i = \pm 1$. Since $(\pi_{i_1} \circ \iota_{(p)}^{-1} \wedge \cdots \wedge \pi_{i_{k+1}} \circ \iota_{(p)}^{-1})^* : [S_{(p)}^{2n_{i_1}-1} \wedge \cdots \wedge S_{(p)}^{2n_{i_{k+1}}-1}, X] \to [\wedge^{k+1} X, X]$ is monic, the above condition is equivalent to that

$$\langle \epsilon_{i_1}^{\nu_1}, \langle \cdots \langle \epsilon_{i_k}^{\nu_k}, \epsilon_{i_{k+1}}^{\nu_{k+1}} \rangle \cdots \rangle\rangle = 0$$

for each $1 \leq i_1, \ldots, i_{k+1} \leq l$ and $\nu_i = \pm 1$. Then, for that Samelson products are bilinear and that $\epsilon_i^{-1} = -\epsilon_i \in \pi_*(X)$, we have established Corollary 2.1. $\square$

Remark 2.1. The reader may compare Corollary 2.1 with the result in the first author's paper [8] concerning the rational homotopy.

In order to proceed the observation on the homotopy nilpotency, let us recall some facts on the p-primary components of the unstable homotopy groups of odd spheres for an odd prime p (see [18] for details).

Fact 2.1. $\pi_{2n-1+k}(S^{2n-1})_{(p)} = \begin{cases} \mathbf{Z}/p & k = 2p-3 \\ 0 & 0 < k < 4p-6, k \neq 2p-3. \end{cases}$

Let $\alpha_1(3)$ denote a generator of $\pi_{2p}(S^3) = \mathbf{Z}/p$ and $\alpha_1(n)$ the suspension $\Sigma^{n-3}\alpha_1(3)$.

Fact 2.2. *The homotopy group* $\pi_{2n+2p-4}(S^{2n-1})_{(p)} = \mathbf{Z}/p$ *is generated by* $\alpha_1(2n-1)$.

Fact 2.3. $\alpha_1(3) \circ \alpha_1(2p) \neq 0$ *and* $\alpha_1(2n-1) \circ \alpha_1(2n+2p-4) = 0$ *for* $n > 2$.

Let X and ϵ_i be as above. Suppose that p is an odd prime and that

$$p > n_l - \frac{n_1}{2} + 1.$$

Then p is a regular prime for X. Now we consider the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$. Since $X_{(0)}$ is homotopy commutative, $\mathbf{r} \circ \langle \epsilon_i, \epsilon_j \rangle = 0$, where $\mathbf{r} : X \to X_{(0)}$ denotes the rationalization. Then $\langle \epsilon_i, \epsilon_j \rangle \in \pi_{2(n_i+n_j-1)}(X)$ is a torsion element. Fact 2.1 and Fact 2.2 yield that

$$\pi_{n_s} \circ \langle \epsilon_i, \epsilon_j \rangle = \begin{cases} N\alpha_1(2n_s - 1) & n_i + n_j = n_s + p - 1 \\ 0 & n_i + n_j \neq n_s + p - 1 \end{cases}$$

for $N \in \mathbf{Z}/p$, not necessarily non-zero. Then it follows from Fact 2.3 that if $\pi_{n_t} \circ \langle \epsilon_k, (\epsilon_s)_{(p)} \circ \pi_{n_s} \circ \langle \epsilon_i, \epsilon_j \rangle\rangle \neq 0$, then

$$n_t = 2,\ n_i + n_j = n_s + p - 1,\ n_k + n_s = p + 1. \tag{2.4}$$

In particular, for a dimensional consideration, if $p > \frac{3}{2} n_l$, then

$$\pi_{n_t} \circ \langle \epsilon_k, (\epsilon_s)_{(p)} \circ \pi_{n_s} \circ \langle \epsilon_i, \epsilon_j \rangle\rangle = 0$$

for each $1 \leq i, j, k, s, t \leq l$. On the other hand, from Fact 2.3, one has

$$\Sigma^2 \pi_{n_t} \circ \langle \epsilon_k, (\epsilon_s)_{(p)} \circ \pi_{n_s} \circ \langle \epsilon_i, \epsilon_j \rangle\rangle = 0$$

for any i,j,k,s,t. Then it follows that

$$\pi_{n_u}\circ\langle\epsilon_m,(\epsilon_t)_{(p)}\circ\pi_{n_t}\circ\langle\epsilon_k,(\epsilon_s)_{(p)}\circ\pi_{n_s}\circ\langle\epsilon_i,\epsilon_j\rangle\rangle\rangle=0$$

for each $1\le i,j,k,m,s,t,u\le l$.

By a quite similar commutator calculus at the beginning of this section, one can see that the k-iterated Samelson product $\langle\epsilon_{i_1},\langle\cdots\langle\epsilon_{i_k},\epsilon_{i_{k+1}}\rangle\cdots\rangle\rangle=0$ for each $1\le i_1,\dots,i_{k+1}\le l$ if and only if $\pi_{j_1}\circ\langle\epsilon_{i_1},(\epsilon_{i_1})_{(p)}\circ\pi_{j_2}\circ\langle\cdots(\epsilon_{i_k})_{(p)}\circ\pi_{j_k}\circ\langle\epsilon_{i_k},\epsilon_{i_{k+1}}\rangle\cdots\rangle\rangle=0$ for each $1\le i_1,\dots,i_{k+1},j_1\dots,j_k\le l$, here we need this discussion for the possibility that $n_i=n_{i+1}$ for some i.

Summarizing the above observation, one has that if an odd prime p satisfies $p>n_l-\frac{n_l}{2}-1$, then $\mathbf{nil}(X)\le 3$ by Corollary 2.1. Moreover, if $p>\frac{3}{2}n_l$, then one has $\mathbf{nil}(X)\le 2$ by Corollary 2.1.

On the other hand, James and Thomas [7] showed that if $n_l-n_1+2\le p<2n_l$, then X is not homotopy commutative, equivalently, $\mathbf{nil}(X)\ge 2$. Hence we have obtained:

Theorem 2.1. *Let X be a p-local, simply connected finite loops space of type $(n_1,\dots,n_l)$ with $n_1\le\cdots\le n_l$ and let p be an odd prime with $p>n_l-\frac{n_1}{2}+1$. Then X is homotopy nilpotent with:*

(1) *If $\frac{3}{2}n_l<p<2n_l$, then $\mathbf{nil}(X)=2$.*
(2) *If $n_l-\frac{n_1}{2}+1<p\le\frac{3}{2}n_l$, then $2\le\mathbf{nil}(X)\le 3$.*

Remark 2.2. Actually, we have $\mathbf{nil}(X)=2$ if $n_l-\frac{n_1}{2}+1<p<2n_l$ and $n_1>2$, where X is as in Theorem 2.1.

Remark 2.3. Let X be as in Theorem 2.1. For a dimensional consideration, one can see that $\langle\epsilon_i,\epsilon_j\rangle=0$ for each $1\le i,j\le l$ if $p>2n_l-n_1+1$ as well. Equivalently, X is homotopy commutative if $p>2n_l-n_1+1$ by Corollary 2.1. Then one can consequently deduce from Theorem 2.1 that the prime p cannot be in the range $2n_l-n_1+1<p<2n_l$. This can be seen also from the observation of James and Thomas [7] using the primary operation $\wp^1$.

In most of cases, Theorem 2.1 reduces the proof of Theorem 1.2 to finding non-zero 2-iterated Samelson products when the prime is in the range in (2) of Theorem 2.1.

3. The case of classical groups

3.1. **The 2-local rank one case.** Let us first consider the case that the rank of a classical group G is one, equivalently, the case that $G\cong S^3$. It is well-known that the Samelson product $\langle 1_G,1_G\rangle$ is a generator of $\pi_6(G)=\mathbf{Z}/12$ (see, for example, the result of Bott below). On the other hand, since $\pi_9(G)=\mathbf{Z}/3$ by Toda [18], we have $l_2\circ\langle 1_G,\langle 1_G,1_G\rangle\rangle=0$, where $l_2:G\to G_{(2)}$ is the 2-localization. Then one has $\mathbf{nil}(G_{(2)})=2$.

For the rest of this section, the ranks of classical groups are assumed to be greater than one.

3.2. **The case of $SU(n)$.** We denote a generator of $\pi_{2i-1}(SU(n)) = \mathbf{Z}$ $(i = 2, \dots, n)$ by ϵ_i. We can deduce from the result of Bott [1] that if $i + j > n$, then the order of the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is a non-zero multiple of

$$\frac{(i+j-1)!}{(i-1)!(j-1)!}.$$

Let p be a prime with $n < p \leq \frac{3}{2}n$. Then one has

$$\langle \bar{\epsilon}_n, \bar{\epsilon}_{p-n+1} \rangle \neq 0, \ \langle \bar{\epsilon}_n, \bar{\epsilon}_{2p-2n} \rangle \neq 0,$$

where $\bar{\epsilon}_{n_i} = l_p \circ \epsilon_{n_i}$ for the p-localization $l_p : SU(n) \to SU(n)_{(p)}$. For Fact 2.1, $\langle \bar{\epsilon}_n, \bar{\epsilon}_{2p-2n} \rangle$ takes values in $S^{2p-2n+1}_{(p)} \subset SU(n)_{(p)}$. Then, from Fact 2.2 and Fact 2.3, it follows that

$$\langle \bar{\epsilon}_n, \langle \bar{\epsilon}_n, \bar{\epsilon}_{2p-2n} \rangle \rangle \neq 0$$

and hence, in this case, Theorem 1.2 follows from Theorem 2.1.

Next, we consider the case that $n = p$ which is not included in Theorem 2.1. Quite similarly to the above calculation, one can see that

$$\langle \bar{\epsilon}_{p-1}, \langle \bar{\epsilon}_{p-1}, \bar{\epsilon}_2 \rangle \rangle \neq 0$$

and then

$$\mathbf{nil}(SU(p)_{(p)}) \geq 3. \tag{3.1}$$

To proceed our observation, let us recall some more facts on the p-primary component of the unstable homotopy groups of odd spheres for an odd prime p (see [18]).

Fact 3.1. $\pi_{2n-1+4p-5}(S^{2n-1})_{(p)} = \mathbf{Z}/p$.

Let $\alpha_2(3)$ denote a generator of $\pi_{4p-2}(S^3)_{(p)} = \mathbf{Z}/p$ and $\alpha_2(n)$ the suspension $\Sigma^{n-3}\alpha_2(3)$.

Fact 3.2. *The homotopy group* $\pi_{2n-1+4p-5}(S^{2n-1})_{(p)} = \mathbf{Z}/p$ *is generated by* $\alpha_2(2n-1)$.

For a dimensional consideration, the only possible non-zero 3-iterated Samelson products in $\pi_*(SU(p))_{(p)}$ are

$$\langle \bar{\epsilon}_{p-1}, \langle \bar{\epsilon}_{p-1}, \langle \bar{\epsilon}_p, \bar{\epsilon}_p \rangle \rangle \rangle, \ \langle \bar{\epsilon}_{p-1}, \langle \bar{\epsilon}_p, \langle \bar{\epsilon}_{p-1}, \bar{\epsilon}_p \rangle \rangle \rangle.$$

By Fact 3.1 and Fact 3.2, the above iterated Samelson products are the non-zero multiple of

$$\alpha_1(3) \circ \alpha_1(2p) \circ \alpha_2(4p-3), \ \alpha_1(3) \circ \alpha_2(2p) \circ \alpha_1(6p-5)$$

respectively. Since $\alpha_1(k) \circ \alpha_2(k+2p-3) = -\alpha_2(k) \circ \alpha_1(k+4p-5)$, one has

$$\alpha_1(3)\circ\alpha_1(2p)\circ\alpha_2(4p-3) = -\alpha_1(3)\circ\alpha_2(2p)\circ\alpha_1(6p-5) = \alpha_2(3)\circ\alpha_1(4p-2)\circ\alpha_1(6p-5) = 0$$

by Fact 2.3 and hence, for Corollary 2.1 and (3.1), we have obtained that

$$\mathbf{nil}(SU(p)_{(p)}) = 3.$$

3.3. **The case of $Sp(n)$.** Let ϵ'_i denote a generator of $\pi_{4i-1}(Sp(n)) = \mathbf{Z}$ $(i = 1, \dots, n)$. We can also deduce from the result of Bott [1] that if $i + j > n$, then the order of the Samelson product $\langle \epsilon'_i, \epsilon'_j \rangle$ is a non-zero multiple of

$$\frac{(2i + 2j - 1)!}{(2i - 1)!(2j - 1)!}.$$

Hence we can find quite similarly to the case of $SU(n)$ a non-zero 2-iterated Samelson product in $Sp(n)_{(p)}$ for $2n < p \leq 3n$ and then, in the case of $Sp(n)$, Theorem 1.2 follows from Theorem 2.1.

3.4. **The case of $Spin(n)$.** Let $i : Spin(2k-1) \to Spin(2k)$ denote the natural inclusion. Harris [3] showed that the fibration

$$Spin(2k-1)_{(p)} \xrightarrow{i_{(p)}} Spin(2k)_{(p)} \to S^{2k-1}_{(p)}$$

splits if p is odd. Then $(i_{(p)})_* : \pi_*(Spin(2k-1)_{(p)}) \to \pi_*(Spin(2k)_{(p)})$ is monic and hence the case of $Spin(2k)$ can be deduced from the case of $Spin(2k-1)$, here we use the fact that p is regular for $Spin(2k-1)$ if and only if so is for $Spin(2k)$. Friedlander [2] gave an A_∞-equivalence

$$Spin(2k-1)_{(p)} \cong Sp(k-1)_{(p)}$$

when p is an odd prime. Then the above consideration of $Sp(n)$ shows that, when p is regular for $Sp(n)$, there exists a non-zero 2-iterated Samelson product in $Spin(2k-1)_{(p)}$ and hence in $Spin(2k)_{(p)}$. Therefore, for Theorem 2.1, the proof of Theorem 1.2 in the case of $Spin(n)$ is completed.

4. The case of E_7 with $p = 23$

In order to prove Theorem 1.2 in the case of E_7 with $p = 23$, we will show that there exists a non-zero 2-iterated Samelson product in $(E_7)_{(p)}$, as in the previous section. To do so, we will exploit the following method of Kono and Ōshima [10].

Let X be a p-local finite loop space of type $(n_1, \dots, n_l)$ and let p be a regular prime for X. As in §1, the mod p cohomology of BX is given by

$$H^*(BX; \mathbf{Z}/p) = \mathbf{Z}/p[y_1, \dots, y_l], \ |y_i| = 2n_i.$$

As in §2, we denote a generator of $\mathbf{Z}_{(p)}$ in $\pi_{2n_i-1}(X)$ corresponding to the entry n_i in the type of X by ϵ_i. The Samelson product $\langle \epsilon_i, \epsilon_j \rangle \in \pi_{2(n_i+n_j-1)}(X)$ can be detected by the primary operation $\wp^1$ as:

Lemma 4.1. *If $\wp^1 y_k$ includes the term $\delta y_i y_j$ with $\delta \neq 0$, then $\langle \epsilon_i, \epsilon_j \rangle \neq 0$.*

Proof. Suppose that $\langle \epsilon_i, \epsilon_j \rangle = 0$, equivalently, the Whitehead product $[\hat{\epsilon}_i, \hat{\epsilon}_j] = 0$, where $\hat{\epsilon}_m : S^{2m} \to BX$ denotes the adjoint of ϵ_m. Then there exists a map $\kappa : S^{2n_i} \times S^{2n_j} \to BX$

satisfying the following homotopy commutative diagram.

$$\begin{tikzcd} S^{2n_i} \vee S^{2n_j} \arrow[r, "\hat{\epsilon}_i \vee \hat{\epsilon}_j"] \arrow[d, phantom, "\cap"] & BX \vee BX \arrow[d, "\nabla"] \\ S^{2n_i} \times S^{2n_j} \arrow[r, "\kappa"] & BX, \end{tikzcd}$$

where ∇ is the folding map. For (1.1) and (1.2), one has

$$(\hat{\epsilon}_i)^*(\sigma(y_j)) = \delta_{ij} s_{2n_i},$$

where s_n is a generator of $H^*(S^n; \mathbf{Z}/p)$. Hence one can see

$$\kappa^*(\wp^1 y_k) = \delta s_{2n_i} \otimes s_{2n_j} \neq 0.$$

On the other hand,

$$\kappa^*(\wp^1 y_k) = \wp^1(\kappa^*(y_k)) = 0$$

and this is a contradiction. Therefore Lemma 4.1 is accomplished. □

Let us prepare some notations of symmetric polynomials. We consider the polynomial ring $\mathbf{Z}/p[t_1, \ldots, t_n]$. Let c_k $(k = 1, \ldots, n)$ denote the k-th elementary symmetric function in $t_1, \ldots, t_n$, that is,

$$\prod_{i=1}^{n}(1 + t_i) = 1 + c_1 + \cdots + c_n.$$

Define a polynomial p_k $(k = 1, \ldots, n)$ by

$$\prod_{i=1}^{n}(1 - t_i^2) = 1 - p_1 + \cdots + (-1)^n p_n.$$

We denote the k-the power sum in $t_1^2, \ldots, t_n^2$ by T_i $(k = 1, \ldots, n)$. Namely,

$$T_k = t_1^{2k} + \cdots + t_n^{2k}.$$

Then one has the Girard's formula

$$T_k = (-1)^k k \sum_{i_1+2i_2+\cdots+ni_n=k} (-1)^{i_1+\cdots+i_n} \frac{(i_1 + \cdots + i_n - 1)!}{i_1! \cdots i_n!} p_1^{i_1} \cdots p_n^{i_n} \tag{4.1}$$

(see [12]).

Note that, by taking a maximal torus in $Spin(2n)$, we can regard the above c_n and p_i $(i = 1, \ldots, n-1)$ the universal Euler class and the universal i-th Pontrjagin class in $H^*(BSpin(2n); \mathbf{Z}/p)$ respectively.

Hereafter, p is fixed to 23 throughout this section. We calculate the action of $\wp^1$ on $H^*(BE_7; \mathbf{Z}/p)$ in virtue of Lemma 4.1. To do so, we make use of the following commutative diagram.

$$\begin{tikzcd} Spin(10) \arrow[r, equal] \arrow[d, "i_1"'] & Spin(10) \arrow[d, "i_2"] \\ E_6 \arrow[r, "j"] & E_7, \end{tikzcd}$$

where i_1, i_2 and j are the natural inclusions. It is well-known that the mod p cohomology of $BSpin(10)$, BE_6 and BE_7 are given by

$$H^*(BSpin(10);\mathbf{Z}/p) = \mathbf{Z}/p[p_1,p_2,p_3,p_4,c_5],$$
$$H^*(BE_6;\mathbf{Z}/p) = \mathbf{Z}/p[\bar{y}_4,\bar{y}_{10},\bar{y}_{12},\bar{y}_{16},\bar{y}_{18},\bar{y}_{24}],\ |\bar{y}_i| = i$$
$$H^*(BE_7;\mathbf{Z}/p) = \mathbf{Z}/p[y_4,y_{12},y_{16},y_{20},y_{24},y_{28},y_{36}],\ |y_i| = i.$$

Hamanaka and Kono [6] showed that we can choose generators $\bar{y}_i$ and y_i such that

$$j^*(y_4) = \bar{y}_4,\ j^*(y_{12}) = \bar{y}_{12},\ j^*(y_{16}) = \bar{y}_{16},\ j^*(y_{20}) = \bar{y}_{10}^2,\ j^*(y_{28}) = \bar{y}_{10}\bar{y}_{18},$$
$$i_1^*(\bar{y}_4) = p_1,\ i_1^*(\bar{y}_{10}) = c_5,\ i_1^*(\bar{y}_{12}) = -6p_3 + p_1p_2,\ i_1^*(\bar{y}_{16}) = 12p_4 + p_2^2 - \tfrac{1}{2}p_1^2p_2.$$

Then one has

$$i_2^*(y_4) = p_1,\ i_2^*(y_{12}) = -6p_3 + p_1p_2,\ i_2^*(y_{16}) = 12p_4 + p_2^2 - \tfrac{1}{2}p_1^2p_2,\ i_2^*(y_{20}) = c_5^4. \tag{4.2}$$

For a dimensional reason, one has

$$i_1^*(\bar{y}_{18}) = \delta_1 p_1^2 c_5 - \delta_2 p_2 c_5$$

and hence, for (4.2),

$$i_2^*(y_{28}) = \delta_1 p_1^2 c_5^2 - \delta_2 p_2 c_5^2. \tag{4.3}$$

From the above facts, we shall prove:

Proposition 4.1. *$\wp^1 y_4$ includes the term $\delta y_{12}y_{36}$ ($\delta \neq 0$).*

Proposition 4.2. *$\wp^1 y_{12}$ includes the term $\delta_1 y_{20}y_{36}$ ($\delta_1 \neq 0$) or $\delta_2 y_{28}y_{28}$ ($\delta_2 \neq 0$).*

From Lemma 4.1 and Proposition 4.1, it follows that

$$\langle \epsilon_{18}, \epsilon_6 \rangle \neq 0,$$

where ϵ_i denotes a generator of $\pi_{2i-1}((E_7)_{(p)}) = \mathbf{Z}_{(p)}$. Similarly, from Lemma 4.1 and Proposition 4.2, it follows that

$$\langle \epsilon_{10}, \epsilon_{18} \rangle \neq 0 \text{ or } \langle \epsilon_{14}, \epsilon_{14} \rangle \neq 0.$$

For Fact 2.1, $\langle \epsilon_{10}, \epsilon_{18} \rangle$ and $\langle \epsilon_{14}, \epsilon_{14} \rangle$ take values in $S^{11}_{(p)} \subset (E_7)_{(p)}$. Then, by Fact 2.2 and Fact 2.3, we obtain

$$\langle \epsilon_{18}, \langle \epsilon_{10}, \epsilon_{18} \rangle \rangle \neq 0 \text{ or } \langle \epsilon_{18}, \langle \epsilon_{14}, \epsilon_{14} \rangle \rangle \neq 0.$$

Therefore Theorem 2.1 completes the proof of Theorem 1.2 in the case of E_7 with $p = 23$.

Proof of Proposition 4.1. We define a ring homomorphism

$$\pi : \mathbf{Z}/p[p_1,\ldots,p_4,c_5] \to \mathbf{Z}/p[a_2,\ldots,a_4,b_5]/(a_2^3, a_3^2, a_4^2, b_5^3, 12a_4 + a_2^2)$$

by

$$\pi(p_1) = 0,\ \pi(p_i) = a_i\ (i = 2,3,4),\ \pi(c_5) = b_5.$$

Then, for (4.2) and (4.3), one has

$$\pi(i_2^*(y_4)) = \pi(i_2^*(y_{16})) = \pi(i_2^*(y_{12}^2)) = \pi(i_2^*(y_{28}y_{20})) = 0. \tag{4.4}$$

Put $\wp^1 y_4 = \delta y_{12}y_{36}$+other terms. Then one can see that

$$\pi(i_2^*(\wp^1 y_4)) = \delta\pi(i_2^*(y_{12}y_{36})).$$

On the other hand, since $p_1 = T_1$ and $\wp^1 T_1 = 2T_{12}$, Girard's formula (4.1) yields that

$$\pi(\wp^1 p_1) = -15a_3a_4b_5^2 \neq 0.$$

Thus $\delta \neq 0$ and this completes the proof. □

Proof of Proposition 4.2. We define a ring homomorphism

$$\pi : \mathbf{Z}/p[p_1, \ldots, p_4, c_5] \to \mathbf{Z}/p[a_2, a_4, b_5]/(a_2^3, a_4^2, b_5^5, 12a_4 + a_2^2)$$

by

$$\pi(p_i) = 0\ (i = 1, 3), \pi(p_j) = a_j\ (j = 2, 4),\ \pi(c_5) = b_5.$$

Then, for (4.2) and (4.3), we have

$$\pi(i_2^*(y_4)) = \pi(i_2^*(y_{12})) = \pi(i_2^*(y_{16})) = 0. \tag{4.5}$$

Put $\wp^1 y_{12} = \delta_1 y_{20}y_{36} + \delta_2 y_{28}y_{28}$+other terms. Then one has

$$\pi(i_2^*(\wp^1 y_{12})) = \delta_1\pi(i_2^*(y_{20}y_{36})) + \delta_2\pi(i_2^*(y_{28}y_{28})).$$

Let us calculate $\pi(i_2^*(\wp^1 y_{12}))$ directly. From Girard's formula (4.1), it follows that

$$\pi(\wp^1 p_1) = \pi(\wp^1 T_1) = \pi(2T_{12}) = -a_2b_5^4,\ \pi(\wp^1 T_3) = \pi(6T_{14}) = -9a_4b_5^4.$$

Since $T_3 = p_1^3 - 3p_1p_2 + 3p_3$, one has

$$\pi(\wp^1 p_3) = 9a_4b_5^4.$$

For (4.2), one can see

$$\pi(\wp^1(i_2^*(y_{12}))) = -19a_4b_5^4 \neq 0.$$

Then we have obtained $\delta_1 \neq 0$ or $\delta_2 \neq 0$ and this completes the proof. □

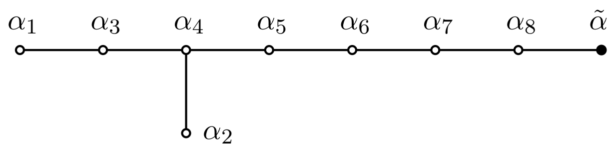
5. The case of E_8 with $p = 37$

We employ Proposition 4.1 to find a non-zero 2-iterated Samelson product in $(E_8)_{(p)}$ as well as in the previous section, where $p = 37$ throughout this section.

The mod p cohomology of BE_8 is given by

$$H^*(BE_8; \mathbf{Z}/p) = \mathbf{Z}/p[y_4, y_{16}, y_{24}, y_{28}, y_{36}, y_{40}, y_{48}, y_{60}],\ |y_i| = i.$$

In order to calculate the action of $\wp^1$ on $H^*(BE_8; \mathbf{Z}/p)$, we shall arrange generators y_i. Let α_i $(i = 1, \ldots, 8)$ and $\tilde{\alpha}$ be respectively the simple roots and the dominant root of E_8 as indicated in the following extended Dynkin diagram of E_8 (see [13] for details).



Let $W(E_8)$ denote the Weyl group of E_8. Let W and φ be the subgroup of $W(E_8)$ generated by the reflections corresponding to α_i $(i = 2, \dots, 8)$ and $\tilde{\alpha}$, and the elements of $W(E_8)$ corresponding to α_1 respectively. Then, by choosing appropriate generators $t_1, \dots, t_8 \in H^*(BT; \mathbf{Z}/p)$, Hamanaka and Kono [6] showed that

$$H^*(BT; \mathbf{Z}/p)^W = \mathbf{Z}/p[p_1, \dots, p_7, c_8],$$

where p_i and c_i are as in the previous section. Since $W(E_8)$ is generated by W and φ, one has

$$H^*(BT; \mathbf{Z}/p)^{W(E_8)} = H^*(BT; \mathbf{Z}/p)^\varphi \cap \mathbf{Z}/p[p_1, \dots, p_7, c_8].$$

Then the projection $\rho : BT \to BE_8$ induces an isomorphism

$$\rho^* : H^*(BE_8; \mathbf{Z}/p) \xrightarrow{\cong} H^*(BT; \mathbf{Z}/p)^\varphi \cap \mathbf{Z}/p[p_1, \dots, p_7, c_8].$$

With the above choice of generators $t_1, \dots, t_8 \in H^*(BT; \mathbf{Z}/p)$, Hamanaka and Kono [6] showed that the automorphism φ is given by

$$\varphi(c_1) = -c_1,\ \varphi(c_2) = c_2,\ \varphi(c_8) = c_8 - \tfrac{1}{4}c_1c_7,\ \varphi(p_1) = p_1$$

and

$$\varphi(p_i) \equiv p_i + c_1h_i \mod (c_1^2) \tag{5.1}$$

for $i = 2, \dots, 7$, where

$$h_2 = \tfrac{3}{2}c_3,\ h_3 = -\tfrac{1}{2}(5c_5 + c_2c_3),\ h_4 = \tfrac{1}{2}(7c_7 + 3c_2c_5 - c_3c_4),$$

$$h_5 = -\tfrac{1}{2}(5c_2c_7 - 3c_3c_6 + c_4c_5),\ h_6 = -\tfrac{1}{2}(5c_3c_8 - 3c_4c_7 + c_5c_6),\ h_7 = \tfrac{1}{2}(3c_5c_8 - c_6c_7).$$

It is obvious that we can choose a generator y_4 of $H^*(BE_8; \mathbf{Z}/p)$ such that

$$\rho^*(y_4) = p_1. \tag{5.2}$$

Moreover, we have:

Proposition 5.1 (Hamanaka and Kono [6]). *If $\tilde{f}_{16} \in H^{16}(BT; \mathbf{Z}/p)$ and $\tilde{f}_{24} \in H^{24}(BT; \mathbf{Z}/p)$ satisfy $\varphi(\tilde{f}_{16}) \equiv \tilde{f}_{16} \mod (c_1^2)$ and $\varphi(\tilde{f}_{24}) \equiv \tilde{f}_{24} \mod (c_1^2, c_2^2)$, then*

$$f_{16} \equiv a_1 f_{16} \mod (p_1^4),\ f_{24} \equiv a_2 \tilde{f}_{24} \mod (p_1^2),$$

where $a_1, a_2 \in \mathbf{Z}/p$ and

$$f_{16} = 120p_4 + 1680c_8 + p_1^2p_2 - 36p_1p_3 + 10p_2^2,$$
$$f_{24} = 60p_6 - p_1p_2p_3 - 5p_1p_5 + \tfrac{5}{36}p_2^3 - 5p_2p_4 + 110p_2c_8 + 3p_3^2$$

Corollary 5.1 (Hamanaka and Kono [6]). *Let f_{16} and f_{24} be as in Proposition 5.1. We can choose generators y_{16} and y_{24} of $H^*(BE_8; \mathbf{Z}/p)$ such that*

$$\rho^*(y_{16}) = f_{16},\ \rho^*(y_{24}) \equiv f_{24} \mod (p_1^2).$$

Now let us proceed to calculate invariant polynomials in $H^*(BT;\mathbf{Z}/p)$. For a dimensional consideration, invariant homogeneous polynomials of degree 28 is given by

$$\delta_1 p_7+\delta_2 p_2^2p_3+\delta_3 p_2p_5+\delta_4 p_3c_8+\delta_5 p_3p_4 \mod (p_1)$$

for $\delta_i\in\mathbf{Z}/p$. It is straightforward to check that

$$\begin{aligned}
&\varphi(p_2^2p_3)\equiv p_2^2p_3+6c_1c_3^3c_4-12c_1c_3c_4c_6-10c_1c_4^2c_5,\\
&\varphi(p_2p_5)\equiv p_2p_5+3c_1c_3^2c_7+\tfrac{3}{2}c_1c_3c_5^2-c_1c_4^2c_5,\\
&\varphi(p_3p_4)\equiv p_3p_4-\tfrac{1}{2}c_1c_3^3c_4+\tfrac{7}{2}c_1c_3^2c_7+c_1c_3c_4c_6+5c_1c_3c_5^2-\tfrac{5}{2}c_1c_4^2c_5-5c_1c_5c_8-7c_1c_6c_7,\\
&\varphi(p_3c_8)\equiv p_3c_8-\tfrac{1}{4}c_1c_3^2c_7-\tfrac{5}{2}c_1c_5c_8+\tfrac{1}{2}c_1c_6c_7 \mod (c_1^2,c_2).
\end{aligned}$$

Then it follows that:

Proposition 5.2. *If $\tilde f_{28}\in H^{28}(BT;\mathbf{Z}/p)$ satisfy $\varphi(\tilde f_{28})\equiv \tilde f_{28} \mod (c_1^2,c_2)$, then*

$$\tilde f_{28}\equiv \delta f_{28} \mod (p_1),$$

where $\delta\in\mathbf{Z}/p$ and

$$f_{28}=480p_7-p_2^2p_3+40p_2p_5-12p_3p_4+312p_3c_8$$

Corollary 5.2. *Let f_{28} be as in Proposition 5.2. We can choose a generator y_{28} of $H^*(BE_8;\mathbf{Z}/p)$ such that*

$$\rho^*(y_{28})\equiv f_{28} \mod (p_1).$$

From the above arrangement of generators of $H^*(BE_8;\mathbf{Z}/p)$, we can deduce the action of $\wp^1$ on $H^*(BE_8;\mathbf{Z}/p)$ as follows.

Proposition 5.3. *$\wp^1 y_4$ includes the term $\delta y_{16}y_{60}$ ($\delta\neq 0$).*

Proposition 5.4. *$\wp^1 y_{16}$ includes the term $\delta y_{40}y_{48}$ ($\delta\neq 0$).*

From Lemma 4.1, Proposition 5.3 and Proposition 5.4, it follows that

$$\langle\epsilon_{30},\epsilon_8\rangle\neq 0,\ \langle\epsilon_{20},\epsilon_{24}\rangle\neq 0,$$

where ϵ_i denotes a generator of $\pi_{2i-1}((E_8)_{(p)})=\mathbf{Z}_{(p)}$. For Fact 2.1, $\langle\epsilon_{20},\epsilon_{24}\rangle$ takes values in $S^{15}_{(p)}\subset (E_8)_{(p)}$. Then, for Fact 2.2 and Fact 2.3, one has

$$\langle\epsilon_{30},\langle\epsilon_{20},\epsilon_{24}\rangle\rangle\neq 0.$$

Therefore Theorem 2.1 completes the proof of Theorem 1.2 in the case of E_8 with $p=37$

Proof of Proposition 5.3. We define a ring homomorphism

$$\pi:\mathbf{Z}/p[p_1,\dots,p_7,c_8]\to\mathbf{Z}/p[a_3,a_4,a_7,b_8]/(a_3^2,a_4^2,a_7^2,b_8^4,-a_3a_4+26a_3b_8+40a_7)$$

by

$$\pi(p_i)=0\ (i=1,2,5,6),\ \pi(p_j)=a_j\ (j=3,4,7),\ \pi(c_8)=b_8.$$

Then it follows from Corollary 5.2 and a dimensional consideration that

$$\pi(i_2^*(y_4)) = \pi(i_2^*(y_{24})) = \pi(i_2^*(y_{28})) = \pi(i_2^*(y_{36})) = 0.$$

Hence, by Setting $\wp^1 y_4 = \delta y_{16}y_{60}$+other terms, one has

$$\pi(i_2^*(\wp^1 y_4)) = \delta\pi(i_2^*(y_{16}y_{60})).$$

On the other hand, it follows from Girard's formula (4.1) that

$$\pi(i_2^*(\wp^1 y_4)) = \pi(\wp^1 T_1) = \pi(2T_{19}) = 2a_4a_7b_8^2 \neq 0.$$

Then $\delta \neq 0$ and the proof is completed. □

Proof of Proposition 5.4. Define a ring homomorphism

$$\pi : \mathbf{Z}/p[p_1, \dots, p_7, c_8] \to \mathbf{Z}/p[a_2, a_4, b_8]/(a_2^2, a_4^6, b_8^6, a_4 + 14b_8)$$

by

$$\pi(p_i) = 0 \ (i = 1, 3, 5, 6, 7),\ \pi(p_j) = a_j \ (j = 2, 4),\ \pi(c_8) = b_8.$$

Then, for Corollary 5.1 and a dimensional reason, we have

$$\pi(i_2^*(y_4)) = \pi(i_2^*(y_{16})) = \pi(i_2^*(y_{28})) = \pi(i_2^*(y_{24}^2)) = 0.$$

Put $\wp^1 y_{16} = \delta y_{40}y_{48}$+other terms. We can deduce that

$$\pi(i_2^*(\wp^1 y_{16})) = \delta\pi(i_2^*(y_{40}y_{48})).$$

Let us make a direct calculation of $\pi(i_2^*(\wp^1 y_{16}))$. From Girard's formula (4.1), it follows that

$$\pi(\wp^1 T_2) = \pi(4T_{20}) = -6b_8^5,\ \pi(\wp^1 T_4) = \pi(8T_{22}) = 16a_2b_8^5,\ \pi(\wp^1 c_8) = \pi(T_{18}c_8) = 26a_2b_8^5.$$

Since $T_2 = p_1^2 - 2p_2$ and $T_4 = p_1^4 - 4p_1^2p_2 + 4p_1p_3 + 2p_2^2 - 4p_4$, we have

$$\pi(\wp^1 p_2) = 3b_8^5,\ \pi(\wp^1 p_4) = -a_2b_8^5.$$

Then, for Corollary 5.1, one can see that

$$\pi(i_2^*(\wp^1 y_{16})) = 120\pi(\wp^1 p_4) + 1680\pi(\wp^1 c_8) + 20a_2\pi(\wp^1 p_2) = -3a_2b_8^5 \neq 0.$$

Hence $\delta \neq 0$ and this completes the proof. □

6. The remaining cases

In the cases of $(G, p) = (G_2, 7), (F_4, 13), (E_6, 13), (E_7, 19), (E_8, 31)$, Hamanaka and Kono [6] showed that there exist non-zero 2-iterated Samelson products in $G_{(p)}$. Then Theorem 2.1 completes the proof of Theorem 1.2 in these cases.

In other remaining cases, (2.4) does not hold for the entries of types and then there is no non-zero 3-iterated Samelson product. Hence, for Corollary 2.1 and Theorem 2.1, we have established Theorem 1.2.

References

[1] R. Bott, *A note on the Samelson products in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.
[2] E.M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. of Math. **101** (1975), 510-520.
[3] B. Harris, *Suspensions and characteristic maps for symmetric spaces*, Ann. of Math. **76** (1962), 295-305.
[4] M. Hopkins, *Nilpotence and finite H-spaces*, Israel J. Math. **66** (1989), 238-246.
[5] J. Hubbuck, *On homotopy commutative H-spaces*, Topology **8** (1969), 119-126.
[6] H. Hamanaka and A. Kono, *A note on the Samelson products in homotopy groups of exceptional Lie groups*, preprint.
[7] I.M. James and E. Thomas, *Homotopy-abelian topological groups*, Topology **1** (1962), 237-240.
[8] S. Kaji, *On the nilpotency of rational H-spaces*, J. Math. Soc. Japan **57** (2005), 1153-1165.
[9] P.G. Kumpel, *Mod p-equivalences of mod p H-spaces*, Quart. J. Math. **23** (1972), 173-178.
[10] A. Kono and H. Ōshima, *Commutativity of the group of self-homotopy classes of Lie groups*, Bull. London Math. Soc. **36** (2004), 37-52.
[11] C. McGibbon, *Homotopy commutativity in localized groups*, Amer. J. Math. **106** (1984), 665-687.
[12] J. Milnor and J.D. Stasheff, *Characteristic Classes*, Ann. of Math. Studies **76**, Princeton Univ. Press, Princeton N.J., 1974.
[13] M. Mimura and H. Toda, *Topology of Lie Groups I, II*, Translations of Math. Monographs **91**, AMS Providence, RI, 1991.
[14] V. Rao, *Spin(n) is not homotopy nilpotent for* $n \geq 7$, Topology **32** (1993), 239-249.
[15] L. Saumell, *Higher homotopy commutativity in localized groups*, Math. Z. **219** (1995), no. 2, 203-213.
[16] J.-P. Serre, *Groupes d'homotopie et classes des groupes abeliens*, Ann. of Math. **58** (1953), 258-294.
[17] M. Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960/1961), 257-269.
[18] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.
[19] F.D. Williams, *Higher homotopy-commutativity*, Trans. Amer. Math. Soc. **139** (1969), 191-206.
[20] N. Yagita, *Homotopy nilpotency for simply connected Lie groups*, Bull. London Math. Soc. **25** (1993), 481-486.
[21] A. Zabrodsky, *Hopf Spaces*, North-Holland Mathematics Studies **22**, North-Holland, Amsterdam (1976).

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