

CONTINUITY OF VOLUMES ON ARITHMETIC VARIETIES

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ABSTRACT. We introduce the volume function for C^∞ -hermitian invertible sheaves on an arithmetic variety as an analogue of the geometric volume function. The main result of this paper is the continuity of the arithmetic volume function. As a consequence, we have the arithmetic Hilbert-Samuel formula for a nef C^∞ -hermitian invertible sheaf. We also give another applications, for example, a generalized Hodge index theorem, an arithmetic Bogomolov-Gieseker's inequality, etc.

INTRODUCTION

Let X be a d -dimensional projective arithmetic variety and $\widehat{\text{Pic}}(X)$ the group of isomorphism classes of C^∞ -hermitian invertible sheaves on X . For $\bar{L} \in \widehat{\text{Pic}}(X)$, the volume $\widehat{\text{vol}}(\bar{L})$ of \bar{L} is defined by

$$\widehat{\text{vol}}(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{\text{sup}} \leq 1\}}{m^d/d!}.$$

For example, if \bar{L} is ample, then $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\widehat{c}(\bar{L})^d)$ (cf. Lemma 3.1). This is an arithmetic analogue of the volume function for invertible sheaves on a projective variety over a field. The geometric volume function plays a crucial role for the birational geometry via big invertible sheaves. In this sense, to introduce the arithmetic analogue of it is very significant.

The first important property of the volume function is the characterization of a big C^∞ -hermitian invertible sheaf by the positivity of its volume (cf. Theorem 4.5). The second one is the homogeneity of the volume function, namely, $\widehat{\text{vol}}(n\bar{L}) = n^d \widehat{\text{vol}}(\bar{L})$ for all non-negative integers n (cf. Proposition 4.7). By this property, it can be extended to $\widehat{\text{Pic}}(X) \otimes \mathbb{Q}$. From viewpoint of arithmetic analogue, the most important and fundamental question is the continuity of

$$\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \otimes \mathbb{Q} \rightarrow \mathbb{R},$$

that is, the validity of the formula:

$$\lim_{\substack{\epsilon_1, \dots, \epsilon_n \in \mathbb{Q} \\ \epsilon_1 \rightarrow 0, \dots, \epsilon_n \rightarrow 0}} \widehat{\text{vol}}(\bar{L} + \epsilon_1 \bar{A}_1 + \dots + \epsilon_n \bar{A}_n) = \widehat{\text{vol}}(\bar{L})$$

for any $\bar{L}, \bar{A}_1, \dots, \bar{A}_n \in \widehat{\text{Pic}}(X) \otimes \mathbb{Q}$. The main purpose of this paper is to give an affirmative answer for the above question (cf. Theorem 5.4). As a consequence, we have the following arithmetic Hilbert-Samuel formula for a nef C^∞ -hermitian invertible sheaf:

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Theorem A (cf. Corollary 5.5). *Let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . If \bar{L} is nef, then*

$$\log \#\{s \in H^0(X, mL + N) \mid \|s\|_{\text{sup}} \leq 1\} = \frac{\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d})}{d!} m^d + o(m^d) \quad (m \gg 1).$$

In particular, $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d})$, and \bar{L} is big if and only if $\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d}) > 0$.

In a more general setting, we have the following generalized Hodge index theorem:

Theorem B (cf. Theorem 6.2). *Let \bar{L} be a C^∞ -hermitian invertible sheaf on X . We assume the following:*

- (i) $L_{\mathbb{Q}}$ is nef on $X_{\mathbb{Q}}$.
- (ii) $c_1(\bar{L})$ is semipositive on $X(\mathbb{C})$.
- (iii) L has moderate growth of positive even cohomologies, that is, there are a generic resolution of singularities $\mu : Y \rightarrow X$ and an ample invertible sheaf A on Y such that, for any positive integer n , there is a positive integer m_0 such that

$$\log \#(H^{2i}(Y, m(n\mu^*(L) + A))) = o(m^d)$$

for all $m \geq m_0$ and for all $i > 0$.

Then we have an inequality $\widehat{\text{vol}}(\bar{L}) \geq \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d})$.

Theorem B implies that if L is nef on every geometric fiber of $X \rightarrow \text{Spec}(\mathbb{Z})$, $c_1(\bar{L})$ is semipositive on $X(\mathbb{C})$, and $\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d}) > 0$, then \bar{L} is big (cf. Corollary 6.4). This is a generalization of [17, Corollary (1.9)]. Moreover we can see the arithmetic Bogomolov-Gieseker's inequality as an application of Theorem B (cf. Corollary 6.5).

In the geometric case, the above Theorem A can be proved by using the Riemann-Roch formula and Fujita's vanishing theorem. In the arithmetic case, the proof in terms of the arithmetic Riemann-Roch theorem seems to be difficult. Instead of it, we prove the continuity of the volume function by direct estimates. For this purpose, the technical core is the following theorem, which was inspired by Yuan's paper [16].

Theorem C (cf. Theorem 3.4). *Let X be a projective and generically smooth arithmetic variety of dimension $d \geq 2$. Let \bar{L} and \bar{A} be C^∞ -hermitian invertible sheaves on X . We assume the following:*

- (i) A and $L + A$ are very ample over \mathbb{Q} .
- (ii) The first Chern forms $c_1(\bar{A})$ and $c_1(\bar{L} + \bar{A})$ on $X(\mathbb{C})$ are positive.
- (iii) There is a non-zero section $s \in H^0(X, A)$ such that the vertical component of $\text{div}(s)$ is contained in the regular locus of X and that the horizontal component of $\text{div}(s)$ is smooth over \mathbb{Q} .

Then there are positive constants a_0 , C and D depending only on X , \bar{L} and \bar{A} such that

$$\begin{aligned} \log \#\{s \in H^0(X, aL + (b - c)A) \mid \|s\|_{\text{sup}} \leq 1\} \\ \leq \log \#\{s \in H^0(X, aL - cA) \mid \|s\|_{\text{sup}} \leq 1\} \\ + Cba^{d-1} + Da^{d-1} \log(a) \end{aligned}$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$.

In order to explain the technical aspects of the above theorem, let us consider it in the geometric case, namely, we assume that X is a projective smooth variety over \mathbb{C} , and we try to estimate

$$\Delta = h^0(X, aL + (b - c)A) - h^0(X, aL - cA).$$

The first elegant way: Let us choose an infinite sequence $\{Y_i\}_{i=1}^{\infty}$ of distinct smooth members of $|A|$ such that

$$h^0(Y_i, nL + mA|_{Y_i}) = h^0(Y_j, nL + mA|_{Y_j})$$

for all i, j and all integers n, m . Then an exact sequence

$$0 \rightarrow H^0(X, aL - cA) \rightarrow H^0(X, aL + (b - c)A) \rightarrow \bigoplus_{i=1}^b H^0(Y_i, aL + (b - c)A|_{Y_i})$$

gives rise to $\Delta \leq b \cdot h^0(Y_1, a(L + A)|_{Y_1})$. This argument does not work in the arithmetic situation.

The second way: In the paper [16], for a fixed smooth member $Y \in |A|$, Yuan considered an exact sequence

$$0 \rightarrow aL + (k - 1 - c)A \rightarrow aL + (k - c)A \rightarrow aL + (k - c)A|_Y \rightarrow 0$$

for each $1 \leq k \leq b$, which yields

$$\Delta \leq \sum_{k=1}^b h^0(Y, aL + (k - c)A|_Y) \leq b \cdot h^0(Y, a(L + A)|_Y).$$

This second way works if we consider the arithmetic $\hat{\chi}$ instead of the number of small sections. In this way, Yuan [16] obtained an arithmetic analogue of a theorem of Siu. However, if we estimate the number of small sections by using the above way, the growth of the contribution from error terms is larger than the main term.

The third way: An exact sequence

$$0 \rightarrow aL - cA \rightarrow aL + (b - c)A \rightarrow aL + (b - c)A|_{bY} \rightarrow 0$$

gives rise to

$$\Delta \leq h^0(bY, aL + (b - c)A|_{bY}).$$

On the other hand, using exact sequences

$$0 \rightarrow aL + (b - c - k)A|_Y \rightarrow aL + (b - c)A|_{(k+1)Y} \rightarrow aL + (b - c)A|_{kY} \rightarrow 0,$$

we have

$$\begin{aligned} h^0(bY, aL + (b - c)A|_{bY}) &\leq \sum_{k=0}^{b-1} h^0(Y, aL + (b - c - k)A|_Y) \\ &\leq b \cdot h^0(Y, a(L + A)|_Y). \end{aligned}$$

In the arithmetic context, the behavior of the error terms by this way is better than the second way, so that we could get the desired estimate. Of course, this way is very complicated because it involves non-reduced schemes.

The paper is organized as follows: In Section 1, we prepare several estimates of norms on complex manifolds. In Section 2, many formulae concerning the number of small sections are discussed. Through Section 3, we give the proof of the main technical estimate of the number of small sections. In Section 4, we introduce the volume function on an arithmetic variety and consider several basic properties. In Section 5, we prove the continuity of the volume function and the arithmetic Hilbert-Samuel formula for a nef C^∞ -hermitian invertible sheaf. Finally, in Section 6, we consider the generalized Hodge index theorem and the arithmetic Bogomolov-Gieseker's inequality.

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Conventions and terminology. We fix several conventions and terminology of this paper.

1. For a real number $x \in \mathbb{R}$, the *round-up* $\lceil x \rceil$, the *round-down* $\lfloor x \rfloor$ and the *fractional part* $\{x\}$ are defined by

$$\lceil x \rceil := \min\{k \in \mathbb{Z} \mid x \leq k\}, \quad \lfloor x \rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\} \quad \text{and} \quad \{x\} = x - \lfloor x \rfloor.$$

2. For a complex vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, two norms $|z|$ and $|z|'$ are defined by

$$|z| = \sqrt{|z_1|^2 + \dots + |z_n|^2} \quad \text{and} \quad |z|' = |z_1| + \dots + |z_n|.$$

Note that $|z| \leq |z|' \leq \sqrt{n}|z|$ for all $z \in \mathbb{C}^n$.

3. Let (V, σ) be a finite dimensional normed vector space over \mathbb{R} . The norm σ is sometimes denoted by $\|\cdot\|$. Let $f : W \rightarrow V$ be an injective homomorphism of vector spaces over \mathbb{R} . Then the norm σ on V yields a norm σ' on W given by $\sigma'(x) = \sigma(f(x))$. This norm σ' is denoted by $\sigma_{W \hookrightarrow V}$ and is called the *subnorm* of σ . Let $g : V \rightarrow Q$ be a surjective homomorphism of vector spaces over \mathbb{R} . Then a norm σ'' on Q is defined by

$$\sigma''(y) = \inf\{\sigma(x) \mid x \in g^{-1}(y)\}.$$

This norm σ'' is denoted by $\sigma_{V \rightarrow Q}$ and is called the *quotient norm* of σ . Let

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

be an exact sequence of finite dimensional vector spaces over \mathbb{R} . Let σ', σ and σ'' be norms of V', V and V'' respectively. We say

$$0 \rightarrow (V', \sigma') \rightarrow (V, \sigma) \rightarrow (V'', \sigma'') \rightarrow 0$$

is an *exact sequence of normed vector spaces* if $\sigma' = \sigma_{V' \hookrightarrow V}$ and $\sigma'' = \sigma_{V \rightarrow V''}$. Let V^\vee be the dual space of V , that is, $V^\vee = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The *dual norm* σ^\vee of V^\vee is given by

$$\sigma^\vee(\phi) = \sup\{|\phi(x)| \mid x \in V \text{ and } \sigma(x) \leq 1\}.$$

4. Let X be either a scheme or a complex space. Let L_1, \dots, L_n be invertible sheaves on X and m_1, \dots, m_n integers. In this paper, the tensor product $L_1^{\otimes m_1} \otimes \dots \otimes L_n^{\otimes m_n}$ of invertible sheaves is usually denoted by

$$m_1 L_1 + \dots + m_n L_n$$

in the additive way like divisors.

5. Let X be a compact complex manifold and Ω a volume form on X . Let $\bar{L} = (L, |\cdot|_L)$ be a C^∞ -hermitian invertible sheaf on X . Then the natural L^2 -norm $\|\cdot\|_{L^2, \Omega}$ and the *sup-norm* $\|\cdot\|_{\text{sup}, \bar{L}}$ on $H^0(X, L)$ are defined by

$$\|s\|_{L^2, \Omega} = \left(\int_X |s|_L^2 \Omega \right)^{1/2} \quad \text{and} \quad \|s\|_{\text{sup}, \bar{L}} = \sup\{|s|_L(x) \mid x \in X\}$$

for $s \in H^0(X, L)$. For simplicity, $\|\cdot\|_{L^2, \Omega}$ (resp. $\|\cdot\|_{\text{sup}, \bar{L}}$) is often denoted by $\|\cdot\|_{L^2}$ or $\|\cdot\|_{L^2}$ (resp. $\|\cdot\|_{\text{sup}}$). For a real number λ , a C^∞ -hermitian invertible sheaf $(L, \exp(-\lambda)|\cdot|_L)$ is denoted by \bar{L}^λ . Let \bar{A} be a positive C^∞ -hermitian invertible sheaf on X . The *normalized volume form* $\Omega(\bar{A})$ associated with \bar{A} is given by

$$\Omega(\bar{A}) = \frac{c_1(\bar{A})^{\wedge d}}{\int_X c_1(\bar{A})^{\wedge d}},$$

where $c_1(\bar{A})$ is the first Chern form of \bar{A} and $d = \dim X$. Note that $\int_X \Omega(\bar{A}) = 1$.

6. A quasi-projective scheme over \mathbb{Z} is called an *arithmetic variety* if X is an integral scheme and flat over \mathbb{Z} . We say X is *generically smooth* if X is smooth over \mathbb{Q} . By Hironaka's resolution of singularities [9], there is a projective birational morphism $\mu : X' \rightarrow X$ of arithmetic varieties such that X' is generically smooth. This $\mu : X' \rightarrow X$ is called a *generic resolution of singularities of X* .

7. Let X be a projective arithmetic variety and \bar{L} a C^∞ -hermitian invertible sheaf on X . According to [14], we define three kinds of the positivity of \bar{L} as follows:

- *ample* : \bar{L} is ample if L is ample on X , the first Chern form $c_1(\bar{L})$ is positive on $X(\mathbb{C})$ and nA is generated by sections $s \in H^0(X, nA)$ with $\|s\|_{\text{sup}} < 1$ for a sufficiently large n .

- *nef* : \bar{L} is nef if the first Chern form $c_1(\bar{L})$ is semipositive and $\widehat{\text{deg}}(\bar{H}|_\Gamma) \geq 0$ for any 1-dimensional closed subscheme Γ in X .

- *big* : \bar{L} is big if $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$ and there are a positive integer n and a non-zero section s of $H^0(X, nL)$ with $\|s\|_{\text{sup}} < 1$.

By [17, Corollary (5.7)], if \bar{L} is ample, then, for a sufficiently large integer n , $H^0(X, nL)$ has a basis s_1, \dots, s_N as a \mathbb{Z} -module with $\|s_i\|_{\text{sup}} < 1$ for all $i = 1, \dots, N$.

8. Let X be a projective arithmetic variety, and let \bar{L} and \bar{M} be C^∞ -hermitian invertible sheaves on X . We say \bar{L} is *less than or equal to \bar{M}* , denoted by $\bar{L} \leq \bar{M}$, if there is an injective homomorphism $\phi : L \rightarrow M$ such that $|\phi_{\mathbb{C}}(\cdot)|_M \leq |\cdot|_L$ on $X(\mathbb{C})$, where $|\cdot|_L$ and $|\cdot|_M$ are hermitian norms of \bar{L} and \bar{M} respectively. The following properties are easily checked (for the proof, see Remark 5.3):

- (1) $\bar{L} \leq \bar{M}$ if and only if $-\bar{M} \leq -\bar{L}$.
- (2) If $\bar{L} \leq \bar{M}$ and $\bar{L}' \leq \bar{M}'$, then $\bar{L} + \bar{L}' \leq \bar{M} + \bar{M}'$.

1. SEVERAL ESTIMATES OF NORMS ON COMPLEX MANIFOLDS

1.1. **Gromov's inequality.** In this subsection, we consider Gromov's inequality and its variants. Let us begin with the local version of Gromov's inequality.

Lemma 1.1.1 (Local Gromov's inequality). *Let a, b, c be real numbers with $a > b > c > 0$. We set $U = \{z \in \mathbb{C}^n \mid |z| < a\}$, $V = \{z \in \mathbb{C}^n \mid |z| < b\}$ and $W = \{z \in \mathbb{C}^n \mid |z| < c\}$. Let Ω be a volume form on U , and let $\bar{H}_1, \dots, \bar{H}_l$ be C^∞ -hermitian invertible sheaves on U . Let $\omega_1, \dots, \omega_l$ be free bases of H_1, \dots, H_l over U respectively. Then there is a constant C depending only on $\bar{H}_1, \dots, \bar{H}_l, \omega_1, \dots, \omega_l, \Omega, a, b, c$ and n such that, for any positive real number p , all non-negative integers m_1, \dots, m_l and all $s \in H^0(U, m_1 H_1 + \dots + m_l H_l)$,*

$$\max_{x \in \bar{W}} \{|s|_{(m_1, \dots, m_l)}^p(x)\} \leq C(\lceil p \rceil)^{2n} (m_1 + \dots + m_l + 1)^{2n} \left(\int_V |s|_{(m_1, \dots, m_l)}^p \Omega \right),$$

where $|\cdot|_{(m_1, \dots, m_l)}$ is the hermitian norm of $m_1 \bar{H}_1 + \dots + m_l \bar{H}_l$ and $\lceil p \rceil$ is the round-up of p (cf. Conventions and terminology 1).

Proof. Let $|\cdot|_i$ be the hermitian norm of \bar{H}_i and $u_i = |\omega_i|_i$ on U . Considering an upper bound of the partial derivatives of u_i over \bar{V} , we can find a positive constant K_i such that

$$|u_i(x) - u_i(y)| \leq K_i |x - y|'$$

for all $x, y \in \bar{V}$ (for the definition of $|\cdot|'$, see Conventions and terminology 2). We set

$$D = \max \left\{ \max_{x \in \bar{V}} \left\{ \frac{K_1}{u_1(x)} \right\}, \dots, \max_{x \in \bar{V}} \left\{ \frac{K_l}{u_l(x)} \right\}, \frac{1}{b-c} \right\} \quad \text{and} \quad R = 1/D.$$

Then, for $x_0, x \in \bar{V}$,

$$\begin{aligned} u_i(x) &\geq u_i(x_0) - K_i |x - x_0|' = u_i(x_0) \left(1 - \frac{K_i}{u_i(x_0)} |x - x_0|' \right) \\ &\geq u_i(x_0) (1 - D |x - x_0|'). \end{aligned}$$

We set $B(x_0, R) = \{x \in \mathbb{C}^n \mid |x - x_0|' \leq R\}$. Then $1 - D|x - x_0|' \geq 0$ for all $x \in B(x_0, R)$. Moreover, if $x_0 \in \bar{W}$, then $B(x_0, R) \subseteq \bar{V}$ because

$$|x - x_0| \leq |x - x_0|' \leq R \leq b - c.$$

Here we claim the following:

Claim 1.1.1.1. *For a non-negative real number m ,*

$$\int_0^1 \cdots \int_0^1 x_1 \cdots x_n \left(1 - \frac{1}{n}(x_1 + \cdots + x_n) \right)^m dx_1 \cdots dx_n \geq \frac{1}{(\lceil m \rceil + 1)^n (\lceil m \rceil + 2)^n}.$$

First let us consider the case where m is an integer. If $m = 0$, then the assertion is obvious, so that we assume $m \geq 1$. Since

$$\begin{aligned} \left(1 - \frac{1}{n}(x_1 + \cdots + x_n) \right)^m &= \frac{1}{n^m} \left(\sum_{i=1}^n (1 - x_i) \right)^m \\ &= \frac{1}{n^m} \sum_{\substack{m_1 + \cdots + m_n = m \\ m_1 \geq 0, \dots, m_n \geq 0}} \frac{m!}{m_1! \cdots m_n!} (1 - x_1)^{m_1} \cdots (1 - x_n)^{m_n} \end{aligned}$$

and

$$\int_0^1 x(1-x)^d dx = \frac{1}{(d+1)(d+2)}$$

for a non-negative integer d , the integral I in the claim is equal to

$$\frac{1}{n^m} \sum_{\substack{m_1 + \cdots + m_n = m \\ m_1 \geq 0, \dots, m_n \geq 0}} \frac{m!}{m_1! \cdots m_n!} \frac{1}{(m_1+1)(m_1+2) \cdots (m_n+1)(m_n+2)}.$$

Thus

$$I \geq \frac{1}{(m+1)^n (m+2)^n n^m} \sum_{\substack{m_1 + \cdots + m_n = m \\ m_1 \geq 0, \dots, m_n \geq 0}} \frac{m!}{m_1! \cdots m_n!} = \frac{1}{(m+1)^n (m+2)^n}.$$

If m is not integer, then

$$\left(1 - \frac{1}{n}(x_1 + \cdots + x_n) \right)^m \geq \left(1 - \frac{1}{n}(x_1 + \cdots + x_n) \right)^{\lceil m \rceil}$$

because $0 \leq 1 - \frac{1}{n}(x_1 + \cdots + x_n) \leq 1$. Thus the claim follows.

We choose a positive constant e with $\Omega \geq e\Omega_{can}$ on \bar{V} , where

$$\Omega_{can} = \left(\frac{\sqrt{-1}}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

Let s be an element of $H^0(U, m_1 H_1 + \cdots + m_l H_l)$. Then we can find a holomorphic function f over U with $s = f \omega_1^{\otimes m_1} \otimes \cdots \otimes \omega_l^{\otimes m_l}$. We also choose $x_0 \in \overline{W}$ such that the continuous function $|s|_{(m_1, \dots, m_l)}$ on \overline{W} takes the maximum value at x_0 . Then

$$\begin{aligned} \int_V |s|_{(m_1, \dots, m_l)}^p \Omega &\geq e \int_{B(x_0, R)} |s|_{(m_1, \dots, m_l)}^p \Omega_{can} \\ &= e \int_{B(x_0, R)} |f|^p u_1^{pm_1} \cdots u_l^{pm_l} \Omega_{can} \\ &\geq e u_1(x_0)^{pm_1} \cdots u_l(x_0)^{pm_l} \int_{B(x_0, R)} |f|^p (1 - D|x - x_0|)^m \Omega_{can}, \end{aligned}$$

where $m = p(m_1 + \cdots + m_l)$. Moreover, if we set

$$x - x_0 = (r_1 \exp(\sqrt{-1}\theta_1), \dots, r_n \exp(\sqrt{-1}\theta_n)),$$

then

$$\begin{aligned} &\int_{B(x_0, R)} |f|^p (1 - D|x - x_0|)^m \Omega_{can} \\ &= \int_{\substack{r_1 + \cdots + r_n \leq R \\ r_1 \geq 0, \dots, r_n \geq 0}} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |f|^p d\theta_1 \cdots d\theta_n \right) r_1 \cdots r_n (1 - D(r_1 + \cdots + r_n))^m dr_1 \cdots dr_n. \end{aligned}$$

Since $|f|^p$ is subharmonic, we have

$$\int_0^{2\pi} \cdots \int_0^{2\pi} |f|^p d\theta_1 \cdots d\theta_n \geq (2\pi)^n |f(x_0)|^p.$$

Therefore, using Claim 1.1.1.1,

$$\begin{aligned} &\int_{B(x_0, R)} |f|^p (1 - D|x - x_0|)^m \Omega_{can} \\ &\geq (2\pi)^n |f(x_0)|^p \int_{\substack{r_1 + \cdots + r_n \leq R \\ r_1 \geq 0, \dots, r_n \geq 0}} r_1 \cdots r_n (1 - D(r_1 + \cdots + r_n))^m dr_1 \cdots dr_n \\ &\geq (2\pi)^n |f(x_0)|^p \int_{[0, R/n]^n} r_1 \cdots r_n (1 - D(r_1 + \cdots + r_n))^m dr_1 \cdots dr_n \\ &\geq \frac{(2\pi)^n |f(x_0)|^p}{(nD)^{2n}} \frac{1}{([\overline{m}] + 1)^n ([\overline{m}] + 2)^n}. \end{aligned}$$

Gathering all calculations, if we set $C' = e(2\pi)^n / (nD)^{2n}$, then

$$\int_V |s|_{(m_1, \dots, m_l)}^p \Omega \geq \frac{C' |s(x_0)|_{(m_1, \dots, m_l)}^p}{([\overline{m}] + 1)^n ([\overline{m}] + 2)^n}.$$

Further, since $[\overline{m}] \leq [p](m_1 + \cdots + m_l)$,

$$\begin{aligned} ([\overline{m}] + 1)^n ([\overline{m}] + 2)^n &\leq ([p](m_1 + \cdots + m_l) + 1)^n ([p](m_1 + \cdots + m_l) + 2)^n \\ &\leq ([p](m_1 + \cdots + m_l + 1))^n (2[p](m_1 + \cdots + m_l + 1))^n \\ &= 2^n ([p])^{2n} (m_1 + \cdots + m_l + 1)^{2n}. \end{aligned}$$

Thus we get the lemma. \square

The partial results of the following corollary are found in [12] and [11].

Corollary 1.1.2 (Gromov's inequality). *Let M be an n -dimensional compact complex manifold, Ω a volume form on M , and let $\overline{H}_1, \dots, \overline{H}_l$ be C^∞ -hermitian invertible sheaves on M . Then there is a constant C depending only on $\overline{H}_1, \dots, \overline{H}_l$, Ω and M such that, for any positive real number p , all integers m_1, \dots, m_l with $m_1 \geq 0, \dots, m_l \geq 0$, and all $s \in H^0(M, m_1 H_1 + \dots + m_l H_l)$,*

$$\max_{x \in M} \{|s|_{(m_1, \dots, m_l)}^p(x)\} \leq C([\![p]\!]^{2n} (m_1 + \dots + m_l + 1)^{2n} \left(\int_M |s|_{(m_1, \dots, m_l)}^p \Omega \right)).$$

Proof. We take a finite covering $\{U_i\}_{i=1, \dots, m}$ of M with the following properties:

- (1) U_i is isomorphic to $\{z \in \mathbb{C}^n \mid |z| < 1\}$ by using a local coordinate $z_i(x) = (z_{i1}(x), \dots, z_{in}(x))$. We set $V_i = \{x \in U_i \mid |z_i(x)| < 1/2\}$ and $W_i = \{x \in U_i \mid |z_i(x)| < 1/4\}$.
- (2) There are local bases $\omega_{i1}, \dots, \omega_{il}$ of H_1, \dots, H_l over U_i respectively.
- (3) $\bigcup_{i=1}^m W_i = M$.

Then our corollary follows from the local Gromov's inequality. \square

Corollary 1.1.3. *Let M be an n -dimensional compact complex manifold, and let $\overline{H}_1, \dots, \overline{H}_l$ be C^∞ -hermitian invertible sheaves on M . Let V be a closed complex submanifold of M . Let Ω_M and Ω_V be volume forms on M and V respectively. Then there is a constant C such that*

$$C(m_1 + \dots + m_l + 1)^{2n} \int_M |s|^2 \Omega_M \geq \int_V |s|_V|^2 \Omega_V$$

for all non-negative integers m_1, \dots, m_l and all $s \in H^0(X, m_1 H_1 + \dots + m_l H_l)$.

Proof. Note that

$$\|s\|_{\text{sup}}^2 \geq \|s|_V\|_{\text{sup}}^2 \geq \frac{\int_V |s|_V|^2 \Omega_V}{\int_V \Omega_V}.$$

Thus the corollary follows from Gromov's inequality. \square

The following lemma is due to Takuro Mochizuki, who kindly tell us its proof. This is a variant of Gromov's inequality.

Lemma 1.1.4. *Let X be an n -dimensional compact complex manifold and ω a positive $(1, 1)$ -form on X . Let $\overline{H}_1, \dots, \overline{H}_l$ be C^∞ -hermitian invertible sheaves on X . Then, for an open set U of X , there are positive constants C , C' and D' such that*

$$\sup_{x \in X} \{|s|_{(m_1, \dots, m_l)}(x)\} \leq C^{m_1 + \dots + m_l} \sup_{x \in U} \{|s|_{(m_1, \dots, m_l)}(x)\}.$$

and

$$\int_X |s|_{(m_1, \dots, m_l)}^2 \omega^{\wedge n} \leq D' \cdot C'^{m_1 + \dots + m_l} \int_U |s|_{(m_1, \dots, m_l)}^2 \omega^{\wedge n}$$

for all non-negative integers m_1, \dots, m_l and all $s \in H^0(X, m_1 H_1 + \dots + m_l H_l)$, where $|\cdot|_{(m_1, \dots, m_l)}$ is the hermitian norm of $m_1 \overline{H}_1 + \dots + m_l \overline{H}_l$.

Proof. Shrinking U if necessarily, we may identify U with $\{x \in \mathbb{C}^n \mid |x| < 1\}$. We set $W = \{x \in \mathbb{C}^n \mid |x| < 1/2\}$. In this proof, we define a Laplacian \square_ω by the formula:

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial}(g) \wedge \omega^{\wedge (n-1)} = \square_\omega(g) \omega^{\wedge n}.$$

Let a_i be a C^∞ -function given by $c_1(\overline{H}_i) \wedge \omega^{\wedge(n-1)} = a_i \omega^{\wedge n}$, where $c_1(\overline{H}_i)$ is the first Chern form of \overline{H}_i . We choose a C^∞ -function ϕ_i on X such that

$$\int_X a_i \omega^{\wedge n} = \int_X \phi_i \omega^{\wedge n}$$

and that ϕ_i is identically zero on $X \setminus W$. Thus we can find a C^∞ -function F_i with $\square_\omega(F_i) = a_i - \phi_i$. Note that $\square_\omega(F_i) = a_i$ on $X \setminus W$.

Let $s \in H^0(X, m_1 H_1 + \cdots + m_l H_l)$ and we set

$$f = |s|_{(m_1, \dots, m_l)}^2 \exp(-(m_1 F_1 + \cdots + m_l F_l)).$$

Claim 1.1.4.1. $\max_{x \in X \setminus W} \{f(x)\} = \max_{x \in \partial(W)} \{f(x)\}$.

If f is a constant over $X \setminus W$, then our assertion is obvious, so that we assume that f is not a constant over $X \setminus W$. In particular, $s \neq 0$. Since

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (\log(|s|_{(m_1, \dots, m_l)}^2)) = c_1(m_1 \overline{H}_1 + \cdots + m_l \overline{H}_l) = m_1 c_1(\overline{H}_1) + \cdots + m_l c_1(\overline{H}_l),$$

we have $\square_\omega(\log(f)) = 0$ on $X \setminus (W \cup \text{Supp}(\text{div}(s)))$. Let us choose $x_0 \in X \setminus W$ such that the C^∞ -function f over $X \setminus W$ takes the maximum value at x_0 . Note that

$$x_0 \in X \setminus (W \cup \text{Supp}(\text{div}(s))).$$

For, if $\text{Supp}(\text{div}(s)) = \emptyset$, then our assertion is obvious. Otherwise, f is zero at any point of $\text{Supp}(\text{div}(s))$.

Since $\log(f)$ is harmonic over $X \setminus (W \cup \text{Supp}(\text{div}(s)))$, $\log(f)$ takes the maximum value at x_0 and $\log(f)$ is not a constant, we have $x_0 \in \partial(W)$ by virtue of the maximum principle of harmonic functions. Thus the claim follows.

We set

$$d_i = \min_{x \in X \setminus W} \{\exp(-F_i)\}, \quad D_i = \max_{x \in \partial(W)} \{\exp(-F_i)\} \quad \text{and} \quad C = \max_{i=1, \dots, l} \{D_i/d_i\}.$$

Then

$$d_1^{m_1} \cdots d_l^{m_l} |s|_{(m_1, \dots, m_l)}^2 \leq f$$

over $X \setminus W$ and

$$f \leq D_1^{m_1} \cdots D_l^{m_l} |s|_{(m_1, \dots, m_l)}^2$$

over $\partial(W)$. Hence

$$\begin{aligned} \max_{x \in X \setminus W} \{|s|_{(m_1, \dots, m_l)}^2\} &\leq C^{m_1 + \cdots + m_l} \max_{x \in \partial(W)} \{|s|_{(m_1, \dots, m_l)}^2\} \\ &\leq C^{m_1 + \cdots + m_l} \max_{x \in \overline{W}} \{|s|_{(m_1, \dots, m_l)}^2\}. \end{aligned}$$

which implies that

$$\max_{x \in X} \{|s|_{(m_1, \dots, m_l)}^2\} \leq C^{m_1 + \cdots + m_l} \max_{x \in \overline{W}} \{|s|_{(m_1, \dots, m_l)}^2\}.$$

This is the first part of the lemma. Note that $e^x \geq x + 1$ for $x \geq 0$. Thus, by the local Gromov's inequality (cf. Lemma 1.1.1), there are constants C_1 and D_1 such that

$$\max_{x \in \overline{W}} \{|s|_{(m_1, \dots, m_l)}^2\} \leq D_1 \cdot C_1^{m_1 + \cdots + m_l} \int_U |s|_{(m_1, \dots, m_l)}^2 \Omega$$

for all non-negative integers m_1, \dots, m_l and all $s \in H^0(X, m_1 H_1 + \cdots + m_l H_l)$. Therefore the second assertion follows. \square

1.2. Distorsion functions. Let X be an n -dimensional projective complex manifold and Ω a volume form of X with $\int_X \Omega = 1$. Let $\overline{H} = (H, h)$ be a C^∞ -hermitian invertible sheaf on X . For $s, s' \in H^0(X, H)$, we set

$$\langle s, s' \rangle_{\overline{H}, \Omega} = \int_X h(s, s') \Omega.$$

Let s_1, \dots, s_N be an orthonormal basis of $H^0(X, H)$ with respect to $\langle \cdot, \cdot \rangle_{\overline{H}, \Omega}$. We define

$$\text{dist}(\overline{H}, \Omega)(x) = \sum_{i=1}^N h(s_i, s_i)(x).$$

Note that $\text{dist}(\overline{H}, \Omega)$ does not depend on the choice of an orthonormal basis. In the case of $H^0(X, H) = \{0\}$, $\text{dist}(\overline{H}, \Omega)$ is defined to be the constant function 0. The function $\text{dist}(\overline{H}, \Omega)$ is called the *distorsion function of \overline{H} with respect to Ω* .

Let \overline{A} be a positive C^∞ -hermitian invertible sheaf on X . Due to Bouche [3] and Tian [15], we know that

$$\sup_{x \in X} \left| \frac{\text{dist}(a\overline{A}, \Omega(A))(x)}{\dim H^0(aA)} - 1 \right| = O(1/a)$$

for $a \gg 1$, where $\Omega(\overline{A})$ is the normalized volume form associated with \overline{A} (cf. Conventions and terminology 5). Using this result, Yuan [16, Theorem 3.3] proved the following:

Theorem 1.2.1. *Let $\overline{A} = (A, h_A)$ and $\overline{B} = (B, h_B)$ be positive C^∞ -hermitian invertible sheaves on X . Then there are positive constants C_1 and C_2 such that*

$$\text{dist}(a\overline{A} - b\overline{B}, \Omega(\overline{A}))(x) \leq \dim H^0(aA) \left(1 + \frac{2C_1}{a} + \frac{3C_2}{b} \right)$$

for all $x \in X$, $a \geq 1$ and $b \geq 3C_2$.

Proof. For reader's convenience, we reprove it here. By Bouche-Tian's theorem, there are constants C_1 and C_2 such that

$$\dim H^0(aA) \left(1 - \frac{C_1}{a} \right) \leq \text{dist}(a\overline{A}, \Omega(A))(z) \leq \dim H^0(aA) \left(1 + \frac{C_1}{a} \right)$$

and

$$\dim H^0(bB) \left(1 - \frac{C_2}{b} \right) \leq \text{dist}(b\overline{B}, \Omega(B))(z) \leq \dim H^0(bB) \left(1 + \frac{C_2}{b} \right)$$

for all $z \in X$, $a \gg 1$ and $b \gg 1$. By taking larger C_1 and C_2 if necessarily, we may assume that the above inequalities hold for all $z \in X$ and all $a, b \geq 1$.

Let us fix an arbitrary $x \in X$. Let us choose an orthonormal basis of $H^0(bB)$ with respect to $\langle \cdot, \cdot \rangle_{b\overline{B}, \Omega(\overline{B})}$ such that only one section is non-zero at x . We denote this section by $s(b)$. Then

$$h_{b\overline{B}}(s(b), s(b))(x) = \text{dist}(b\overline{B}, \Omega(\overline{B}))(x) \geq \dim H^0(bB)(1 - C_2/b).$$

On the other hand,

$$\|s(b)\|_{\text{sup}}^2 \leq \sup_{z \in X} \text{dist}(b\overline{B}, \Omega(\overline{B}))(z) \leq \dim H^0(bB)(1 + C_2/b).$$

Therefore

$$\frac{h_{b\overline{B}}(s(b), s(b))(x)}{\|s(b)\|_{\text{sup}}^2} \geq \frac{1 - C_2/b}{1 + C_2/b}.$$

We choose an orthonormal basis t_1, \dots, t_r of $H^0(aA - bB)$ with respect to $\langle \cdot, \cdot \rangle_{a\bar{A} - b\bar{B}, \Omega(\bar{A})}$ such that $s(b)t_1, \dots, s(b)t_r$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{a\bar{A}, \Omega(\bar{A})}$ in $H^0(aA)$. This is possible because a hermitian matrix is diagonalizable by a unitary matrix. Then

$$\{s(b)t_i / \|s(b)t_i\|_{a\bar{A}, \Omega(\bar{A})}\}_{i=1, \dots, r}$$

is a part of an orthonormal basis of $H^0(aA)$. Thus

$$\sum_{i=1}^r \frac{h_{a\bar{A}}(s(b)t_i, s(b)t_i)(x)}{\|s(b)t_i\|_{a\bar{A}, \Omega(\bar{A})}^2} \leq \text{dist}(aA, \Omega(\bar{A}))(x) \leq \dim H^0(aA)(1 + C_1/a).$$

On the other hand,

$$\|s(b)t_i\|_{a\bar{A}, \Omega(\bar{A})}^2 \leq \|s(b)\|_{\text{sup}}^2.$$

Therefore

$$\begin{aligned} \frac{1 - C_2/b}{1 + C_2/b} \text{dist}(a\bar{A} - b\bar{B}, \Omega(\bar{A}))(x) &\leq \frac{h_{b\bar{B}}(s(b), s(b))(x)}{\|s(b)\|_{\text{sup}}^2} \sum_{i=1}^r h_{a\bar{A} - b\bar{B}}(t_i, t_i)(x) \\ &\leq \sum_{i=1}^r \frac{h_{b\bar{B}}(s(b), s(b))(x)}{\|s(b)t_i\|_{a\bar{A}, \Omega(\bar{A})}^2} h_{a\bar{A} - b\bar{B}}(t_i, t_i)(x) \\ &= \sum_{i=1}^r \frac{h_{a\bar{A}}(s(b)t_i, s(b)t_i)(x)}{\|s(b)t_i\|_{a\bar{A}, \Omega(\bar{A})}^2} \leq \dim H^0(aA)(1 + C_1/a). \end{aligned}$$

Thus, if $b \geq 3C_2$, then

$$\text{dist}(a\bar{A} - b\bar{B}, \Omega(\bar{A}))(x) \leq \dim H^0(aA) \frac{(1 + C_1/a)(1 + C_2/b)}{1 - C_2/b}.$$

It is easy to see that

$$\frac{(1 + C_1/a)(1 + C_2/b)}{1 - C_2/b} = 1 + \frac{2C_1}{a} + \frac{3C_2}{b} - \frac{b - 3C_2}{b - C_2} \left(\frac{C_1}{a} + \frac{C_2}{b} \right).$$

Therefore, if $b \geq 3C_2$, then

$$\frac{(1 + C_1/a)(1 + C_2/b)}{1 - C_2/b} \leq 1 + \frac{2C_1}{a} + \frac{3C_2}{b}.$$

□

Let \bar{L} and \bar{A} be C^∞ -hermitian invertible sheaves on a projective complex manifold X . Assume that \bar{A} and $\bar{L} + \bar{A}$ are positive. We set $\Omega = \Omega(\bar{L} + \bar{A})$. Let a, b, c be non-negative integers. Let s be a non-zero element of $H^0(bA)$ with $\|s\|_{\text{sup}} \leq 1$. Let $\langle \cdot, \cdot \rangle_{a\bar{L} - c\bar{A}}$ and $\langle \cdot, \cdot \rangle_{a\bar{L} + (b-c)\bar{A}}$ be the natural hermitian metric of $H^0(aL - cA)$ and $H^0(aL + (b-c)A)$ with respect to Ω . We set

$$\begin{cases} B_{L^2} = \{t \in H^0(aL - cA) \mid \langle t, t \rangle_{a\bar{L} - c\bar{A}} \leq 1\} \\ B_{\text{sub}} = \{t \in H^0(aL - cA) \mid \langle st, st \rangle_{a\bar{L} + (b-c)\bar{A}} \leq 1\}. \end{cases}$$

Then we have the following corollary, which is a variant of [16, Proposition 3.1]

Corollary 1.2.2. *There are positive constants C_1 and C_2 such that*

$$\log \left(\frac{\text{vol}(B_{L^2})}{\text{vol}(B_{\text{sub}})} \right) \geq \dim H^0(a(L + A)) \left(\int_X \log(|s|) \Omega \right) \left(1 + \frac{2C_1}{a} + \frac{3C_2}{a+c} \right)$$

for all $a > 3C_2$ and $c \geq 0$.

Proof. Since $a\bar{L} - c\bar{A} = a(\bar{L} + \bar{A}) - (a+c)\bar{A}$, by Theorem 1.2.1, there are positive constants C_1 and C_2 such that

$$\text{dist}(a\bar{L} - c\bar{A}, \Omega)(x) \leq \dim H^0(a(L + A)) \left(1 + \frac{2C_1}{a} + \frac{2C_2}{a+c}\right)$$

for all $x \in X$, $a \geq 1$ and $a+c \geq 3C_2$. Note that if $a > 3C_2$ and $c \geq 0$, then $a+c \geq 3C_2$ and $a \geq 1$.

We choose an orthonormal basis t_1, \dots, t_r of $H^0(aL - cA)$ with respect to $\langle \cdot, \cdot \rangle_{a\bar{L} - c\bar{A}}$ such that st_1, \dots, st_r are orthogonal with respect to $\langle \cdot, \cdot \rangle_{a\bar{L} + (b-c)\bar{A}}$. Then, using Jensen's inequality, for $a > 3C_2$ and $c \geq 0$,

$$\begin{aligned} \log \left(\frac{\text{vol}(B_{L^2})}{\text{vol}(B_{\text{sub}})} \right) &= \sum_{i=1}^r \log \|st_i\|_{a\bar{L} + (b-c)\bar{A}, \Omega} = \frac{1}{2} \sum_{i=1}^r \log \int_X |s|^2 |t_i|^2 \Omega \\ &\geq \frac{1}{2} \sum_{i=1}^r \int_X \log(|s|^2) |t_i|^2 \Omega \\ &= \frac{1}{2} \int_X \log(|s|^2) \text{dist}(a\bar{L} - c\bar{A}, \Omega) \Omega \\ &\geq \dim H^0(a(L + A)) \left(\int_X \log(|s|) \Omega \right) \left(1 + \frac{2C_1}{a} + \frac{3C_2}{a+c} \right). \end{aligned}$$

□

2. NORMED \mathbb{Z} -MODULE AND ITS INVARIANTS \hat{h}^0 , \hat{h}^1 AND $\hat{\chi}$

Let $(M, \|\cdot\|)$ be a normed finitely generated \mathbb{Z} -module, namely, M is a finitely generated \mathbb{Z} -module and $\|\cdot\|$ is a norm on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We define $\hat{H}^0(M, \|\cdot\|)$ and $\hat{h}^0(M, \|\cdot\|)$ to be

$$\hat{H}^0(M, \|\cdot\|) = \{x \in M \mid \|x\| \leq 1\} \quad \text{and} \quad \hat{h}^0(M, \|\cdot\|) = \log \# \hat{H}^0(M, \|\cdot\|).$$

It is easy to see that

$$\hat{h}^0(M, \|\cdot\|) = \hat{h}^0(M/M_{\text{tor}}, \|\cdot\|) + \log \#(M_{\text{tor}}),$$

where M_{tor} is the torsion part of M . We set

$$B(M, \|\cdot\|) = \{x \in M_{\mathbb{R}} \mid \|x\| \leq 1\}.$$

Then $\hat{\chi}(M, \|\cdot\|)$ is defined by

$$\hat{\chi}(M, \|\cdot\|) = \log \left(\frac{\text{vol}(B(M, \|\cdot\|))}{\text{vol}(M_{\mathbb{R}}/(M/M_{\text{tor}}))} \right) + \log \#(M_{\text{tor}}).$$

Note that $\hat{\chi}(M, \|\cdot\|)$ does not depend on the choice of a Lebesgue measure of $M_{\mathbb{R}}$ arising from a basis of $M_{\mathbb{R}}$. Let M^{\vee} be the dual of M , that is, $M^{\vee} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Note that M^{\vee} is torsion free. Since $(M^{\vee})_{\mathbb{R}}$ is naturally isomorphic to $(M_{\mathbb{R}})^{\vee}$, we denote $(M^{\vee})_{\mathbb{R}}$ by $M_{\mathbb{R}}^{\vee}$. The norm $\|\cdot\|$ of $M_{\mathbb{R}}$ yields the dual norm $\|\cdot\|^{\vee}$ of $M_{\mathbb{R}}^{\vee}$ as follows: for $\phi \in M_{\mathbb{R}}^{\vee}$,

$$\|\phi\|^{\vee} = \sup\{|\phi(x)| \mid x \in B(M, \|\cdot\|)\}.$$

Then $\hat{H}^1(M, \|\cdot\|)$ and $\hat{h}^1(M, \|\cdot\|)$ are defined by

$$\hat{H}^1(M, \|\cdot\|) = \hat{H}^0(M^{\vee}, \|\cdot\|^{\vee}) \quad \text{and} \quad \hat{h}^1(M, \|\cdot\|) = \hat{h}^0(M^{\vee}, \|\cdot\|^{\vee}).$$

Let $\Sigma = \{e_1, \dots, e_r\}$ be a free basis of M/M_{tor} and let $\langle \cdot, \cdot \rangle_\Sigma$ be the standard inner product of M/M_{tor} in terms of the basis Σ , that is,

$$\langle x, y \rangle_\Sigma = a_1 b_1 + \dots + a_r b_r$$

for $x = a_1 e_1 + \dots + a_r e_r, y = b_1 e_1 + \dots + b_r e_r \in M/M_{tor}$. Then we can see

$$\hat{h}^1(M, \|\cdot\|) = \log \#\{x \in M/M_{tor} \mid |\langle x, y \rangle_\Sigma| \leq 1 \text{ for all } y \in B(M, \|\cdot\|)\}.$$

In the case where $M = \{0\}$, $\hat{h}^0(M, \|\cdot\|)$, $\hat{h}^1(M, \|\cdot\|)$ and $\hat{\chi}(M, \|\cdot\|)$ are defined to be 0. The following proposition is very useful to estimate \hat{h}^0 of normed \mathbb{Z} -module. This is essentially the results in Gillet-Soulé [6]. The following formulae are also pointed out in Yuan's paper [16].

Proposition 2.1. (1) *For a normed finitely generated \mathbb{Z} -module $(M, \|\cdot\|)$,*

$$\begin{aligned} -\log(6) \operatorname{rk} M &\leq \hat{h}^0(M, \|\cdot\|) - \hat{h}^1(M, \|\cdot\|) - \hat{\chi}(M, \|\cdot\|) \\ &\leq \log(3/2) \operatorname{rk} M + 2 \log((\operatorname{rk} M)!). \end{aligned}$$

(2) *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms of a finitely generated \mathbb{Z} -module M with $\|\cdot\|_1 \leq \|\cdot\|_2$. Then*

$$\hat{h}^0(M, \|\cdot\|_1) \geq \hat{h}^0(M, \|\cdot\|_2) \quad \text{and} \quad \hat{h}^1(M, \|\cdot\|_1) \leq \hat{h}^1(M, \|\cdot\|_2).$$

Moreover,

$$\begin{aligned} \hat{\chi}(M, \|\cdot\|_2) - \hat{\chi}(M, \|\cdot\|_1) &\leq \hat{h}^0(M, \|\cdot\|_2) - \hat{h}^0(M, \|\cdot\|_1) \\ &\quad + \log(9) \operatorname{rk} M + 2 \log((\operatorname{rk} M)!). \end{aligned}$$

(3) *For a non-negative real number λ ,*

$$\begin{aligned} 0 &\leq \hat{h}^0(M, \exp(-\lambda)\|\cdot\|) - \hat{h}^0(M, \|\cdot\|) \\ &\leq \lambda \operatorname{rk} M + \log(9) \operatorname{rk} M + 2 \log((\operatorname{rk} M)!). \end{aligned}$$

(4) *Let*

$$0 \rightarrow (M', \|\cdot\|') \xrightarrow{f} (M, \|\cdot\|) \xrightarrow{g} (M'', \|\cdot\|'') \rightarrow 0$$

be an exact sequence of normed finitely generated \mathbb{Z} -modules, that is,

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

is an exact sequence of finitely generated \mathbb{Z} -modules and

$$0 \rightarrow (M'_\mathbb{R}, \|\cdot\|') \xrightarrow{f_\mathbb{R}} (M_\mathbb{R}, \|\cdot\|) \xrightarrow{g_\mathbb{R}} (M''_\mathbb{R}, \|\cdot\|'') \rightarrow 0$$

is an exact sequence of normed vector spaces over \mathbb{R} . Then

$$\begin{aligned} \hat{h}^0(M, \|\cdot\|) &\leq \hat{h}^0(M', \|\cdot\|') + \hat{h}^0(M'', \|\cdot\|'') + \log(18) \operatorname{rk} M' \\ &\quad + 2 \log((\operatorname{rk} M)!). \end{aligned}$$

(5) *If there is a basis $\{e_1, \dots, e_{\operatorname{rk} M}\}$ of M/M_{tor} with $\|e_i\| \leq 1$ for all i , then*

$$\hat{h}^1(M, \|\cdot\|) \leq \log(3) \operatorname{rk} M.$$

Proof. First we would like to give remarks on the paper [6] due to Gillet-Soulé. We use the same notation as in [6]. Let K be a convex centrally symmetric bounded and absorbing set in \mathbb{R}^n . Let K^* be the polar body of K , i.e.,

$$K^* = \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| \leq 1 \text{ for all } y \in K\}.$$

We denote the volume of K by $V(K)$ and $\#(K \cap \mathbb{Z}^n)$ by $M(K)$. We assume an inequality

$$(2.1.1) \quad V(K)V(K^*) \geq f(n),$$

where $f(n)$ is a constant depending only on n . If we read the paper [6] carefully (especially Theorem 1 and Proposition 4), we can easily realize that the above inequality implies the following inequalities:

$$(2.1.2) \quad 6^{-n} \leq \frac{M(K)}{M(K^*)V(K)} \leq \frac{6^n}{f(n)}$$

and

$$(2.1.3) \quad M(K) \leq M(aK) \leq \frac{a^n M(K) 36^n}{f(n)} \quad (\text{for } a \in \mathbb{R} \text{ with } a > 1).$$

Mahler showed (2.1.1) holds for $f(n) = 4^n (n!)^{-2}$ (cf. [8, §14, Theorem 4]). Bourgain and Milman [4] also proved (2.1.1) for $f(n) = c^n V_n$, where c is an absolute constant and V_n is the volume of the unit sphere in \mathbb{R}^n . Here we use Mahler's result for its simplicity.

(1) Since

$$\begin{aligned} \hat{h}^0(M, \|\cdot\|) - \hat{h}^1(M, \|\cdot\|) - \hat{\chi}(M, \|\cdot\|) \\ = \hat{h}^0(M/M_{tor}, \|\cdot\|) - \hat{h}^1(M/M_{tor}, \|\cdot\|) - \hat{\chi}(M/M_{tor}, \|\cdot\|), \end{aligned}$$

we may assume that M is torsion free. Thus (1) is a consequence of (2.1.2).

(2) The inequalities $\hat{h}^0(M, \|\cdot\|_1) \geq \hat{h}^0(M, \|\cdot\|_2)$ and $\hat{h}^1(M, \|\cdot\|_1) \leq \hat{h}^1(M, \|\cdot\|_2)$ are obvious by their definitions. The third inequality is a consequence of (1).

(3) Since

$$\begin{aligned} \hat{h}^0(M, \exp(-\lambda)\|\cdot\|) - \hat{h}^0(M, \|\cdot\|) \\ = \hat{h}^0(M/M_{tor}, \exp(-\lambda)\|\cdot\|) - \hat{h}^0(M/M_{tor}, \|\cdot\|), \end{aligned}$$

we may assume that M is torsion free. Thus it follows from (2.1.3).

(4) We may assume M' is a sub-module of M . Let us choose $x_1, \dots, x_l \in M$ with the following properties:

- (i) $\|x_i\| \leq 1$ for all i .
- (ii) $g(x_i) \neq g(x_j)$ for all $i \neq j$.
- (iii) For any $x \in M$ with $\|x\| \leq 1$, there is x_i such that $g(x) = g(x_i)$.

By using (i) and (ii), for any $x \in M$ with $\|x\| \leq 1$, there is a unique x_i with $g(x) = g(x_i)$. Moreover $x - x_i \in M'$ and $\|x - x_i\| \leq 2$. On the other hand, $\log(l) \leq \hat{h}^0(M'', \|\cdot\|'')$ because $\|g(x_i)\|'' \leq 1$ for all i . Therefore,

$$\hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M'', \|\cdot\|'') + \log \#\{x' \in M' \mid \|x'\| \leq 2\}$$

Hence (4) follows from (3).

(5) Let $\langle \cdot, \cdot \rangle$ be an inner product of M/M_{tor} with respect to the basis $\{e_1, \dots, e_{\text{rk } M}\}$. Then, for $x = a_1 e_1 + \dots + a_{\text{rk } M} e_{\text{rk } M}$, if $|\langle x, e_i \rangle| \leq 1$ for all i , then $|a_i| \leq 1$ for all i . Thus (5) follows. \square

Remark 2.2. Note that

$$\begin{cases} (x+1)\log(x+1) \geq x \text{ for all } x \geq 0, \\ \log(n!) \leq (n+1)\log(n+1) \text{ for all non-negative integer } n. \end{cases}$$

Therefore, we have simpler inequalities for each case of Proposition 2.1 as follows. The inequalities (2.2.1), (2.2.2), (2.2.3) and (2.2.4) are simpler versions of the corresponding inequalities in (1), (2), (3) and (4) of Proposition 2.1 respectively.

$$(2.2.1) \quad \left| \hat{h}^0(M, \|\cdot\|) - \hat{h}^1(M, \|\cdot\|) - \hat{\chi}(M, \|\cdot\|) \right| \leq (\log(3/2) + 2)(\text{rk } M + 1) \log(\text{rk } M + 1).$$

$$(2.2.2) \quad \hat{\chi}(M, \|\cdot\|_2) - \hat{\chi}(M, \|\cdot\|_1) \leq \hat{h}^0(M, \|\cdot\|_2) - \hat{h}^0(M, \|\cdot\|_1) + (\log(9) + 2)(\text{rk } M + 1) \log(\text{rk } M + 1).$$

$$(2.2.3) \quad 0 \leq \hat{h}^0(M, \exp(-\lambda)\|\cdot\|) - \hat{h}^0(M, \|\cdot\|) \leq \lambda \text{rk } M + (\log(9) + 2)(\text{rk } M + 1) \log(\text{rk } M + 1).$$

$$(2.2.4) \quad \hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M', \|\cdot\|') + \hat{h}^0(M'', \|\cdot\|'') + (\log(18) + 2)(\text{rk } M' + 1) \log(\text{rk } M' + 1).$$

3. APPROXIMATION OF THE NUMBER OF SMALL SECTIONS

In this section, we prove the main technical tool of this paper. First we consider the following three lemmas. The first one is an upper estimate of the number of small sections.

Lemma 3.1. *Let X be a projective arithmetic variety of dimension d , and let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . Then we have the following:*

(1) *If \bar{L} is ample, then*

$$\hat{h}^0\left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}}\right) = \frac{\widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d})}{d!} m^d + o(m^d)$$

for $m \gg 1$.

(2) *In general, there is a constant C with*

$$\hat{h}^0\left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}}\right) \leq Cm^d$$

for all $m \geq 1$.

(3) *Let $\mu : Y \rightarrow X$ be a generic resolution of singularities of X . Let Ω be a volume form on $Y(\mathbb{C})$. An L^2 -norm of $H^0(X, mL + N)$ is given in the following way: for $t \in H^0(X, mL + N)$,*

$$\|t\|_{L^2, \Omega}^{m\bar{L} + \bar{N}} := \left(\int_{Y(\mathbb{C})} \mu^*(|t|_{m\bar{L} + \bar{N}}^2) \Omega \right)^{1/2},$$

where $|\cdot|_{m\bar{L}+\bar{N}}$ is the hermitian norm of $m\bar{L} + \bar{N}$. Then there is a constant C with

$$\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{L^2, \Omega}^{m\bar{L}+\bar{N}} \right) \leq Cm^d$$

for all $m \geq 1$.

Proof. (1) It is well-known that

$$\hat{\chi} \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) = \frac{\widehat{\text{deg}}(\hat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d)$$

for $m \gg 1$ (cf. [6], [1] and [17, Theorem (1.4)]). Thus, by (2) of Proposition 2.1,

$$\begin{aligned} \hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) - \hat{h}^1 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) \\ = \frac{\widehat{\text{deg}}(\hat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d). \end{aligned}$$

Since \bar{L} is ample, by [17, Theorem (4.2)], $H^0(X, mL + N)$ is generated by sections t with $\|t\|_{\text{sup}} < 1$. Thus, by (5) of Proposition 2.1,

$$\hat{h}^1 \left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}} \right) = o(m^d).$$

Hence we get (1).

(2) Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . Then there are a positive integer n and a non-zero section t of $H^0(n\bar{A} - L)$ with $\|s\|_{\text{sup}} \leq 1$. Let $\phi : L \rightarrow n\bar{A}$ be an injective homomorphism given by $\phi(t) = s \otimes t$. Then since $|s \otimes t| = |s||t| \leq |t|$, ϕ yields $\bar{L} \leq n\bar{A}$ (cf. Conventions and terminology 8). Therefore $m\bar{L} \leq mn\bar{A}$ for all $m \geq 1$. Thus (2) follows from (1).

(3) By using Gromov's inequality on $Y(\mathbb{C})$, there is a constant C_1 such that

$$\|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \leq C_1 m^{d-1} \|\cdot\|_{L^2, \Omega}^{m\bar{L}+\bar{N}}$$

for $m \gg 1$. Thus

$$\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) \geq \hat{h}^0 \left(H^0(X, mL + N), C_1 m^{d-1} \|\cdot\|_{L^2, \Omega}^{m\bar{L}+\bar{N}} \right).$$

Moreover, by (3) of Proposition 2.1,

$$\begin{aligned} \hat{h}^0 \left(H^0(X, mL + N), C_1 m^{d-1} \|\cdot\|_{L^2, \Omega}^{m\bar{L}+\bar{N}} \right) \\ = \hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{L^2, \Omega}^{m\bar{L}+\bar{N}} \right) + o(m^d). \end{aligned}$$

Therefore we get (3). \square

Next we consider formulae concerning subnorms and quotient norms (cf. Conventions and terminology 3).

Lemma 3.2. (1) Let $f : V \rightarrow W$ and $g : W \rightarrow U$ be surjective homomorphisms of finite dimensional vector spaces over \mathbb{R} . For a norm σ of V , $(\sigma_{V \rightarrow W})_{W \rightarrow U} = \sigma_{V \rightarrow U}$ as norms of U .

(2) Let

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ g' \downarrow & & \downarrow g \\ P & \xrightarrow{f'} & Q \end{array}$$

be a commutative diagram of finite dimensional vector spaces over \mathbb{R} such that f and f' are injective and that g and g' are surjective. Let σ be a norm of V . Then

$$(\sigma_{W \hookrightarrow V})_{W \rightarrow P} \geq (\sigma_{V \rightarrow Q})_{P \hookrightarrow Q}$$

as norms of P . Moreover, if $\ker(g) \subseteq f(W)$, then

$$(\sigma_{W \hookrightarrow V})_{W \rightarrow P} = (\sigma_{V \rightarrow Q})_{P \hookrightarrow Q}.$$

Proof. (1) Let us fix $u \in U$. For $v \in (g \circ f)^{-1}(u)$,

$$\sigma(v) \geq \sigma_{V \rightarrow W}(f(v)) \geq (\sigma_{V \rightarrow W})_{W \rightarrow U}(u)$$

Therefore $\sigma_{V \rightarrow U}(u) \geq (\sigma_{V \rightarrow W})_{W \rightarrow U}(u)$.

Pick up $v_0 \in (g \circ f)^{-1}(u)$ with $\sigma_{V \rightarrow U}(u) = \sigma(v_0)$. Then, for any $w \in g^{-1}(u)$, $\sigma(v_0) \leq \sigma_{V \rightarrow W}(w)$ because $f^{-1}(w) \subseteq (g \circ f)^{-1}(u)$. Hence

$$\sigma_{V \rightarrow U}(u) = \sigma(v_0) \leq (\sigma_{V \rightarrow W})_{W \rightarrow U}(u).$$

(2) Since $f(\ker(g')) = f(W) \cap \ker(g)$, for $w \in W$,

$$\begin{cases} (\sigma_{W \hookrightarrow V})_{W \rightarrow P}(g'(w)) = \inf\{\sigma(x) \mid x \in f(w) + f(W) \cap \ker(g)\} \\ (\sigma_{V \rightarrow Q})_{g(W) \hookrightarrow Q}(g'(w)) = \inf\{\sigma(x) \mid x \in f(w) + \ker(g)\}. \end{cases}$$

Thus $(\sigma_{W \hookrightarrow V})_{W \rightarrow P}(g(w)) \geq (\sigma_{V \rightarrow Q})_{P \hookrightarrow Q}(g(w))$. Moreover, if $\ker(g) \subseteq f(W)$ (or, equivalently $f(W) \cap \ker(g) = \ker(g)$), then $(\sigma_{W \hookrightarrow V})_{W \rightarrow P} = (\sigma_{V \rightarrow Q})_{P \hookrightarrow Q}$. \square

The following lemma is needed to find a good \bar{A} in the proof of Theorem 3.4.

Lemma 3.3. *Let X be a projective and generically smooth arithmetic variety of dimension d , and let Ω be a volume form on $X(\mathbb{C})$. Let \bar{L} and \bar{A} be C^∞ -hermitian invertible sheaves on X . Let us consider the following assertion $\Sigma(X, \bar{L}, \bar{A})$:*

There are positive constants a_0, C and D depending only on X, \bar{L} and \bar{A} such that

$$\begin{aligned} \hat{h}^0\left(H^0(aL + (b-c)A), \|\cdot\|_{L^2, \Omega}^{a\bar{L} + (b-c)\bar{A}}\right) \\ \leq \hat{h}^0\left(H^0(aL - cA), \|\cdot\|_{L^2, \Omega}^{a\bar{L} - c\bar{A}}\right) + Cba^{d-1} + Da^{d-1} \log(a) \end{aligned}$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$.

Then we have the following:

- (1) *Let \bar{A}' be another C^∞ -hermitian invertible sheaf on X with $\bar{A}' \leq \bar{A}$ (cf. Conventions and terminology 8). If $\Sigma(X, \bar{L}, \bar{A})$ holds, then so does $\Sigma(X, \bar{L}, \bar{A}')$.*
- (2) *We assume that $\text{rk } H^0(X, A) \neq 0$. Let $|\cdot|_A$ be the hermitian norm of \bar{A} . If $\Sigma(X, \bar{L}, \bar{A})$ holds, then so does $\Sigma(X, \bar{L}, (A, \exp(-\lambda)|\cdot|_A))$ for all $\lambda \geq 0$.*
- (3) *We assume that $\text{rk } H^0(X, A) \neq 0$. Let \bar{A}' be another C^∞ -hermitian invertible sheaf on X such that A' is isomorphic to A over \mathbb{Q} . Then $\Sigma(X, \bar{L}, \bar{A})$ holds if and only if so does $\Sigma(X, \bar{L}, \bar{A}')$.*

Proof. (1) Since

$$\bar{L} + (b-c)\bar{A}' \leq \bar{L} + (b-c)\bar{A} \quad \text{and} \quad a\bar{L} - c\bar{A} \leq a\bar{L} - c\bar{A}',$$

(1) follows.

(2) We set $\bar{A}' = (A, \exp(-\lambda)|\cdot|_A)$. Let us fix constants C_1 and C_2 such that

$$\text{rk } H^0(a(L+A)) \leq C_1 a^{d-1}$$

for all $a \geq 1$ and that

$$(\log(18) + 2) (\operatorname{rk} H^0(a(L + A)) + 1) \log (\operatorname{rk} H^0(a(L + A)) + 1) \leq C_2 a^{d-1} \log(a)$$

for all $a \geq 2$. It is easy to see that

$$\begin{cases} \|\cdot\|_{L^2, \Omega}^{a\bar{L}+(b-c)\bar{A}'} = \exp(-(b-c)\lambda) \|\cdot\|_{L^2, \Omega}^{a\bar{L}+(b-c)\bar{A}}, \\ \|\cdot\|_{L^2, \Omega}^{a\bar{L}-c\bar{A}'} = \exp(c\lambda) \|\cdot\|_{L^2, \Omega}^{a\bar{L}-c\bar{A}}. \end{cases}$$

Since

$$\operatorname{rk} H^0(aL - cA) \leq \operatorname{rk} H^0(aL + (b-c)A) \leq \operatorname{rk} H^0(a(L + A)),$$

using (2.2.3), we have

$$\begin{aligned} 0 &\leq \hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2, \Omega}^{a\bar{L}+(b-c)\bar{A}'} \right) \\ &\quad - \hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2, \Omega}^{a\bar{L}+(b-c)\bar{A}} \right) \\ &\leq C_1 \lambda (b-c) a^{d-1} + C_2 a^{d-1} \log(a) \leq C_1 \lambda b a^{d-1} + C_2 a^{d-1} \log(a) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, \Omega}^{a\bar{L}-c\bar{A}'} \right) - \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, \Omega}^{a\bar{L}-c\bar{A}} \right) \\ &\leq C_1 \lambda c a^{d-1} + C_2 a^{d-1} \log(a) \leq C_1 \lambda b a^{d-1} + C_2 a^{d-1} \log(a). \end{aligned}$$

Thus we have

$$\begin{aligned} \hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2, \Omega}^{a\bar{L}+(b-c)\bar{A}'} \right) &\leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, \Omega}^{a\bar{L}-c\bar{A}'} \right) \\ &\quad + (C + 2\lambda C_1) b a^{d-1} + (D + 2C_2) a^{d-1} \log(a). \end{aligned}$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$.

(3) It is sufficient to show that if $\Sigma(X, \bar{L}, \bar{A})$ holds, then so does $\Sigma(X, \bar{L}, \bar{A}')$. Since A' is isomorphic to A over \mathbb{Q} , there is a Cartier divisor F such that $A' \otimes \mathcal{O}_X(F) \simeq A$ and $\operatorname{Supp}(F)$ is vertical. Thus there is a positive integer N such that $\mathcal{O}_X \cdot N \subseteq \mathcal{O}_X(F)$. Hence we have a natural injective homomorphism $\alpha : A' \cdot N \rightarrow A$. Let $|\cdot|$ and $|\cdot|'$ be C^∞ -hermitian norms of \bar{A} and \bar{A}' . Then $(A' \cdot N, |\cdot|')$ is a C^∞ -hermitian invertible sheaf on X . Since $\alpha : A' \cdot N \rightarrow A$ is isomorphism over \mathbb{Q} , there is a positive number λ such that $|\alpha_\mathbb{C}(\cdot)| \leq \exp(\lambda) |\cdot|'$. Then $(A' \cdot N, \exp(\lambda) |\cdot|') \leq (A, |\cdot|)$. Hence, by (1), $\Sigma(X, \bar{L}, (A' \cdot N, \exp(\lambda) |\cdot|'))$ holds. Note that the homomorphism $A' \rightarrow A' \cdot N$ given by $a \mapsto a \cdot N$ yields to an isometry $(A', N \exp(\lambda) |\cdot|') \rightarrow (A' \cdot N, \exp(\lambda) |\cdot|')$. Therefore $\Sigma(X, \bar{L}, (A', N \exp(\lambda) |\cdot|'))$ holds, so that so does $\Sigma(X, \bar{L}, (A', |\cdot|'))$ by (2). \square

Let X be a compact complex manifold, and let $\bar{L} = (L, |\cdot|_L)$ and $\bar{M} = (M, |\cdot|_M)$ be C^∞ -hermitian invertible sheaves on X . Let t be a non-zero global section of $H^0(X, M)$. We denote by $\|\cdot\|_{L^2, t, \text{sub}}^{\bar{L}, L-M}$ the subnorm of $H^0(X, L - M)$ induced by the natural injective homomorphism $H^0(X, L - M) \xrightarrow{\otimes t} H^0(X, L)$ and the L^2 -norm of $\|\cdot\|_{L^2}^{\bar{L}}$ of $H^0(X, L)$ for a fixed volume form on X . For simplicity, $\|\cdot\|_{L^2, t, \text{sub}}^{\bar{L}, L-M}$ is often denoted by $\|\cdot\|_{L^2, t, \text{sub}}^{\bar{L}}$.

The following theorem is the technical core of this paper. The similar result for an arithmetic curve will be treated in Proposition 3.5.

Theorem 3.4. *Let X be a projective and generically smooth arithmetic variety of dimension $d \geq 2$. Let \bar{L} and \bar{A} be C^∞ -hermitian invertible sheaves on X . We assume the following:*

- (i) A and $L + A$ are very ample over \mathbb{Q} .
- (ii) The first Chern forms $c_1(\bar{A})$ and $c_1(\bar{L} + \bar{A})$ on $X(\mathbb{C})$ are positive.
- (iii) There is a non-zero section $s \in H^0(X, A)$ such that the vertical component of $\text{div}(s)$ is contained in the regular locus of X and that the horizontal component of $\text{div}(s)$ is smooth over \mathbb{Q} .

Then there are positive constants a_0 , C and D depending only on X , \bar{L} and \bar{A} such that

$$\hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}} \right) \leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2}^{a\bar{L}-c\bar{A}} \right) + Cba^{d-1} + Da^{d-1} \log(a)$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$, where the volume form Ω to define L^2 -norms is $\Omega(\bar{L} + \bar{A})$ (cf. Conventions and terminology 5). Moreover the sup-version of the above estimate holds as follows: there are positive constants a'_0 , C' and D' depending only on X , \bar{L} and \bar{A} such that

$$\hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{\text{sup}}^{a\bar{L}+(b-c)\bar{A}} \right) \leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{\text{sup}}^{a\bar{L}-c\bar{A}} \right) + C'ba^{d-1} + D'a^{d-1} \log(a)$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a'_0$

Proof. First let us fix constants C_1 and C_2 such that

$$\text{rk } H^0(a(L + A)) \leq C_1 a^{d-1}$$

for all $a \geq 1$ and that

$$(\log(18) + 2) (\text{rk } H^0(a(L + A)) + 1) \log (\text{rk } H^0(a(L + A)) + 1) \leq C_2 a^{d-1} \log(a)$$

for all $a \geq 2$.

Let $|\cdot|_A$ be the C^∞ -hermitian norm of \bar{A} . As in Conventions and terminology 5, for $\lambda \in \mathbb{R}$, we set

$$\bar{A}^\lambda = (A, \exp(-\lambda)|\cdot|_A).$$

First we claim the following:

Claim 3.4.1. *We may assume that there is a non-zero section $s \in H^0(X, A)$ such that $\|s\|_{\text{sup}} \leq 1$, $\text{div}(s)$ is smooth over \mathbb{Q} and that $\text{div}(s)$ has no vertical components. We may further assume that there are a positive integer n and a non-zero section t of $H^0(X, nA - L)$ such that $\|t\|_{\text{sup}} \leq 1$ and t is not zero on $\text{div}(s)$.*

By our assumption (iii), there is a non-zero section $s \in H^0(X, A)$ such that the vertical component of $\text{div}(s)$ is contained in the regular locus of X and that the horizontal component of $\text{div}(s)$ is smooth over \mathbb{Q} . Let Y and F be the horizontal component of $\text{div}(s)$ and the vertical component of $\text{div}(s)$ respectively. Note that Y and F are effective Cartier divisors because F is contained in the regular locus of X . We define a C^∞ -hermitian invertible sheaf \bar{A}_1 by the equation

$$\bar{A} = \bar{A}_1 \otimes (\mathcal{O}_X(F), |\cdot|_{\text{can}}).$$

Then there is a non-zero section $s_1 \in H^0(X, A_1)$ such that $s = s_1 \otimes 1_F$ and $\text{div}(s_1) = Y$, where 1_F is the canonical section of $\mathcal{O}_X(F)$. Let λ be a non-negative real number with

$\exp(-\lambda)\|s_1\|_{\text{sup}} \leq 1$. Then, by (3) of Lemma 3.3, if the assertion holds for \bar{L} and \bar{A}_1^λ , then so does for \bar{L} and \bar{A} .

Moreover, since A is very ample over \mathbb{Q} and $\text{div}(s)$ has no vertical components, there are a positive integer n and a non-zero section t of $H^0(nA - L)$ such that t is not zero on $\text{div}(s)$. Let λ' be a non-negative real number with $\exp(-\lambda'/n)\|t\|_{\text{sup}} \leq 1$. Then t is a small section of a C^∞ -hermitian invertible sheaf $n\bar{A}^{\lambda'} - \bar{L}$. Thus, by (1) of Lemma 3.3, the claim follows.

For a coherent sheaf \mathcal{F} on X and a subscheme Z of X , the image $H^i(X, \mathcal{F}) \rightarrow H^i(Z, \mathcal{F}|_Z)$ is denoted by $I^i(Z, \mathcal{F}|_Z)$.

If $b = 0$, then $c = 0$. Thus, in this case, the assertion is obvious, so that we may assume $b \geq 1$. As in Claim 3.4.1, let s be a non-zero section $H^0(X, A)$ such that $\|s\|_{\text{sup}} \leq 1$, $Y := \text{div}(s)$ is smooth over \mathbb{Q} and that Y has no vertical components. Let us choose positive numbers C_3 and C_4 such that

$$\text{rk } H^0(Y, a(L + A)|_Y) \leq C_3 a^{d-2}$$

for all $a \geq 1$ and that

$$\begin{aligned} (\log(18) + 2) (\text{rk } H^0(Y, a(L + A)|_Y) + 1) \log (\text{rk } H^0(Y, a(L + A)|_Y) + 1) \\ \leq C_4 a^{d-2} \log(a) \end{aligned}$$

for all $a \geq 2$

Let $\|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}}$ be the quotient norm of $I^0(aL + (b-c)A|_{bY})$ induced by the surjective homomorphism $H^0(aL + (b-c)A) \rightarrow I^0(aL + (b-c)A|_{bY})$ and the L^2 -norm $\|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}}$ of $H^0(aL + (b-c)A)$. Note that $I^0(aL + (b-c)A|_{bY})$ is torsion free because bY is flat over \mathbb{Z} .

Claim 3.4.2. *For all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq 2$,*

$$\begin{aligned} \hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}} \right) \leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{a\bar{L}+(b-c)\bar{A}} \right) \\ + \hat{h}^0 \left(I^0(aL + (b-c)A|_{bY}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) + C_2 a^{d-1} \log(a). \end{aligned}$$

Using an exact sequence

$$0 \rightarrow H^0(aL - cA) \xrightarrow{s^b} H^0(aL + (b-c)A) \rightarrow I^0(aL + (b-c)A|_{bY}) \rightarrow 0,$$

we have a normed exact sequence

$$\begin{aligned} 0 \rightarrow \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{a\bar{L}+(b-c)\bar{A}} \right) \rightarrow \left(H^0(aL + (b-c)A), \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}} \right) \\ \rightarrow \left(I^0(aL + (b-c)A|_{bY}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) \rightarrow 0, \end{aligned}$$

where $\|\cdot\|_{L^2, s^b, \text{sub}}^{a\bar{L}+(b-c)\bar{A}}$ is the subnorm of $H^0(aL - cA)$ induced by the injective homomorphism $H^0(aL - cA) \xrightarrow{s^b} H^0(aL + (b-c)A)$ and the L^2 -norm $\|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}}$ of $H^0(aL + (b-c)A)$. Thus, by (2.2.4), it yields the claim because

$$\text{rk } H^0(aL - cA) \leq \text{rk } H^0(a(L + A)).$$

Next we claim the following:

Claim 3.4.3. *There are constants a_0 and C_5 depending only on \bar{L} and \bar{A} such that*

$$\begin{aligned} \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \right) &\leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2}^{\bar{a}\bar{L} - c\bar{A}} \right) \\ &\quad + C_5 b a^{d-1} + C_2 a^{d-1} \log(a). \end{aligned}$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$.

Note that $\|\cdot\|_{L^2, s^b, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \leq \|\cdot\|_{L^2}^{\bar{a}\bar{L} - c\bar{A}}$. Thus, by (2.2.2),

$$\begin{aligned} \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2}^{\bar{a}\bar{L} - c\bar{A}} \right) &- \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \right) + C_2 a^{d-1} \log(a) \\ &\geq \hat{\chi} \left(H^0(aL - cA), \|\cdot\|_{L^2}^{\bar{a}\bar{L} - c\bar{A}} \right) - \hat{\chi} \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \right). \end{aligned}$$

Therefore it is sufficient to find positive constants a_0 and C_5 such that

$$\hat{\chi} \left(H^0(aL - cA), \|\cdot\|_{L^2}^{\bar{a}\bar{L} - c\bar{A}} \right) - \hat{\chi} \left(H^0(aL - cA), \|\cdot\|_{L^2, s^b, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \right) \geq -C_5 b a^{d-1}$$

for all a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$. This is nothing more than a consequence of Corollary 1.2.2.

Let k be an integer with $0 \leq k < b$. Let $\|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{\bar{a}\bar{L} + (b-c)\bar{A}}$ be the quotient norm of $I^0(Y, aL + (b-c-k)A|_Y)$ induced by a surjective homomorphism

$$H^0(aL + (b-c-k)A) \rightarrow I^0(Y, aL + (b-c-k)A|_Y)$$

and $\|\cdot\|_{L^2, s^k, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}}$ of $H^0(aL + (b-c-k)A)$.

Claim 3.4.4. *There is a constant C_6 and C_7 depending only on \bar{L} and \bar{A} such that*

$$\hat{h}^0 \left(I^0(Y, aL + (b-c-k)A|_Y), \|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \right) \leq C_6 a^{d-1} + C_7 a^{d-2} \log(a)$$

for all integers a, b, c, k with $a \geq b \geq c \geq 0$, $a \geq 2$ and $0 \leq k < b$.

Let us choose a small open set U of $X(\mathbb{C})$ such that the closure of U does not meet with $Y(\mathbb{C})$ and U is not empty on each connected component of $X(\mathbb{C})$. Then, applying Lemma 1.1.4 to the cases $L_{\mathbb{C}}, A_{\mathbb{C}}$ and $L_{\mathbb{C}}, -A_{\mathbb{C}}$, there are constant $D_1 \geq 1$ and $D'_1 \geq 1$ such that

$$D'_1 D_1^{l+|m|} \int_U |u|^2 \Omega \geq \int_{X(\mathbb{C})} |u|^2 \Omega$$

for all integers l, m with $l \geq 0$ and all $u \in H^0(X(\mathbb{C}), lL + mA)$. Since $0 < \inf_{x \in U} \{|s|(x)\} < 1$, if we set

$$D_2 = 1 / \inf_{x \in U} \{|s|(x)\},$$

then $D_2 > 1$. Thus, if we set $D_3 = \max\{D_2, D_1\}$, then, for $u \in H^0(X, aL + (b-c-k)A)$,

$$\begin{aligned} \int_{X(\mathbb{C})} |s^k \otimes u|^2 \Omega &\geq \int_U |s^k \otimes u|^2 \Omega \geq D_2^{-2k} \int_U |u|^2 \Omega \\ &\geq D_2^{-2k} D_1^{-1} D_1^{-(a+|b-c-k|)} \int_{X(\mathbb{C})} |u|^2 \Omega \geq D_1^{-1} D_3^{-4a} \int_{X(\mathbb{C})} |u|^2 \Omega, \end{aligned}$$

which means that

$$\|\cdot\|_{L^2, s^k, \text{sub}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \geq D_1^{-1/2} D_3^{-2a} \|\cdot\|_{L^2}^{\bar{a}\bar{L} + (b-c-k)\bar{A}}.$$

Hence

$$\|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{\bar{a}\bar{L} + (b-c)\bar{A}} \geq D_1^{-1/2} D_3^{-2a} \|\cdot\|_{L^2, \text{quot}}^{\bar{a}\bar{L} + (b-c-k)\bar{A}},$$

where $\|\cdot\|_{L^2, \text{quot}}^{\overline{aL}+(b-c-k)\overline{A}}$ is the quotient norm of $I^0(Y, aL + (b-c-k)A|_Y)$ induced by a surjective homomorphism

$$H^0(X, aL + (b-c-k)A) \rightarrow I^0(Y, aL + (b-c-k)A|_Y).$$

Note that $e^x \geq x + 1$ for $x \geq 0$. Thus, applying Corollary 1.1.3 to the cases L_C, A_C and $L_{\mathbb{C}}, -A_C$, there are constants $D_4, D'_4 \geq 1$ such that

$$\begin{aligned} \|\cdot\|_{L^2, \text{quot}}^{\overline{aL}+(b-c-k)\overline{A}} &\geq D_4'^{-1/2} D_4^{-(a+|b-c-k|)/2} \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y \\ &\geq D_4'^{-1/2} D_4^{-a} \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y \end{aligned}$$

on $I^0(Y, aL + (b-c-k)A|_Y)$, where the volume form on Y is given by the C^∞ -hermitian invertible sheaf $\overline{L} + \overline{A}|_Y$. Therefore, if we set $D_5 = \max\{D_3, D_4\}$ and $D'_5 = \max\{D'_1, D'_4\}$, then

$$\|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{\overline{aL}+(b-c)\overline{A}} \geq D_5'^{-1} D_5^{-3a} \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y$$

on $I^0(Y, aL + (b-c-k)A|_Y)$. Thus, by (2.2.3),

$$\begin{aligned} &\hat{h}^0 \left(I^0(Y, aL + (b-c-k)A|_Y), \|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{\overline{aL}+(b-c)\overline{A}} \right) \\ &\leq \hat{h}^0 \left(I^0(Y, aL + (b-c-k)A|_Y), \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y \right) \\ &\quad + \log(D_5' D_5^{3a}) C_3 a^{d-2} + C_4 a^{d-2} \log(a) \\ &\leq \hat{h}^0 \left(H^0(Y, aL + (b-c-k)A|_Y), \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y \right) \\ &\quad + \log(D_5' D_5^{3a}) C_3 a^{d-2} + C_4 a^{d-2} \log(a). \end{aligned}$$

Let \tilde{Y} be the normalization of Y . Let t be a non-zero section as in Claim 3.4.1. Then t gives rise to a relation $\overline{L}|_{\tilde{Y}} \leq n\overline{A}|_{\tilde{Y}}$ (cf. Conventions and terminology 8). Thus

$$\overline{aL} + (b-c-k)\overline{A}|_{\tilde{Y}} \leq (an+b-c-k)\overline{A}|_{\tilde{Y}}.$$

Therefore,

$$\begin{aligned} &\hat{h}^0 \left(H^0(Y, aL + (b-c-k)A|_Y), \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_Y \right) \\ &\leq \hat{h}^0 \left(H^0(\tilde{Y}, aL + (b-c-k)A|_{\tilde{Y}}), \|\cdot\|_{L^2}^{\overline{aL}+(b-c-k)\overline{A}}|_{\tilde{Y}} \right) \\ &\leq \hat{h}^0 \left(H^0(\tilde{Y}, (an+b-c-k)A|_{\tilde{Y}}), \|\cdot\|_{L^2}^{(an+b-c-k)\overline{A}}|_{\tilde{Y}} \right). \end{aligned}$$

Further, by Lemma 3.1, there is a positive constant D_6 with

$$\hat{h}^0 \left(H^0(\tilde{Y}, nA|_{\tilde{Y}}), \|\cdot\|_{L^2}^{n\overline{A}}|_{\tilde{Y}} \right) \leq D_6 n^{d-1}$$

for all $n \geq 1$. Thus the claim follows.

Finally we claim the following:

Claim 3.4.5. *There is a constant C_7 depending only on \overline{L} and \overline{A} such that*

$$\hat{h}^0 \left(I^0((aL + (b-c)A)|_{bY}), \|\cdot\|_{L^2, \text{quot}}^{\overline{aL}+(b-c)\overline{A}} \right) \leq C_6 b a^{d-1} + (C_4 + C_7) a^{d-1} \log(a)$$

for all integers a, b, c with $a \geq b \geq c \geq 0$, $a \geq 2$.

A commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & -(k+1)A & \xrightarrow{s^{k+1}} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{(k+1)Y} \longrightarrow 0 \\
& & \downarrow s & & \parallel & & \downarrow \\
0 & \longrightarrow & -kA & \xrightarrow{s^k} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{kY} \longrightarrow 0 \\
& & \downarrow & & & & \\
& & -kA|_Y & & & &
\end{array}$$

yields an injective homomorphism $\alpha_k : -kA|_Y \rightarrow \mathcal{O}_{(k+1)Y}$ together with a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & -kA & \xrightarrow{s^k} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{kY} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & -kA|_Y & \xrightarrow{\alpha_k} & \mathcal{O}_{(k+1)Y} & \longrightarrow & \mathcal{O}_{kY} \longrightarrow 0,
\end{array}$$

where two horizontal sequences are exact. Thus, tensoring the above diagram with $aL + (b-c)A$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & aL + (b-c-k)A & \xrightarrow{s^k} & aL + (b-c)A & \longrightarrow & aL + (b-c)A|_{kY} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & aL + (b-c-k)A|_Y & \xrightarrow{\alpha_k} & aL + (b-c)A|_{(k+1)Y} & \longrightarrow & aL + (b-c)A|_{kY} \longrightarrow 0
\end{array}$$

Therefore we have an exact sequence

$$\begin{aligned}
0 \rightarrow I^0((aL + (b-c-k)A)|_Y) &\rightarrow I^0((aL + (b-c)A)|_{(k+1)Y}) \\
&\rightarrow I^0((aL + (b-c)A)|_{kY}) \rightarrow 0
\end{aligned}$$

Note that in the commutative diagram

$$\begin{array}{ccc}
H^0(aL + (b-c-k)A) & \xrightarrow{s^k} & H^0(aL + (b-c)A) \\
\downarrow & & \downarrow \\
I^0((aL + (b-c-k)A)|_Y) & \xrightarrow{\alpha_k} & I^0((aL + (b-c)A)|_{(k+1)Y}),
\end{array}$$

the two vertical arrows have the same kernel. Thus, by Lemma 3.2,

$$\begin{aligned}
0 \rightarrow & \left(I^0((aL + (b-c-k)A)|_Y), \|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{a\bar{L} + (b-c)\bar{A}} \right) \\
& \rightarrow \left(I^0((aL + (b-c)A)|_{(k+1)Y}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L} + (b-c)\bar{A}} \right) \\
& \rightarrow \left(I^0((aL + (b-c)A)|_{kY}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L} + (b-c)\bar{A}} \right) \rightarrow 0
\end{aligned}$$

is a normed exact sequence, where for each $1 \leq i \leq b$, the norm $\|\cdot\|_{L^2, \text{quot}}^{a\bar{L} + (b-c)\bar{A}}$ of $I^0((aL + (b-c)A)|_{iY})$ is the quotient norm induced by the surjective homomorphism

$H^0(aL + (b - c)A) \rightarrow I^0((aL + (b - c)A)|_{iY})$ and the L^2 -norm $\|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}}$ of $H^0(aL + (b - c)A)$. Therefore, by (2.2.4),

$$\begin{aligned} & \hat{h}^0 \left(I^0((aL + (b - c)A)|_{(k+1)Y}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) \\ & \quad - \hat{h}^0 \left(I^0((aL + (b - c)A)|_{kY}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) \\ & \leq \hat{h}^0 \left(I^0((aL + (b - c - k)A)|_Y), \|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) + C_4 a^{d-2} \log(a). \end{aligned}$$

Thus, taking $\sum_{k=1}^{b-1}$, the above yields

$$\begin{aligned} & \hat{h}^0 \left(I^0((aL + (b - c)A)|_{bY}), \|\cdot\|_{L^2, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) \\ & \leq \sum_{k=0}^{b-1} \hat{h}^0 \left(I^0((aL + (b - c - k)A)|_Y), \|\cdot\|_{L^2, s^k, \text{sub}, \text{quot}}^{a\bar{L}+(b-c)\bar{A}} \right) \\ & \quad + (b - 1)C_4 a^{d-2} \log(a). \end{aligned}$$

Therefore, using Claim 3.4.4, we have the claim.

Gathering Claim 3.4.2, Claim 3.4.3 and Claim 3.4.5, if we set $C = C_5 + C_6$ and $D = 2C_2 + C_4 + C_7$, then

$$\begin{aligned} & \hat{h}^0 \left(H^0(aL + (b - c)A), \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}} \right) \\ & \leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2}^{a\bar{L}-c\bar{A}} \right) + Cba^{d-1} + Da^{d-1} \log(a) \end{aligned}$$

for all $a \geq b \geq c \geq 0$ and $a \geq a_0$.

Finally let us consider the sup-version of our estimate. First of all, since

$$\|\cdot\|_{\text{sup}}^{a\bar{L}+(b-c)\bar{A}} \geq \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}},$$

we have

$$\hat{h}^0 \left(H^0(aL + (b - c)A), \|\cdot\|_{\text{sup}}^{a\bar{L}+(b-c)\bar{A}} \right) \leq \hat{h}^0 \left(H^0(aL + (b - c)A), \|\cdot\|_{L^2}^{a\bar{L}+(b-c)\bar{A}} \right).$$

Moreover, by virtue of Gromov's inequality, there is a constant $C_8 \geq 1$

$$\|\cdot\|_{L^2}^{a\bar{L}-c\bar{A}} \geq C_8^{-1}(a + c + 1)^{-(d-1)} \|\cdot\|_{\text{sup}}^{a\bar{L}-c\bar{A}}$$

for all $a, c \geq 0$. Thus, since $a \geq c$,

$$\begin{aligned} & \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{L^2}^{a\bar{L}-c\bar{A}} \right) \\ & \leq \hat{h}^0 \left(H^0(aL - cA), C_8^{-1}(a + c + 1)^{-(d-1)} \|\cdot\|_{\text{sup}}^{a\bar{L}-c\bar{A}} \right) \\ & \leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{\text{sup}}^{a\bar{L}-c\bar{A}} \right) \\ & \quad + \log(C_8(2a + 1)^{d-1})C_1 a^{d-1} + C_2 a^{d-1} \log(a) \end{aligned}$$

for all $a \geq c \geq 0$. Therefore we obtain the sup-version. \square

Let R be an integral domain such that R is flat and finite over \mathbb{Z} . Let K be a quotient field of R . Note that K is a number field. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$ of fields. Let L be a finitely generated and free R -module of rank 1. For each $\sigma \in K(\mathbb{C})$, the tensor product $L \otimes_R \mathbb{C}$ in terms of the embedding $\sigma : K \hookrightarrow \mathbb{C}$ is denoted by L_σ . For each $\sigma \in K(\mathbb{C})$, let $|\cdot|_\sigma$ be a norm of L_σ . The collection $(L, \{|\cdot|_\sigma\}_{\sigma \in K(\mathbb{C})})$ is

called a normed invertible R -module. For simplicity, $(L, \{|\cdot|_\sigma\}_{\sigma \in K(\mathbb{C})})$ is often denoted by $(L, |\cdot|)$ or \bar{L} . We define $\|\cdot\|_{\text{sup}}^{\bar{L}}$ by

$$\|s\|_{\text{sup}}^{\bar{L}} = \max\{|s|_\sigma \mid \sigma \in K(\mathbb{C})\}.$$

Then $(L, \|\cdot\|_{\text{sup}}^{\bar{L}})$ is a normed finitely generated free \mathbb{Z} -module.

Proposition 3.5. *Let \bar{L} and \bar{A} be normed invertible R -modules of rank 1. We assume that there is $s \in A$ with $s \neq 0$ and $\|s\|_{\text{sup}}^{\bar{A}} \leq 1$. Then there are positive constants C and D depending only on \bar{L} and \bar{A} such that*

$$\hat{h}^0\left(aL + (b-c)A, \|\cdot\|_{\text{sup}}^{\bar{L}+(b-c)\bar{A}}\right) \leq \hat{h}^0\left(aL - cA, \|\cdot\|_{\text{sup}}^{\bar{L}-c\bar{A}}\right) + Cb + D$$

for all non-negative integers a, b, c .

Proof. Let $\|\cdot\|_{\text{sup}, s^b, \text{sub}}^{\bar{L}+(b-c)\bar{A}}$ be the subnorm of $aL - cA$ induced by the injective homomorphism $aL - cA \xrightarrow{s^b} aL + (b-c)A$ and the norm $\|\cdot\|_{\text{sup}}^{\bar{L}+(b-c)\bar{A}}$ of $aL + (b-c)A$. Then, by (4) of Proposition 2.1, we have

$$\begin{aligned} \hat{h}^0\left(aL + (b-c)A, \|\cdot\|_{\text{sup}, s^b, \text{sub}}^{\bar{L}+(b-c)\bar{A}}\right) &\leq \hat{h}^0\left(aL - cA, \|\cdot\|_{\text{sup}, s^b, \text{sub}}^{\bar{L}+(b-c)\bar{A}}\right) \\ &\quad + \log \#(\text{Coker}(aL - cA \rightarrow aL + (b-c)A)) + D, \end{aligned}$$

where $d = [K : \mathbb{Q}]$ and $D = (\log(18) + 2)(d+1) \log(d+1)$. Note that

$$\begin{aligned} \log \#(\text{Coker}(aL - cA \xrightarrow{s^b} aL + (b-c)A)) \\ &= \log \#(\text{Coker}(R \xrightarrow{s^b} bA) \otimes (aL - cA)) \\ &= \log \#(\text{Coker}(R \xrightarrow{s^b} bA)). \end{aligned}$$

Let us consider a sequence of injective homomorphisms:

$$R \xrightarrow{s} A \xrightarrow{s} \dots \xrightarrow{s} bA.$$

Then

$$\begin{aligned} \log \#(\text{Coker}(R \xrightarrow{s^b} bA)) &= \sum_{i=1}^b \log \#(\text{Coker}((i-1)A \xrightarrow{s} iA)) \\ &= b \cdot \log \#(\text{Coker}(R \xrightarrow{s} A)). \end{aligned}$$

On the other hand, for all $t \in aL - cA$,

$$\|s^b \otimes t\|_{\text{sup}}^{\bar{L}+(b-c)\bar{A}} \geq (\min\{|s|_\sigma \mid \sigma \in K(\mathbb{C})\})^b \|t\|_{\text{sup}}^{\bar{L}-c\bar{A}}.$$

Thus, by (3) of Proposition 2.1,

$$\hat{h}^0\left(aL - cA, \|\cdot\|_{\text{sup}, s^b, \text{sub}}^{\bar{L}+(b-c)\bar{A}}\right) \leq \hat{h}^0\left(aL - cA, \|\cdot\|_{\text{sup}}^{\bar{L}-c\bar{A}}\right) + b \log(C'(s))d + D,$$

where $C'(s) = \min\{|s|_\sigma \mid \sigma \in K(\mathbb{C})\}$. Therefore,

$$\begin{aligned} \hat{h}^0\left(aL + (b-c)A, \|\cdot\|_{\text{sup}}^{\bar{L}+(b-c)\bar{A}}\right) &\leq \hat{h}^0\left(aL - cA, \|\cdot\|_{\text{sup}}^{\bar{L}-c\bar{A}}\right) \\ &\quad + b(\log \#(\text{Coker}(R \xrightarrow{s} A)) + \log(C'(s)))d + 2D. \end{aligned}$$

□

Finally we consider the following lemma which guarantees the existence of a good C^∞ -hermitian invertible sheaf \bar{A} satisfying the assumptions (i), (ii) and (iii) of Theorem 3.4.

Lemma 3.6. *Let X be a projective and generically smooth arithmetic variety of dimension $d \geq 2$, and let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . Then, for any C^∞ -hermitian invertible sheaf \bar{L} on X , there is a positive integer n_0 such that, for all $n \geq n_0$, $n\bar{A}$ satisfies the assumptions (i), (ii) and (iii) of Theorem 3.4.*

Proof. This is a consequence of arithmetic Bertini's theorem (cf. [13]). We can give however an easy and direct proof of the lemma as follows: It is easy to find n_0 for the assumptions (i) and (ii). In addition to (i) and (ii), we choose n_0 such that nA is very ample for all $n \geq n_0$. Let $\pi : X \rightarrow \text{Spec}(\mathbb{Z})$ be the structure morphism and S the minimal finite set of $\text{Spec}(\mathbb{Z}) \setminus \{0\}$ such that $\pi^{-1}(\text{Spec}(\mathbb{Z}) \setminus S)$ is regular. Let Z_1, \dots, Z_r be all irreducible components of $\pi^{-1}(S)$, and let x_1, \dots, x_r be closed points of X with $x_i \in Z_i$ for all i . Let m_1, \dots, m_r be the maximal ideals corresponding to x_1, \dots, x_r . Then there is a positive integer n_1 such that, for all $n \geq n_1$, $H^1(X, nA \otimes m_1 \cdots m_r) = 0$, which means that the natural homomorphism

$$H^0(X, nA) \rightarrow \bigoplus_{i=1}^n nA \otimes (\mathcal{O}_X/m_i)$$

is surjective. Thus if $n \geq \max\{n_0, n_1\}$, then nA is very ample and there is a non-zero section t_n of $H^0(X, nA)$ with $t_n(x_i) \neq 0$ for all x_i . We set $\gamma(s) = t_n + ls$ for $s \in H^0(X, nA)$, where $l = \prod_{s \in S} \text{char}(\kappa(s))$ and $\kappa(s)$ is the residue field of \mathbb{Z} at s . Note that $\gamma(s)(x_i) \neq 0$ for all i . In particular, every vertical component of $\text{div}(\gamma(s))$ is contained in $\pi^{-1}(\text{Spec}(\mathbb{Z}) \setminus S)$. On the other hand, it is easy to see that the set $\{\gamma(s) \mid s \in H^0(X, nA)\}$ is Zariski dense in a vector space $H^0(X_{\mathbb{Q}}, nA_{\mathbb{Q}}) = H^0(X, nA) \otimes \mathbb{Q}$. Thus, by Bertini's theorem, there is $s \in H^0(X, nA)$ such that $\text{div}(\gamma(s))$ is smooth over \mathbb{Q} . \square

4. VOLUME FUNCTION FOR C^∞ -HERMITIAN INVERTIBLE SHEAVES AND ITS BASIC PROPERTIES

Let X be a projective arithmetic variety of dimension d . For a C^∞ -hermitian invertible sheaf \bar{L} on X , the *arithmetic volume of \bar{L}* is defined by

$$\widehat{\text{vol}}(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{\bar{L}})}{m^d/d!}.$$

This number is a finite real number by Lemma 3.1. Moreover, if \bar{L} is ample, then

$$\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d).$$

First let us consider elementary properties of volume function:

Proposition 4.1. *Let \bar{L} and \bar{M} be C^∞ -hermitian invertible sheaves on X . Then we have the following:*

- (1) *If $\bar{L} \leq \bar{M}$ (Conventions and terminology 8), then $\widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{M})$.*
- (2) *Let $|\cdot|_L$ be the hermitian norm of \bar{L} . For a real number λ , we set*

$$\bar{L}^\lambda = (L, \exp(-\lambda)|\cdot|_L).$$

If $\lambda \geq 0$, then we have

$$\begin{cases} \widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{L}^\lambda) \leq \widehat{\text{vol}}(\bar{L}) + d\lambda \text{vol}(L_{\mathbb{Q}}), \\ \widehat{\text{vol}}(\bar{L}) - d\lambda \text{vol}(L_{\mathbb{Q}}) \leq \widehat{\text{vol}}(\bar{L}^{-\lambda}) \leq \widehat{\text{vol}}(\bar{L}), \end{cases}$$

where $\text{vol}(L_{\mathbb{Q}})$ is the geometric volume of $L_{\mathbb{Q}}$ on $X_{\mathbb{Q}}$.

$$(3) \widehat{\text{vol}}(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{\text{sup}}^{m\bar{L}} < 1\}}{m^d/d!}.$$

Proof. (1) Since $\bar{L} \leq \bar{M}$, we have $m\bar{L} \leq m\bar{M}$ for all $m \geq 1$. Thus

$$\hat{h}^0\left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}\right) \leq \hat{h}^0\left(H^0(X, m\bar{M}), \|\cdot\|_{\text{sup}}^{m\bar{M}}\right)$$

for all $m \geq 1$. Hence $\widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{M})$.

(2) Since $\|\cdot\|_{\text{sup}}^{m\bar{L}^\lambda} = \exp(-m\lambda)\|\cdot\|_{\text{sup}}^{m\bar{L}}$, by using (2.2.3), there is a positive constant C such that

$$\begin{aligned} 0 &\leq \hat{h}^0\left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{\bar{L}^\lambda}\right) - \hat{h}^0\left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{\bar{L}}\right) \\ &\leq \lambda m \dim H^0(X_{\mathbb{Q}}, mL_{\mathbb{Q}}) + Cm^{d-1} \log(m) \end{aligned}$$

for $m \gg 1$. Thus we obtain the first inequalities. These implies that

$$\widehat{\text{vol}}(\bar{L}^{-\lambda}) \leq \widehat{\text{vol}}\left(\left(\bar{L}^{-\lambda}\right)^\lambda\right) \leq \widehat{\text{vol}}(\bar{L}^{-\lambda}) + d\lambda \text{vol}(L_{\mathbb{Q}}),$$

which is nothing more than the second inequalities because $\left(\bar{L}^{-\lambda}\right)^\lambda = \bar{L}$.

(3) For a positive real number λ ,

$$\begin{aligned} \hat{H}^0\left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}^{-\lambda}}\right) \\ \subseteq \left\{s \in H^0(X, mL) \mid \|s\|_{\text{sup}}^{m\bar{L}} < 1\right\} \\ \subseteq \hat{H}^0\left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}\right) \end{aligned}$$

because $\|\cdot\|_{\text{sup}}^{m\bar{L}^{-\lambda}} = \exp(m\lambda)\|\cdot\|_{\text{sup}}^{m\bar{L}}$. Thus, using (2), we have

$$\widehat{\text{vol}}(\bar{L}) - d\lambda \text{vol}(L_{\mathbb{Q}}) \leq \limsup_{m \rightarrow \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{\text{sup}}^{m\bar{L}} < 1\}}{m^d/d!} \leq \widehat{\text{vol}}(\bar{L}),$$

which shows the assertion because λ is an arbitrary positive number. \square

The following theorem shows that the volume function is a birational invariant.

Theorem 4.2. *Let $\pi : X' \rightarrow X$ be a birational morphism of projective arithmetic varieties, and let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . Then*

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0\left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}}\right)}{m^d} \\ = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0\left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L} + \bar{N})}\right)}{m^d}. \end{aligned}$$

In particular, $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{vol}}(\pi^*(\bar{L}))$.

Proof. The proof of this theorem is similar to [16, Theorem 2.2]. First of all, note that

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}} \right)}{m^d} \\ \leq \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L} + \bar{N})} \right)}{m^d}. \end{aligned}$$

Thus, considering a generic resolution of singularities of X' , we may assume that X' is generically smooth.

Let us consider an exact sequence:

$$0 \rightarrow mL + N \rightarrow \pi_*(\pi^*(mL + N)) \rightarrow (mL + N) \otimes (\pi_*(\mathcal{O}_{X'})/\mathcal{O}_X) \rightarrow 0.$$

The image of the natural homomorphism

$$H^0(X', \pi^*(mL + N)) \rightarrow H^0(X, (mL + N) \otimes (\pi_*(\mathcal{O}_{X'})/\mathcal{O}_X))$$

is denoted by $\Gamma(X'/X, mL + N)$. Let $\|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L} + \bar{N})}$ be the quotient norm of $\Gamma(X'/X, mL + N)$ induced by the surjective homomorphism

$$H^0(X', \pi^*(mL + N)) \rightarrow \Gamma(X'/X, mL + N)$$

and the sup-norm $\|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L} + \bar{N})}$ of $H^0(X', \pi^*(mL + N))$.

Claim 4.2.1. *If π is finite and \bar{L} is ample, then*

$$\hat{h}^0 \left(\Gamma(X'/X, mL + N), \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L} + \bar{N})} \right) \leq o(m^d).$$

We fix a normalized volume form Ω on $X'(\mathbb{C})$. Using Ω on $X'(\mathbb{C})$, as in Lemma 3.1, we can define L^2 -norms of $H^0(X, mL + N)$ and $H^0(X', \pi^*(mL + N))$ as follows: for $t \in H^0(X, mL + N)$ and $t' \in H^0(X', \pi^*(mL + N))$,

$$\|t\|_{L^2, \Omega}^{m\bar{L} + \bar{N}} = \left(\int_{X'(\mathbb{C})} \pi^*(|t|_{m\bar{L} + \bar{N}}^2) \Omega \right)^{1/2}$$

and

$$\|t'\|_{L^2, \Omega}^{\pi^*(m\bar{L} + \bar{N})} = \left(\int_{X'(\mathbb{C})} |t'|_{\pi^*(m\bar{L} + \bar{N})}^2 \Omega \right)^{1/2},$$

where $|\cdot|_{m\bar{L} + \bar{N}}$ and $|\cdot|_{\pi^*(m\bar{L} + \bar{N})}$ are the hermitian norms of $m\bar{L} + \bar{N}$ and $\pi^*(m\bar{L} + \bar{N})$ respectively. Note that $\pi^*(|\cdot|_{m\bar{L} + \bar{N}}) = |\cdot|_{\pi^*(m\bar{L} + \bar{N})}$. Let $\|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L} + \bar{N})}$ be the quotient norm of $\Gamma(X'/X, mL + N)$ induced by $H^0(X', \pi^*(mL + N)) \rightarrow \Gamma(X'/X, mL + N)$ and the L^2 -norm $\|\cdot\|_{L^2, \Omega}^{\pi^*(m\bar{L} + \bar{N})}$ of $H^0(X', \pi^*(mL + N))$. Then we have a normed exact sequence

(4.2.2)

$$\begin{aligned} 0 \rightarrow \left(H^0(X, mL + N), \|\cdot\|_{L^2}^{m\bar{L} + \bar{N}} \right) &\rightarrow \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L} + \bar{N})} \right) \\ &\rightarrow \left(\Gamma(X'/X, mL + N), \|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L} + \bar{N})} \right) \rightarrow 0. \end{aligned}$$

Since $\|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L} + \bar{N})} \leq \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L} + \bar{N})}$, it is sufficient to show that

$$\hat{h}^0 \left(\Gamma(X'/X, mL + N), \|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L} + \bar{N})} \right) \leq o(m^d).$$

By virtue of [17, Corollary (4.8)], $H^0(X', \pi^*(mL + N))$ is generated by sections t with

$$\|t\|_{L^2}^{\pi^*(m\bar{L}+\bar{N})} \leq \|t\|_{\sup}^{\pi^*(m\bar{L}+\bar{N})} < 1$$

for $m \gg 1$ because $\pi^*(\bar{L})$ is ample. Thus so does $\Gamma(X'/X, mL + N)$ with respect to $\|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L}+\bar{N})}$. Hence, by using (1) and (5) of Proposition 2.1, it suffices to show that

$$\hat{\chi}\left(\Gamma(X'/X, mL + N), \|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L}+\bar{N})}\right) \leq o(m^d)$$

because $\text{rk } \Gamma(X'/X, mL + N) = o(m^{d-1})$. By using the normed exact sequence (4.2.2) and [16, Theorem 2.1, (1)], we have

$$\begin{aligned} \hat{\chi}\left(\Gamma(X'/X, mL + N), \|\cdot\|_{L^2, \text{quot}}^{\pi^*(m\bar{L}+\bar{N})}\right) &= \hat{\chi}\left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L}+\bar{N})}\right) \\ &\quad - \hat{\chi}\left(H^0(X, mL + N), \|\cdot\|_{L^2}^{\pi^*(m\bar{L}+\bar{N})}\right) + o(m^d). \end{aligned}$$

On the other hand, using [17, Theorem (1.4)] and Gromov's inequality on $X'(\mathbb{C})$, we can see that

$$\begin{cases} \hat{\chi}\left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L}+\bar{N})}\right) = \frac{\widehat{\text{deg}}(\widehat{c}_1(\pi^*(\bar{L}))^d)}{d!} m^d + o(m^d), \\ \hat{\chi}\left(H^0(X, mL + N), \|\cdot\|_{L^2}^{\pi^*(m\bar{L}+\bar{N})}\right) = \frac{\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d) \end{cases}$$

as in the proof of Lemma 3.1. Moreover, by the projection formula,

$$\widehat{\text{deg}}(\widehat{c}_1(\pi^*(\bar{L}))^d) = \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d).$$

Thus the claim follows.

Claim 4.2.3. *If π is finite, then*

$$\hat{h}^0\left(\Gamma(X'/X, mL + N), \|\cdot\|_{\sup, \text{quot}}^{\pi^*(m\bar{L}+\bar{N})}\right) \leq o(m^d).$$

Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . Replacing \bar{A} by a higher multiple of \bar{A} if necessarily, we may assume that there is a non-zero section s of $H^0(X, A - L)$ such that $\|s\|_{\sup} \leq 1$ and s does not vanish at any associated point of $\pi_*(\mathcal{O}_{X'})/\mathcal{O}_X$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^0(X', \pi^*(mL + N)) & \xrightarrow{\pi^*(s)} & H^0(X', \pi^*(mA + N)) \\ \downarrow & & \downarrow \\ \Gamma(X'/X, mL + N) & \xrightarrow{s} & \Gamma(X'/X, mA + N). \end{array}$$

By our choice of s , the horizontal arrows are injective. Let $\|\cdot\|_{\sup, \pi^*(s), \text{sub}}^{\pi^*(m\bar{A}+\bar{N})}$ be the subnorm of $H^0(X', \pi^*(mL + N))$ induced by

$$H^0(X', \pi^*(mL + N)) \xrightarrow{\pi^*(s)} H^0(X', \pi^*(mA + N))$$

and $\|\cdot\|_{\sup}^{\pi^*(m\bar{A}+\bar{N})}$. Moreover, let $\|\cdot\|_{\sup, \pi^*(s), \text{sub}, \text{quot}}^{\pi^*(m\bar{A}+\bar{N})}$ be the quotient norm of $\Gamma(X'/X, mL + N)$ induced by

$$H^0(X', \pi^*(mL + N)) \rightarrow \Gamma(X'/X, mL + N),$$

and let $\|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{A}+\bar{N})}$ be the quotient norm of $\Gamma(X'/X, mA + N)$ induced by

$$H^0(X', \pi^*(mA + N)) \rightarrow \Gamma(X'/X, mA + N).$$

Then, by (2) of Lemma 3.2,

$$\|\cdot\|_{\text{sup}, \pi^*(s), \text{sub,quot}}^{\pi^*(m\bar{A}+\bar{N})} \geq \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{A}+\bar{N})}$$

on $\Gamma(X'/X, mL + N)$. Therefore, by the previous claim,

$$\begin{aligned} \hat{h}^0 \left(\Gamma(X'/X, mL + N), \|\cdot\|_{\text{sup}, \pi^*(s), \text{sub,quot}}^{\pi^*(m\bar{A}+\bar{N})} \right) \\ \leq \hat{h}^0 \left(\Gamma(X'/X, mA + N), \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{A}+\bar{N})} \right) \leq o(m^d). \end{aligned}$$

On the other hand, since

$$\|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L}+\bar{N})} \geq \|\cdot\|_{\text{sup}, \pi^*(s), \text{sub,quot}}^{\pi^*(m\bar{A}+\bar{N})},$$

we have

$$\begin{aligned} \hat{h}^0 \left(\Gamma(X'/X, mL + N), \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L}+\bar{N})} \right) \\ \leq \hat{h}^0 \left(\Gamma(X'/X, mL + N), \|\cdot\|_{\text{sup}, \pi^*(s), \text{sub,quot}}^{\pi^*(m\bar{A}+\bar{N})} \right). \end{aligned}$$

Thus the claim follows.

Claim 4.2.4. *If π is finite, then*

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right)}{m^d} \\ = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L}+\bar{N})} \right)}{m^d}. \end{aligned}$$

By using (4) of Proposition 2.1 and Claim 4.2.3, the normed exact sequence

$$\begin{aligned} 0 \rightarrow \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) \rightarrow \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L}+\bar{N})} \right) \\ \rightarrow \left(\Gamma(X'/X, mL + N), \|\cdot\|_{\text{sup,quot}}^{\pi^*(m\bar{L}+\bar{N})} \right) \rightarrow 0 \end{aligned}$$

gives rise to

$$\begin{aligned} \hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) \leq \hat{h}^0 \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L}+\bar{N})} \right) \\ \leq \hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}} \right) + o(m^d). \end{aligned}$$

This shows the claim.

Let us consider a general case. We set $X'' = \text{Spec}(\pi_*(\mathcal{O}_{X'}))$. Then $\pi : X' \rightarrow X$ can be factorized $\pi_1 : X' \rightarrow X''$ and $\pi_2 : X'' \rightarrow X$ such that $\pi = \pi_2 \circ \pi_1$, $(\pi_1)_*(\mathcal{O}_{X'}) = \mathcal{O}_{X''}$

and π_2 is finite. Thus, by Claim 4.2.4,

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}} \right)}{m^d} \\ = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X'', \pi_2^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi_2^*(m\bar{L} + \bar{N})} \right)}{m^d}. \end{aligned}$$

On the other hand, since $(\pi_1)_*(\mathcal{O}_{X'}) = \mathcal{O}_{X''}$,

$$H^0(X', \pi^*(mL + N)) = H^0(X'', \pi_2^*(mL + N))$$

for all $m \geq 1$. Thus

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X'', \pi_2^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi_2^*(m\bar{L} + \bar{N})} \right)}{m^d} \\ = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X', \pi^*(mL + N)), \|\cdot\|_{\text{sup}}^{\pi^*(m\bar{L} + \bar{N})} \right)}{m^d}. \end{aligned}$$

Hence the theorem follows. \square

Next let us consider the following theorem.

Theorem 4.3. *Let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . Then*

$$\limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}} \right)}{m^d} = \frac{\widehat{\text{vol}}(\bar{L})}{d!}.$$

Proof. By Theorem 4.2, we may assume that X is generically smooth. By using Lemma 3.6, there are ample C^∞ -hermitian invertible sheaves \bar{A} and \bar{B} such that $-\bar{B} \leq \bar{N} \leq \bar{A}$ and that \bar{A} and \bar{B} satisfy the assumptions (i), (ii) and (iii) of Theorem 3.4. The inequalities $-\bar{B} \leq \bar{N} \leq \bar{A}$ gives rise to

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL - B), \|\cdot\|_{\text{sup}}^{m\bar{L} - \bar{B}} \right)}{m^d} \\ \leq \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}} \right)}{m^d} \\ \leq \limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + A), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{A}} \right)}{m^d}. \end{aligned}$$

Applying Theorem 3.4 to the case where $b = 1$ and $c = 0$, we have

$$\hat{h}^0 \left(H^0(X, mL + A), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{A}} \right) \leq \hat{h}^0 \left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}} \right) + o(m^d)$$

for $m \gg 1$, which yields

$$\limsup_{m \rightarrow \infty} \frac{\hat{h}^0 \left(H^0(X, mL + A), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{A}} \right)}{m^d} \leq \frac{\widehat{\text{vol}}(\bar{L})}{d!}.$$

Further, applying Theorem 3.4 to the case where $b = c = 1$,

$$\hat{h}^0 \left(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}} \right) \leq \hat{h}^0 \left(H^0(X, mL - B), \|\cdot\|_{\text{sup}}^{m\bar{L} - \bar{B}} \right) + o(m^d)$$

for $m \gg 1$, which implies

$$\frac{\widehat{\text{vol}}(\bar{L})}{d!} \leq \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL - B), \|\cdot\|_{\text{sup}}^{m\bar{L}-\bar{B}})}{m^d}.$$

Thus we get the theorem. \square

The following lemma is need to see the characterization of bigness and the homogeneity of the arithmetic volume function.

Lemma 4.4. *Let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . We assume that \bar{L} is big. Then, for a fixed positive integer p ,*

$$\limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L}+\bar{N}})}{(pn)^d} = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}})}{m^d}$$

and

$$\liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L}+\bar{N}})}{(pn)^d} = \liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L}+\bar{N}})}{m^d}$$

Proof. First we claim the following:

Claim 4.4.1. *There is a positive integer m_0 such that $\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}) \neq 0$ for all $m \geq m_0$.*

Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X such that

$$\hat{h}^0(H^0(X, A), \|\cdot\|_{\text{sup}}^{\bar{A}}) \neq 0 \quad \text{and} \quad \hat{h}^0(H^0(X, L + A), \|\cdot\|_{\text{sup}}^{\bar{L}+\bar{A}}) \neq 0.$$

Since \bar{L} is big, we can find a positive integer a with $\hat{h}^0(H^0(X, aL - A), \|\cdot\|_{\text{sup}}^{a\bar{L}-\bar{A}}) \neq 0$ (cf. [14, Proposition 2.2]). Note that

$$aL = (aL - A) + A \quad \text{and} \quad (a+1)L = (aL - A) + (L + A).$$

Thus

$$\hat{h}^0(H^0(X, aL), \|\cdot\|_{\text{sup}}^{a\bar{L}}) \neq 0 \quad \text{and} \quad \hat{h}^0(H^0(X, (a+1)L), \|\cdot\|_{\text{sup}}^{(a+1)\bar{L}}) \neq 0.$$

Let m be an integer with $m \geq a^2 + a$. We set $m = aq + r$, where $0 \leq r < a$. Then $q \geq a$. Thus we can find $b > 0$ with $q = b + r$. Therefore $m\bar{L} = b(a\bar{L}) + r((a+1)\bar{L})$, which means that

$$\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}) \neq 0.$$

Next we claim the following:

Claim 4.4.2. *There is a positive integer n_0 such that*

$$\begin{aligned} & \hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L}+\bar{N}}) \\ & \leq \hat{h}^0(H^0(X, (p(n+n_0)+i)L + N), \|\cdot\|_{\text{sup}}^{(p(n+n_0)+i)\bar{L}+\bar{N}}) \\ & \leq \hat{h}^0(H^0(X, p(n+2n_0+1)L + N), \|\cdot\|_{\text{sup}}^{p(n+2n_0+1)\bar{L}+\bar{N}}) \end{aligned}$$

for all $n \geq 1$ and all $i = 0, \dots, p$.

We choose n_0 with $pn_0 \geq m_0$. For each $i = 0, \dots, p$, there is a non-zero section s_i of $H^0(X, (pn_0 + i)L)$ with $\|s_i\|_{\text{sup}} \leq 1$. Therefore we have injective homomorphisms

$$H^0(X, pnL + N) \xrightarrow{s_i} H^0(X, (p(n + n_0) + i)L + N) \xrightarrow{s_{p-i}} H^0(X, (p(n + 2n_0 + 1)L + N)).$$

Thus our the claim follows.

Let us go back to the proof of the lemma. By the above claim,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L} + \bar{N}})}{(pn)^d} \\ \leq \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, (p(n + n_0) + i)L), \|\cdot\|_{\text{sup}}^{(p(n+n_0)+i)\bar{L} + \bar{N}})}{(pn)^d} \\ \leq \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, p(n + 2n_0 + 1)L), \|\cdot\|_{\text{sup}}^{p(n+2n_0+1)\bar{L} + \bar{N}})}{(pn)^d}. \end{aligned}$$

Note that

$$\lim_{n \rightarrow \infty} \frac{(pn)^d}{(p(n + n_0) + i)^d} = \lim_{n \rightarrow \infty} \frac{(pn)^d}{(p(n + 2n_0 + 1))^d} = 1.$$

This shows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L} + \bar{N}})}{(pn)^d} \\ = \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, (pn + i)L + N), \|\cdot\|_{\text{sup}}^{(pn+i)\bar{L} + \bar{N}})}{(pn + i)^d} \end{aligned}$$

for all $i = 0, \dots, p - 1$. Hence

$$\limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L} + \bar{N}})}{(pn)^d} = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}})}{m^d}.$$

In the same way, we can see

$$\liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, pnL + N), \|\cdot\|_{\text{sup}}^{pn\bar{L} + \bar{N}})}{(pn)^d} = \liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}})}{m^d}.$$

□

The following theorem is a characterization of a big C^∞ -hermitian invertible sheaf. The similar property is observed in [16].

Theorem 4.5. *For a C^∞ -hermitian invertible sheaf \bar{L} on X , the following are equivalent:*

- (1) $\widehat{\text{vol}}(\bar{L}) > 0$.
- (2) \bar{L} is big.
- (3) $\liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}})}{m^d} > 0$.
- (4) $\liminf_{m \rightarrow \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \|s\|_{\text{sup}} < 1\}}{m^d} > 0$.

Proof. Obviously (3) \implies (1) and (4) \implies (1), so that it is sufficient to show that (1) \implies (2), (2) \implies (3) and (2) \implies (4).

(1) \implies (2): We assume that $\widehat{\text{vol}}(\bar{L}) > 0$. By (3) of Proposition 4.1, there is a positive integer m and a non-zero section s of $H^0(X, mL)$ with $\|s\|_{\text{sup}}^{m\bar{L}} < 1$. Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . By Theorem 4.3,

$$\limsup_{m \rightarrow \infty} \frac{\hat{h}^0\left(H^0(X, mL - A), \|\cdot\|_{\text{sup}}^{m\bar{L} - \bar{A}}\right)}{m^d} = \frac{\widehat{\text{vol}}(\bar{L})}{d!} > 0,$$

which implies that there is a positive integer n with $\hat{h}^0\left(H^0(X, nL - A), \|\cdot\|_{\text{sup}}^{n\bar{L} - \bar{A}}\right) \neq 0$. Hence $n\bar{L} \geq \bar{A}$. In particular, $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$.

(2) \implies (3): Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . Since \bar{L} is big, there is a positive integer p with $p\bar{L} \geq \bar{A}$. Therefore,

$$\liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(pnL), \|\cdot\|_{\text{sup}}^{pn\bar{L}})}{(pn)^d} \geq \frac{1}{p^d} \liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(nA), \|\cdot\|_{\text{sup}}^{n\bar{A}})}{n^d} > 0.$$

Hence, by Lemma 4.4,

$$\liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d} > 0.$$

(2) \implies (4): We choose a sufficiently small positive number λ such that $\bar{L}^{-\lambda}$ is big. Since (2) \implies (3), we have

$$\liminf_{m \rightarrow \infty} \frac{\log \#\{s \in H^0(X, mL) \mid \exp(m\lambda)\|s\|_{\text{sup}} \leq 1\}}{m^d} > 0,$$

which yields (4). \square

Remark 4.6. In the paper [16], Yuan uses the condition (4) of the above theorem as a definition of a big C^∞ -hermitian invertible sheaf. By the above theorem, Yuan's definition is equivalent to our bigness.

Proposition 4.7. $\widehat{\text{vol}}$ is homogeneous of degree d , that is, $\widehat{\text{vol}}(p\bar{L}) = p^d \widehat{\text{vol}}(\bar{L})$ for every non-negative integer p .

Proof. Since

$$\limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, npL), \|\cdot\|_{\text{sup}}^{np\bar{L}})}{(np)^d} \leq \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d},$$

we have $\widehat{\text{vol}}(p\bar{L}) \leq p^d \widehat{\text{vol}}(\bar{L})$. Thus, if $\widehat{\text{vol}}(\bar{L}) = 0$, then the assertion is obvious. Therefore we may assume that $\widehat{\text{vol}}(\bar{L}) > 0$, namely, by Theorem 4.5, \bar{L} is big. Hence, by Lemma 4.4,

$$\limsup_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, npL), \|\cdot\|_{\text{sup}}^{np\bar{L}})}{(np)^d} = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d},$$

which means that $\widehat{\text{vol}}(p\bar{L}) = p^d \widehat{\text{vol}}(\bar{L})$. \square

5. CONTINUITY OF THE VOLUME FUNCTION

Let X be a d -dimensional projective arithmetic variety and $\widehat{\text{Pic}}(X)$ the group of isomorphism classes of C^∞ -hermitian invertible sheaves on X . An element of $\widehat{\text{Pic}}(X) \otimes \mathbb{Q}$ is called a C^∞ -hermitian \mathbb{Q} -invertible sheaf on X . For $\bar{L} \in \widehat{\text{Pic}}(X)$, the image of \bar{L} via the canonical homomorphism $\widehat{\text{Pic}}(X) \rightarrow \widehat{\text{Pic}}(X) \otimes \mathbb{Q}$ is denoted by $[\bar{L}]$. Note that $[\bar{L}] = [(\mathcal{O}_X, |\cdot|_{\text{can}})]$ if and only if \bar{L} is a torsion in $\widehat{\text{Pic}}(X)$, that is, there is a positive integer n with $n\bar{L} = (\mathcal{O}_X, |\cdot|_{\text{can}})$. We say a C^∞ -hermitian \mathbb{Q} -invertible sheaf \bar{L} is represented by $\bar{M} \in \widehat{\text{Pic}}(X)$ if $[\bar{M}] = \bar{L}$. Moreover a C^∞ -hermitian \mathbb{Q} -invertible sheaf \bar{L} on X is said to be ample if there is a positive integer n such that $n\bar{L}$ is represented by an ample C^∞ -hermitian invertible sheaf on X . Similarly we say \bar{L} is nef (resp. big) if $n\bar{L}$ is represented by a nef (resp. big) C^∞ -hermitian invertible sheaf for some positive integer n . Let us begin with the following lemma.

Lemma 5.1. $\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \rightarrow \mathbb{R}$ extends to a homogeneous map

$$\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \otimes \mathbb{Q} \rightarrow \mathbb{R}$$

of degree d , that is, $\widehat{\text{vol}}(a\bar{L}) = a^d \widehat{\text{vol}}(\bar{L})$ for every non-negative rational number a .

Proof. Let \bar{L} be a C^∞ -hermitian \mathbb{Q} -invertible sheaf on X . Let n be a positive integer such that $n\bar{L}$ is represented by a C^∞ -hermitian invertible sheaf \bar{M} . Then we would like to define $\widehat{\text{vol}}(\bar{L})$ to be $\widehat{\text{vol}}(\bar{M})/n^d$. Indeed this is well-defined. Let n' be another positive integer such that $n'\bar{L}$ is represented by a C^∞ -hermitian invertible sheaf \bar{M}' . Then, since $[n'\bar{M}] = [n\bar{M}']$, there is a positive integer m with $mn'\bar{M} = mn\bar{M}'$. On the other hand,

$$\widehat{\text{vol}}(mn'\bar{M}) = (mn')^d \widehat{\text{vol}}(\bar{M}) \quad \text{and} \quad \widehat{\text{vol}}(mn\bar{M}') = (mn)^d \widehat{\text{vol}}(\bar{M}')$$

Thus $\widehat{\text{vol}}(\bar{M})/n^d = \widehat{\text{vol}}(\bar{M}')/n'^d$.

Next let us see that $\widehat{\text{vol}}(a\bar{L}) = a^d \widehat{\text{vol}}(\bar{L})$ for every non-negative rational number a . Let n and m be positive integers such that $ma \in \mathbb{Z}$ and $n\bar{L}$ is represented by $\bar{M} \in \widehat{\text{Pic}}(X)$. Then, since $(mn)a\bar{L}$ is represented by $(ma)\bar{M}$,

$$\widehat{\text{vol}}(a\bar{L}) = \widehat{\text{vol}}((ma)\bar{M})/(mn)^d = a^d \widehat{\text{vol}}(\bar{M})/n^d = a^d \widehat{\text{vol}}(\bar{L}).$$

□

In Conventions and terminology 8, we define the order \leq on the group $\widehat{\text{Pic}}(X)$. We would like to extend it to $\widehat{\text{Pic}}(X) \otimes \mathbb{Q}$. For $\bar{L}, \bar{M} \in \widehat{\text{Pic}}(X) \otimes \mathbb{Q}$, if there is a positive integer n such that $n\bar{L}$ and $n\bar{M}$ are represented by a C^∞ -hermitian invertible sheaf \bar{L}' and \bar{M}' respectively with $\bar{L}' \leq \bar{M}'$, then we denote this by $\bar{L} \leq_{\mathbb{Q}} \bar{M}$.

Lemma 5.2. For $\bar{L}, \bar{L}', \bar{M}, \bar{M}' \in \widehat{\text{Pic}}(X) \otimes \mathbb{Q}$, we have the following:

- (1) $\bar{L} \leq_{\mathbb{Q}} \bar{M}$ if and only if $-\bar{M} \leq_{\mathbb{Q}} -\bar{L}$.
- (2) If $\bar{L} \leq_{\mathbb{Q}} \bar{M}$ and $\bar{L}' \leq_{\mathbb{Q}} \bar{M}'$, then $\bar{L} + \bar{L}' \leq_{\mathbb{Q}} \bar{M} + \bar{M}'$.
- (3) If $\bar{L} \leq_{\mathbb{Q}} \bar{M}$ and a is a non-negative rational number, then $a\bar{L} \leq_{\mathbb{Q}} a\bar{M}$.
- (4) If $\bar{L} \leq_{\mathbb{Q}} \bar{M}$, then $\widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{M})$.

Proof. (1), (2) and (3) are consequence of the properties in Conventions and terminology 8. Let us consider (4). Let n be a positive integer such that $n\bar{L}$ and $n\bar{M}$ are represented by C^∞ -hermitian invertible sheaves \bar{L}' and \bar{M}' with $\bar{L}' \leq \bar{M}'$. Then $\widehat{\text{vol}}(\bar{L}') \leq \widehat{\text{vol}}(\bar{M}')$ by (1) of Proposition 4.1. Hence we have (4). □

Remark 5.3. For reader's convenience, let us give a sketch of the proof of the properties (1) and (2) in Conventions and terminology 8. Let (V, σ) and (W, τ) be normed \mathbb{C} -vector spaces of dimension one. We denote $(V, \sigma) \leq (W, \tau)$ if there is an isomorphism $\phi : V \rightarrow W$ over \mathbb{C} such that $\tau(\phi(x)) \leq \sigma(x)$ for all $x \in V$. Then, in order to see the properties (1) and (2), it is sufficient to show the following:

- (a) $(V, \sigma) \leq (W, \tau)$ if and only if $(W^\vee, \tau^\vee) \leq (V^\vee, \sigma^\vee)$.
- (b) If $(V, \sigma) \leq (W, \tau)$ and $(V', \sigma') \leq (W', \tau')$, then

$$(V \otimes V', \sigma \otimes \sigma') \leq (W \otimes W', \tau \otimes \tau').$$

(a) Let $\phi : V \rightarrow W$ be an isomorphism over \mathbb{C} , v a basis of V and $w = \phi(v)$. Let v^\vee and w^\vee be the dual bases of v and w respectively. Since $\sigma(v/\sigma(v)) = 1$,

$$\sigma^\vee(v^\vee) = \max\{|v^\vee(x)| \mid \sigma(x) = 1\} = 1/\sigma(v).$$

In the same way, $\tau^\vee(w^\vee) = 1/\tau(w)$. Note that $\phi^\vee(w^\vee) = v^\vee$. Thus (a) follows.

(b) Let $\phi : V \rightarrow W$ and $\phi' : V' \rightarrow W'$ be isomorphisms over \mathbb{C} such that $\tau(\phi(x)) \leq \sigma(x)$ and $\tau'(\phi'(x')) \leq \sigma'(x')$ for all $x \in V$ and $x' \in V'$. Then

$$(\tau \otimes \tau')((\phi \otimes \phi')(x \otimes x')) = \tau(\phi(x))\tau'(\phi'(x')) \leq \sigma(x)\sigma'(x') = (\sigma \otimes \sigma')(x \otimes x').$$

Therefore $(V \otimes V', \sigma \otimes \sigma') \leq (W \otimes W', \tau \otimes \tau')$.

The following theorem is the main result of this paper.

Theorem 5.4 (Continuity of volume). *Let \bar{L} and \bar{A} be C^∞ -hermitian \mathbb{Q} -invertible sheaves on X . Then*

$$\lim_{\substack{\epsilon \in \mathbb{Q} \\ \epsilon \rightarrow 0}} \widehat{\text{vol}}(\bar{L} + \epsilon \bar{A}) = \widehat{\text{vol}}(\bar{L}).$$

More generally, for C^∞ -hermitian \mathbb{Q} -invertible sheaves $\bar{A}_1, \dots, \bar{A}_n$ on X ,

$$\lim_{\substack{\epsilon_1, \dots, \epsilon_n \in \mathbb{Q} \\ \epsilon_1 \rightarrow 0, \dots, \epsilon_n \rightarrow 0}} \widehat{\text{vol}}(\bar{L} + \epsilon_1 \bar{A}_1 + \dots + \epsilon_n \bar{A}_n) = \widehat{\text{vol}}(\bar{L}).$$

Proof. First let us consider the case $n = 1$. Let $\mu : X' \rightarrow X$ be a generic resolution of singularities of X . Then, by Theorem 4.2, $\widehat{\text{vol}}(\bar{L} + \epsilon \bar{A}) = \widehat{\text{vol}}(\mu^*(\bar{L}) + \epsilon \mu^*(\bar{A}))$ and $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{vol}}(\mu^*(\bar{L}))$. Thus we may assume that X is generically smooth.

Claim 5.4.1. *We may further assume that \bar{A} is ample.*

Let \bar{B} be an ample C^∞ -hermitian \mathbb{Q} -invertible sheaf on X such that $\bar{A} + \bar{B}$ is ample. Then, for $\epsilon \geq 0$,

$$\bar{L} - \epsilon(\bar{A} + \bar{B}) \leq_{\mathbb{Q}} \bar{L} - \epsilon \bar{A} \leq_{\mathbb{Q}} \bar{L} + \epsilon \bar{B} \quad \text{and} \quad \bar{L} - \epsilon \bar{B} \leq_{\mathbb{Q}} \bar{L} + \epsilon \bar{A} \leq_{\mathbb{Q}} \bar{L} + \epsilon(\bar{A} + \bar{B}).$$

Thus, by (4) of Lemma 5.2,

$$\begin{cases} \widehat{\text{vol}}(\bar{L} - \epsilon(\bar{A} + \bar{B})) \leq \widehat{\text{vol}}(\bar{L} - \epsilon \bar{A}) \leq \widehat{\text{vol}}(\bar{L} + \epsilon \bar{B}), \\ \widehat{\text{vol}}(\bar{L} - \epsilon \bar{B}) \leq \widehat{\text{vol}}(\bar{L} + \epsilon \bar{A}) \leq \widehat{\text{vol}}(\bar{L} + \epsilon(\bar{A} + \bar{B})). \end{cases}$$

Hence the claim follows.

From now on, we assume that \bar{A} is ample. It is obvious that

$$\lim_{\substack{\epsilon \in \mathbb{Q} \\ \epsilon \rightarrow 0}} \widehat{\text{vol}}(\bar{L} + \epsilon \bar{A}) = \widehat{\text{vol}}(\bar{L}) \quad \iff \quad \lim_{\substack{\epsilon \in \mathbb{Q} \\ \epsilon \rightarrow 0}} \widehat{\text{vol}}(\bar{L} + \epsilon a \bar{A}) = \widehat{\text{vol}}(\bar{L})$$

for any positive rational number a . Moreover,

$$\widehat{\text{vol}}(n\bar{L} + \epsilon\bar{A}) = n^d \text{vol}(\bar{L} + (\epsilon/n)\bar{A}) \quad \text{and} \quad \widehat{\text{vol}}(n\bar{L}) = n^d \text{vol}(\bar{L}).$$

Therefore, we may assume that \bar{L} is C^∞ -hermitian invertible sheaf. Further, by Lemma 3.6, we may assume that \bar{A} is a C^∞ -hermitian invertible sheaf and that \bar{A} satisfies the assumptions (i), (ii) and (iii) of Theorem 3.4.

Since

$$\widehat{\text{vol}}(\bar{L} - \epsilon'\bar{A}) \leq \widehat{\text{vol}}(\bar{L} - \epsilon\bar{A}) \leq \widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{L} + \epsilon\bar{A}) \leq \widehat{\text{vol}}(\bar{L} + \epsilon'\bar{A})$$

for $0 \leq \epsilon \leq \epsilon'$, it is sufficient to show that

$$\widehat{\text{vol}}(\bar{L}) = \lim_{p \rightarrow \infty} \widehat{\text{vol}}(\bar{L} + (1/p)\bar{A}) = \lim_{p \rightarrow \infty} \widehat{\text{vol}}(\bar{L} - (1/p)\bar{A}).$$

By Theorem 3.4 (or Proposition 3.5 for $d = 1$), there are positive constants a'_0 , C' and D' depending only on X , \bar{L} and \bar{A} such that

$$\begin{aligned} \hat{h}^0 \left(H^0(aL + (b-c)A), \|\cdot\|_{\text{sup}}^{a\bar{L} + (b-c)\bar{A}} \right) &\leq \hat{h}^0 \left(H^0(aL - cA), \|\cdot\|_{\text{sup}}^{a\bar{L} - c\bar{A}} \right) \\ &\quad + C'ba^{d-1} + D'a^{d-1} \log(a) \end{aligned}$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a'_0$.

First we set $a = pm$, $b = m$ and $c = 0$ for a fixed positive integer p . Then

$$\begin{aligned} \hat{h}^0 \left(H^0(pmL + mA), \|\cdot\|_{\text{sup}}^{pm\bar{L} + m\bar{A}} \right) &\leq \hat{h}^0 \left(H^0(pmL), \|\cdot\|_{\text{sup}}^{pm\bar{L}} \right) \\ &\quad + C'p^{d-1}m^d + D'p^{d-1}m^{d-1} \log(pm) \end{aligned}$$

for $m \gg 1$. This implies that

$$\widehat{\text{vol}}(p\bar{L} + \bar{A}) \leq \widehat{\text{vol}}(p\bar{L}) + C'p^{d-1}$$

for all $p \geq 1$, which means that

$$\widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{L} + (1/p)\bar{A}) \leq \widehat{\text{vol}}(\bar{L}) + C'(1/p).$$

Hence

$$\lim_{p \rightarrow \infty} \widehat{\text{vol}}(\bar{L} + (1/p)\bar{A}) = \widehat{\text{vol}}(\bar{L}).$$

Next we set $a = pm$ and $b = c = m$. Then

$$\begin{aligned} \hat{h}^0 \left(H^0(pmL), \|\cdot\|_{\text{sup}}^{pm\bar{L}} \right) &\leq \hat{h}^0 \left(H^0(pmL - mA), \|\cdot\|_{\text{sup}}^{pm\bar{L} - m\bar{A}} \right) \\ &\quad + C'p^{d-1}m^d + D'p^{d-1}m^{d-1} \log(pm) \end{aligned}$$

for $m \gg 1$. This implies that

$$\widehat{\text{vol}}(\bar{L} - (1/p)\bar{A}) \leq \widehat{\text{vol}}(\bar{L}) \leq \widehat{\text{vol}}(\bar{L} - (1/p)\bar{A}) + C'(1/p).$$

Thus

$$\lim_{p \rightarrow \infty} \widehat{\text{vol}}(\bar{L} - (1/p)\bar{A}) = \widehat{\text{vol}}(\bar{L}).$$

Let us consider a general case. We can find C^∞ -hermitian \mathbb{Q} -invertible sheaves \bar{A}'_i and \bar{A}''_i such that $0 \leq_{\mathbb{Q}} \bar{A}'_i$, $0 \leq_{\mathbb{Q}} \bar{A}''_i$ and $\bar{A}_i = \bar{A}'_i - \bar{A}''_i$ for each i . Then

$$\bar{L} + \epsilon_1 \bar{A}_1 + \cdots + \epsilon_n \bar{A}_n = \bar{L} + \epsilon_1 \bar{A}'_1 + \cdots + \epsilon_n \bar{A}'_n + (-\epsilon_1) \bar{A}''_1 + \cdots + (-\epsilon_n) \bar{A}''_n.$$

Thus we may assume that $0 \leq_{\mathbb{Q}} \bar{A}_1, \dots, 0 \leq_{\mathbb{Q}} \bar{A}_n$. Find an ample C^∞ -hermitian \mathbb{Q} -invertible sheaf \bar{B} such that $\bar{A}_i \leq_{\mathbb{Q}} \bar{B}$ for all $i = 1, \dots, n$. Then

$$-|\epsilon_i| \bar{B} \leq_{\mathbb{Q}} -|\epsilon_i| \bar{A}_i \leq_{\mathbb{Q}} \epsilon_i \bar{A}_i \leq_{\mathbb{Q}} |\epsilon_i| \bar{A}_i \leq_{\mathbb{Q}} |\epsilon_i| \bar{B}$$

for each i , which implies

$$\bar{L} - (|\epsilon_1| + \dots + |\epsilon_n|) \bar{B} \leq_{\mathbb{Q}} \bar{L} + \epsilon_1 \bar{A}_1 + \dots + \epsilon_n \bar{A}_n \leq_{\mathbb{Q}} \bar{L} + (|\epsilon_1| + \dots + |\epsilon_n|) \bar{B}.$$

Therefore

$$\begin{aligned} \widehat{\text{vol}}(\bar{L} - (|\epsilon_1| + \dots + |\epsilon_n|) \bar{B}) \\ \leq \widehat{\text{vol}}(\bar{L} + \epsilon_1 \bar{A}_1 + \dots + \epsilon_n \bar{A}_n) \\ \leq \widehat{\text{vol}}(\bar{L} + (|\epsilon_1| + \dots + |\epsilon_n|) \bar{B}). \end{aligned}$$

Thus the general assertion follows from the case $n = 1$. \square

As a corollary, we can show the following arithmetic Hilbert-Samuel theorem for a nef C^∞ -hermitian invertible sheaf.

Corollary 5.5 (Arithmetic Hilbert-Samuel formula). *Let \bar{L} and \bar{N} be C^∞ -hermitian invertible sheaves on X . If \bar{L} is nef, then*

$$\hat{h}^0 \left(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}} \right) = \frac{\widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d})}{d!} m^d + o(m^d) \quad (m \gg 1).$$

In particular, $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d})$, and \bar{L} is big if and only if $\widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d}) > 0$.

Proof. First let us see the following claim:

Claim 5.5.1. $\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d})$.

Let \bar{A} be an ample C^∞ -hermitian invertible sheaf on X . Then $\bar{L} + \epsilon \bar{A}$ is ample for all $\epsilon > 0$. Thus

$$\widehat{\text{vol}}(\bar{L} + \epsilon \bar{A}) = \widehat{\text{deg}} \left((\hat{c}_1(\bar{L}) + \epsilon \hat{c}_1(\bar{A}))^{\cdot d} \right).$$

Therefore our claim follows from the continuity of volumes.

Let us go back to the proof of the corollary. It is sufficient to show

$$\frac{\widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d})}{d!} = \lim_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\text{sup}}^{m\bar{L} + \bar{N}})}{m^d}.$$

If \bar{L} is not big, then, by Claim 5.5.1,

$$\widehat{\text{vol}}(\bar{L}) = \widehat{\text{deg}}(\hat{c}_1(\bar{L})^{\cdot d}) = 0.$$

Thus our assertion is obvious by Theorem 4.3, so that we may assume that \bar{L} is big. Then there is a positive integer k with $k\bar{L} \geq \bar{A}$. We set $\bar{E} = k\bar{L} - \bar{A}$. Since

$$p\bar{L} - \bar{E} = (p - k)\bar{L} + \bar{A},$$

$p\bar{L} - \bar{E}$ is ample if $p \geq k$. On the other hand, since $p\bar{L} \geq p\bar{L} - \bar{E}$, we have

$$\hat{h}^0(H^0(X, npL + N), \|\cdot\|_{\text{sup}}^{np\bar{L} + \bar{N}}) \geq \hat{h}^0(H^0(X, n(pL - E) + N), \|\cdot\|_{\text{sup}}^{n(p\bar{L} - \bar{E}) + \bar{N}})$$

for $n \geq 1$, which implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, npL + N), \|\cdot\|_{\sup}^{\overline{L} + \overline{N}})}{(np)^d} \\ \geq \liminf_{n \rightarrow \infty} \frac{\hat{h}^0(H^0(X, n(pL - E) + N), \|\cdot\|_{\sup}^{n(\overline{L} - \overline{E}) + \overline{N}})}{(np)^d}. \end{aligned}$$

Therefore, for a fixed p with $p \geq k$, by using Lemma 3.1 and Lemma 4.4,

$$\liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\sup}^{\overline{L} + \overline{N}})}{m^d} \geq \frac{\widehat{\deg}(\hat{c}_1(p\overline{L} - \overline{E}) \cdot d)}{p^d d!}.$$

Thus, taking $p \rightarrow \infty$,

$$\liminf_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\sup}^{\overline{L} + \overline{N}})}{m^d} \geq \frac{\widehat{\deg}(\hat{c}_1(\overline{L}) \cdot d)}{d!}.$$

On the other hand, by Theorem 4.3 and Claim 5.5.1,

$$\limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL + N), \|\cdot\|_{\sup}^{\overline{L} + \overline{N}})}{m^d} = \frac{\widehat{\deg}(\hat{c}_1(\overline{L}) \cdot d)}{d!},$$

which proves the corollary. \square

Finally let us consider the volume of the difference of nef C^∞ -hermitian \mathbb{Q} -invertible sheaves, which is essentially the main result of Yuan's paper [16].

Theorem 5.6. *Let \overline{L} and \overline{M} be nef C^∞ -hermitian \mathbb{Q} -invertible sheaves on X . Then*

$$\widehat{\text{vol}}(\overline{L} - \overline{M}) \geq \widehat{\deg}(\hat{c}_1(\overline{L}) \cdot d) - d \cdot \widehat{\deg}(\hat{c}_1(\overline{L}) \cdot (d-1) \cdot \hat{c}_1(\overline{M})).$$

Proof. First we assume that \overline{L} and \overline{M} are ample C^∞ -hermitian invertible sheaves on X . Then, by [16],

$$\begin{aligned} \widehat{\text{vol}}(\overline{L} - \overline{M}) &\geq \limsup_{m \rightarrow \infty} \frac{\chi\left(H^0(m(L - M)), \|\cdot\|_{\sup}^{m(\overline{L} - \overline{M})}\right)}{m^d/d!} \\ &\geq \widehat{\deg}(\hat{c}_1(\overline{L}) \cdot d) - d \cdot \widehat{\deg}(\hat{c}_1(\overline{L}) \cdot (d-1) \cdot \hat{c}_1(\overline{M})). \end{aligned}$$

Thus, using the homogeneity of $\widehat{\text{vol}}$, the inequality holds for ample C^∞ -hermitian \mathbb{Q} -invertible sheaves on X . Let \overline{A} be an ample C^∞ -hermitian invertible sheaf on X . Then, for a small positive number ϵ , $\overline{L} + \epsilon\overline{A}$ and $\overline{M} + \epsilon\overline{A}$ are ample. Thus,

$$\begin{aligned} \widehat{\text{vol}}(\overline{L} - \overline{M}) &= \widehat{\text{vol}}((\overline{L} + \epsilon\overline{A}) - (\overline{M} + \epsilon\overline{A})) \\ &\geq \widehat{\deg}((\hat{c}_1(\overline{L}) + \epsilon\hat{c}_1(\overline{A})) \cdot d) \\ &\quad - d \cdot \widehat{\deg}((\hat{c}_1(\overline{L}) + \epsilon\hat{c}_1(\overline{A})) \cdot (d-1) \cdot (\hat{c}_1(\overline{M}) + \epsilon\hat{c}_1(\overline{A}))). \end{aligned}$$

Therefore the theorem follows. \square

Remark 5.7 (Arithmetic analogue of Fujita's approximation theorem). It is very natural to ask the following arithmetic analogue of Fujita's approximation theorem: Let \overline{L} be a big C^∞ -hermitian \mathbb{Q} -invertible sheaf on X . For any positive number ϵ , do there exist a birational morphism $\mu : X' \rightarrow X$ and an ample C^∞ -hermitian \mathbb{Q} -invertible sheaf \overline{A} on X' such that $\overline{A} \leq_{\mathbb{Q}} \mu^*(\overline{L})$ and $\widehat{\text{vol}}(\overline{L}) \leq \widehat{\text{vol}}(\overline{A}) + \epsilon$?

6. GENERALIZED HODGE INDEX THEOREM

In this section, we consider a generalized Hodge index theorem as an application of the continuity of the volume function. First let us introduce a technical definition.

Let X be a projective arithmetic variety of dimension d . Let L be an invertible sheaf on X such that L is nef on the generic fiber $X_{\mathbb{Q}}$ of $X \rightarrow \text{Spec}(\mathbb{Z})$. We say L has *moderate growth of positive even cohomologies* if there are a generic resolution of singularities $\mu : Y \rightarrow X$ and an ample invertible sheaf A on Y such that, for any positive integer n , there is a positive integer m_0 such that

$$\log \#(H^{2i}(Y, m(n\mu^*(L) + A))) = o(m^d)$$

for all $m \geq m_0$ and for all $i > 0$. Here we consider examples of invertible sheaves with moderate growth of positive even cohomologies.

Example 6.1. (1) We assume that $d = 2$. Then L has obviously moderate growth of positive even cohomologies.

(2) If L is nef on each geometric fiber of $X \rightarrow \text{Spec}(\mathbb{Z})$, then L has moderate growth of positive even cohomologies. Indeed, let $\mu : Y \rightarrow X$ be a generic resolution of singularities and A an ample invertible sheaf on Y . Then, for all $n \geq 1$, $n\mu^*(L) + A$ is ample. Thus $H^i(Y, m(n\mu^*(L) + A)) = 0$ for $m \gg 1$ and $i > 0$.

(3) We assume that $d = 2$ and X is generically smooth. Let E be a rank r locally free sheaf on X . Let $\pi : P = \text{Proj}(\bigoplus_{m \geq 0} \text{Sym}^m(E)) \rightarrow X$ be the projective bundle of E and $\mathcal{O}_P(1)$ the tautological invertible sheaf of P . We set $L = r \cdot \mathcal{O}_P(1) - \pi^*(\det E)$. If E is semistable on the generic fiber $X_{\mathbb{Q}}$, then it is well-known that L is nef on the generic fiber $P_{\mathbb{Q}}$. Moreover L has moderate growth of positive even cohomologies. This fact can be checked as follows: Let B be an ample invertible sheaf on X such that $A = \mathcal{O}_P(1) + \pi^*(B)$ is ample. Then

$$\begin{aligned} H^i(P, m(nL + A)) &= H^i(P, \mathcal{O}_P(mnr + m) + \pi^*(mB - mn \det(E))) \\ &= H^i(X, \text{Sym}^{mnr+m}(E) \otimes (mB - mn \det(E))) \end{aligned}$$

because $R^j \pi_* \mathcal{O}_P(l) = 0$ for $l \geq 0$ and $j > 0$. In particular,

$$H^i(P, m(nL + A)) = 0$$

for $i \geq 2$.

The main result of this section is the following generalized Hodge index theorem.

Theorem 6.2 (Generalized Hodge index theorem). *Let X be a d -dimensional projective arithmetic variety and \bar{L} a C^∞ -hermitian invertible sheaf on X . We assume the following:*

- (1) $L_{\mathbb{Q}}$ is nef on $X_{\mathbb{Q}}$.
- (2) $c_1(\bar{L})$ is semipositive on $X(\mathbb{C})$.
- (3) L has moderate growth of positive even cohomologies.

Then we have an inequality $\widehat{\text{vol}}(\bar{L}) \geq \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)$.

Proof. First we assume that X is generically smooth. Moreover, instead of the properties (1), (2) and (3) as above, we assume the following (a), (b) and (c):

- (a) $L_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$.
- (b) $c_1(\bar{L})$ is positive on $X(\mathbb{C})$.

(c) There is a positive number m_0 such that

$$\log \#(H^{2i}(X, mL)) = o(m^d)$$

for $m \geq m_0$.

Then let us see the following:

Claim 6.2.1. $\widehat{\text{vol}}(\bar{L}) \geq \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)$.

By virtue of the arithmetic Riemann-Roch theorem [7] and the asymptotic estimate of analytic torsions [2], we obtain

$$\begin{aligned} \hat{\chi}(H^0(X, mL), \|\cdot\|_{\bar{L}_2}) \\ + \sum_{i \geq 1} \log \#(H^{2i}(X, mL)) - \sum_{i \geq 1} \log \#(H^{2i-1}(X, mL)) \\ = \frac{\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d) \end{aligned}$$

for $m \gg 1$. Thus, using the assumption (c) and (1) of Proposition 2.1,

$$\hat{h}^0(H^0(X, mL), \|\cdot\|_{\bar{L}_2}) \geq \frac{\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d)$$

for $m \gg 1$. By Gromov's inequality and (3) of Proposition 2.1, the above inequality implies

$$\hat{h}^0(H^0(X, mL), \|\cdot\|_{\bar{L}_{\text{sup}}}) \geq \frac{\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^d)}{d!} m^d + o(m^d)$$

for $m \gg 1$. Hence the claim follows.

Let us go back to a general case. Since L has moderate growth of positive even cohomologies, there is a generic resolution of singularities $\mu : Y \rightarrow X$ and an ample invertible sheaf A on Y such that, for any positive integer n , there is a positive integer m_0 such that $\log \#(H^{2i}(Y, m(n\mu^*(L) + A))) = o(m^d)$ for all $m \geq m_0$ and for all $i > 0$. Let us give a C^∞ -hermitian metric $|\cdot|_A$ to A such that $\bar{A} = (A, |\cdot|_A)$ is ample as a C^∞ -hermitian invertible sheaf. Then, by Claim 6.2.1,

$$\widehat{\text{vol}}(n\mu^*(\bar{L}) + \bar{A}) \geq \widehat{\text{deg}}(\widehat{c}_1(n\mu^*(\bar{L}) + \bar{A})^d),$$

which implies

$$\widehat{\text{vol}}(\mu^*(\bar{L}) + (1/n)\bar{A}) \geq \widehat{\text{deg}}((\widehat{c}_1(\mu^*(\bar{L})) + (1/n)\widehat{c}_1(\bar{A}))^d)$$

by Proposition 4.7. Hence, using the continuity of the volume function,

$$\widehat{\text{vol}}(\mu^*(\bar{L})) \geq \widehat{\text{deg}}(\widehat{c}_1(\mu^*(\bar{L}))^d).$$

This gives rise to our assertion by Theorem 4.2 and the projection formula. \square

According to (1), (2) and (3) of Example 6.1, we have the following corollaries.

Corollary 6.3. *Let X be a projective arithmetic surface and \bar{L} a C^∞ -hermitian invertible sheaf on X such that L is nef on the generic fiber of $X \rightarrow \text{Spec}(\mathbb{Z})$ and $c_1(\bar{L})$ is semipositive on $X(\mathbb{C})$. Then*

$$\widehat{\text{vol}}(\bar{L}) \geq \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^2).$$

Corollary 6.4. *Let X be a projective arithmetic variety of dimension d and \bar{L} a C^∞ -hermitian invertible sheaf on X such that L is nef on every geometric fiber of $X \rightarrow \text{Spec}(\mathbb{Z})$ and $c_1(\bar{L})$ is semipositive on $X(\mathbb{C})$. Then*

$$\widehat{\text{vol}}(\bar{L}) \geq \widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d}).$$

In particular, if $\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot d}) > 0$, then \bar{L} is big.

Corollary 6.5. *Let X be a projective and generically smooth arithmetic surface and \bar{E} a C^∞ -hermitian locally free sheaf on X . If the metric of \bar{E} is Einstein-Hermitian, then*

$$\widehat{\text{deg}}\left(\widehat{c}_2(\bar{E}) - \frac{r-1}{2r}\widehat{c}_1(\bar{E})^2\right) \geq 0,$$

where $r = \text{rk } E$.

Proof. Let $\pi : P = \text{Proj}(\bigoplus_{m \geq 0} \text{Sym}^m(E)) \rightarrow X$ be the projective bundle of E and $\mathcal{O}_P(1)$ the tautological invertible sheaf of P . Using the surjective homomorphism $\pi^*(E) \rightarrow \mathcal{O}_P(1)$ and the hermitian metric of E , we give the quotient metric $|\cdot|_P$ to $\mathcal{O}_P(1)$. We set $\bar{\mathcal{O}}_P(1) = (\mathcal{O}_P(1), |\cdot|_P)$ and $\bar{L} = r \cdot \bar{\mathcal{O}}_P(1) - \pi^*(\det \bar{E})$. Note that $L_{\mathbb{Q}}$ is nef and not big and that $c_1(\bar{L})$ is semipositive (cf. [12, Lemma 8.7.1]). Moreover L has moderate growth of positive even cohomologies. Thus, by Theorem 6.2,

$$\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot r+1}) \leq 0$$

because \bar{L} is not big. Note that

$$\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot r+1}) = r^{r+1} \cdot \widehat{\text{deg}}\left(\frac{r-1}{2r}\widehat{c}_1(\bar{E})^2 - \widehat{c}_2(\bar{E})\right)$$

(cf. [12, Section 8]). Thus

$$\widehat{\text{deg}}\left(\widehat{c}_2(\bar{E}) - \frac{r-1}{2r}\widehat{c}_1(\bar{E})^2\right) \geq 0.$$

□

Remark 6.6. (1) In Corollary 6.3, if $\widehat{\text{deg}}(L_{\mathbb{Q}}) = 0$, then $\widehat{\text{deg}}(\widehat{c}_1(\bar{L})^{\cdot 2}) \leq 0$. This is nothing more than the Hodge index theorem due to Faltings and Hriljac (cf. [5] and [10]). In this sense, we call Theorem 6.2 the generalized Hodge index theorem.

(2) The second assertion of Corollary 6.4 is a generalization of [17, Corollary (1.9)].

(3) Corollary 6.5 is valid even if $E_{\mathbb{Q}}$ is semistable on $X_{\mathbb{Q}}$. The case where the metric is Einstein-Hermitian is however essential and crucial for a general case. For details, see [12].

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