# FIXED-POINT PROPERTY FOR CAT(0) SPACES

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Abstract. We investigate the fixed-point property for the class of CAT(0) spaces with Izeki-Nayatani invariant $\delta \leq \delta_0$ for some $\delta_0 < 1/2$. It turns out that we can generalize many theorems which are known to hold for Property (T) to this much stronger fixed-point property.

## 1. Introduction

A finitely generated group $\Gamma$ is said to have the fixed-point property for $Y$ or have Property $FY$ if any isometric action of $\Gamma$ on $Y$ admits a fixed point. Similarly the fixed-point property for a class $\mathcal{Y}$ of metric spaces means that any isometric action on any $Y \in \mathcal{Y}$ has a fixed point, and we denote this property by $F\mathcal{Y}$.

It is well-known that Kazhdan's Property (T) for a discrete group $\Gamma$ is equivalent to the fixed-point property for Hilbert spaces ([3], [7]). The distribution of Property (T) in the space $\mathcal{G}_m$ of marked groups with $m$ generators was studied by Champetier [2] and Shalom [13].

In this paper, we consider the fixed-point property $F\mathcal{Y}_{\leq \delta_0}$ for the class $\mathcal{Y}_{\leq \delta_0}$ of CAT(0) spaces with the invariant $\delta$ less than or equal to $\delta_0$, where $\delta$ is Izeki-Nayatani invariant introduced in [9]. Since the class $\mathcal{Y}_{\leq \delta_0}$ contains Hilbert spaces and all Hadamard manifolds, Property $F\mathcal{Y}_{\leq \delta_0}$ is much stronger than Property (T).

The following is our main result.

**Theorem 1.1.** (1) *Let $\Gamma$ be a finitely generated group and $\Gamma'$ be any finite index subgroup of $\Gamma$, then $\Gamma$ has $F\mathcal{Y}_{\leq \delta_0}$ if and only if $\Gamma'$ has $F\mathcal{Y}_{\leq \delta_0}$.*

(2) *The product $\Gamma_1 \times \Gamma_2$ has $F\mathcal{Y}_{\leq \delta_0}$ if and only if both $\Gamma_1$ and $\Gamma_2$ have $F\mathcal{Y}_{\leq \delta_0}$.*

(3) *$F\mathcal{Y}_{\leq \delta_0}$ is an invariant of measure equivalence.*

(4) *The set of marked groups with $F\mathcal{Y}_{\leq \delta_0}$ is open in $\mathcal{G}_m$.*

(5) *There exists a dense $G_\delta$ subset $X$ of $\mathcal{H}_m$ such that each element in $X$ has $F\mathcal{Y}_{<1/2}$ and all homomorphism into $GL(n, \mathbf{C})$ has finite image.*

Here, $\mathcal{Y}_{<1/2}$ denotes the class of CAT(0) spaces with Izeki-Nayatani invariant less than $1/2$ and $\mathcal{H}_m$ denotes the closure in $\mathcal{G}_m$ of the set of marked groups which are torsion-free non-elementary hyperbolic.

However, there are few examples of CAT(0) spaces whose value of $\delta$ are known. Hence it is unclear how many CAT(0) spaces are or which building is in the class $\mathcal{Y}_{<1/2}$, and it is unknown whether there is a hyperbolic group or even finitely presented group with the fixed-point property $F\mathcal{Y}_{<1/2}$.

This theorem generalizes many results known for the case of Property (T). See [4], [13], [2] for the corresponding result for Property (T).

As a corollary of (4), any finitely generated group with the fixed-point property $F\mathcal{Y}_{\leq\delta_0}$ can be expressed as a quotient of a certain finitely presented group with Property $F\mathcal{Y}_{\leq\delta_0}$ (see also [5], p.117). The argument here is referred to Silberman's note [14].

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## 2. Izeki-Nayatani invariant

In this section, we give the definition of the invariant $\delta$ of CAT(0) spaces introduced by Izeki-Nayatani in [9].

**Definition 2.1** ([9]). *Let $Y$ be a* CAT(0) *space. Izeki-Nayatani invariant for $Y$ is defined as*

$$\delta(Y) = \sup_{\mu} \inf_{\phi} \frac{\| \int_Y \phi d\mu \|^2}{\int_Y \|\phi\|^2 d\mu}$$

*where $\mu$ runs over all finitely supported probability measures on $Y$, and $\phi$ runs over all 1-Lipschitz maps $\phi : \mathrm{Supp}(\mu) \to \mathcal{H}$ with $\|\phi(y)\| = \mathrm{dist}_Y(y, \mathrm{bar}(\mu))$ for any $y \in \mathrm{Supp}(\mu)$. Here $\mathcal{H}$ denotes any Hilbert space and $\mathrm{bar}(\mu)$ denotes the barycenter of $\mu$. We call a map $\phi$ satisfying the above condition a realization of $\mu$.*

$\delta \in [0, 1]$ follows immediately from the definition. The computation of $\delta$ is hard and except in the case where $\delta = 0$, only estimation for some specific spaces is known.

**Example 2.2.** (1) *Assume that $Y$ is a finite or infinite dimensional Hadamard manifold or an $\mathbf{R}$-tree. Then we have $\delta(Y) = 0$.*

(2) *Assume that $Y_p$ is the building $PSL(3, \mathbf{Q}_p)/PSL(3, \mathbf{Z}_p)$. Then*

$$\delta(Y_p) \geq \frac{(\sqrt{p} - 1)^2}{2(p - \sqrt{p} + 1)}.$$

*When $p = 2$, we have $\delta(Y_2) \leq 0.4122\ldots$.*

The following conjecture is due to Izeki and Nayatani.

**Conjecture 2.3.** *For any prime $p \in \mathbf{N}$,*

$$\delta(PSL(3, \mathbf{Q}_p)/PSL(3, \mathbf{Z}_p)) < 1/2.$$

The followings are shown in [9], [10].

**Proposition 2.4.** (1) *For any convex closed subspace $Y'$ of a* CAT(0) *space $Y$, we have $\delta(Y') \leq \delta(Y)$.*
(2) *For the product of two* CAT(0) *spaces $Y, Y'$, we have $\delta(Y \times Y') = \max\{\delta(Y), \delta(Y')\}$.*
(3) *Let $(Y_n, d_n)$ be a sequence of* CAT(0) *spaces, $\omega$ a non-principal ultrafilter on $\mathbf{N}$ and $(Y_\omega, d_\omega)$ the ultralimit $(Y_\omega, d_\omega) = \omega\text{-}\lim_n (Y_n, d_n)$. Then,*

$$\delta(Y_\omega) \leq \omega\text{-}\lim_n \delta(Y_n)$$

*holds* ([10, Proposition 3.2]).

This proposition shows that the class $\mathcal{Y}_{\leq \delta_0}$ is closed under the operation taking a direct product, a convex closed subspace and a ultralimit. Furthermore this class is closed under taking a mapping space from a finite measure space.

**Proposition 2.5.** *Let $Y$ be a* CAT(0) *space, $X$ a standard Borel space (i.e., a Polish space equipped with its associated $\sigma$-algebra of Borel subsets) with a probability measure. Then for the space $L^2(X, Y)$ of $L^2$-maps from $X$ to $Y$, we have $\delta(L^2(X, Y)) = \delta(Y)$.*

To show this proposition, we first show the next lemma.

**Lemma 2.6.** *Let $Y$ be a* CAT(0) *space and $Y' \subset Y$ be a dense subset. Then, we have $\delta(Y') = \delta(Y)$. Here, $\delta(Y')$ is defined similar to $\delta(Y)$.*

*Proof.* Obviously, $\delta(Y') \leq \delta(Y)$. Therefore we show the opposite inequality. Let

$$\mu = \sum_{k=1}^{n} t_k \delta_{y_k} \quad (\sum_{k=1}^{n} t_k = 1)$$

be a finitely supported probability measure on $Y$.

We take sequences $y_k^i \to y_k (i \to \infty)$ in $Y'$ and consider the sequence of probability measures $\mu_i = \sum_{k=1}^{n} t_k \delta_{y_k^i}$.

For a fixed probability measure $\mu$, we denote the functional

$$\phi \mapsto \frac{\| \int_Y \phi d\mu \|^2}{\int_Y \|\phi\|^2 d\mu}$$

by $F_\mu$. For each $\mu_i$, we take a realization $\phi_i \colon \mathrm{Supp}(\mu_i) \to \mathbf{R}^N$ which attains the infimum of the functional $F_{\mu_i}$. Then $\{\phi_i\}_{i=1}^{\infty}$ forms a precompact set and we can choose a subsequence which converge to $\phi_\infty$.

Then $\phi_\infty \colon \mathrm{Supp}(\mu) \to \mathbf{R}^N$ is also a 1-Lipschitz map and for any $y_k \in \mathrm{Supp}(\mu)$, $\|\phi_\infty(y_k)\| = \mathrm{dist}_Y(y_k, \mathrm{bar}(\mu))$ hold. Thus $\phi_\infty$ gives a realization of $\mu$.

Hence,

$$\begin{aligned} \delta(Y, \mu) &\leq F_\mu(\phi_\infty) \\ &= \lim_{i \to \infty} F_{\mu_i} \\ &\leq \delta(Y') \end{aligned}$$

for any $\mu$, which implies $\delta(Y) \leq \delta(Y')$. $\square$

*Proof of Proposition 2.5.* Any $L^2$-map can be approximated by simple functions, and each simple function is considered to be an element of a weighted product of finite copies of $Y$. Hence its $\delta$ is equal to $\delta(Y)$. Therefore the dense subset of $L^2(X,Y)$ satisfies $\delta = \delta(Y)$ and by the above lemma, $\delta(L^2(X,Y)) = \delta(Y)$. $\square$

**Remark 2.7.** *Assume that $X$ is a standard Borel space with Borel measure with a measure preserving $\Gamma$-action, $Y$ has an isometric $\Gamma$-action, and $X$ admits a finite measure fundamental domain. Then similarly we can show that the space $L^2_\Gamma(X,Y)$ of $\Gamma$-equivariant $L^2$-maps from $X$ to $Y$ satisfies $\delta(L^2_\Gamma(X,Y)) \leq \delta(Y)$.*

## 3. Hereditary properties

Throughout this section, $\Gamma$ denotes a finitely generated group and $\Gamma'$ denotes any finite index subgroup of $\Gamma$.

**Proposition 3.1.** *Let $Y$ be any* CAT(0) *space. If $\Gamma'$ has Property $FY$, then $\Gamma$ also has Property $FY$. Thus for any* CAT(0) *space $Y$, Property $FY$ is inherited from finite index subgroup to the ambient group.*

*Proof.* Fix an action $\rho\colon \Gamma \to \mathrm{Isom}(Y)$. If we consider the restriction of this action to $\Gamma'$, there exists a fixed point since $\Gamma'$ has Property $FY$. Hence let $p \in Y$ be a fixed point of $\Gamma'$-action. Then, the $\Gamma$-orbit of $p$ $\{\rho(\gamma)p|\gamma \in \Gamma\}$ is a finite set, thus its barycenter is a $\Gamma$-fixed point. $\square$

**Proposition 3.2.** *If $\Gamma$ has Property $F\mathcal{Y}_{\leq\delta_0}$, then $\Gamma'$ also has Property $F\mathcal{Y}_{\leq\delta_0}$.*

*Proof.* Assume that $Y$ is a CAT(0) space in $\mathcal{Y}_{\leq\delta_0}$, and $\rho\colon \Gamma' \to \mathrm{Isom}(Y)$ is any isometric action of $\Gamma'$ on $Y$. Let $L^2_{\Gamma'}(\Gamma,Y)$ denote the set of $\Gamma'$-equivariant maps from $\Gamma$ to $Y$. Then, there is a natural $\Gamma$-action on $L^2_{\Gamma'}(\Gamma,Y)$ given by $(\rho(\gamma)f)(x) := f(x\gamma)$.

Since we have $\delta(Y) \geq \delta(L^2_{\Gamma'}(\Gamma,Y))$, the action of $\Gamma$ on $L^2_{\Gamma'}(\Gamma,Y)$ admits a fixed point. Let $f$ be the fixed point. Then $f$ is a $\Gamma'$-equivariant constant map. Thus $f(e) \in Y$ is a fixed point of $\Gamma'$. $\square$

**Remark 3.3.** *Proposition 3.2 also follows from Theorem 4.2.*

**Remark 3.4.** *More generally, if a class $\mathcal{Y}$ of metric spaces satisfies the condition that $L^2_{\Gamma'}(\Gamma,Y) \in \mathcal{Y}$ whenever $Y \in \mathcal{Y}$, then Property $F\mathcal{Y}$ of $\Gamma$ is inherited to $\Gamma'$. The class $\mathcal{Y}$ need not consist of* CAT(0) *spaces.*

**Proposition 3.5.** *Let $N \hookrightarrow \Gamma \to \Gamma/N$ be a short exact sequence. Here $\Gamma$ is a finitely generated group and $N$ is a normal subgroup of $\Gamma$. If $N$ and $\Gamma/N$ have Property $F\mathcal{Y}_{\leq\delta_0}$, then $\Gamma$ has Property $F\mathcal{Y}_{\leq\delta_0}$.*

*Proof.* Assume that $Y$ is in $\mathcal{Y}_{\leq\delta_0}$. Give any isometric action $\rho\colon \Gamma \to \mathrm{Isom}(Y)$ of $\Gamma$ on $Y$. First consider the restriction of $\Gamma$-action to $N$. Then there are fixed points since $N$ has a fixed-point property for $\mathcal{Y}_{\leq\delta_0}$. Let $Y^N$ denote the fixed point set. Then the subspace $Y^N$ is also a CAT(0) space and satisfies $\delta(Y^N) \leq \delta(Y)$ since $Y^N$ is a convex closed set of $Y$.

Next, consider $\gamma y$ for $y \in Y^N$ and $\gamma \in \Gamma$. Then $N\gamma y = \gamma N y = \gamma y$ since $N$ is a normal subgroup. Hence $Y^N$ is closed under the action of $\Gamma$.

Therefore if we consider the action of $\Gamma$ on $Y^N$, this action is trivial on $N$. Thus we can regard it as $\Gamma/N$-action. Then from the fixed-point property of $\Gamma/N$ there is a fixed point $p \in Y^N$ and $p$ is a $\Gamma$-fixed point. $\square$

**Remark 3.6.** *In particular, Property $F\mathcal{Y}_{\leq\delta_0}$ is inherited to the direct product.*

**Remark 3.7.** *Let $G$ be a group and $H$ be a subgroup of $G$. Then relative Property (T) for the pair $(G, H)$ is equivalent to the existence of an $H$-fixed point for any isometric action of $G$ on a Hilbert space.*

*Similarly, we can define the relative fixed-point property for the pair $(G, H)$ on $\mathcal{Y}$ or relative Property $F\mathcal{Y}$ as the existence of an $H$-fixed point for any isometric action of $G$ on $\mathcal{Y}$. Then from the above proof, the following conditions are equivalent.*

(1) $(\Gamma, N)$ *has relative Property $F\mathcal{Y}_{\leq\delta_0}$ and $\Gamma/N$ has Property $F\mathcal{Y}_{\leq\delta_0}$.*
(2) $\Gamma$ *has Property $F\mathcal{Y}_{\leq\delta_0}$.*

## 4. Invariance under measure equivalence

Furman showed in [4] that Property (T) is an invariant of measure equivalence.

In this section, we prove that the fixed-point property $F\mathcal{Y}_{\leq\delta_0}$ for the family of CAT(0) spaces $Y$ satisfying $\delta(Y) \leq \delta_0$ is also an invariant of measure equivalence, which extends Furman's theorem.

Measure equivalence is defined as follows.

**Definition 4.1** (ME)**.** *Two countable groups $\Gamma$ and $\Lambda$ are called measure equivalent (ME) when there exist an infinite Lebesgue measure space $(\Omega, m)$ and commutative measure-preserving free actions of $\Gamma$ and $\Lambda$ each of which admits a finite measure fundamental domain. The space $(\Omega, m)$ is called a ME coupling of $\Gamma$ with $\Lambda$.*

Then, the following holds.

**Theorem 4.2.** *The fixed-point property $F\mathcal{Y}_{\leq\delta_0}$ is a ME invariant. Namely, if two countable groups $\Gamma$ and $\Lambda$ are ME and $\Lambda$ has Property $F\mathcal{Y}_{\leq\delta_0}$, then $\Gamma$ also has Property $F\mathcal{Y}_{\leq\delta_0}$.*

To prove this theorem, we introduce the notion of an ergodic ME coupling.

**Definition 4.3** (Ergodicity of ME coupling)**.** *The ME coupling $(\Omega, m)$ of $\Gamma$ with $\Lambda$ is called ergodic if the $\Gamma$-action on $\Lambda\backslash\Omega$ and the $\Lambda$-action on $\Gamma\backslash\Omega$ are both ergodic.*

*Proof.* Let $Y$ be a CAT(0) space with $\delta(Y) \le \delta_0$, and let a $\Gamma$-action on $Y$ be given. Let $(\Omega, m)$ be an ergodic ME coupling of $\Gamma$ with $\Lambda$. The existence of an ergodic ME coupling is guaranteed by Lemma 2.3 of [4].

We first induce an isometric $\Lambda$-action on the space $L^2_\Gamma(\Omega, Y)$ of all $\Gamma$-equivariant $L^2$-maps. Let $F$ be a $\Gamma$-fundamental domain of $(\Omega, m)$ and $\beta \colon \Lambda \times F \to \Gamma$ be an associated cocycle : i.e. $\beta(\lambda, \omega)$ is defined to be a unique element in $\Gamma$ which satisfies $\beta(\lambda, \omega)\lambda\omega \in F$.

We generalize the domain of the cocycle to $\Lambda \times \Omega$ by defining $\beta(\lambda, \gamma\omega_0) = \gamma\beta(\lambda, \omega_0)\gamma^{-1}$ for any $\gamma \in \Gamma$ and $\omega_0 \in \Omega$.

Then, the induced $\Lambda$-action on $L^2_\Gamma(\Omega, Y)$ is defined as $(\lambda \cdot f)(\omega) = \beta(\lambda^{-1}, \omega)f(\lambda^{-1}\omega)$. The map $\lambda \cdot f$ is also $\Gamma$-equivariant because

$$\begin{aligned}
(\lambda \cdot f)(\gamma\omega) =& \beta(\lambda^{-1}, \gamma\omega) f(\lambda^{-1}\gamma\omega) \\
=& \gamma\beta(\lambda, \omega_0)\gamma^{-1} \cdot \gamma f(\lambda^{-1}\omega) \\
=& \gamma\beta(\lambda, \omega_0) f(\lambda^{-1}\omega) \\
=& \gamma((\lambda \cdot f)(\omega)).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_F \mathrm{dist}_Y(y_0, (\lambda \cdot f)(\omega))^2 dm(\omega) \\
=&\int_F \mathrm{dist}_Y(y_0, \beta(\lambda^{-1}, \omega) f(\lambda^{-1}\omega))^2 dm(\omega) \\
=&\int_F \mathrm{dist}_Y(y_0, f(\beta(\lambda^{-1}, \omega)\lambda^{-1}\omega))^2 dm(\omega) \\
=&\int_F \mathrm{dist}_Y(y_0, f(\omega))^2 dm(\omega) \\
<&\infty,
\end{aligned}$$

the $L^2$-condition is preserved under this action. This is in fact a left $\Lambda$-action on $L^2_\Gamma(\Omega, Y)$ because

$$\begin{aligned}
(\lambda_2 \cdot (\lambda_1 \cdot f))(\omega) &= \beta(\lambda_2^{-1}, \omega)(\lambda_1 \cdot f)(\lambda_2^{-1}\omega) \\
&= \beta(\lambda_2^{-1}, \omega)\beta(\lambda_1^{-1}, \lambda_2^{-1}\omega) f(\lambda_1^{-1}\lambda_2^{-1}\omega) \\
&= \beta(\lambda_2^{-1}, \omega)\beta(\lambda_2^{-1}, \omega)^{-1}\beta(\lambda_1^{-1}, \beta(\lambda_2^{-1}, \omega)\lambda_2^{-1}\omega)\beta(\lambda_2^{-1}, \omega) f(\lambda_1^{-1}\lambda_2^{-1}\omega) \\
&= \beta(\lambda_1^{-1}\lambda_2^{-1}, \omega) f(\lambda_1^{-1}\lambda_2^{-1}\omega) \\
&= ((\lambda_2\lambda_1) \cdot f)(\omega).
\end{aligned}$$

Here we used the cocycle condition

$$\beta(\lambda_2\lambda_1, \omega) = \beta(\lambda_2, \beta(\lambda_1, \omega)\lambda_1\omega)\beta(\lambda_1, \omega).$$

Furthermore, this action is isometric because

$$\begin{aligned}\mathrm{dist}_{L^2_\Gamma}(\lambda\cdot f,\lambda\cdot g)^2 &= \int_F \mathrm{dist}_Y((\lambda\cdot f)(\omega),(\lambda\cdot g)(\omega))^2 dm(\omega)\\ &= \int_F \mathrm{dist}_Y(\beta(\lambda^{-1},\omega)f(\lambda^{-1}\omega),\beta(\lambda^{-1},\omega)g(\lambda^{-1}\omega))^2 dm(\omega)\\ &= \int_F \mathrm{dist}_Y(f(\beta(\lambda^{-1},\omega)\lambda^{-1}\omega),g(\beta(\lambda^{-1},\omega)\lambda^{-1}\omega))^2 dm(\omega)\\ &= \int_F \mathrm{dist}_Y(f(\omega),g(\omega))^2 dm(\omega)\\ &= \mathrm{dist}_{L^2_\Gamma}(f,g)^2.\end{aligned}$$

This induced action of $\Lambda$ on $L^2_\Gamma(\Omega,Y)$ admits a fixed point since $\delta(L^2_\Gamma(\Omega,Y))\le\delta(Y)\le\delta_0$. Namely, there exists a map $f\in L^2_\Gamma(\Omega,Y)$ such that $f(\omega)=f(\beta(\lambda^{-1},\omega)\lambda^{-1}\omega)$ for any $\lambda\in\Lambda$. Since the $\Lambda$-action on $\Gamma\backslash\Omega$ is ergodic, $f$ is a constant map. Thus $f$ is a $\Gamma$-equivariant constant map and this means that there is a $\Gamma$-fixed point in $Y$. □

**Remark 4.4.** *The proof above gives an alternative proof of Furman's theorem* [4] *since* $L^2_\Gamma(\Omega,\mathcal{H})$ *is a Hilbert space for any Hilbert space* $\mathcal{H}$.

## 5. Distribution in $\mathcal{G}_m$

In this section, we introduce the space $\mathcal{G}_m$ of marked groups and show that the set of marked groups with certain strong fixed-point property is generic in $\mathcal{H}_m$, which is the closure in $\mathcal{G}_m$ of torsion-free non-elementary hyperbolic groups.

Let $m$ be a positive integer and $\mathbb{F}_m$ the free group on $m$ elements. A marked group is a pair $(\Gamma,S)$, where $\Gamma$ is a finitely generated group and $S$ is a generating subset of $\Gamma$ consisting of $m$ elements. We fix an order of $S$. Two marked groups $(\Gamma_1,S_1)$, $(\Gamma_2,S_2)$ are defined to be isomorphic if the order preserving bijection from $S_1$ to $S_2$ extends to an isomorphism from $\Gamma_1$ to $\Gamma_2$. The space $\mathcal{G}_m$ consists of all isomorphism class of marked groups.

Then $\mathcal{G}_m$ can be naturally identified with the set of normal subgroups in the free group $\mathbb{F}_m$.

Now we endow $\mathcal{G}_m$ with a metric. For two normal subgroups $N_1,N_2$ of $\mathbb{F}_m$, put

$$v(N_1,N_2)=\sup\{R\in\mathbf{N}\,|\,N_1\cap B_{\mathbb{F}_m}(R)=N_2\cap B_{\mathbb{F}_m}(R)\},$$

and define a metric on $\mathcal{G}_m$ as

$$\mathrm{dist}(N_1,N_2)=\exp(-v(N_1,N_2)).$$

Here $B_{\mathbb{F}_m}(R)$ denotes the ball of radius $R$ in the Cayley graph of $\mathbb{F}_m$. With this metric, $\mathcal{G}_m$ becomes a compact metric space. Furthermore $\mathcal{G}_m$ is totally disconnected because this metric satisfies the ultra-metric inequality

$$\mathrm{dist}(N_1, N_3) \le \max\{\mathrm{dist}(N_1, N_2), \mathrm{dist}(N_2, N_3)\}.$$

**Lemma 5.1** ([2]). *Let $(\Gamma, S)$ be a marked group with $\Gamma$ finitely presented. Then, there exists a neighborhood $U$ of $(\Gamma, S)$ in $\mathcal{G}_m$ consisting of quotients of $\Gamma$.*

There is a natural equivalence relation $\mathcal{R}$ on $\mathcal{G}_m$ given by an abstract isomorphism of groups.

Any group generated by $m$ elements can be viewed as a group generated by $m+1$ elements by adding trivial element as a last generator. Hence we have a canonical embedding of $\mathcal{G}_m$ into $\mathcal{G}_{m+1}$. Using this embedding, $\mathcal{G}_m$ can be viewed as a subset of $\mathcal{G}_{2m}$ where $Aut(\mathbb{F}_{2m})$ act as homeomorphism.

**Proposition 5.2** ([2]). *The equivalence class of $\mathcal{R}$ in $\mathcal{G}_m$ is the restriction of orbits of the action of $Aut(\mathbb{F}_{2m}) \subset Homeo(\mathcal{G}_{2m})$ on $\mathcal{G}_m$. Thus if $U$ is an open subset of $\mathcal{G}_m$, then the set of all the orbits which intersect $U$ is also open.*

**Theorem 5.3.** *Let $m \ge 2$ be an integer and $\mathcal{H}_m$ denotes the closure in $\mathcal{G}_m$ of the set of marked groups which are torsion-free non-elementary hyperbolic. Then, $\mathcal{H}_m$ contains a dense $G_\delta$ (i.e. countable intersection of open) set $X$ of marked groups $(\Gamma, S)$ with the following properties.*

1. *For any element $(\Gamma, S)$ of $X$, the equivalence class of $(\Gamma, S)$ with respect to $\mathcal{R}$ is dense in $\mathcal{H}_m$.*
2. *Any element $(\Gamma, S)$ has Property $F\mathcal{Y}_{\delta<1/2}$.*
3. *Any homomorphism from any element $(\Gamma, S)$ into $GL(n, \mathbf{C})$ has finite image.*

To prove this theorem, we prepare the following lemma.

**Lemma 5.4.** *For any $\delta_0 < 1/2$, there exists a torsion-free non-elementary hyperbolic group $\Gamma$ with the fixed-point property for $\mathcal{Y}_{\le\delta_0}$.*

*Proof.* By [10], random groups in the Żuk's model [16] with density grater than 1/3 are non-elementary hyperbolic with fixed-point property for $\mathcal{Y}_{\le\delta_0}$. Furthermore, as Ollivier showed in [12], there is no spherical diagram for random groups. Thus random groups are torsion-free. □

*Proof of Theorem 5.3.* We first note that it suffices to prove each class contains a dense $G_\delta$ set since the intersection of finitely many dense $G_\delta$ sets is also a dense $G_\delta$ set.

(1) is due to [2].

(2) Consider the group $\Gamma_1$ given by the above lemma. Since hyperbolic groups are finitely presented, there is a neighborhood $U$ of $(\Gamma_1, S)$ consisting of quotients of $\Gamma_1$. Hence the set of marked groups with $F\mathcal{Y}_{\le\delta_0}$ contains a certain open set. By (1), such a set contains a dense open subset of $\mathcal{H}_m$.

Let $\{\delta_{i_k}\}_{k=1}^{\infty}$ be an increasing sequence with $\lim_{k\to\infty}\delta_{i_k}=1/2$. Taking an intersection of the subset with $F\mathcal{Y}_{\leq\delta_{i_k}}$ shows that the set of marked groups with $F\mathcal{Y}_{<1/2}$ contains a dense $G_\delta$ subset of $\mathcal{H}_m$.

(3) Take a torsion-free uniform lattice $\Gamma_2$ of the isometry group of the quoternionic hyperbolic space, then by the non-Archimedian superrigidity by Gromov-Schoen [6], any homomorphism from $\Gamma_2$ to $PSL(n,\mathbf{Q}_p)$ has bounded image. Since this property is inherited to its quotient, by using the former part of the argument for (2) again, we have a dense open set $X_1$ of marked groups with this property.

On the other hand, we have already seen that the set of marked groups with $F\mathcal{Y}_{\leq\delta_0}$ for some $\delta_0<1/2$ contains a dense open subset $X_2$ of $\mathcal{H}_m$. Because the class $\mathcal{Y}_{\leq\delta_0}$ contains all Hadamard manifolds, any homomorphism from an element of $X_2$ to $PSL(n,\mathbf{R})$ has bounded image.

Then taking intersection of these two classes $X_1$ and $X_2$, we have a dense $G_\delta$ set of marked groups such that any homomorphism from the group in this set into $GL(n,\mathbf{C})$ has finite image. □

Since the above set $X$ contains uncountably many elements and each equivalence class of $\mathcal{R}$ contains at most countably many elements, we have the following

**Corollary 5.5.** *There exist uncountably many finitely generated groups up to isomorphism with the property (2) and (3) in Theorem 5.3.*

But at this time, it is unclear if the above set $X$ contains a finitely presented group. This suggests the following

**Problem 5.6.** *Is there a hyperbolic group with $F\mathcal{Y}_{\delta<1/2}$ ?*

**Remark 5.7.** *If we start with M. Kapovich's example* [11] *of nonlinear hyperbolic group which is torsion-free in the proof of Theorem 5.3 (3), the statement can be strengthened to the following (3'): Any homomorphism into $GL(n,F)$ has finite image for any field $F$.*

## 6. UNIFORM ACTION

In this section, we generalize a theorem of Shalom, which says that the subset of marked groups having Property (T) is open in $\mathcal{G}_m$, to a general setting. As a corollary, we prove that any finitely generated group with $F\mathcal{Y}_{\leq\delta_0}$ can be expressed as a quotient of a finitely presented group with $F\mathcal{Y}_{\leq\delta_0}$. It should be mentioned that this theorem is essentially expressed in Gromov's paper ([5], p.117) and we refer to Silberman's note [14].

**Definition 6.1** (uniformity constant)**.** *For an action of a marked group* $(\Gamma,S)$ *on* $Y$*, a uniformity constant* $\epsilon(y)$ *at* $y\in Y$ *is defined as* $\epsilon(y)=\max_{\gamma\in S}\mathrm{dist}(\gamma y,y)$.

**Definition 6.2.** *An isometric action of* $\Gamma$ *on* $Y$ *is called uniform if for any* $y\in Y$*, we have* $\epsilon(y)\geq\epsilon>0$*. Namely,* $\inf_{y\in Y}\epsilon(y)>0$ *holds.*

Note that an isometric $\Gamma$-action on $Y$ admits a fixed point if and only if $\epsilon(y)=0$ for some $y\in Y$.

**Theorem 6.3.** *Let $\mathcal{Y}$ be a class of complete metric spaces closed under scaling and taking a ultralimit. The subset of marked groups having $F\mathcal{Y}$ is open in $\mathcal{G}_m$.*

To show this theorem, we first prove the next lemma.

**Lemma 6.4.** *Assume that the action of $\Gamma$ on $Y$ does not admit any fixed point. Let $a$ be a positive number. Then, for any $y \in Y$, there exists $y' \in B(y, a\epsilon(y))$ such that $\epsilon(y'') \geq \epsilon(y')/2$ holds for any $y'' \in B(y', \frac{a}{2}\epsilon(y'))$.*

*Proof.* If there is no such $y'$ for $y \in Y$, then for any $y'_i \in B(y, a\epsilon(y))$, there exist $y'_{i+1} \in B(y'_i, \frac{a}{2}\epsilon(y'_i))$ with $\epsilon(y'_{i+1}) < \frac{1}{2}\epsilon(y'_i)$.

If we put $y'_1 = y$, then $\epsilon(y'_{i+1}) < \frac{1}{2^i}\epsilon(y)$. Hence,

$$d(y'_1, y'_{i+1}) < a\Sigma_{j=1}^{i}\frac{\epsilon(y)}{2^j}.$$

Therefore,

$$\begin{aligned} d(y'_1, y'_{i+1}) &\leq d(y'_1, y'_2) + \cdots + d(y'_i, y'_{i+1}) \\ &\leq \frac{a}{2}\epsilon(y'_1) + \cdots + \frac{a}{2}\epsilon(y'_i) \\ &\leq a(\frac{\epsilon(y'_1)}{2} + \frac{\epsilon(y'_1)}{2^2} + \cdots + \frac{\epsilon(y'_1)}{2^i}) \\ &< a\epsilon(y'_1). \end{aligned}$$

Hence $y'_{i+1} \in B(y, a\epsilon(y))$. Thus we can continue the above construction. Since

$$d(y'_i, y'_{i+1}) < \frac{a}{2^i}\epsilon(y),$$

$\{y'_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since $\epsilon$ is a continuous function, it converges to a fixed point, which is a contradiction. $\square$

*Proof of Theorem 6.3.* Take a sequence of groups $\Gamma_i$ each of which admits an action with no global fixed point on a space in $\mathcal{Y}$. Assume that they converge to a group $\Gamma$ in $\mathcal{G}_m$. Then, it suffices to prove that $\Gamma$ also has an action with no global fixed point on a space in $\mathcal{Y}$.

Let $N_i$ (resp. $N$) be the normal subgroups of $\mathbb{F}_m$ corresponding to $\Gamma_i$ (resp. $\Gamma$). Assume that each $\Gamma_i$ act on $Y_i \in \mathcal{Y}$ without a global fixed point.

We take a sequence $a_i \to \infty$. From the above lemma, for each $y_i \in Y_i$, there exists $x_i \in B(y_i, a_i\epsilon(y_i))$ such that $\epsilon(z) \geq \frac{1}{2}\epsilon(x_i)$ for each $z \in B(x_i, \frac{a_i}{2}\epsilon(x_i))$. Let $\omega$ be an any non-principal ultrafilter, and put $Y_\infty = \lim_\omega(Y_i, \frac{2}{\epsilon(x_i)}\mathrm{dist}_{Y_i}, x_i)$. Then, $Y_\infty$ still belongs to $\mathcal{Y}$.

By lifting the action of $\Gamma_i$ on $Y_i$ to $\overline{\varphi}_i : \mathbb{F}_m \to \mathrm{Isom}(Y_i)$, we have the action $\prod_{i\in\mathbb{N}}\overline{\varphi}_i$ of $\mathbb{F}_m$ on $\prod_{i\in\mathbb{N}} Y_i$. This induces the isometric action $\overline{\varphi}_\infty : \mathbb{F}_m \to \mathrm{Isom}(Y_\infty)$. Then, $\overline{\varphi}_\infty : \mathbb{F}_m \to \mathrm{Isom}(Y_\infty)$ sends $N$ to the identity, since each $\overline{\varphi}_i$ sends $N_i$ to the identity of $\mathrm{Isom}(Y_i)$. Thus we have $\varphi_\infty : \mathbb{F}_m/N \to \mathrm{Isom}(Y_\infty)$. This $\Gamma$-action on $Y_\infty$ is uniform because the uniformity constant of this action satisfies

$$\begin{aligned}\epsilon_{\varphi_\infty}(y) &= \max_{\gamma\in S} \mathrm{dist}_{Y_\infty}(\varphi_\infty(\gamma)y, y)\\ &= \max_{\gamma\in S} \mathrm{dist}_{Y_\infty}(\overline{\varphi}_\infty(\gamma)y, y)\\ &= \epsilon_{\overline{\varphi}_\infty}(y)\\ &\geq 1.\end{aligned}$$

In particular, the limit group $\Gamma$ also has an action with no global fixed point on a space in $\mathcal{Y}$. Thus the set of marked groups with fixed-point property $F\mathcal{Y}$ is an open set. □

**Corollary 6.5** ([5], p.117)**.** *Assume that $\Gamma$ is a finitely generated group with Property $F\mathcal{Y}$. Then there exist a finitely presented group $\Gamma_0$ with Property $F\mathcal{Y}$ and a surjective homomorphism $\Gamma_0 \to \Gamma$.*

*Proof.* Fix a presentation $\langle s_1, \dots, s_m | R_1, R_2, \dots\rangle$ of $\Gamma$ and consider the finitely presented groups $\Gamma_i = \langle s_1, \dots, s_m | R_1, R_2, \dots, R_i\rangle$ for $i \geq 1$. Then the sequence $\{\Gamma_i\}_{i=1}^\infty$ converges to $\Gamma$ in $\mathcal{G}_m$. Since the set of marked groups with fixed-point property $F\mathcal{Y}$ is open, there exists some $i$ such that $\Gamma_i$ also has the fixed-point property $F\mathcal{Y}$. □

**Remark 6.6.** *The set of marked groups with Property $F\mathcal{Y}_{\leq\delta_0}$ is also an open set in $\mathcal{G}_m$ because the class $\mathcal{Y}_{\leq\delta_0}$ is closed under scaling and ultralimit. Therefore any finitely generated group with $F\mathcal{Y}_{\leq\delta_0}$ can be expressed as a quotient of a finitely presented group with $F\mathcal{Y}_{\leq\delta_0}$.*

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