SAMELSON PRODUCTS OF $SO(3)$ AND APPLICATIONS

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Abstract. Certain generalized Samelson products of $SO(3)$ are calculated and applications to the homotopy of gauge groups are given.

1. Introduction and statement of results

We work in the category of CW-complexes and continuous maps, and often do not distinguish maps from their homotopy classes.

Let $G$ be a topological group and let $\gamma : G \wedge G \to G$ denote the commutator of $G$. A generalized Samelson product of maps $\alpha : A \to G$ and $\beta : B \to G$ is defined as a homotopy class of the composition

$$A \wedge B \xrightarrow{\alpha \wedge \beta} G \wedge G \xrightarrow{\gamma} G$$

and denoted by $\langle \alpha, \beta \rangle$. We denote the adjoint map $\Sigma A \to BG$ of a map $\alpha : A \to G$ by $\text{ad}(\alpha)$. Regarding the generalized Samelson product $\langle \alpha, \beta \rangle$, Arkowitz [1] showed that $\text{ad}(\langle \alpha, \beta \rangle) = [\text{ad}(\alpha), \text{ad}(\beta)]$, where $[ , ]$ is the generalized Whitehead product.

The purpose of this paper is to calculate certain generalized Samelson products of $SO(3)$ and to give applications to the homotopy of gauge groups. Let $\epsilon_1$ and $\epsilon_3$ be generators of $\pi_1(SO(3)) \cong \mathbb{Z}/2$ and $\pi_3(SO(3)) \cong \mathbb{Z}$ respectively, and let $\bar{\epsilon}$ and $\iota$ be the natural inclusion $\mathbb{R}P^2 \hookrightarrow SO(3)$ ($= \mathbb{R}P^3$) and the identity of $SO(3)$ respectively. Then we will show :

Theorem 1.1. The order of the generalized Samelson product $\langle \epsilon_3, \bar{\epsilon} \rangle$ is 4.

Corollary 1.1. The order of the generalized Samelson product $\langle \epsilon_3, \iota \rangle$ is 12.

Let $G$ be a compact, connected Lie group and let $P$ be a principal $G$-bundle over $S^4$. The gauge group $\mathcal{G}_P$ of $P$ is a group of all $G$-equivariant automorphisms of $P$ covering the identity of $S^4$. Atiyah and Bott [3] showed that

$$B\mathcal{G}_P \simeq \text{Map}_P(S^4, BG),$$

where $\text{Map}_P(S^4, BG)$ denotes the component of $\text{Map}(S^4, BG)$ corresponding to the classifying map of $P$. Then we will often identify $B\mathcal{G}_P$ with $\text{Map}_P(S^4, BG)$. For simplicity, when $G = SO(3)$ and $P$ is classified by $k \in \mathbb{Z} \cong \pi_4(BSO(3))$, we write $\mathcal{G}_P$ by $\mathcal{G}_k$. Let $(n, m)$ be the GCD of $n$ and $m$. As applications of the above results, we will show :

Proposition 1.1. $\mathcal{G}_k \simeq \mathcal{G}_l$ if and only if $(12, k) = (12, l)$.

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Proposition 1.2.

\[ \pi_0(\mathcal{G}_k) \cong \begin{cases} \mathbb{Z}/2 & k \equiv 0 \ (2) \\ 0 & k \equiv 1 \ (2) \end{cases} \]

\[ \pi_1(\mathcal{G}_k) \cong \begin{cases} \mathbb{Z}/2 & k \equiv 1 \ (2) \\ \mathbb{Z}/4 & k \equiv 2 \ (4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & k \equiv 0 \ (4) \end{cases} \]

Remark 1.1. Readers may refer to [7] for the relevant results of the homotopy of \( \mathcal{G}_P \) when \( P \) is a principal \( SU(2) \)-bundle over \( S^4 \). Readers may also refer to [6] for an alternative calculation of \( \pi_0(\mathcal{G}_k) \) and \( \pi_1(\mathcal{G}_k) \) in a different context.

Remark 1.2. Regarding the homotopy of the classifying space \( B\mathcal{G}_k \), we have the following. Let \( P \) be a principal \( SU(2) \)-bundle over \( S^4 \) corresponding to \( k \in \mathbb{Z} \cong \pi_4(BSU(2)) \). Since the natural projection \( \mathcal{G}_P \rightarrow \mathcal{G}_k \) is a double covering, the universal covering group of the identity components of \( \mathcal{G}_P \) and \( \mathcal{G}_k \) are isomorphic. Then it follows from Theorem 1.5 of [9] that \( B\mathcal{G}_k \cong B\mathcal{G}_l \) if and only if \( k = \pm l \).

Remark 1.3. Let \( P \) be as in Remark 1.2. Then it is straightforward to check that \( \pi_2(\mathcal{G}_k) \cong \pi_2(\mathcal{G}_P) \). Hence, by a result of [7], one finds \( \pi_2(\mathcal{G}_k) \cong \mathbb{Z}/(12, k) \).

2. Proofs of Theorem 1.1 and Corollary 1.1

Before starting the proofs, let us recall a result of Bott [4]. Denote a generator of \( \pi_i(U(2)) \) by \( \tilde{e}_i \) for \( i = 1, 3 \). Then Bott [4] showed that the order of the Samelson product \( \langle \tilde{e}_3, \tilde{e}_1 \rangle \) is 2 and hence \( \langle \tilde{e}_3, \tilde{e}_1 \rangle \) is a generator of \( \pi_4(U(2)) \cong \mathbb{Z}/2 \).

Proof of Theorem 1.1. Let \( \pi : U(2) \rightarrow SO(3) \) be the natural projection. It is obvious that \( \pi_*(\tilde{e}_i) = \epsilon_i \) for \( i = 1, 3 \). Then one has

\[ \pi_*(\langle \tilde{e}_3, \tilde{e}_1 \rangle) = \langle \epsilon_3, \epsilon_1 \rangle \in \pi_4(SO(3)). \]

Since \( \pi_* : \pi_4(U(2)) \rightarrow \pi_4(SO(3)) \) is an isomorphism, the order of \( \langle \epsilon_3, \epsilon_1 \rangle \) is 2 and hence \( \langle \epsilon_3, \epsilon_1 \rangle \) is a generator of \( \pi_4(SO(3)) \cong \mathbb{Z}/2 \). Let \( i : S^1 \hookrightarrow \mathbb{R}P^2 \) be the inclusion of the 1-skeleton. Then \( i^*(\epsilon) = \epsilon_1 \) and, by the above observation, one can see that

\[ \langle \epsilon_3, \epsilon \rangle \notin 2([S^3 \wedge \mathbb{R}P^2, SO(3)]). \]

Since \( S^3 \wedge \mathbb{R}P^2 \) is 3-connected we have a group isomorphism

\[ [S^3 \wedge \mathbb{R}P^2, SO(3)] \cong [S^3 \wedge \mathbb{R}P^2, Sp(1)]. \]

By applying \( [S^3 \wedge \mathbb{R}P^2, \_] \) to the fiber sequence

\[ \Omega(Sp(\infty)/Sp(1)) \rightarrow Sp(1) \rightarrow Sp(\infty) \rightarrow Sp(\infty)/Sp(1), \]

we can derive an exact sequence

\[ [S^3 \wedge \mathbb{R}P^2, \Omega(Sp(\infty)/Sp(1))] \rightarrow [S^3 \wedge \mathbb{R}P^2, Sp(\infty)] \rightarrow [S^3 \wedge \mathbb{R}P^2, Sp(1)] \rightarrow [S^3 \wedge \mathbb{R}P^2, Sp(\infty)/Sp(1)] \]

Since \( S^3 \wedge \mathbb{R}P^2 \) is 5-dimensional and \( Sp(\infty)/Sp(1) \) is 6-connected, we obtain a group isomorphism

\[ [S^3 \wedge \mathbb{R}P^2, Sp(1)] \cong [S^3 \wedge \mathbb{R}P^2, Sp(\infty)]. \]

On the other hand, one has a sequence of group isomorphisms

\[ [S^3 \wedge \mathbb{R}P^2, Sp(\infty)] \cong [S^4 \wedge \mathbb{R}P^2, BSp(\infty)] \cong KO^0(\mathbb{R}P^2) \cong \mathbb{Z}/4, \]
where the second and the last isomorphisms are due to Bott periodicity and a result of Adams [2] respectively. Therefore we obtain

\[ [S^3 \wedge \mathbb{R}P^2, SO(3)] \cong \mathbb{Z}/4 \]

and, by (2.1), the proof is completed. □

**Proof of Corollary 1.1.** Note that the cofibration \( S^3 \wedge \mathbb{R}P^2 \xrightarrow{1 \wedge i} S^3 \wedge SO(3) \to S^6 \) splits as \( S^3 \wedge SO(3) \cong (S^3 \wedge \mathbb{R}P^2) \vee S^6 \). Then the Samelson product \( \langle \epsilon_3, \iota \rangle \) is factored as

\[ S^3 \wedge SO(3) \cong (S^3 \wedge \mathbb{R}P^2) \vee S^6 \xrightarrow{\langle \epsilon_3, \iota \rangle \vee \alpha} SO(3) \]

by a map \( \alpha : S^6 \to SO(3) \). One can see that \( \alpha = \pi_*(\langle \hat{\epsilon}_3, \hat{\iota} \rangle) \), when localized at any primes but 2. It is well-known that the Samelson product \( \langle \epsilon_3, \iota \rangle \) is a generator of \( \pi_6(U(2)) \cong \mathbb{Z}/12 \). Then we obtain that the order of \( \alpha \) is a divisor of 12 and it is divisible by 3, since \( \pi_6 : \pi_6(U(2)) \to \pi_6(SO(3)) \) is an isomorphism. Hence, by Theorem 1.1, the order of \( \langle \epsilon_3, \iota \rangle = \langle \epsilon_3, \hat{\iota} \rangle \vee \alpha \) is found to be 12. □

### 3. Proofs of Proposition 1.1 and Proposition 1.2

**Proof of Proposition 1.1.** The idea of the proof is due to [7]. Let \( e : B\mathbb{Z}_k \cong \text{Map}_k(S^1, BSO(3)) \to BSO(3) \) denote the evaluation at the basepoint of \( BSO(3) \). By the fibration

\[ \mathcal{G}_k \cong \Omega B\mathbb{Z}_k \xrightarrow{\Omega e} \Omega SO(3) \xrightarrow{\Gamma_k} \Omega SO_3(3), \]

\( \mathcal{G}_k \) can be considered as a homotopy fiber of the above map \( \Gamma_k \). Then we shall analyze the map \( \Gamma_k \).

By Lang [8], it is shown that a homotopy class of \( \Gamma_k \) is \( \text{ad}^3(\langle k\epsilon_3, \iota \rangle) \). Since Samelson products are bilinear, we have \( \Gamma_k \cong k\Gamma_1 \). By Corollary 1.1, the order of \( \Gamma_1 \) is 12. Since \( \pi_*(\Omega SO_3(3)) \) is finite for all \( * \), it follows from Lemma 3.2 of [5] that \( \mathcal{G}_k \cong \mathcal{G}_1 \) if and only if \( (12, k) = (12, l) \). Thus Proposition 1.1 is accomplished. □

**Proof of Proposition 1.2.** Consider the following homotopy sequence of the evaluation fibration \( \Omega SO_3(3) \to B\mathbb{Z}_k \xrightarrow{e} BSO(3) \). Then we have an exact sequence

\[ 0 = \pi_3(BSO(3)) \to \pi_2(\Omega SO_3(3)) \cong \mathbb{Z}/2 \to \pi_2(B\mathbb{Z}_k) \to \pi_2(BSO(3)) \cong \mathbb{Z}/2 \to \pi_2(B\mathbb{Z}_k) \to \pi_1(BSO(3)) = 0. \]

(3.1)

Since the order of the Samelson product \( \langle \epsilon_3, \epsilon_1 \rangle \) is 2, the order of its adjoint \( \text{ad}(\langle \epsilon_3, \epsilon_1 \rangle) = [\text{ad}(\epsilon_3), \text{ad}(\epsilon_1)] \) is 2 as well. Then the generalized Whitehead product \([k\text{ad}(\epsilon_3), \text{ad}(\epsilon_1)] = 0 \) if and only if \( k \equiv 0 \pmod{2} \). Hence, by the exponential law, there exists a map \( \alpha : S^2 \to B\mathbb{Z}_k \) satisfying the homotopy commutative diagram

\[ \xymatrix{ S^2 \ar[d]_{\text{ad}(\epsilon_1)} \ar[dr]^\alpha & \ar[l] \ar[d] \ar[r] \mathbb{Z}/2 \ar[r] & \pi_2(B\mathbb{Z}_k) \ar[l] \ar[d] \ar[r] & \pi_2(BSO(3)) = \mathbb{Z}/2 \ar[r] & 0. } \]

if and only if \( k \equiv 0 \pmod{2} \). Since \( \text{ad}(\epsilon_1) \) is the inclusion of the 2-skeleton of \( BSO(3) \), \( \pi_0(\mathcal{G}_k) \cong \pi_1(B\mathbb{Z}_k) \) is obtained as in the statement by the exact sequence (3.1).

By the above argument, we have obtained \( \pi_1(\mathcal{G}_k) \cong \pi_2(B\mathbb{Z}_k) \cong \mathbb{Z}/2 \) if \( k \equiv 1 \pmod{2} \). Then we shall consider the case that \( k \equiv 0 \pmod{2} \) and have an exact sequence

\[ 0 \to \mathbb{Z}/2 \to \pi_2(B\mathbb{Z}_k) \xrightarrow{\cong} \pi_2(BSO(3)) = \mathbb{Z}/2 \to 0. \]

(3.2)
By Theorem 1.1, we see that the order of $\text{ad}(\epsilon_3, \hat{\epsilon}) = [\text{ad}(\epsilon_3), \text{ad}(\hat{\epsilon})]$ is 4. Then the generalized Whitehead product $[k\text{ad}(\epsilon_3), \text{ad}(\epsilon_1)] = 0$ if and only if $k \equiv 0 \pmod{4}$. Hence there exists a map $\hat{\alpha} : \Sigma \mathbb{RP}^2 \to B\mathbb{G}_k$ satisfying the homotopy commutative diagram

$$\begin{array}{ccc}
\Sigma \mathbb{RP}^2 & \xrightarrow{\text{ad}(\hat{\epsilon})} & B\mathbb{G}_k \\
\downarrow & & \downarrow \\
B\mathbb{G}_k & \xrightarrow{e} & BSO(3)
\end{array}$$

if and only if $k \equiv 0 \pmod{4}$. Since $\text{ad}(\hat{\epsilon})$ is the inclusion of the 3-skeleton of $BSO(3)$, we obtain, by (3.2), that $\pi_1(\mathbb{G}_k) \cong \pi_2(B\mathbb{G}_k) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ when $k \equiv 0 \pmod{4}$. On the other hand, $\Sigma \mathbb{RP}^2$ is the Moore space $S^2 \cup_2 e^3$. Then one can see that if the order of each element of $\pi_2(B\mathbb{G}_k)$ is 2, then there exists the above map $\hat{\alpha}$. Hence, by (3.2), we obtain $\pi_1(\mathbb{G}_k) \cong \pi_2(B\mathbb{G}_k) \cong \mathbb{Z}/4$ when $k \equiv 2 \pmod{4}$ and this completes the proof. □

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