SAMELSON PRODUCTS OF $SO(3)$ AND APPLICATIONS

YASUHIKO KAMIYAMA, DAISUKE KISHIMOTO, AKIRA KONO, AND SHUICHI TSUKUDA

Abstract. Certain generalized Samelson products of $SO(3)$ are calculated and applications to the homotopy of gauge groups are given.

1. Introduction and statement of results

We work in the category of CW-complexes and continuous maps, and often do not distinguish maps from their homotopy classes.

Let G be a topological group and let $\gamma : G \wedge G \to G$ denote the commutator of G. A generalized Samelson product of maps $\alpha : A \to G$ and $\beta : B \to G$ is defined as a homotopy class of the composition

$$A \wedge B \xrightarrow{\alpha\wedge\beta} G \wedge G \xrightarrow{\gamma} G$$

and denoted by $\langle \alpha, \beta \rangle$. We denote the adjoint map $\Sigma A \to BG$ of a map $\alpha : A \to G$ by $\mathrm{ad}(\alpha)$. Regarding the generalized Samelson product $\langle \alpha, \beta \rangle$, Arkowitz [1] showed that

$$\mathrm{ad}(\langle \alpha, \beta \rangle) = [\mathrm{ad}(\alpha), \mathrm{ad}(\beta)],$$

where $[\ ,\ ]$ is the generalized Whitehead product.

The purpose of this paper is to calculate certain generalized Samelson products of $SO(3)$ and to give applications to the homotopy of gauge groups. Let ϵ_1 and ϵ_3 be generators of $\pi_1(SO(3)) \cong \mathbf{Z}/2$ and $\pi_3(SO(3)) \cong \mathbf{Z}$ respectively, and let $\hat{\epsilon}$ and ι be the natural inclusion $\mathbf{R}P^2 \hookrightarrow SO(3)$ $(= \mathbf{R}P^3)$ and the identity of $SO(3)$ respectively. Then we will show :

Theorem 1.1. *The order of the generalized Samelson product $\langle \epsilon_3, \hat{\epsilon} \rangle$ is 4.*

Corollary 1.1. *The order of the generalized Samelson product $\langle \epsilon_3, \iota \rangle$ is 12.*

Let G be a compact, connected Lie group and let P be a principal G-bundle over S^4. The gauge group $\mathscr{G}_P$ of P is a group of all G-equivariant automorphisms of P covering the identity of S^4. Atiyah and Bott [3] showed that

$$B\mathscr{G}_P \simeq \mathrm{Map}_P(S^4, BG),$$

where $\mathrm{Map}_P(S^4, BG)$ denotes the component of $\mathrm{Map}(S^4, BG)$ corresponding to the classifying map of P. Then we will often identify $B\mathscr{G}_P$ with $\mathrm{Map}_P(S^4, BG)$. For simplicity, when $G = SO(3)$ and P is classified by $k \in \mathbf{Z} \cong \pi_4(BSO(3))$, we write $\mathscr{G}_P$ by $\mathscr{G}_k$. Let (n, m) be the GCD of n and m. As applications of the above results, we will show :

Proposition 1.1. *$\mathscr{G}_k \simeq \mathscr{G}_l$ if and only if $(12, k) = (12, l)$.*

2000 *Mathematics Subject Classification.* Primary 55Q15, Secondary 55P15, 54C35.
Key words and phrases. Samelson product, Whitehead product, gauge group.
Supported by Grant-in-Aid for Scientific Research (B) 18340016.

Proposition 1.2.

$$\pi_0(\mathscr{G}_k) \cong \begin{cases} \mathbf{Z}/2 & k \equiv 0\ (2) \\ 0 & k \equiv 1\ (2) \end{cases} \quad \pi_1(\mathscr{G}_k) \cong \begin{cases} \mathbf{Z}/2 & k \equiv 1\ (2) \\ \mathbf{Z}/4 & k \equiv 2\ (4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & k \equiv 0\ (4) \end{cases}$$

Remark 1.1. Readers may refer to [7] for the relevant results of the homotopy of $\mathscr{G}_P$ when P is a principal $SU(2)$-bundle over S^4. Readers may also refer to [6] for an alternative calculation of $\pi_0(\mathscr{G}_k)$ and $\pi_1(\mathscr{G}_k)$ in a different context.

Remark 1.2. Regarding the homotopy of the classifying space $B\mathscr{G}_k$, we have the following. Let P be a principal $SU(2)$-bundle over S^4 corresponding to $k \in \mathbf{Z} \cong \pi_4(BSU(2))$. Since the natural projection $\mathscr{G}_P \to \mathscr{G}_k$ is a double covering, the universal covering group of the identity components of $\mathscr{G}_P$ and $\mathscr{G}_k$ are isomorphic. Then it follows from Theorem 1.5 of [9] that $B\mathscr{G}_k \simeq B\mathscr{G}_l$ if and only if $k = \pm l$.

Remark 1.3. Let P be as in Remark 1.2. Then it is straightforward to check that $\pi_2(\mathscr{G}_k) \cong \pi_2(\mathscr{G}_P)$. Hence, by a result of [7], one finds $\pi_2(\mathscr{G}_k) \cong \mathbf{Z}/(12, k)$.

2. Proofs of Theorem 1.1 and Corollary 1.1

Before starting the proofs, let us recall a result of Bott [4]. Denote a generator of $\pi_i(U(2))$ by $\tilde{\epsilon}_i$ for $i = 1, 3$. Then Bott [4] showed that the order of the Samelson product $\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle$ is 2 and hence $\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle$ is a generator of $\pi_4(U(2)) \cong \mathbf{Z}/2$.

Proof of Theorem 1.1. Let $\pi : U(2) \to SO(3)$ be the natural projection. It is obvious that $\pi_*(\tilde{\epsilon}_i) = \epsilon_i$ for $i = 1, 3$. Then one has

$$\pi_*(\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle) = \langle \epsilon_3, \epsilon_1 \rangle \in \pi_4(SO(3)).$$

Since $\pi_* : \pi_4(U(2)) \to \pi_4(SO(3))$ is an isomorphism, the order of $\langle \epsilon_3, \epsilon_1 \rangle$ is 2 and hence $\langle \epsilon_3, \epsilon_1 \rangle$ is a generator of $\pi_4(SO(3)) \cong \mathbf{Z}/2$. Let $i : S^1 \hookrightarrow \mathbf{R}P^2$ be the inclusion of the 1-skeleton. Then $i^*(\hat{\epsilon}) = \epsilon_1$ and, by the above observation, one can see that

$$\langle \epsilon_3, \hat{\epsilon} \rangle \notin 2([S^3 \wedge \mathbf{R}P^2, SO(3)]). \tag{2.1}$$

Since $S^3 \wedge \mathbf{R}P^2$ is 3-connected we have a group isomorphism

$$[S^3 \wedge \mathbf{R}P^2, SO(3)] \cong [S^3 \wedge \mathbf{R}P^2, Sp(1)].$$

By applying $[S^3 \wedge \mathbf{R}P^2,\ ]$ to the fiber sequence

$$\Omega(Sp(\infty)/Sp(1)) \to Sp(1) \to Sp(\infty) \to Sp(\infty)/Sp(1),$$

we can derive an exact sequence

$$\begin{aligned}[S^3 \wedge \mathbf{R}P^2, \Omega(Sp(\infty)/Sp(1))] &\to [S^3 \wedge \mathbf{R}P^2, Sp(\infty)] \\ &\to [S^3 \wedge \mathbf{R}P^2, Sp(1)] \to [S^3 \wedge \mathbf{R}P^2, Sp(\infty)/Sp(1)]\end{aligned}$$

Since $S^3 \wedge \mathbf{R}P^2$ is 5-dimensional and $Sp(\infty)/Sp(1)$ is 6-connected, we obtain a group isomorphism

$$[S^3 \wedge \mathbf{R}P^2, Sp(1)] \cong [S^3 \wedge \mathbf{R}P^2, Sp(\infty)].$$

On the other hand, one has a sequence of group isomorphisms

$$[S^3 \wedge \mathbf{R}P^2, Sp(\infty)] \cong [S^4 \wedge \mathbf{R}P^2, BSp(\infty)] \cong \widetilde{KO}^0(\mathbf{R}P^2) \cong \mathbf{Z}/4,$$

where the second and the last isomorphisms are due to Bott periodicity and a result of Adams [2] respectively. Therefore we obtain

$$[S^3 \wedge \mathbf{R}P^2, SO(3)] \cong \mathbf{Z}/4$$

and, by (2.1), the proof is completed. $\square$

Proof of Corollary 1.1. Note that the cofibration $S^3 \wedge \mathbf{R}P^2 \xrightarrow{1\wedge\hat{\epsilon}} S^3 \wedge SO(3) \to S^6$ splits as $S^3 \wedge SO(3) \simeq (S^3 \wedge \mathbf{R}P^2) \vee S^6$. Then the Samelson product $\langle \epsilon_3, \iota\rangle$ is factored as

$$S^3 \wedge SO(3) \simeq (S^3 \wedge \mathbf{R}P^2) \vee S^6 \xrightarrow{\langle\epsilon_3,\hat{\epsilon}\rangle\vee\alpha} SO(3)$$

by a map $\alpha : S^6 \to SO(3)$. One can see that $\alpha = \pi_*(\langle\tilde{\epsilon}_3, \tilde{\epsilon}_3\rangle)$, when localized at any primes but 2. It is well-known that the Samelson product $\langle\tilde{\epsilon}_3, \tilde{\epsilon}_3\rangle$ is a generator of $\pi_6(U(2)) \cong \mathbf{Z}/12$. Then we obtain that the order of α is a divisor of 12 and it is divisible by 3, since $\pi_* : \pi_6(U(2)) \to \pi_6(SO(3))$ is an isomorphism. Hence, by Theorem 1.1, the order of $\langle\epsilon_3, \iota\rangle = \langle\epsilon_3, \hat{\epsilon}\rangle \vee \alpha$ is found to be 12. $\square$

3. Proofs of Proposition 1.1 and Proposition 1.2

Proof of Proposition 1.1. The idea of the proof is due to [7]. Let $e : B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, BSO(3)) \to BSO(3)$ denote the evaluation at the basepoint of $BSO(3)$. By the fibration

$$\mathcal{G}_k \simeq \Omega B\mathcal{G}_k \xrightarrow{\Omega e} SO(3) \xrightarrow{\Gamma_k} \Omega_0^3 SO(3),$$

$\mathcal{G}_k$ can be considered as a homotopy fiber of the above map Γ_k. Then we shall analyze the map Γ_k.

By Lang [8], it is shown that a homotopy class of Γ_k is $\mathrm{ad}^3(\langle k\epsilon_3, \iota\rangle)$. Since Samelson products are bilinear, we have $\Gamma_k \simeq k\Gamma_1$. By Corollary 1.1, the order of Γ_1 is 12. Since $\pi_*(\Omega_0^3 SO(3))$ is finite for all $*$, it follows from Lemma 3.2 of [5] that $\mathcal{G}_k \simeq \mathcal{G}_l$ if and only if $(12, k) = (12, l)$. Thus Proposition 1.1 is accomplished. $\square$

Proof of Proposition 1.2. Consider the following homotopy sequence of the evaluation fibration $\Omega_0^3 SO(3) \to B\mathcal{G}_k \xrightarrow{e} BSO(3)$. Then we have an exact sequence

$$\begin{aligned}(3.1)\quad 0 = \pi_3(BSO(3)) \to \pi_2(\Omega_0^3 SO(3)) \cong \mathbf{Z}/2 \to \pi_2(B\mathcal{G}_k)\\ \xrightarrow{e_*} \pi_2(BSO(3)) \cong \mathbf{Z}/2 \to \pi_1(\Omega_0^3 SO(3)) \cong \mathbf{Z}/2 \to \pi_1(B\mathcal{G}_k) \to \pi_1(BSO(3)) = 0.\end{aligned}$$

Since the order of the Samelson product $\langle\epsilon_3, \epsilon_1\rangle$ is 2, the order of its adjoint $\mathrm{ad}(\langle\epsilon_3, \epsilon_1\rangle) = [\mathrm{ad}(\epsilon_3), \mathrm{ad}(\epsilon_1)]$ is 2 as well. Then the generalized Whitehead product $[k\mathrm{ad}(\epsilon_3), \mathrm{ad}(\epsilon_1)] = 0$ if and only if $k \equiv 0\ (2)$. Hence, by the exponential law, there exists a map $\alpha : S^2 \to B\mathcal{G}_k$ satisfying the homotopy commutative diagram

$$\begin{tikzcd} & S^2 \arrow[dl, "\alpha"'] \arrow[d, "\mathrm{ad}(\epsilon_1)"] \\ B\mathcal{G}_k \arrow[r, "e"] & BSO(3) \end{tikzcd}$$

if and only if $k \equiv 0\ (2)$. Since $\mathrm{ad}(\epsilon_1)$ is the inclusion of the 2-skeleton of $BSO(3)$, $\pi_0(\mathcal{G}_k) \cong \pi_1(B\mathcal{G}_k)$ is obtained as in the statement by the exact sequence (3.1).

By the above argument, we have obtained $\pi_1(\mathcal{G}_k) \cong \pi_2(B\mathcal{G}_k) \cong \mathbf{Z}/2$ if $k \equiv 1\ (2)$. Then we shall consider the case that $k \equiv 0\ (2)$ and have an exact sequence

$$(3.2)\qquad 0 \to \mathbf{Z}/2 \to \pi_2(B\mathcal{G}_k) \xrightarrow{e_*} \pi_2(BSO(3)) = \mathbf{Z}/2 \to 0.$$

By Theorem 1.1, we see that the order of $\mathrm{ad}(\langle \epsilon_3, \hat{\epsilon}\rangle) = [\mathrm{ad}(\epsilon_3), \mathrm{ad}(\hat{\epsilon})]$ is 4. Then the generalized Whitehead product $[k\mathrm{ad}(\epsilon_3), \mathrm{ad}(\epsilon_1)] = 0$ if and only if $k \equiv 0$ (4). Hence there exists a map $\hat{\alpha} : \Sigma \mathbf{R}P^2 \to B\mathscr{G}_k$ satisfying the homotopy commutative diagram

$$\begin{tikzcd} & \Sigma\mathbf{R}P^2 \arrow[dl, "\hat{\alpha}"'] \arrow[d, "\mathrm{ad}(\hat{\epsilon})"] \\ B\mathscr{G}_k \arrow[r, "e"] & BSO(3) \end{tikzcd}$$

if and only if $k \equiv 0$ (4). Since $\mathrm{ad}(\hat{\epsilon})$ is the inclusion of the 3-skeleton of $BSO(3)$, we obtain, by (3.2), that $\pi_1(\mathscr{G}_k) \cong \pi_2(B\mathscr{G}_k) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ when $k \equiv 0$ (4). On the other hand, $\Sigma\mathbf{R}P^2$ is the Moore space $S^2 \cup_2 e^3$. Then one can see that if the order of each element of $\pi_2(B\mathscr{G}_k)$ is 2, then there exists the above map $\hat{\alpha}$. Hence, by (3.2), we obtain $\pi_1(\mathscr{G}_k) \cong \pi_2(B\mathscr{G}_k) \cong \mathbf{Z}/4$ when $k \equiv 2$ (4) and this completes the proof. □

References

[1] M. Arkowitz, *The generalized Whitehead product*, Pacific J. Math. **12**, no. 1 (1962), 7-23.

[2] J.F. Adams, *Vector fields on spheres*, Ann. Math. **75** (1962), 603-632.

[3] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523-615.

[4] R. Bott, *A note on the Samelson product in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.

[5] H. Hamanaka and A. Kono, *Unstable K^1-group and homotopy type of certain gauge groups*, Proc. Royal Soc. Edinburgh **136** A (2006), 149-155.

[6] Y. Kamiyama and D. Kishimoto, *Spin structures on instanton moduli spaces*, preprint.

[7] A. Kono, *A note on the homotopy types of certain gauge groups*, Proc. Royal Soc. Edinburgh **117** A (1991), 295-297.

[8] G.E. Lang, *The evaluation map and EHP sequences*, Pacific J. Math. **44** (1973), 201-210.

[9] S. Tsukuda, *Comparing the homotopy types of the components of* $\mathrm{Map}(S^4, B\mathrm{SU}(2))$, J. Pure. Appl. Alg. **161** (2001), 235-243.

Department of Mathematics, University of the Ryukyus, Nishihara-Cho, Okinawa 903-0213, Japan

E-mail address: kamiyama@sci.u-ryukyu.ac.jp

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: kishi@math.kyoto-u.ac.jp

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: kono@math.kyoto-u.ac.jp

Department of Mathematics, University of the Ryukyus, Nishihara-Cho, Okinawa 903-0213, Japan

E-mail address: tsukuda@math.u-ryukyu.ac.jp