RIGIDITY OF LOG MORPHISMS

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Dedicated to Professor Hironaka on his 77th birthday

Introduction

In the paper [5], we proved Kato's conjecture, that is, the finiteness of dominant rational maps in the category of log schemes as a generalization of Kobayashi-Ochiai theorem [4]. It guarantees the finiteness of K-rational points of a certain kind of log smooth schemes for a big function field K, which gives rise to an evidence for Lang's conjecture. In the proof of the above theorem, the most essential part is the rigidity theorem of log morphisms. In this paper, we would like to generalize it to a semistable scheme over an arbitrary noetherian scheme.

Let $f : X \to S$ be a scheme of finite type over a locally noetherian scheme S. We assume that $f : X \to S$ is a semistable scheme over S, namely, f is flat and, for any morphism $\operatorname{Spec}(\Omega) \to S$ with Ω an algebraic closed field, the completion of the local ring of $X \times_S \operatorname{Spec}(\Omega)$ at every closed point is isomorphic to a ring of the type

$$\Omega[\![X_1, \ldots, X_n]\!]/(X_1 \cdots X_l).$$

Let $g : Y \to S$ be another semistable scheme over S, and let $\phi : X \to Y$ be a morphism over S. Let M_X, M_Y and M_S be fine log structures on X, Y and S respectively. We assume that (X, M_X) and (Y, M_Y) are log smooth and integral over (S, M_S) and ϕ is admissible with respect to M_Y/M_S, i.e., for all $s \in S$ and any irreducible components V of the geometric fiber $X \times_S \operatorname{Spec}(\overline{\kappa(s)})$ over s,

$$(\phi \times_S \mathrm{id}_{\operatorname{Spec}(\overline{\kappa(s)})})(V) \not\subseteq \operatorname{Supp}(M_Y/M_S)|_{Y \times_S \operatorname{Spec}(\overline{\kappa(s)})},$$

where

$$\operatorname{Supp}(M_Y/M_S) = \{y \in Y \mid M_{S,\overline{g(y)}} \times \mathcal{O}_{Y,\bar{y}}^{\times} \to M_{Y,\bar{y}} \text{ is not surjective}\}.$$

The following theorem is one of the main results of this paper.

Theorem A (Rigidity theorem). *If we have log morphisms*

$$(\phi, h) : (X, M_X) \to (Y, M_Y) \quad \text{and} \quad (\phi, h') : (X, M_X) \to (Y, M_Y)$$

over (S, M_S) as extensions of $\phi : X \to Y$, then $h = h'$.

For the proof of the above theorem, our starting point is the following local structure theorem, which asserts the local description of integral and smooth log morphisms of semistable schemes.

Date: 24/July/2006, 20:15(JP), (Version 1.52).

Theorem B (Local structure theorem)**.** *Let $(f,h) : (X, M_X) \to (S, M_S)$ be a smooth and integral morphism of fine log schemes. Let x be a point of X and $s = f(x)$. We assume that $f : X \to S$ is semistable at x, i.e., f is flat at x and, for any morphism $\eta : \operatorname{Spec}(\Omega) \to S$ with Ω an algebraic closed field and $\eta(0) = s$, the completion of the local ring of $X \times_S \operatorname{Spec}(\Omega)$ at every closed point lying over x is isomorphic to a ring of the type*

$$\Omega[\![X_1, \dots, X_n]\!]/(X_1 \cdots X_l).$$

Then we have the following for each case:

I. *If f is smooth at x, then $\overline{M}_{X,\bar{x}} \simeq \overline{M}_{S,\bar{s}} \times \mathbb{N}^a$ for some non-negative integer a.*

II. *If f is not smooth at x and $\bar{h}_{\bar{x}} : \overline{M}_{S,\bar{s}} \to \overline{M}_{X,\bar{x}}$ splits, then $\overline{M}_{X,\bar{x}} \simeq \overline{M}_{S,\bar{s}} \times N$, where N is the monoid arising from monomials of*

$$\mathbb{Z}[U_1, U_2, \dots, U_a]/(U_1^2 - U_2^2)$$

for some $a \geq 2$.

III. *If f is not smooth at x and $\bar{h}_{\bar{x}} : \overline{M}_{S,\bar{s}} \to \overline{M}_{X,\bar{x}}$ does not split, then there are integers $a \geq 2$ and $b \geq 0$, elements $q_0 \in \overline{M}_{S,\bar{s}} \setminus \{0\}$ and $B \in \mathbb{N}^b$, and homomorphisms $\alpha : \mathbb{N}^a \to \overline{M}_{X,\bar{x}}$ and $\beta : \mathbb{N}^b \to \overline{M}_{X,\bar{x}}$ with the following properties:*

III.1. *The diagram*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{(q_0,B)} & \overline{M}_{S,\bar{s}} \times \mathbb{N}^b \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle \bar{h}_{\bar{x}}+\beta} \\ \mathbb{N}^a & \xrightarrow{\alpha} & \overline{M}_{X,\bar{x}} \end{array}$$

is commutative, where Δ and (q_0, B) are homomorphisms given by $\Delta(n) = (n, \dots, n)$ and $(q_0, B)(n) = (nq_0, nB)$ respectively.

III.2. *The induced homomorphism*

$$\mathbb{N}^a \bar{\otimes}_{\mathbb{N}} \left(\overline{M}_{S,\bar{s}} \times \mathbb{N}^b\right) \to \overline{M}_{X,\bar{x}}$$

is an isomorphism (*For the definition of the integral tensor product $\bar{\otimes}_{\mathbb{N}}$, see Conventions and terminology* 7).

Based on the local structure theorem, the proof of the rigidity theorem is carried out as follows: Clearly we may assume that $S = \operatorname{Spec}(A)$ for some noetherian local ring (A, m). First we establish the theorem in the case where A is an algebraically closed field. This was proved actually in the previous paper [5]. Next, by induction on n, we see that the assertion holds for the case $S = \operatorname{Spec}(A/m^n)$. Finally, using the Krull intersection theorem, we can conclude its proof.

In §1, we give the definition of semistable schemes and show their elementary properties. In §2, we recall several facts concerning log schemes and prove the local structure theorem. §3 contains the proof of the rigidity theorem. Several applications of the rigidity theorem will be treated in the forthcoming paper [6].

Conventions and terminology. We will fix several conventions and terminology of this paper.

1. Throughout this paper, a ring means a commutative ring with the unity.

2. The set of all natural numbers starting from 0 is denoted by $\mathbb{N}$, that is,

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}.$$

3. In this paper, the logarithmic structures of schemes means the sense of J.-M Fontaine, L. Illusie, and K. Kato. For the details, we refer to [3]. For a log structure M_X on a scheme X, we denote the quotient $M_X/\mathcal{O}_X^\times$ by $\overline{M}_X$.

4. Let X be a scheme and F a sheaf in the étale topology. For a point $x \in X$, the germ of F at x with respect to the Zariski topology (resp. the étale topology) is denoted by F_x (resp. $F_{\bar{x}}$).

5. Let $\alpha : M_X \to \mathcal{O}_X$ be a log structure on a scheme X. For $x \in X$, an element $p \in \overline{M}_{X,\bar{x}}$ is said to be *regular* if there is $m \in M_{X,\bar{x}}$ such that p is congruent to m modulo $\mathcal{O}_{X,\bar{x}}^\times$ and $\alpha(m)$ is a regular element of $\mathcal{O}_{X,\bar{x}}$. Note that the regularity of p does not depend on the choice of m.

6. Throughout this paper, a monoid is a commutative monoid with the unity. The binary operation of a monoid is often written additively. We say a monoid P is *finitely generated* if there are $p_1, \ldots, p_n$ such that $P = \mathbb{N}p_1 + \cdots + \mathbb{N}p_r$. Moreover P is said to be *integral* if whenever $x + z = y + z$ for some elements $x, y, z \in P$, we have $x = y$. An integral and finitely generated monoid is said to be *fine*. We say P is *sharp* if whenever $x + y = 0$ for some $x, y \in P$, then $x = y = 0$. For a sharp monoid P, an element x of P is said to be *irreducible* if whenever $x = y + z$ for some $y, z \in P$, then either $y = 0$ or $z = 0$. A homomorphism $f : Q \to P$ of monoids is said to be *integral* if it is injective and an equation

$$f(q) + p = f(q') + p' \quad (p, p' \in P, q, q' \in Q)$$

implies that $p = f(q_1) + p''$ and $p' = f(q_2) + p''$ for some $p'' \in P$ and some $q_1, q_2 \in Q$ with $q + q_1 = q' + q_2$. Further we say an injective homomorphism $f : Q \to P$ *splits* if there is a submonoid N of P such that the homomorphism $f(Q) \times N \to P$ given by $(x, y) \mapsto x + y$ is an isomorphism.

7. Let $f : Q \to P$ and $g : Q \to R$ be homomorphisms of monoids. The *integral tensor product* $P\bar{\otimes}_Q R$ *of* P *and* R *over* Q is defined as follows: Let us consider a relation $\sim$ on $P \times R$ given by

$$(p, r) \sim (p', r') \iff (f(q), g(q')) + (p, r) = (f(q'), g(q)) + (p', r') \text{ for some } q, q' \in Q.$$

It is easy to see that $\sim$ is an equivalence relation on $P \times R$. We set

$$P\bar{\otimes}_Q R = P \times R/\sim .$$

Note that $P\bar{\otimes}_Q R$ is a monoid in the natural way and it is integral if so are P and Q (for more details, see [6]).

8. Let X be a set. We denote the set of all maps $X \to \mathbb{N}$ by $\mathbb{N}^X$. For $T \in \mathbb{N}^X$, $\mathrm{Supp}(T)$ is given by $\{x \in X \mid T(x) > 0\}$. Moreover, for $T, T' \in \mathbb{N}^X$,

$$T \le T' \overset{\text{def}}{\iff} T(x) \le T'(x) \ \forall x \in X.$$

In the case where $X = \{1, \ldots, n\}$, $\mathbb{N}^X$ is sometimes denoted by $\mathbb{N}^n$.

9. Let M be a monoid, X a finite subset of M and $T \in \mathbb{N}^X$. For simplicity, $\sum_{x\in X} T(x)x$ is often denoted by $T \cdot X$. If we use the product symbol for the binary operation of the monoid M, then $\prod_{x\in X} x^{T(x)}$ is written by X^T. In particular, if $X = \{X_1, \ldots, X_n\}$ and $I \in \mathbb{N}^n$, then $I \cdot X$ and X^I means $\sum_{i=1}^n I(i)X_i$ and $\prod_{i=1}^n X_i^{I(i)}$ respectively according to a way of the binary operator of M. For example, let A be a ring and R be either the ring of polynomials of n-variables over A, or the ring of formal power series of n-variables over A, that is, $R = A[X_1, \ldots, X_n]$ or $A[\![X_1, \ldots, X_n]\!]$. Note that R is a monoid with respect to the ring multiplication. As before, for $I \in \mathbb{N}^n$, the monomial $X_1^{I(1)} \cdots X_n^{I(n)}$ is denoted by X^I.

10. Let $f : Q \to P$ be an integral homomorphism of fine and sharp monoids. In the following, the binary operators of monoids are written in the additive way. For a finite subset σ of P, $q_0 \in Q$ and $\Delta, B \in \mathbb{N}^\sigma$, we say P has a *semistable structure* (σ, q_0, Δ, B) *over* Q (or P is *of semistable type* (σ, q_0, Δ, B) *over* Q) if the following conditions are satisfied:

(1) $q_0 \neq 0$, $\mathrm{Supp}(\Delta) \neq \emptyset$ and $\Delta(x)$ is either 0 or 1 for all $x \in \sigma$.
(2) P is generated by σ and $f(Q)$ and the natural homomorphism $\mathbb{N}^\sigma \to P$ given by $T \mapsto T \cdot \sigma$ is injective.
(3) $\mathrm{Supp}(\Delta) \cap \mathrm{Supp}(B) = \emptyset$ and $\Delta \cdot \sigma = f(q_0) + B \cdot \sigma$.
(4) If we have a relation
$$T \cdot \sigma = f(q) + T' \cdot \sigma \quad (T, T' \in \mathbb{N}^\sigma)$$
with $q \neq 0$, then $T(x) > 0$ for all $x \in \mathrm{Supp}(\Delta)$.

Let $\mathbb{N} \to Q \times \mathbb{N}^{\sigma\setminus\mathrm{Supp}(\Delta)}$ and $\mathbb{N} \to \mathbb{N}^{\mathrm{Supp}(\Delta)}$ be homomorphisms given by $1 \mapsto (f(q_0), B|_{\sigma\setminus\mathrm{Supp}(\Delta)})$ and $1 \mapsto \Delta|_{\mathrm{Supp}(\Delta)}$ respectively. It is known that the natural homomorphism
$$(Q \times \mathbb{N}^{\sigma\setminus\mathrm{Supp}(\Delta)})\bar{\otimes}_{\mathbb{N}}\mathbb{N}^{\mathrm{Supp}(\Delta)} \to P$$
is bijective, where $\bar{\otimes}_{\mathbb{N}}$ is the integral tensor product (cf. [5, Proposition 2.2]).

11. Let (A, m) be a local ring. The henselization of A and the completion of A with respect to m are denoted by A^h and $\widehat{A}$ respectively.

1. Semistable schemes over a scheme

1.1. **Algebraic preliminaries.** In this subsection, we consider several lemmas which will be used later. Let us begin with the following lemma.

Lemma 1.1.1. *Let $f : (A, m_A) \to (B, m_B)$ be a local homomorphism of noetherian local rings such that f induces an isomorphism $A/m_A \xrightarrow{\sim} B/m_B$.*

(1) *Let $x_1, \ldots, x_n$ be generator of m_B, i.e., $m_B = Bx_1 + \cdots + Bx_n$. If (B, m_B) is complete, then, for any $b \in B$, there is a sequence*
$$\{\alpha_{(a_1,\ldots,a_n)}\}_{(a_1,\ldots,a_n)\in\mathbb{N}^n}$$
of elements of A indexed by $\mathbb{N}^n$ with
$$b = \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} f(\alpha_{(a_1,\ldots,a_n)})x_1^{a_1} \cdots x_n^{a_n}.$$

(2) *Let $x_1, \ldots, x_n$ be elements of m_B with $m_B = Bx_1 + \cdots + Bx_n + m_A B$. If (A, m_A) and (B, m_B) are complete, then, for any $b \in B$, there is a sequence*
$$\{\alpha_{(a_1,\ldots,a_n)}\}_{(a_1,\ldots,a_n)\in\mathbb{N}^n}$$
of elements of A indexed by $\mathbb{N}^n$ with
$$b = \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} f(\alpha_{(a_1,\ldots,a_n)}) x_1^{a_1}\cdots x_n^{a_n}.$$

Proof. (1) First we claim the following:

Claim 1.1.1.1.
$$m_B^d \subseteq \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{N}^n\\ a_1+\cdots+a_n=d}} f(A)x_1^{a_1}\cdots x_n^{a_n} + m_B^{d+1}$$
for all $d \geq 0$.

We prove this claim by induction on d. Since $A/m_A \simeq B/m_B$, we have $B = f(A) + m_B$, which means that the assertion holds for $d = 0$. Thus
$$\begin{aligned} m_B &= (f(A)+m_B)x_1 + \cdots + (f(A)+m_B)x_n \\ &\subseteq f(A)x_1 + \cdots + f(A)x_n + m_B^2, \end{aligned}$$
which show that the assertion holds for $d = 1$, so that we assume $d \geq 2$. By the hypothesis of induction,
$$\begin{aligned} m_B^d &= m_B \cdot m_B^{d-1} \\ &\subseteq \left(f(A)x_1 + \cdots + f(A)x_n + m_B^2\right)\cdot\left(\sum_{\substack{(a_1',\ldots,a_n')\in\mathbb{N}^n\\ a_1'+\cdots+a_n'=d-1}} f(A)x_1^{a_1'}\cdots x_n^{a_n'} + m_B^d\right) \\ &\subseteq \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{N}^n\\ a_1+\cdots+a_n=d}} f(A)x_1^{a_1}\cdots x_n^{a_n} + m_B^{d+1} \end{aligned}$$
Hence we get the claim.

In order to complete the proof of (1), it is sufficient to see the following claim:

Claim 1.1.1.2. *For all $b \in B$, there is a sequence $\{b_d\}_{d=0}^{\infty}$ of B such that*
$$b_d \in \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{N}^n\\ a_1+\cdots+a_n=d}} f(A)x_1^{a_1}\cdots x_n^{a_n}$$
and
$$b - (b_0 + \cdots + b_d) \in m_B^{d+1}$$
for all $d \geq 0$.

Since $B = f(A) + m_B$, we can set $b = b_0 + c$ with $b_0 \in f(A)$ and $c \in m_B$. We assume that $b_0, \ldots, b_{d-1}$ are given. Then, by Claim 1.1.1.1,
$$b - (b_0 + \cdots + b_{d-1}) = b_d + c',$$
where $b_d \in \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{N}^n\\ a_1+\cdots+a_n=d}} f(A)x_1^{a_1}\cdots x_n^{a_n}$ and $c' \in m_B^{d+1}$. This yields the second claim.

(2) Let us choose $y_1, \ldots, y_r \in A$ with $m_A = y_1 A + \cdots + y_r A$. Then

$$m_B = x_1 B + \cdots + x_n B + f(y_1)B + \cdots + f(y_r)B.$$

Note that

$$x_1^{a_1} \cdots x_n^{a_n} f(y_1)^{b_1} \cdots f(y_r)^{b_r} = f(y_1^{b_1} \cdots y_r^{b_r}) x_1^{a_1} \cdots x_n^{a_n}.$$

Therefore, since (A, m_A) is complete, using (1), for any $b \in B$, there is a sequence $\{\alpha_{(a_1,\ldots,a_n,b_1,\ldots,b_r)}\}_{(a_1,\ldots,a_n,b_1,\ldots,b_r)\in\mathbb{N}^n\times\mathbb{N}^r}$ with

$$\begin{aligned} b &= \sum_{(a_1,\ldots,a_n,b_1,\ldots,b_r)\in\mathbb{N}^n\times\mathbb{N}^r} f(\alpha_{(a_1,\ldots,a_n,b_1,\ldots,b_r)}) x_1^{a_1} \cdots x_n^{a_n} f(y_1)^{b_1} \cdots f(y_r)^{b_r} \\ &= \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} f\left(\sum_{(b_1,\ldots,b_r)\in\mathbb{N}^r} \alpha_{(a_1,\ldots,a_n,b_1,\ldots,b_r)} y_1^{b_1} \cdots y_r^{b_r} \right) x_1^{a_1} \cdots x_n^{a_n}. \end{aligned}$$

Thus we get (2). □

Next let us consider the following lemma.

Lemma 1.1.2. *Let (A, m) be a noetherian local ring and $T \in \mathbb{N}^n \setminus \{(0,\ldots,0)\}$. Let $G \in m[\![X_1,\ldots,X_n]\!]$, $R = A[\![X_1,\ldots,X_n]\!]/(X^T - G)$ and $\pi : A[\![X_1,\ldots,X_n]\!] \to R$ the canonical homomorphism. Then we have the following:*

(1) *Let M be an A-submodule of $A[\![X_1,\ldots,X_n]\!]$ given by*

$$M = \left\{ \sum_{T \not\leq I} a_I X^I \mid a_I \in A \right\}$$

(cf. Conventions and terminology 8 and 9). If (A, m) is complete, then $\pi|_M : M \to R$ is bijective.

(2) *$A[\![X_1,\ldots,X_n]\!]/(X^T - G)$ is flat over A.*

Proof. (1) We denote $\pi(X_i)$ by x_i. First we claim the following:

Claim 1.1.2.1. *For $f \in R$, there is a sequence $\{F_n\}_{n=0}^{\infty}$ in M such that $F_{n+1} - F_n \in m^n[\![X_1,\ldots,X_n]\!]$ and $f - \pi(F_n) \in m^n R$ for all $n \geq 0$.*

We will construct a sequence $\{F_n\}_{n=0}^{\infty}$ inductively. Clearly we may set $F_0 = 0$. We assume that $F_0, F_1, \ldots, F_n$ have been constructed. Then we can set $f - \pi(F_n) = \pi(H) + x^T \pi(H')$ for some $H, H' \in m^n[\![X_1,\ldots,X_n]\!]$ with $H \in M$. Here $x^T\pi(H') = \pi(G)\pi(H') \in m^{n+1}R$. Thus, if we set $F_{n+1} = F_n + H$, then we get our desired F_{n+1}.

The above claim shows that $\pi|_M$ is surjective. Next let us consider the injectivity of $\pi|_M$. We assume

$$\pi\left(\sum_{T \not\leq I} a_I X^I \right) = 0.$$

Then there is $H \in A[\![X_1,\ldots,X_n]\!]$ with

$$\sum_{T \not\leq I} a_I X^I = (X^T - G)H.$$

Here we set

$$G = \sum_{I \in \mathbb{N}^n} g_I X^I \quad \text{and} \quad H = \sum_{I \in \mathbb{N}^n} h_I X^I.$$

Then $g_I \in m$ for all I and

$$\sum_{T \not\leq I} a_I X^I = \sum_{I \in \mathbb{N}^n} h_I X^{T+I} - \sum_{I \in \mathbb{N}^n} \left(\sum_{J+J'=I} g_J h_{J'} \right) X^I.$$

On the left hand side of the above equation, there is no term of a form X^{I+T}. Thus

$$h_I = \sum_{J+J'=I+T} g_J h_{J'}$$

for all I. Here we claim that $h_I \in m^n$ for all n and all I. We see this fact by induction n. First of all, since $g_I \in m$ for all I, we have $h_I \in m$ for all I. We assume that $h_I \in m^n$ for all I. By the above equation, we can see that $h_I \in m^{n+1}$. By this claim, h_I must be zero for all I because $\bigcap_{n \geq 0} m^n = 0$. Therefore $a_I = 0$ for all I.

(2) If (A, m) is complete, then the assertion follows from (1) by Chase's theorem [2]. In general, let $\widehat{A}$ be the completion of A and

$$R' = \widehat{A}[\![X_1, \ldots, X_n]\!]/(X^T - G).$$

Then we have the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "h'"] & R' \\ A \arrow[u, "f"] \arrow[r, "h"] & \widehat{A}. \arrow[u, "f'"'] \end{tikzcd}$$

Note that f', h and h' are faithfully flat. Thus so is f. $\square$

Remark 1.1.3. Let A be a ring and $A[X_1, \ldots, X_n]$ the polynomial ring of n-variables over A. In the same way as in Lemma 1.1.2, we can see that $R = A[X_1, \ldots, X_n]/(X^T - a)$ is flat over A for $T \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$ and $a \in A$. Indeed, if we set

$$M = \left\{ \sum_{T \not\leq I} a_I X^I \in A[X_1, \ldots, X_n] \right\},$$

then the natural homomorphism $\pi : M \to R$ is bijective. The surjectivity of π is obvious. We assume that an element $\sum a_I X^I$ of M is zero in R. Then

$$\sum a_I X^I = (X^T - a) \sum_J b_J X^J$$

for some $\sum_J b_J X^J \in A[X_1, \ldots, X_n]$. Thus $b_J = a b_{J+T}$ for all J. Therefore $b_J = a^m b_{J+mT}$ for all $m > 0$. On the other hand, since $\sum_J b_J X^J \in A[X_1, \ldots, X_n]$, $b_{J+mT} = 0$ if $m \gg 0$. Hence $b_J = 0$ for all J.

We consider an approximation by an étale neighborhood.

Proposition 1.1.4. *Let (A, m_A) be a noetherian local ring essentially of finite type over an excellent discrete valuation ring or a field. Let $f : X \to \operatorname{Spec}(A)$ be a scheme of finite type over A. Let x be a point of X such that $f(x) = m_A$ and A/m_A is naturally isomorphic to $\mathcal{O}_{X,x}/m_{X,x}$. We assume that there are $F_1, \ldots, F_r \in A[X_1, \ldots, X_n]$ (the polynomial ring of n-variables over A) and an isomorphism*

$$\phi : \widehat{A}[\![X_1, \ldots, X_n]\!]/(F_1, \ldots, F_r) \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$$

over $\widehat{A}$ with $\phi(\bar{X}_i) \in \widehat{m}_{X,x}$ for all i, where $\bar{X}_i = X_i \mod (F_1, \ldots, F_r)$. Then there is an étale neighborhood (U, x') of X at x together with an étale morphism

$$\rho : U \to \operatorname{Spec}(A[T_1, \ldots, T_n]/(F_1(T), \ldots, F_r(T)))$$

such that $\rho(x') = (m_A, \bar{T}_1, \ldots, \bar{T}_n)$, where $\bar{T}_i = T_i \mod (F_1(T), \ldots, F_r(T))$.

Proof. First note that

$$F_1(\phi(\bar{X}_1), \ldots, \phi(\bar{X}_n)) = \cdots = F_r(\phi(\bar{X}_1), \ldots, \phi(\bar{X}_n)) = 0.$$

Thus, by Artin's approximation theorem [1], there are $t_1, \ldots, t_n \in \mathcal{O}^h_{X,x}$ such that

$$F_1(t_1, \ldots, t_n) = \cdots = F_r(t_1, \ldots, t_n) = 0$$

and $t_i - \phi(\bar{X}_i) \in \widehat{m}^2_{X,x}$ for all i. Here we claim the following:

Claim 1.1.4.1.

$$\begin{aligned} \widehat{m}_{X,x} &= \phi(\bar{X}_1)\widehat{\mathcal{O}}_{X,x} + \cdots + \phi(\bar{X}_n)\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x} \\ &= t_1\widehat{\mathcal{O}}_{X,x} + \cdots + t_n\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x}. \end{aligned}$$

Clearly

$$\widehat{m}_{X,x} \supseteq \phi(\bar{X}_1)\widehat{\mathcal{O}}_{X,x} + \cdots + \phi(\bar{X}_n)\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x}.$$

Conversely let us pick up $f \in \widehat{m}_{X,x}$. Then we can write $f = \phi\left(\sum_I a_I \bar{X}^I\right)$. If $a_{(0,\ldots,0)} \in \widehat{A}^{\times}$, then f must be a unit because $f \in a_{(0,\ldots,0)} + \widehat{m}_{X,x}$. This is a contradiction. Thus $a_{(0,\ldots,0)} \in m_A$, which means that

$$f \in \phi(\bar{X}_1)\widehat{\mathcal{O}}_{X,x} + \cdots + \phi(\bar{X}_n)\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x}.$$

Therefore we obtain

$$\widehat{m}_{X,x} = \phi(\bar{X}_1)\widehat{\mathcal{O}}_{X,x} + \cdots + \phi(\bar{X}_n)\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x}$$

Moreover, since $t_i - \phi(\bar{X}_i) \in \widehat{m}^2_{X,x}$, we can see that

$$\widehat{m}_{X,x} = t_1\widehat{\mathcal{O}}_{X,x} + \cdots + t_n\widehat{\mathcal{O}}_{X,x} + m_A\widehat{\mathcal{O}}_{X,x} + \widehat{m}^2_{X,x}.$$

Hence, by Nakayama's lemma, we have our desired result.

Let us choose an étale neighborhood (U, x') of X at x with the same residue fields such that $t_1, \ldots, t_n$ are defined over U. Here let us define a homomorphism

$$\psi : A[T_1, \ldots, T_n]/(F_1(T), \ldots, F_r(T)) \to \mathcal{O}_{U,x'}$$

to be $\psi(\bar{T}_i) = t_i$ for all i. By the above claim, we can see

$$\psi^{-1}(m_{U,x'}) = (m_A, \bar{T}_1, \ldots, \bar{T}_n).$$

Thus it is sufficient to show that ψ is étale. Let

$$\mu : \widehat{A}[\![T_1, \ldots, T_n]\!]/(F_1, \ldots, F_r) \to \widehat{A}[\![X_1, \ldots, X_n]\!]/(F_1, \ldots, F_r)$$

be a homomorphism given by the composition of homomorphisms

$$\widehat{A}[\![T_1,\ldots,T_n]\!]/(F_1,\ldots,F_r) \xrightarrow{\widehat{\psi}} \widehat{\mathcal{O}}_{U,x'} = \widehat{\mathcal{O}}_{X,x} \xrightarrow{\phi^{-1}} \widehat{A}[\![X_1,\ldots,X_n]\!]/(F_1,\ldots,F_r).$$

By the above claim and Lemma 1.1.1, μ is surjective. Hence, by the following Lemma 1.1.5, it must be an isomorphism. Therefore so is $\widehat{\psi}$. This means that ψ is étale because x' and $(m_A, \bar{T}_1, \ldots, \bar{T}_n)$ have the same residue field. □

Finally we consider the following lemma concerning the bijectivity of a ring homomorphism.

Lemma 1.1.5. *Let $\phi : A \to A$ be an endomorphism of a noetherian ring. If ϕ is surjective, then ϕ is injective.*

Proof. We set $I_n = \operatorname{Ker}(\phi^n)$ for $n \geq 1$. Since ϕ is surjective, we can see that $\phi(I_{n+1}) = I_n$ for all $n \geq 1$. Moreover there is $N \geq 1$ such that $I_{N+1} = I_N$ because A is noetherian and $I_n \subseteq I_{n+1}$ for all $n \geq 1$. Therefore

$$\operatorname{Ker}(\phi) = I_1 = \phi^N(I_{N+1}) = \phi^N(I_N) = \{0\}.$$

□

1.2. **Semistable varieties and semistable schemes.** Let k be an algebraically closed field and X an algebraic scheme over k. A closed point x of X is called *a semistable point of* X if the completion of the local ring at x is isomorphic to a ring of type

$$k[\![X_1,\ldots,X_n]\!]/(X_1 \cdots X_l).$$

The number l is called *the multiplicity of X at x*, and is denoted by $\operatorname{mult}_x(X)$. Moreover we say X is a semistable variety over k if every closed point is a semistable point. By the following Proposition 1.2.1, the set of all semistable points of X is a Zariski open set. Thus we say a point x of X (x is not necessarily closed) is *a semistable point* if there is a Zariski open set U of X such that $x \in U$ and every closed point of U is a semistable point. Let Ω be an algebraically closed field such that k is a subfield of Ω. Note that if X is a semistable variety over k, then so is $X_\Omega = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\Omega)$ over Ω (cf. Proposition 1.2.2).

Let S be a locally noetherian scheme and $f : X \to S$ a morphism of finite type. First we assume that $S = \operatorname{Spec}(F)$ for some field F. Let $\bar{F}$ be the algebraic closure of F, $X' = X \times_{\operatorname{Spec}(F)} \operatorname{Spec}(\bar{F})$, and $\pi : X' \to X$ the canonical morphism. A point x of X is called *a semistable point of* X if every point x' of X' with $\pi(x') = x$ is a semistable point. For a general S, we say $f : X \to S$ is *semistable at* $x \in X$ if f is flat at x and x is a semistable point of the fiber $f^{-1}(f(x))$ passing through x. Moreover we say X is a semistable scheme over S if f is semistable at all points of X. By Proposition 1.2.2, for a flat morphism $f : X \to S$, X is a semistable scheme over S if and only if, for any algebraically closed field Ω, any morphism $\operatorname{Spec}(\Omega) \to S$ and any closed point $x' \in X \times_S \operatorname{Spec}(\Omega)$, the completion of the local ring at x' is isomorphic to a ring of type

$$\Omega[\![X_1,\ldots,X_n]\!]/(X_1 \cdots X_l).$$

We say a semistable scheme X over S is *proper* if X is proper over S. Moreover a proper semistable scheme X over S is said to be *connected* if $f_*(\mathcal{O}_X) = \mathcal{O}_S$.

In the remaining of this subsection, let us consider elementary properties of semistable varieties.

Proposition 1.2.1. *Let X be an algebraic scheme over an algebraically closed field k. If x is a semistable closed point of X, then there is a Zariski open set U of X such that $x \in U$ and every closed point of U is a semistable point.*

Proof. By Proposition 1.1.4, there are an étale neighborhood $\pi : (U, x') \to (X, x)$ of x and an étale morphism

$$\rho : U \to \operatorname{Spec}(k[T_1, \ldots, T_n]/(T_1 \cdots T_l))$$

with $\rho(x') = (0, \ldots, 0)$. Note that $\operatorname{Spec}(k[T_1, \ldots, T_n]/(T_1 \cdots T_l))$ is a semistable variety over k. Thus so is U over k. Therefore every closed point of $\pi(U)$ is a semistable point. □

Proposition 1.2.2. *Let X be an algebraic scheme over an algebraically closed field k. Let Ω be an algebraically closed field such that k is a subfield of Ω. Let $\pi : X_\Omega = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\Omega) \to X$ be the canonical morphism. For $y \in X_\Omega$, If $x = \pi(y)$ is a semistable point, then so is y.*

Proof. Let U be an open set of X containing x such that every closed point of U is a semistable point.

First we assume that y is a closed point. Let us choose a closed point $o \in \overline{\{x\}} \cap U$. By using Proposition 1.1.4 and shrinking U around o if necessarily, there are étale morphisms

$$f : V \to U \quad \text{and} \quad g : V \to W = \operatorname{Spec}(k[X_1, \ldots, X_n]/(X_1 \cdots X_l))$$

of algebraic schemes over k and closed points $o' \in V$ and $o'' \in W$ such that $f(o') = o$ and $g(o') = o'' = (0, \ldots, 0)$. Since $x \in U$, $o \in \overline{\{x\}} \cap U$ and f is faithfully flat at o', we can find $x' \in V$ with $f(x') = x$ and $o' \in \overline{\{x'\}}$. Here we set

$$\begin{cases} U_\Omega = U \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\Omega), \\ V_\Omega = V \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\Omega), \\ W_\Omega = \operatorname{Spec}(\Omega[X_1, \ldots, X_n]/(X_1 \cdots X_l)) \end{cases}$$

and the induced morphisms $V_\Omega \to U_\Omega$ and $V_\Omega \to W_\Omega$ are denoted by f_Ω and g_Ω respectively. Then $y \in U_\Omega$. Let $\tilde{y} : \operatorname{Spec}(\Omega) \to U_\Omega$ be the morphism induced by y. Let $\kappa(y)$, $\kappa(x)$ and $\kappa(x')$ be the residue fields of y, x and x' respectively. Then there is an embedding $\iota : \kappa(x') \hookrightarrow \Omega$ over k such that the following diagram is commutative:

$$\begin{tikzcd} & & \kappa(x') \arrow[lld, hook', "\iota"'] \\ \Omega & \kappa(y) \arrow[l, "\sim", "\tilde{y}^*"'] & \kappa(x) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

This yields a morphism $\beta : \operatorname{Spec}(\Omega) \to V_\Omega$ such that the diagram

$$\begin{tikzcd} & V_\Omega \arrow[r, "\pi'"] \arrow[d, "f_\Omega"] & V \arrow[d, "f"] \\ \operatorname{Spec}(\Omega) \arrow[ru, "\beta"] \arrow[r, "\tilde{y}"'] & U_\Omega \arrow[r, "\pi"] & U \end{tikzcd}$$

is commutative and the image of $\pi' \circ \beta$ is x'. Let y' be the image of β. Then $f_\Omega(y') = y$. Note that f_Ω and g_Ω are étale and the residue fields of y, y' and $y'' = g_\Omega(y')$ are Ω. Thus we can see that

$$\widehat{\mathcal{O}}_{X_\Omega,y} \simeq \widehat{\mathcal{O}}_{V_\Omega,y'} \simeq \widehat{\mathcal{O}}_{W_\Omega,y''}.$$

We set $y'' = (a_1, \ldots, a_n) \in \mathbb{A}^n(\Omega)$ and $I = \{i \mid a_i = 0 \text{ and } i = 1, \ldots, l\}$. Note that $I \neq \emptyset$ because $y'' \in W_\Omega$. Therefore, if we set $Z_i = X_i - a_i$ and $Z = \prod_{i \in I} Z_i$, then it is easy to see that

$$\widehat{\mathcal{O}}_{W_\Omega,y''} = \Omega[\![Z_1, \ldots, Z_n]\!]/(Z).$$

Thus we get our lemma in the case where y is a closed point.

Next we consider a general case. We set $U_\Omega = \pi^{-1}(U)$. Then, by the previous observation, every closed point of U_Ω is a semistable point. On the other hand, $y \in U_\Omega$. Thus y is a semistable point. □

2. Some facts of log structures

In this section, we consider several facts concerning log structures, which will be used later.

2.1. **Ring extension for a good chart.** Here we consider a ring extension to get a good chart.

Proposition 2.1.1. *Let (A, m) be a noetherian local ring, $S = \operatorname{Spec}(A)$ and s the closed point of S. Let M_S be a fine log structure on S. Then there is a local homomorphism $f : (A, m) \to (B, n)$ of noetherian local rings with the following properties:*

(1) *B/n is algebraic over A/m, and f is flat and quasi-finite.*
(2) *Let $f^a : S' = \operatorname{Spec}(B) \to S = \operatorname{Spec}(A)$ be the induced morphism, s' the closed point of $S' = \operatorname{Spec}(B)$, and $M_{S'} = (f^a)^*(M_S)$. There are a fine and sharp monoid Q and a homomorphism $\pi_Q : Q \to M_{S',s'}$ such that $Q \to M_{S',\bar{s}'} \to \overline{M}_{S',\bar{s}'}$ is bijective.*

Proof. Let us begin with the following lemma:

Lemma 2.1.2. *Let G be a finitely generated abelian group and R a ring. Let us fix an element δ of $\operatorname{Ext}^1(G, R^\times)$. Then there are $u_1, \ldots, u_l \in R^\times$ and integers $a_1, \ldots, a_l \geq 2$ with the following property:*

(1) *The product $a_1 \cdots a_l$ of integers $a_1, \ldots, a_l$ is equal to the order of the torsion part of G.*
(2) *For any homomorphism $f : R \to S$ of rings, if there are $v_1, \ldots, v_l \in S$ with $v_i^{a_i} = f(u_i)$ for all i, then the image of δ via the canonical homomorphism*

$$\operatorname{Ext}^1(G, R^\times) \to \operatorname{Ext}^1(G, S^\times)$$

is zero.

Proof. By the fundamental theorem of abelian groups, we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z}^l \xrightarrow{\ \phi\ } \mathbb{Z}^{l'} \longrightarrow G \longrightarrow 0,$$

where ϕ is given by $\phi(x_1,\ldots,x_l) = (a_1x_1,\ldots,a_lx_l,0,\ldots,0)$ for some integers $a_1,\ldots,a_l \geq 2$. Note that $a_1\cdots a_l$ is equal to the order of the torsion part of G. The above exact sequence yields an exact sequence

$$\mathrm{Hom}(\mathbb{Z}^{l'}, R^\times) \xrightarrow{\phi_R^*} \mathrm{Hom}(\mathbb{Z}^l, R^\times) \xrightarrow{\alpha_R} \mathrm{Ext}^1(G, R^\times) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}^{l'}, R^\times).$$

Note that $\mathrm{Ext}^1(\mathbb{Z}^{l'}, R^\times) = \{0\}$. Thus there is $h \in \mathrm{Hom}(\mathbb{Z}^l, R^\times)$ with $\alpha_R(h) = \delta$. We set $u_i = h(e_i)$ for $i = 1,\ldots,l$, where $\{e_1,\ldots,e_l\}$ is the standard basis of $\mathbb{Z}^l$.

Let $f : R \to S$ be any homomorphism of rings with $v_i^{a_i} = f(u_i)$ $(i = 1,\ldots,l)$ for some $v_1,\ldots,v_l \in S$. Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathbb{Z}^{l'}, R^\times) & \xrightarrow{\phi_R^*} & \mathrm{Hom}(\mathbb{Z}^l, R^\times) & \xrightarrow{\alpha_R} & \mathrm{Ext}^1(G, R^\times) & \longrightarrow & 0 \\
{\scriptstyle g_1}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & \\
\mathrm{Hom}(\mathbb{Z}^{l'}, S^\times) & \xrightarrow{\phi_S^*} & \mathrm{Hom}(\mathbb{Z}^l, S^\times) & \xrightarrow{\alpha_S} & \mathrm{Ext}^1(G, S^\times) & \longrightarrow & 0
\end{array}$$

Note that $g_2(h)(e_i) = f(u_i)$ for $i = 1,\ldots,l$. Thus, if we set $h' \in \mathrm{Hom}(\mathbb{Z}^{l'}, S^\times)$ by

$$h'(e_i) = \begin{cases} v_i & \text{if } i = 1,\ldots .l \\ 0 & \text{if } i > l \end{cases}$$

then $\phi_S^*(h') = g_2(h)$. Therefore

$$g_3(\delta) = g_3(\alpha_R(h)) = \alpha_S(g_2(h)) = \alpha_S(\phi_S^*(h')) = 0.$$

□

Let us start the proof of Proposition 2.1.1. Let $\delta \in \mathrm{Ext}^1(\overline{M}^{gr}_{S,\bar{s}}, \mathcal{O}^\times_{S,\bar{s}})$ be the extension class of

$$0 \to \mathcal{O}^\times_{S,\bar{s}} \to M^{gr}_{S,\bar{s}} \to \overline{M}^{gr}_{S,\bar{s}} \to 0.$$

Then, by Lemma 2.1.2, there are $u_1,\ldots,u_l \in \mathcal{O}^\times_{S,\bar{s}}$ and integers $a_1,\ldots,a_l$ with the properties as in Lemma 2.1.2. Let us choose an étale neighborhood (U,u) of s such that $u_1,\ldots,u_l \in \mathcal{O}^\times_{U,u}$. Let B be the localization of

$$\mathcal{O}_{U,u}[X_1,\ldots,X_l]/(X_1^{a_1} - u_1,\ldots,X_l^{a_l} - u_l).$$

at a closed point over u. Then B is flat and quasi-finite over A. Let v_i be the class of X_i in B. Note that $v_i^{a_i} = u_i$ in B for all i. Let s' be the closed point of $S' = \mathrm{Spec}(B)$, $\pi : S' \to S$ the canonical morphism, and $M_{S'} = \pi^*(M_S)$. Then we have an exact sequence

$$0 \to \mathcal{O}^\times_{S',\bar{s}'} \to M^{gr}_{S',\bar{s}'} \to \overline{M}^{gr}_{S',\bar{s}'} \to 0.$$

Since $M^{gr}_{S',\bar{s}'}$ is the push-out $\mathcal{O}^\times_{S',\bar{s}'} \bar{\otimes}_{\mathcal{O}^\times_{S,\bar{s}}} M^{gr}_{S,\bar{s}}$ (cf. Conventions and terminology 7), we can see that $\overline{M}^{gr}_{S',\bar{s}'} = \overline{M}^{gr}_{S,\bar{s}}$ and the extension class δ' of the above exact sequence is the image of the canonical homomorphism $\mathrm{Ext}^1(\overline{M}^{gr}_{S,\bar{s}}, \mathcal{O}^\times_{S,\bar{s}}) \to \mathrm{Ext}^1(\overline{M}^{gr}_{S',\bar{s}'}, \mathcal{O}^\times_{S',\bar{s}'})$. Thus, by Lemma 2.1.2, $\delta' = 0$. Therefore we have a splitting $s : \overline{M}^{gr}_{S',\bar{s}'} \to M^{gr}_{S',\bar{s}'}$ of $M^{gr}_{S',\bar{s}'} \to \overline{M}^{gr}_{S',\bar{s}'}$. Here we set $Q = \overline{M}_{S',\bar{s}'}$. Let us see that $s(q) \in M_{S',\bar{s}'}$ for all $q \in Q$. Indeed, if we denote $M^{gr}_{S',\bar{s}'} \to \overline{M}^{gr}_{S',\bar{s}'}$ by π, then $\pi(s(q)) = q$. Thus there are $u \in \mathcal{O}^\times_{S',\bar{s}'}$ and $m \in M_{S',\bar{s}'}$ with $s(q) = m \cdot u$, which implies $s(q) \in M_{S',\bar{s}'}$. Moreover $Q \to M_{S',\bar{s}'} \to \overline{M}_{S',\bar{s}'}$ is the identity map. Further, changing S' by an étale neighborhood of S', we may assume that $Q \to M_{S',\bar{s}}$ is defined on S'. □

2.2. **The support of log structures.** In this subsection, we consider the support of log structures. The main result of this subsection is the following proposition:

Proposition 2.2.1. *Let X be a scheme and let M and N be fine log structures on X. Let $h : N \to M$ be a homomorphism of log structures, i.e., a homomorphism of sheaves of monoids with the following diagram commutative:*

$$\begin{tikzcd} N \arrow[rr, "h"] \arrow[dr] & & M \arrow[dl] \\ & \mathcal{O}_X & \end{tikzcd}$$

Then the set $\{x \in X \mid h_{\bar{x}} : N_{\bar{x}} \to M_{\bar{x}} \text{ is surjective}\}$ is open.

Proof. It is sufficient to show that if $h_{\bar{x}} : N_{\bar{x}} \to M_{\bar{x}}$ is surjective, then there is an étale neighborhood U of x such that, for all $y \in U$, $h_{\bar{y}} : N_{\bar{y}} \to M_{\bar{y}}$ is surjective. By virtue of [3, (2.8)], for a suitable étale neighborhood U of x, there are finitely generated monoids P and Q together with homomorphisms $\pi : P \to M|_U$, $\mu : Q \to N|_U$ and $f : Q \to P$ such that π and μ give rise to local charts of M and N respectively and the diagram

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "\mu"'] & P \arrow[d, "\pi"] \\ N|_U \arrow[r, "h_U"] & M|_U \end{tikzcd}$$

is commutative. Let $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_r\}$ be generators of P and Q respectively. Renumbering $p_1, \ldots, p_n$ and $q_1, \ldots, q_r$, we may assume that

$$\begin{cases} \pi(p_1), \ldots, \pi(p_{n'}) \in \mathcal{O}_{X,\bar{x}}^{\times}, \ \pi(p_{n'+1}), \ldots, \pi(p_n) \notin \mathcal{O}_{X,\bar{x}}^{\times}, \\ \mu(q_1), \ldots, \mu(q_{r'}) \in \mathcal{O}_{X,\bar{x}}^{\times}, \ \mu(q_{r'+1}), \ldots, \mu(q_r) \notin \mathcal{O}_{X,\bar{x}}^{\times}. \end{cases}$$

Let P_0 and Q_0 be submonoids of P and Q generated by $p_1, \ldots, p_{n'}$ and $q_1, \ldots, q_{r'}$ respectively. Let us see the following:

Claim 2.2.1.1. $\pi_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times}) = P_0$ *and* $\mu_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times}) = Q_0$.

Clearly $P_0 \subseteq \pi_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times})$. Conversely we assume that $w \in \pi_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times})$. We set $w = p_1^{a_1} \cdots p_n^{a_n}$. Then

$$\pi(w) = \pi(p_1)^{a_1} \cdots \pi(p_n)^{a_n} \in \mathcal{O}_{X,\bar{x}}^{\times}.$$

Therefore, if $a_i > 0$, then $\pi(p_i) \in \mathcal{O}_{X,\bar{x}}^{\times}$. Hence $a_i = 0$ for all $i > n'$, which means that $w \in P_0$. In the same way, we can see that $\mu_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times}) = Q_0$.

Claim 2.2.1.2. *For $z \in X$, $h_{\bar{z}} : N_{\bar{z}} \to M_{\bar{z}}$ is surjective if and only if $\bar{h}_{\bar{z}} : \overline{N}_{\bar{z}} \to \overline{M}_{\bar{z}}$ is surjective.*

Clearly, if $h_{\bar{z}} : N_{\bar{z}} \to M_{\bar{z}}$ is surjective, then so is $\bar{h}_{\bar{z}} : \overline{N}_{\bar{z}} \to \overline{M}_{\bar{z}}$. Conversely we assume that $\bar{h}_{\bar{z}} : \overline{N}_{\bar{z}} \to \overline{M}_{\bar{z}}$ is surjective. Let m be an element of $M_{\bar{z}}$. Then there is $n \in N_{\bar{z}}$ such that $m \equiv h_{\bar{z}}(n) \mod \mathcal{O}_{X,\bar{z}}^{\times}$, i.e., $m = uh_{\bar{z}}(n)$ for some $u \in \mathcal{O}_{X,\bar{z}}^{\times}$. Thus $m = uh_{\bar{z}}(n) = h_{\bar{z}}(un)$.

Shrinking U if necessarily, we may assume that

$$\pi(p_1), \ldots, \pi(p_{n'}) \in \mathcal{O}_{X,\bar{y}}^{\times} \quad \text{and} \quad \mu(q_1), \ldots, \mu(q_{r'}) \in \mathcal{O}_{X,\bar{y}}^{\times}$$

for all $y \in U$. Let us check that $h_{\bar{y}} : N_{\bar{y}} \to M_{\bar{y}}$ is surjective for all $y \in U$, which is equivalent to show that $\bar{h}_{\bar{y}} : \overline{N}_{\bar{y}} \to \overline{M}_{\bar{y}}$ is surjective by Claim 2.2.1.2. Note that the commutative diagram

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "\bar{\mu}_{\bar{y}}"'] & P \arrow[d, "\bar{\pi}_{\bar{y}}"] \\ \overline{N}_{\bar{y}} \arrow[r, "\bar{h}_{\bar{y}}"] & \overline{M}_{\bar{y}} \end{tikzcd}$$

gives rise to the commutative diagram

$$\begin{tikzcd} Q/\mu_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}) \arrow[r] \arrow[d] & P/\pi_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}) \arrow[d] \\ \overline{N}_{\bar{y}} \arrow[r, "\bar{h}_{\bar{y}}"] & \overline{M}_{\bar{y}} \end{tikzcd}$$

such that the vertical homomorphisms are bijective (cf. [3] and [5]). Therefore it is sufficient to see that

$$Q/\mu_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}) \to P/\pi_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times})$$

is surjective. Note that $Q_0 \subseteq \mu_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times})$ and $P_0 \subseteq \pi_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times})$. Thus we get the following commutative diagram:

$$\begin{tikzcd} Q/Q_0 \arrow[r] \arrow[d] & P/P_0 \arrow[d] \\ Q/\mu_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}) \arrow[r] & P/\pi_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}). \end{tikzcd}$$

Here, by Claim 2.2.1.1, $Q/Q_0 \to P/P_0$ is surjective because $\overline{N}_{\bar{x}} \to \overline{M}_{\bar{x}}$ is surjective. Hence so is $Q/\mu_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times}) \to P/\pi_{\bar{y}}^{-1}(\mathcal{O}_{X,\bar{y}}^{\times})$. □

Corollary 2.2.2. *Let X be a scheme and M a fine log structure on X. Then the set* $\operatorname{Supp}(M) = \{x \in X \mid M_{\bar{x}}$ *is not trivial*$\}$ *is closed.*

Proof. There is a natural homomorphism $\mathcal{O}_X^{\times} \to M$. Thus this is a consequence of the above proposition. □

Corollary 2.2.3. *Let X and Y be schemes and let M and N be fine log structures on X and Y respectively. Let $(f,h) : (X,M) \to (Y,N)$ be a log morphism.*

(1) *The set*

$$\operatorname{Supp}(M/N) = \{x \in X \mid \ N_{\overline{f(x)}} \times \mathcal{O}_{X,\bar{x}}^{\times} \to M_{\bar{x}} \text{ is not surjective}\}$$

is closed.

(2) *Let $\rho : S' \to S$ be a morphism of schemes and $X' = X \times_S S'$. We set the induced morphisms as follows:*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\rho'"'] \arrow[d, "f'"] \\ S & S'. \arrow[l, "\rho"'] \end{tikzcd}$$

Then ${\rho'}^{-1}(\operatorname{Supp}(M/N)) = \operatorname{Supp}({\rho'}^{*}(M)/\rho^{*}(N))$.

Proof. (1) Note that the surjectivity of $N_{\overline{f(x)}} \times \mathcal{O}^{\times}_{X,\bar{x}} \to M_{\overline{x}}$ is equivalent to the surjectivity of $f^*(N)_{\bar{x}} \to M_{\bar{x}}$. Thus it follows from Proposition 2.2.1.

(2) For $x' \in X'$, we set $x = \rho'(x')$. Note that $\rho'^*(f^*(N)) = f'^*(\rho^*(N))$. Thus we have a commutative diagram:

$$\begin{array}{ccc} \overline{M}_{\bar{x}} & \xrightarrow{\nu'} & \overline{\rho'^*(M)}_{\bar{x}'} \\ \uparrow & & \uparrow \\ \overline{f^*(N)}_{\bar{x}} & \xrightarrow{\nu} & \overline{f'^*(\rho^*(N))}_{\bar{x}'}. \end{array}$$

Then, by Lemma 3.2, the horizontal homomorphisms ν' and ν are bijective. Hence, by using Claim 2.2.1.2 of Proposition 2.2.1, we have (2). □

3. Local structure theorem

In this section, we consider the following fundamental structure theorem of this paper.

Theorem 3.1 (Local structure theorem)**.** *Let $(f, h) : (X, M_X) \to (S, M_S)$ be a smooth and integral morphism of fine log schemes. Let x be a point of X and $s = f(x)$. We assume that $f : X \to S$ is semistable at x. Then we have the following:*

1. *If f is smooth at x, then there is a submonoid N of $\overline{M}_{X,\bar{x}}$ such that $\overline{M}_{X,\bar{x}} = \bar{h}_{\bar{x}}(\overline{M}_{S,\bar{s}}) \times N$ and N is isomorphic to $\mathbb{N}^a$ for some non-negative integer a. Moreover every element of N is regular* (*For the definition of regularity, see Conventions and terminology* 5)*.*
2. *If f is not smooth at x and $\bar{h}_{\bar{x}} : \overline{M}_{S,\bar{s}} \to \overline{M}_{X,\bar{x}}$ splits, there is a submonoid N of $\overline{M}_{X,\bar{x}}$ such that $\overline{M}_{X,\bar{x}} = \bar{h}_{\bar{x}}(\overline{M}_{S,\bar{s}}) \times N$ and N is isomorphic to the monoid arising from monomials of*
$$\mathbb{Z}[U_1, U_2, \ldots, U_a]/(U_1^2 - U_2^2)$$
for some $a \geq 2$. In this case, the characteristic of the residue field of $\mathcal{O}_{X,\bar{x}}$ is not equal to 2, and every element of N is regular.
3. *If f is not smooth at x and $\bar{h}_{\bar{x}} : \overline{M}_{S,\bar{s}} \to \overline{M}_{X,\bar{x}}$ does not split, then $\overline{M}_{X,\bar{x}}$ has a semistable structure (σ, q_0, Δ, B) over $\overline{M}_{S,\bar{s}}$ for some $\sigma \subseteq \overline{M}_{X,\bar{x}}$ with $\#(\sigma) \geq 2$, $q_0 \in \overline{M}_{S,\bar{s}}$ and $\Delta, B \in \mathbb{N}^{\sigma}$* (*For the definition of semistable structure, see Conventions and terminology* 10)*. More precisely, σ is the set of all irreducible elements of $\overline{M}_{X,\bar{x}}$ not lying in $\bar{h}_{\bar{x}}(\overline{M}_{S,\bar{s}})$. Further every element of $\sigma \setminus \mathrm{Supp}(\Delta)$ is regular.*

Proof. Let us begin with the following lemma.

Lemma 3.2. *Let $f : X \to Y$ be a morphism of schemes and M_Y a fine log structure on Y. If we set $M_X = f^*(M_Y)$, then, for any $x \in X$ and $y \in Y$ with $y = f(x)$, the induced homomorphism $\overline{M}_{Y,\bar{y}} \to \overline{M}_{X,\bar{x}}$ is bijective.*

Proof. Let P be a chart of $M_{Y,\bar{y}}$. Then $\overline{M}_{Y,\bar{y}}$ and $\overline{M}_{X,\bar{x}}$ are given by
$$P/\pi^{-1}(\mathcal{O}^{\times}_{Y,\bar{y}}) \quad \text{and} \quad P/\pi'^{-1}(\mathcal{O}^{\times}_{X,\bar{x}})$$
respectively, where $\pi : P \to \mathcal{O}_{Y,\bar{y}}$ is the canonical morphism and $\pi' : P \to \mathcal{O}_{Y,\bar{y}} \to \mathcal{O}_{X,\bar{x}}$ (cf. [3] and [5]). Thus it is sufficient to see that $\pi^{-1}(\mathcal{O}^{\times}_{Y,\bar{y}}) = \pi'^{-1}(\mathcal{O}^{\times}_{X,\bar{x}})$.

Indeed, letting $m_{\bar{x}}$ and $m_{\bar{y}}$ be the maximal ideals of $\mathcal{O}_{X,\bar{x}}$ and $\mathcal{O}_{Y,\bar{y}}$, and $\alpha : \mathcal{O}_{Y,\bar{y}} \to \mathcal{O}_{X,\bar{x}}$ the canonical homomorphism,

$$p \in \pi^{-1}(\mathcal{O}_{Y,\bar{y}}^{\times}) \iff \pi(p) \in \mathcal{O}_{Y,\bar{y}}^{\times} \iff \alpha(\pi(p)) \in \mathcal{O}_{X,\bar{x}}^{\times} \iff p \in {\pi'}^{-1}(\mathcal{O}_{X,\bar{x}}^{\times})$$

because $\alpha(m_{\bar{y}}) \subseteq m_{\bar{x}}$ and $\alpha(\mathcal{O}_{Y,\bar{y}}^{\times}) \subseteq \mathcal{O}_{X,\bar{x}}^{\times}$. □

Let us go back to the proof of Theorem 3.1. Let us consider the geometric fiber $X \times_S \operatorname{Spec}(\overline{\kappa(s)})$ over s. Then, by using Lemma 3.2, we may assume that $S = \operatorname{Spec}(k)$ for some algebraically closed field k. Thus the theorem follows from [5, Theorem 3.1] except the following facts:

(i) In the case (2), N is isomorphic to the monoid T arising from monomials of
$$\mathbb{Z}[U_1, U_2, \ldots, U_a]/(U_1^2 - U_2^2).$$

(ii) In the case (2) or (3), the regularity of elements of either N or $\sigma \setminus \operatorname{Supp}(\Delta)$.

(i) Let T_k be the the monoid arising from monomials of
$$k[U_1, U_2, \ldots, U_a]/(U_1^2 - U_2^2).$$
In order to see (i), we need to show the natural homomorphism $T \to T_k$ is bijective. Let $\bar{U}_1^{e_1}\bar{U}_2^{e_2}\cdots\bar{U}_a^{e_a}$ and $\bar{U}_1^{e'_1}\bar{U}_2^{e'_2}\cdots\bar{U}_a^{e'_a}$ be elements of T. Clearly we may assume that $e_1, e'_1 \in \{0,1\}$. We suppose that $\bar{U}_1^{e_1}\bar{U}_2^{e_2}\cdots\bar{U}_a^{e_a} = \bar{U}_1^{e'_1}\bar{U}_2^{e'_2}\cdots\bar{U}_a^{e'_a}$ in $k[U_1, U_2, \ldots, U_a]/(U_1^2 - U_2^2)$. Then there is $\phi \in k[U_1, \ldots. U_a]$ with
$$U_1^{e_1}U_2^{e_2}\cdots U_a^{e_a} - U_1^{e'_1}U_2^{e'_2}\cdots U_a^{e'_a} = (U_1^2 - U_2^2)\phi.$$
Comparing the degrees with respect to U_1 of both sides, we can see that $\phi = 0$. Therefore $(e_1, \ldots, e_a) = (e'_1, \ldots, e'_a)$.

(ii) Let $(\mathcal{O}_{S,s}, m_{S,s}) \to (A, m)$ be a flat local homomorphism of local rings. We set $S' = \operatorname{Spec}(A)$, $X' = X \times_S S'$ and the induced morphisms as follows:

$$\begin{array}{ccc} X' & \xrightarrow{\pi'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\pi} & S. \end{array}$$

Let us choose $x' \in X'$ with $f'(x') = m$ and $\pi'(x') = x$. Then, since $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, using Lemma 3.2, if regularity holds at x', then so does at x.

Let k be the algebraic closure of the residue field at x. Note that by virtue of [EGA III, Chapter 0, 10.3.1], there are a noetherian local ring (A, m) and a local homomorphism $(\mathcal{O}_{S,s}, m_{S,s}) \to (A, m)$ such that $m_{S,s}A = m$, A/m is isomorphic to k over $\mathcal{O}_{S,s}/m_{S,s}$ and that A is flat over $\mathcal{O}_{S,s}$. Therefore we may assume that $\mathcal{O}_{S,s}/m_{S,s}$ is algebraically closed and x is a closed point. Moreover, by using Proposition 2.1.1, we may further assume that there are a fine and sharp monoid Q and a homomorphism $\pi_Q : Q \to M_{S,s}$ such that $Q \to M_{S,\bar{s}} \to \overline{M}_{S,\bar{s}}$ is bijective. Hence, by [5], there is a fine and sharp monoid P together with homomorphisms $f : Q \to P$ and $\pi_P : P \to M_{X,\bar{s}}$ such that the following properties are satisfied:

(a) The diagram

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ {\scriptstyle \pi_Q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_P} \\ M_{S,\bar{s}} & \longrightarrow & M_{X,\bar{x}}, \end{array}$$

is commutative.

(b) The induced homomorphism $P \to M_{X,\bar{x}} \to \overline{M}_{X,\bar{x}}$ is bijective.

(c) The natural homomorphism

$$\mathcal{O}_{S,\bar{s}} \otimes_{\mathcal{O}_{S,\bar{s}}[Q]} \mathcal{O}_{S,\bar{s}}[P] \to \mathcal{O}_{X,\bar{s}}$$

is smooth.

Since $\mathcal{O}_{S,\bar{s}} \otimes_{\mathcal{O}_{S,\bar{s}}[Q]} \mathcal{O}_{S,\bar{s}}[P] \to \mathcal{O}_{X,\bar{s}}$ is smooth, it is sufficient to see the regularity of each element in $\mathcal{O}_{S,\bar{s}} \otimes_{\mathcal{O}_{S,\bar{s}}[Q]} \mathcal{O}_{S,\bar{s}}[P]$.

If there is a submonoid N of P with $P = f(Q) \times N$, then

$$\mathcal{O}_{S,\bar{s}} \otimes_{\mathcal{O}_{S,\bar{s}}[Q]} \mathcal{O}_{S,\bar{s}}[P] = \mathcal{O}_{S,\bar{s}}[N].$$

Thus the assertions follow from Lemma 3.3 below.

Next we assume that $f : Q \to P$ does not splits. Let us set $\sigma = \{p_1, \ldots, p_r\}$ such that $\mathrm{Supp}(\Delta) = \{p_1, \ldots, p_l\}$. Moreover we set $x_i = \alpha(\pi_P(p_i))$ and $t = \beta(\pi_Q(q_0))$, where $\alpha : M_X \to \mathcal{O}_X$ and $\beta : M_{S,\bar{s}} \to \mathcal{O}_{S,\bar{s}}$ are the canonical homomorphisms. Then

$$\mathcal{O}_{S,\bar{s}} \otimes_{\mathcal{O}_{S,\bar{s}}[Q]} \mathcal{O}_{S,\bar{s}}[P] = \mathcal{O}_{S,\bar{s}}[X_1, \ldots, X_r]/(X_1 \cdots X_l - tX_{l+1}^{b_{l+1}} \cdots X_r^{b_r}),$$

where $b_i = B(p_i)$ and x_i is the class of X_i. Thus the assertions follow from Lemma 3.3 below. □

Lemma 3.3. *Let A be a ring. Then we have the following:*

(1) *Let $A[X]$ be the polynomial ring of one variable over A. For a regular element $a \in A$, X is regular in $A[X]/(X^2 - a)$, that is, the multiplication of X in $A[X]/(X^2 - a)$ is injective.*

(2) *We assume A is a local ring with the maximal ideal m. Let $A[X_1, \ldots, X_l]$ be the polynomial ring of l-variables over A. For $a \in m$, let us consider a ring R given by $R = A[X_1, \ldots, X_l]/(X_1 \cdots X_l - a)$. If α is a regular element of A, then so is α in R.*

Proof. (1) We assume that $Xf(X) = (X^2 - a)g(X)$ for some $f(X), g(X) \in A[X]$. We set $g(X) = Xh(X) + c$ for some $h(X) \in A[X]$ and $c \in A$. Then

$$ca = X(h(X)(X^2 - a) + cX - f(X)).$$

Thus, $ca = 0$. Since a is regular, c must be zero. Therefore

$$Xf(X) = X(X^2 - a)h(X),$$

which implies $f(X) = (X^2 - a)h(X)$ because X is regular in $A[X]$.

(2) Let $\widehat{R}$ be the completion with respect to $(m, X_1, \ldots, X_n)$. Since $R \to \widehat{R}$ is faithfully flat, it is sufficient to see the homomorphism $\widetilde{\alpha} : \widehat{R} \to \widehat{R}$ given by the multiplication of α is injective. Note that $\widehat{R}$ is the direct products of many copies of $\widehat{A}$ by Lemma 1.1.2. Thus $\widetilde{\alpha}$ is injective. □

Remark 3.4. The semistable structure of $\bar{h}_{\bar{x}} : \overline{M}_{S,\bar{s}} \to \overline{M}_{X,\bar{x}}$ in the case (3) of Theorem 3.1 is uniquely determined by virtue of a result in [6], which is not needed in this paper.

4. Rigidity theorem

First of all, we would like to define the admissibility of morphisms. Let k be an algebraically closed field, and let $\phi : X \to Y$ be a morphism of algebraic schemes over k. Let Z be a subscheme of Y. We say ϕ is *admissible with respect to* Z if, for any irreducible component X' of X, $\phi(X') \not\subseteq Z$.

Let $f : X \to S$ and $g : Y \to S$ be schemes of finite type over a locally noetherian scheme S, and let M_Y and M_S be fine log structures of Y and S such that g extends to a log morphism $(Y, M_Y) \to (S, M_S)$. As in Corollary 2.2.3, the closed set $\mathrm{Supp}(M_Y/M_S)$ is given by

$$\{y \in Y \mid M_{S,\overline{g(y)}} \times \mathcal{O}_{Y,\bar{y}}^{\times} \to M_{Y,\bar{y}} \text{ is not surjective}\}.$$

Let $\phi : X \to Y$ be a morphism over S. For a point $s \in S$, we say $\phi : X \to Y$ is *admissible over* s *with respect to* M_Y/M_S, if

$$\phi \times_S \mathrm{id}_{\mathrm{Spec}(\overline{\kappa(s)})} : X \times_S \mathrm{Spec}(\overline{\kappa(s)}) \to Y \times_S \mathrm{Spec}(\overline{\kappa(s)})$$

is admissible with respect to $\mathrm{Supp}(M_Y/M_S)|_{Y \times_S \mathrm{Spec}(\overline{\kappa(s)})}$. If $\phi : X \to Y$ is admissible over any points of S with respect to M_Y/M_S, then ϕ is said to be *admissible with respect to* M_Y/M_S. By (2) of Corollary 2.2.3, ϕ is admissible over s with respect to M_X/M_S if and only if

$$\phi \times_S \mathrm{id}_{\mathrm{Spec}(\overline{\kappa(s)})} : X \times_S \mathrm{Spec}(\overline{\kappa(s)}) \to Y \times_S \mathrm{Spec}(\overline{\kappa(s)})$$

is admissible with respect to $(M_Y|_{Y \times_S \mathrm{Spec}(\overline{\kappa(s)})})/(M_S|_{\mathrm{Spec}(\overline{\kappa(s)})})$.

The following theorem is the main theorem of this paper.

Theorem 4.1. *Let X, Y and S be locally noetherian schemes, and let M_X, M_Y and M_S be fine log structures of X, Y and S respectively. Let $(X, M_X) \to (S, M_S)$ and $(Y, M_Y) \to (S, M_S)$ be integral and log smooth morphisms, and let $\phi : X \to Y$ be a morphism over S. Let us fix a point $s \in S$. We assume that $X \to S$ and $Y \to S$ are semistable at any points lying over s and that $\phi : X \to Y$ is admissible over s with respect to M_Y/M_S. If*

$$(\phi, h) : (X, M_X) \to (Y, M_Y) \quad \textit{and} \quad (\phi, h') : (X, M_X) \to (Y, M_Y)$$

are extensions of $\phi : X \to Y$ as log morphisms over (S, M_S), then, for all closed points x lying over s, $h_{\bar{x}} = h'_{\bar{x}}$ as homomorphisms $M_{Y,\overline{\phi(x)}} \to M_{X,\bar{x}}$ of the germs of étale topology.

Proof. Since this is a local problem, we may assume that $S = \mathrm{Spec}(A)$ for a noetherian local ring (A, m). Let $\rho : (A, m) \to (B, n)$ be a local homomorphism of local rings such that B/n is algebraic over A/m. We denote the closed point of S by s and the closed point of $S' = \mathrm{Spec}(B)$ by s'. We set $X' = X \times_S S'$, $Y' = Y \times_S S'$, $M_{X'} = \pi_X^*(M_X)$, $M_{Y'} = \pi_Y^*(M_Y)$, and $M_{S'} = \pi_S^*(M_S)$, where $\pi_X : X' \to X$, $\pi_Y : Y' \to Y$ and $\pi_S : S' \to S$ are the canonical morphisms. Let $\phi_{S'} : X' \to Y'$ be the morphism given by $\phi_{S'} = \phi \times_S \mathrm{id}_{S'}$.

$$\begin{tikzcd}
X' \arrow[rr, "\pi_X"] \arrow[d, "\phi_{S'}"'] \arrow[ddr] & & X \arrow[d, "\phi"'] \arrow[ddr] & \\
Y' \arrow[rr, "\pi_Y"] \arrow[dr] & & Y \arrow[dr] & \\
& S' \arrow[rr, "\pi_S"] & & S
\end{tikzcd}$$

Then we have log morphisms

$$(\phi_{S'}, h_{S'}), (\phi_{S'}, h'_{S'}) : (X', M_{X'}) \to (Y', M_{Y'})$$

over $(S', M_{S'})$, where $h_{S'}$ and $h'_{S'}$ are the homomorphisms induced by h and h' respectively.

Claim 4.1.1. *If ρ is flat and $h_{S',\bar{x}'} = h'_{S',\bar{x}'}$ for all closed points x' lying over s', then $h_{\bar{x}} = h'_{\bar{x}}$ for all closed points x lying over s.*

Let us choose a closed point $x \in X$ over s. Then there is a closed point $x' \in X'$ such that $\pi_X(x') = x$ and x' is lying over s'. If we set $y = \phi(x)$ and $y' = \phi_{S'}(x')$, then $\pi_Y(y') = y$. Here we consider the natural commutative diagram:

$$\begin{tikzcd}
\overline{M}_{Y,\bar{y}} \arrow[r] \arrow[d, shift right, "\bar{h}_{\bar{x}}"'] \arrow[d, shift left, "\bar{h}'_{\bar{x}}"] & \overline{M}_{Y',\bar{y}'} \arrow[d, "\bar{h}_{S',\bar{x}'} = \bar{h}'_{S',\bar{x}'}"] \\
\overline{M}_{X,\bar{x}} \arrow[r] & \overline{M}_{X',\bar{x}'}
\end{tikzcd}$$

By Lemma 3.2, $\overline{M}_{Y,\bar{y}} \to \overline{M}_{Y',\bar{y}'}$ and $\overline{M}_{X,\bar{x}} \to \overline{M}_{X',\bar{x}'}$ are bijective. Thus we can see that $\bar{h}_{\bar{x}} = \bar{h}'_{\bar{x}}$. Let us pick up $w \in M_{Y,\bar{y}}$. Then, since $\bar{h}_{\bar{x}} = \bar{h}'_{\bar{x}}$, there is $u \in \mathcal{O}_{X,\bar{x}}^{\times}$ with $h_{\bar{x}}(w) = h_{\bar{x}}(w) \cdot u$. Here $h_{S',\bar{x}'} = h'_{S',\bar{x}'}$. Thus u must be 1 in $\mathcal{O}_{X',\bar{x}'}$. Note that $\mathcal{O}_{X',\bar{x}'}$ is flat over $\mathcal{O}_{X,\bar{x}}$. Therefore u is the identity in $\mathcal{O}_{X,\bar{x}}$.

Let I be an ideal of A with $I^2 = \{0\}$, and $B = A/I$. Next we consider a case where ρ is given by the natural homomorphism $A \to B$.

Claim 4.1.2. *We assume that* (i) *$k = A/m$ is algebraically closed and* (ii) *there are a fine and sharp monoid Q and a homomorphism $\pi_Q : Q \to M_{S,s}$ such that $Q \to M_{S,\bar{s}} \to \overline{M}_{S,\bar{s}}$ is bijective. If $h_{S',\bar{x}'} = h'_{S',\bar{x}'}$ for all closed points x' lying over s', then $h_{\bar{x}} = h'_{\bar{x}}$ for all closed points x lying over s.*

Let x be a closed point of X lying over s, and $y = \phi(x)$. First of all, by [5], there are finite and sharp monoids P and P' and homomorphisms $P \to M_{X,\bar{x}}$, $Q \to P$, $P' \to M_{Y,\bar{y}}$, $Q \to P'$ with the following properties:

(1) The induced homomorphisms $P \to M_{X,\bar{x}} \to \overline{M}_{X,\bar{x}}$ and $P' \to M_{Y,\bar{y}} \to \overline{M}_{Y,\bar{y}}$ are bijective.

(2) The following diagrams are commutative:

$$\begin{tikzcd}
Q \arrow[r, "f"] \arrow[d] & P \arrow[d] \\
M_{S,\bar{s}} \arrow[r] & M_{X,\bar{x}},
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "f'"] \arrow[d] & P' \arrow[d] \\
M_{S,\bar{s}} \arrow[r] & M_{Y,\bar{y}}.
\end{tikzcd}$$

(3) There are étale neighborhoods (U, x') and (V, y') of x and y such that $P \to M_{X,\bar{x}}$ and $P' \to M_{Y,\bar{y}}$ are defined over U and V respectively, and that the natural morphisms
$$U \to \operatorname{Spec}(A \otimes_{A[Q]} A[P]) \quad \text{and} \quad V \to \operatorname{Spec}(A \otimes_{A[Q']} A[P'])$$
are smooth at x' and y' respectively.

Clearly we may assume that P, P' and Q are submonoids of $M_{X,\bar{x}}$, $M_{Y,\bar{y}}$ and $M_{S,\bar{s}}$ respectively. We set $U_s = U \times_S \operatorname{Spec}(\kappa(s))$, $V_s = V \times_S \operatorname{Spec}(\kappa(s))$, $\phi_s = \phi \times_S \mathrm{id}_{\operatorname{Spec}(\kappa(s))}$, $M_{U_s} = M_X|_{U_s}$, $M_{V_s} = M_Y|_{V_s}$ and $M_k = M_S|_{\operatorname{Spec}(\kappa(s))}$. By Lemma 4.3 below, the admissibility of ϕ_s guarantees that for any irreducible components T of U_s, $\phi_s(T) \not\subseteq \operatorname{Supp}(M_{V_s}/M_k)$.

Let σ (resp. σ') be the set of all irreducible elements of P not lying in $f(Q)$ (resp. the set of all irreducible elements of P' not lying in $f'(Q)$). For $j \in \sigma$ and $i \in \sigma'$, we denote $\alpha(j)$ by x_j and $\alpha'(i)$ by y_i, where $\alpha : M_{X,\bar{x}} \to \mathcal{O}_{X,\bar{x}}$ and $\alpha' : M_{Y,\bar{y}} \to \mathcal{O}_{Y,\bar{y}}$ are the canonical homomorphisms. Moreover $x_j|_{U_s}$ and $y_i|_{V_s}$ are denoted by x_{js} and y_{is} respectively. Let us consider h and h' on the fibers $X_s = X \times_S \operatorname{Spec}(\kappa(s))$ and $Y_s = Y \times_S \operatorname{Spec}(\kappa(s))$ over s. Using Lemma 3.2 and [5, Theorem 4.1], $\bar{h}_{\bar{x}} = \bar{h}'_{\bar{x}}$ as $P' \to P$. Thus we can set as follows:

$$h_{\bar{x}}(i) = u_i \cdot (I_i \cdot \sigma + f(q_i)) \quad \text{and} \quad h'_{\bar{x}}(i) = u'_i \cdot (I_i \cdot \sigma + f(q_i)), \tag{4.1.3}$$

where $q_i \in Q$, $I_i \in \mathbb{N}^{\sigma}$ and $u_i, u'_i \in \mathcal{O}_{X,\bar{x}}^{\times}$. Then we have

$$\phi^*(y_i) = \beta(q_i) \cdot x^{I_i} \cdot u_i = \beta(q_i) \cdot x^{I_i} \cdot u'_i, \tag{4.1.4}$$

where $\beta : M_{S,\bar{s}} \to \mathcal{O}_{S,\bar{s}}$ is the canonical homomorphism. We claim the following:

(4.1.5) If $\phi_s^*(y_{is}) \neq 0$ for some $i \in \sigma'$, then $q_i = 0$ and $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$.

Indeed, by (4.1.4), $\phi_s^*(y_{is}) = \beta_s(q_i) \cdot x_s^{I_i} \cdot u_{is}$ on U_s, where $\beta_s : Q \to k$ is a homomorphism given by
$$\beta_s(q) = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$
and $u_{is} = u_i|_{U_s}$. Thus $q_i = 0$, which yields $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$.

Here we consider the following four cases:

(A) $f : Q \to P$ splits and $f' : Q \to P'$ splits.
(B) $f : Q \to P$ does not split and $f' : Q \to P'$ splits.
(C) $f : Q \to P$ splits and $f' : Q \to P'$ does not split.
(D) $f : Q \to P$ does not split and $f' : Q \to P'$ does not split.

(Case A): In this case, there are submonoids N and N' of P and P' respectively such that $P = f(Q) \times N$ and $P' = f'(Q) \times N'$. Note that σ and σ' are nothing more than the set of all irreducible elements of N and N' respectively. Then, by the local structure theorem (cf. Theorem 3.1),
$$\operatorname{Supp}(M_{V_s}/M_k) = \bigcup_{i \in \sigma'} \{y_{is} = 0\}.$$
around y' on V_s. Thus, using the admissibility of ϕ_s, $\phi_s^*(y_{is}) \neq 0$. Hence, by (4.1.5), $q_i = 0$ and $x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$ for all $i \in \sigma'$. Therefore $u_i = u'_i$ for all $i \in \sigma'$ because x_j's are regular elements (cf. Theorem 3.1).

(Case B): In this case, there is a submonoid N' of P' such that $P' = f'(Q) \times N'$. Moreover P is of semistable type

$$(\sigma, q_0, \Delta, B)$$

over Q for some $q_0 \in Q$ and $\Delta, B \in \mathbb{N}^{\sigma}$. By the local structure theorem (cf. Theorem 3.1),

$$\operatorname{Supp}(M_{V_s}/M_k) = \bigcup_{i \in \sigma'} \{y_{is} = 0\}$$

around y' on V_s. Thus, by the admissibility of ϕ_s, $\phi_s^*(y_{is}) \neq 0$. Therefore, by (4.1.5), $q_i = 0$ and $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$ for all $i \in \sigma'$. Since U_s is given by $\prod_{j \in \operatorname{Supp}(\Delta)} x_{js} = 0$, if $j \in \operatorname{Supp}(I_i) \cap \operatorname{Supp}(\Delta)$, then $\phi_s^*(y_{is}) = 0$ on the irreducible component $\{x_{js} = 0\}$ of U_s. This contradicts to the admissibility of ϕ_s. Hence $\operatorname{Supp}(I_i) \cap \operatorname{Supp}(\Delta) = \emptyset$ for all $i \in \sigma'$. Thus x^{I_i}'s are regular elements (cf. Theorem 3.1). Therefore $u_i = u'_i$ for all $i \in \sigma'$.

(Case C): In this case, there is a submonoid N of P with $P = f(Q) \times N$. P' is of semistable type

$$(\sigma', q'_0, \Delta', B')$$

over Q for some $q'_0 \in Q$ and $\Delta', B' \in \mathbb{N}^{\sigma'}$. Note that

$$\operatorname{Supp}(M_{V_s}/M_k) = \operatorname{Sing}(V_s) \cup \bigcup_{i \in \sigma' \setminus \operatorname{Supp}(\Delta')} \{y_{is} = 0\}.$$

around y' on V_s (cf. Theorem 3.1).

Let us see that if $\phi_s^*(y_{is}) \neq 0$ for some $i \in \sigma'$, then $q_i = 0$ and $u_i = u'_i$. Indeed, by (4.1.5), we have $q_i = 0$ and $x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$. Thus $u_i = u'_i$ because x^{I_i}'s are regular elements (cf. Theorem 3.1).

Therefore we may assume that there is $i_0 \in \sigma'$ with $\phi_s^*(y_{i_0 s}) = 0$. By using the admissibility of ϕ_s, $\phi_s^*(y_{is}) \neq 0$ for $i \in \sigma' \setminus \operatorname{Supp}(\Delta')$. Thus $i_0 \in \operatorname{Supp}(\Delta')$. Moreover, if $\phi_s^*(y_{i_1 s}) = 0$ for $i_1 \in \operatorname{Supp}(\Delta') \setminus \{i_0\}$, then

$$\phi_s(U_s) \subseteq \{y_{i_0 s} = y_{i_1 s} = 0\} \subseteq \operatorname{Sing}(V_s),$$

which contradicts to the admissibility of ϕ_s. Thus $\phi_s^*(y_{is}) \neq 0$ for all $i \in \sigma' \setminus \{i_0\}$. Hence $u_i = u'_i$ for all $i \in \sigma' \setminus \{i_0\}$. Let us consider a relation

$$\Delta' \cdot \sigma' = f'(q'_0) + B' \cdot \sigma'.$$

Then we have

$$\begin{cases} \sum_{i \in \operatorname{Supp}(\Delta')} h_{\bar{x}}(i) = f(q'_0) + \sum_{i \in \operatorname{Supp}(B')} B'(i) h_{\bar{x}}(i) \\ \sum_{i \in \operatorname{Supp}(\Delta')} h'_{\bar{x}}(i) = f(q'_0) + \sum_{i \in \operatorname{Supp}(B')} B'(i) h'_{\bar{x}}(i) \end{cases}$$

Here $h_{\bar{x}}(i) = h'_{\bar{x}}(i)$ for all $i \neq i_0$. Thus we can see that $h_{\bar{x}}(i_0) = h'_{\bar{x}}(i_0)$.

(Case D): In the final case, P and P' are of semistable type

$$(\sigma, q_0, \Delta, B) \quad \text{and} \quad (\sigma', q'_0, \Delta', B')$$

over Q for some $q_0, q'_0 \in Q$, $\Delta, B \in \mathbb{N}^{\sigma}$ and $\Delta', B' \in \mathbb{N}^{\sigma'}$. For $j \in \operatorname{Supp}(\Delta)$ and $i \in \operatorname{Supp}(\Delta')$, let U_{js} and V_{is} be the irreducible components of U_s and V_s given by $x_{js} = 0$ and $y_{is} = 0$ respectively. By the admissibility of ϕ_s, for each $j \in \operatorname{Supp}(\Delta)$, there is a unique $i \in \mathrm{Supp}(\Delta')$ with $\phi_s(U_{js}) \subseteq V_{is}$. This i is denoted by $\mu(j)$. Note that

$$\mathrm{Supp}(M_{V_s}/M_k) = \mathrm{Sing}(V_s) \cup \bigcup_{i \in \sigma' \setminus \mathrm{Supp}(\Delta')} \{y_{is} = 0\}$$

around y' on V_s. Here we claim the following:

(i) If $i \neq \mu(j)$ for $i \in \sigma'$ and $j \in \mathrm{Supp}(\Delta)$, then $\phi^*(y_i)|_{U_{js}} \neq 0$.
(ii) If there is $j \in \mathrm{Supp}(\Delta)$ with $i \neq \mu(j)$, then $q_i = 0$ and $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$.
(iii) If $i \notin \mu(\mathrm{Supp}(\Delta))$, then $q_i = 0$ and $u_i = u'_i$.
(iv) If $i, i' \in \mathrm{Supp}(\Delta')$ and $i \neq i'$, then $\mathrm{Supp}(I_i) \cap \mathrm{Supp}(I_{i'}) = \emptyset$.

(i) is obvious by the admissibility of ϕ_s. (ii) is a consequence of (i) and (4.1.5). Let us see (iii). By (ii), $q_i = 0$ and $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$. Using (i), $\phi^*(y_i)|_{U_{js}} \neq 0$ for all $j \in \mathrm{Supp}(\Delta)$. Thus $\mathrm{Supp}(I_i) \cap \mathrm{Supp}(\Delta) = \emptyset$. Hence x^{I_i} is a regular element (cf. Theorem 3.1). Therefore $u_i = u'_i$. Finally we consider (iv). We assume that $j \in \mathrm{Supp}(I_i) \cap \mathrm{Supp}(I_{i'})$. Then, since $\phi^*(y_l) = \beta(q_l) \cdot x^{I_l} \cdot u_l$ for all $l \in \sigma'$,

$$\phi(U_{js}) \subseteq \{y_{is} = y_{i's} = 0\} \subseteq \mathrm{Sing}(V_s),$$

which contradicts to the admissibility of ϕ_s.

First we consider the case where $\#\mu(\mathrm{Supp}(\Delta)) = 1$, i.e., $\mu(\mathrm{Supp}(\Delta)) = \{i_0\}$ for some $i_0 \in \mathrm{Supp}(\Delta')$. Then, by (iii), for $i \neq i_0$, $q_i = 0$ and $u_i = u'_i$. Considering a relation:

$$\Delta' \cdot \sigma' = f'(q'_0) + B' \cdot \sigma',$$

we have

$$\begin{cases} \sum_{i \in \mathrm{Supp}(\Delta')} h_{\bar{x}}(i) = f(q'_0) + \sum_{i \in \mathrm{Supp}(B')} B'(i) h_{\bar{x}}(i) \\ \sum_{i \in \mathrm{Supp}(\Delta')} h'_{\bar{x}}(i) = f(q'_0) + \sum_{i \in \mathrm{Supp}(B')} B'(i) h'_{\bar{x}}(i) \end{cases}$$

Since $h_{\bar{x}}(i) = h'_{\bar{x}}(i)$ for all $i \neq i_0$, we can see that $h_{\bar{x}}(i_0) = h'_{\bar{x}}(i_0)$.

Next let us consider the case where $\#\mu(\mathrm{Supp}(\Delta)) \geq 2$. In this case, by (ii), $q_i = 0$ and $\phi^*(y_i) = x^{I_i} \cdot u_i = x^{I_i} \cdot u'_i$ for all $i \in \sigma'$. Moreover, by (iii), $u_i = u'_i$ for all $i \in \sigma' \setminus \mathrm{Supp}(\Delta')$. By our assumption, $u_i \equiv u'_i \mod I\mathcal{O}_{X,\bar{x}}$. Note that x_j ($j \notin \mathrm{Supp}(\Delta)$) is regular. Thus, if we set $I'_i = I_i|_{\mathrm{Supp}(\Delta)} \in \mathbb{N}^{\mathrm{Supp}(\Delta)}$, then

$$x^{I'_i} \cdot u_i = x^{I'_i} \cdot u'_i$$

for all $i \in \mathrm{Supp}(\Delta')$. By (iv), $\mathrm{Supp}(I'_i) \cap \mathrm{Supp}(I'_{i'}) = \emptyset$ for all $i \neq i' \in \mathrm{Supp}(\Delta')$. Further let us consider a relation

$$\Delta' \cdot \sigma' = f'(q'_0) + B' \cdot \sigma'.$$

Since $h_{\bar{x}}(i) = h'_{\bar{x}}(i)$ for all $\sigma' \setminus \mathrm{Supp}(\Delta')$, we have

$$\sum_{i \in \mathrm{Supp}(\Delta')} h_{\bar{x}}(i) = \sum_{i \in \mathrm{Supp}(\Delta')} h'_{\bar{x}}(i),$$

which implies $\prod_{i \in \mathrm{Supp}(\Delta')} u_i = \prod_{i \in \mathrm{Supp}(\Delta')} u'_i$. Here we set $v_i = u_i/u'_i$ for $i \in \mathrm{Supp}(\Delta)$. Then, gathering the above observations, we have seen that

$$\begin{cases} x^{I'_i} = x^{I'_i} \cdot v_i \text{ for all } i \in \mathrm{Supp}(\Delta'), \\ v_i \equiv 1 \mod I\mathcal{O}_{X,\bar{x}} \text{ for all } i \in \mathrm{Supp}(\Delta'), \\ \prod_{i \in \mathrm{Supp}(\Delta')} v_i = 1, \\ \mathrm{Supp}(I'_i) \cap \mathrm{Supp}(I'_{i'}) = \emptyset \text{ for all } i \neq i' \in \mathrm{Supp}(\Delta'). \end{cases}$$

Since $A \otimes_{A[Q]} A[P] \to \mathcal{O}_{U,u}$ is smooth, $A \otimes_{A[Q]} A[P \times \mathbb{N}^e] \to \mathcal{O}_{U,x'}$ is étale for some $e \geq 0$. Let o be the origin of $\operatorname{Spec}(A \otimes_{A[Q]} A[P \times \mathbb{N}^e])$. Then the residue field of $A \otimes_{A[Q]} A[P \times \mathbb{N}^e]$ at o is k. Moreover the residue fields of $\mathcal{O}_{U,x'}$ and $\mathcal{O}_{X,\bar{x}}$ are k because k is algebraically closed. Therefore the completion of $A \otimes_{A[Q]} A[P \times \mathbb{N}^e]$ at o is isomorphic to the completion of $\mathcal{O}_{X,\bar{x}}$. Thus, by Lemma 4.4 below, $v_i = 1$, that is, $u_i = u'_i$ for all $i \in \operatorname{Supp}(\Delta')$.

Let $k = A/m$ and $\bar{k}$ the algebraic closure of k. By virtue of [EGA III, Chapter 0, 10.3.1], there are a noetherian local ring (B, n) and a local homomorphism $A \to B$ such that $mB = n$, B/n is isomorphic to $\bar{k}$ over $k = A/m$ and that B is flat over A. Thus, by Claim 4.1.1, we may assume that the residue field $k = A/m$ is algebraically closed. Moreover, by Proposition 2.1.1, we may further assume that there are a find and sharp monoid Q and a homomorphism $\pi_Q : Q \to M_{S,s}$ such that $Q \to M_{S,\bar{s}} \to \overline{M}_{S,\bar{s}}$ is bijective.

Let $A_i = A/m^{i+1}$, $\rho_i : A_i \to A_{i-1}$ the canonical homomorphism and $I_i = \operatorname{Ker}(\rho_i)$. Then $A_0 = k$ and $I_i^2 = \{0\}$ for $i \geq 1$. We set $X_i = X \times_S \operatorname{Spec}(A_i)$, $M_{X_i} = M_X|_{X_i}$, $Y_i = Y \times_S \operatorname{Spec}(A_i)$, $M_{Y_i} = M_Y|_{Y_i}$. Moreover the induced morphisms $M_{Y_i} \to M_{X_i}$ and $M_{Y_i} \to M_{X_i}$ via h and h' are denoted by h_i and h'_i respectively. Note that $h_0 = h'_0$ at any closed points of X_s by [5]. By Claim 4.1.2, $h_n = h_n$ at any closed points lying over s implies that $h_{n+1} = h'_{n+1}$ at any closed points lying over s. Therefore we have $h_n = h'_n$ at any closed points of X_s for all $n \geq 0$. Let x be a closed point of X over s and $y = \phi(x)$. Since $\bar{h}_{\bar{x}} = \bar{h}'_{\bar{x}}$ as a homomorphism $\overline{M}_{Y,\bar{y}} \to \overline{M}_{X,\bar{x}}$, for $w \in M_{Y,\bar{y}}$, there is $u \in \mathcal{O}^{\times}_{X,\bar{x}}$ with $h_{\bar{x}}(w) = h'_{\bar{x}}(w) \cdot u$. Since $h_n = h'_n$, we can see that $u - 1 \in m^{n+1}\mathcal{O}_{X,\bar{x}}$. Note that $\mathcal{O}_{X,\bar{x}}$ is noetherian, which implies that $\bigcap_{n=0} m^{n+1}\mathcal{O}_{X,\bar{x}} = \{0\}$. Therefore $u = 1$. $\square$

As corollary of Theorem 4.1, we have the following:

Corollary 4.2 (Rigidity theorem). *Let $f : X \to S$ and $g : Y \to S$ be semistable schemes over a locally noetherian scheme S, and let $\phi : X \to Y$ be a morphism over S. Let M_X, M_Y and M_S be fine log structures on X, Y and S respectively. We assume that (X, M_X) and (Y, M_Y) are log smooth and integral over (S, M_S) and ϕ is admissible with respect to M_Y/M_S. If we have log morphisms*

$$(\phi, h) : (X, M_X) \to (Y, M_Y) \quad \text{and} \quad (\phi, h') : (X, M_X) \to (Y, M_Y)$$

over (S, M_S) as extensions of $\phi : X \to Y$, then $h = h'$.

The following two lemmas was needed for the proof of Theorem 4.1.

Lemma 4.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\pi'} & Y' \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\ X & \xrightarrow{\pi} & Y \end{array}$$

be a commutative diagram of reduced algebraic schemes over an algebraically closed field such that X and X' is equi-dimensional and μ is flat. Let Z be a closed subset of Y. If $\pi(T) \not\subseteq Z$ for any irreducible components T of X, then $\pi'(T') \not\subseteq \nu^{-1}(Z)$ for any irreducible components T' of X'.

Proof. We assume that $\pi'(T') \subseteq \nu^{-1}(Z)$ for an irreducible component T' of X'. Then

$$\pi(\mu(T')) = \nu(\pi'(T')) \subseteq \nu(\nu^{-1}(Z)) \subseteq Z.$$

Let T be the Zariski closure of $\mu(T')$. If $\dim T < \dim X$, then

$$\dim \mu^{-1}(x) \geq \dim T' - \dim T > \dim X' - \dim X$$

for $x \in \mu(T')$, which is a contradiction because μ is flat. Thus we have $\dim T = \dim X$, which means that T is an irreducible component of X. On the other hand, we know $\pi(T) \subseteq Z$. This is a contradiction to our assumption. Therefore we get our lemma. $\square$

Lemma 4.4. *Let (A, m) be a noetherian complete local ring and $A[\![X_1, \ldots, X_n]\!]$ the ring of formal power series of n-variables over A. For a fixed $a \in m$, let*

$$R = A[\![X_1, \ldots, X_n]\!]/(X_1 \cdots X_n - a)$$

and J an ideal of R with $J^2 = 0$. Let $u_1, \ldots u_l$ be elements of R and $I_1, \ldots, I_l$ elements of $\mathbb{N}^n$ with $\operatorname{Supp}(I_i) \cap \operatorname{Supp}(I_j) = \emptyset$ for $i \neq j$. We assume that (1) *$u_1 \cdots u_l = 1$,* (2) *$X^{I_i} u_i = X^{I_i}$ in R for all i, and that* (3) *$u_i \equiv 1 \mod J$. Then we have $u_1 = \cdots = u_l = 1$.*

Proof. We set $\Sigma = \{I \in \mathbb{N}^n \mid \Delta \not\leq I\}$ and

$$A[\![X_1, \ldots, X_n]\!]_\Sigma = \left\{ \sum_{I \in \Sigma} a_I X^I \mid a_I \in A \right\},$$

where $\Delta = (1, \ldots, 1)$. Then, by Lemma 1.1.2, the natural map $A[\![X_1, \ldots, X_n]\!]_\Sigma \to R$ is bijective. Here we claim the following:

Claim 4.4.1. *Let T be an element of $\mathbb{N}^n$. We set $\Sigma_T = \{I \in \Sigma \mid I + T \geq \Delta\}$. Then, for $f \in A[\![X_1, \ldots, X_n]\!]_\Sigma$, if $X^T f = 0$ in R, then f can be written by a form*

$$f = \sum_{I \in \Sigma_T} b_I X^I.$$

If either $T = (0, \ldots, 0)$ or $T \geq \Delta$, then our assertion is trivial. Thus we may assume that $T \neq (0, \ldots, 0)$ and $T \not\geq \Delta$. For $I \in \mathbb{N}^n$, we can find a non-negative integer a and $J \in \Sigma$ with $I = a\Delta + J$. We denote a and J by $a(I)$ and $J(I)$ respectively. Here let us see that $J(I + T) \notin \{S + T \mid S \in \Sigma \setminus \Sigma_T\}$ for $I \in \Sigma_T$. Indeed, since $I \in \Sigma_T$, we can find i with $I(i) = 0$ and $T(i) > 0$. Thus

$$J(I + T)(i) = T(i) - a(I + T) < T(i).$$

Hence $J(I + T) \notin \{S + T \mid S \in \Sigma \setminus \Sigma_T\}$.

We set $f = \sum_{I \in \Sigma} a_I X^I$. Then

$$\begin{aligned} X^T f &= \sum_{I \in \Sigma_T} a_I X^{I+T} + \sum_{I \in \Sigma \setminus \Sigma_T} a_I X^{I+T} \\ &= \sum_{I \in \Sigma_T} a_I a^{a(I+T)} X^{J(I+T)} + \sum_{I \in \Sigma \setminus \Sigma_T} a_I X^{I+T}. \end{aligned}$$

Thus $a_I = 0$ for $I \in \Sigma \setminus \Sigma_T$.

Since $u_i \equiv 1 \mod J$, there is $a_i \in J$ with $u_i = 1 + a_i$. Then $X^{I_i} a_i = 0$. Moreover, since $J^2 = 0$,

$$u_1 \cdots u_l = 1 + a_1 + \cdots + a_l = 1.$$

Hence $a_1 + \cdots + a_l = 0$. Since $X^{I_i} a_i = 0$, by the above claim, $a_i = \sum_{I \in \Sigma_{I_i}} c_{i,I} X^I$, where $\Sigma_{I_i} = \{I \in \Sigma \mid I + I_i \geq \Delta\}$. Therefore

$$\sum_{i=1}^{l} \sum_{I \in \Sigma_{I_i}} c_{i,I} X^I = 0.$$

Note that if $I \in \Sigma_{I_i}$ and $I' \in \Sigma_{I_j}$ for $i \neq j$, then $I \neq I'$ because $\operatorname{Supp}(I_i) \cap \operatorname{Supp}(I_j) = \emptyset$. Thus we can see that $c_{i,I} = 0$, which shows us $a_i = 0$ for all i. □

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