ON HOLOMORPHIC CURVES IN ALGEBRAIC TORUS

MASAKI TSUKAMOTO

Abstract. We study entire holomorphic curves in the algebraic torus, and show that they can be characterized by the "growth rate" of their derivatives.

1. Introduction

Let $z = x+y\sqrt{-1}$ be the natural coordinate in the complex plane $\mathbb{C}$, and let $f(z)$ be an entire holomorphic function in the complex plane. Suppose that there are a non-negative integer m and a positive constant C such that

$$|f(z)| \leq C|z|^m, \quad (|z| \geq 1).$$

Then $f(z)$ becomes a polynomial with $\deg f(z) \leq m$. This is a well-known fact in the complex analysis in one variable. In this paper, we prove an analogous result for entire holomorphic curves in the algebraic torus $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$.

Let $[z_0 : z_1 : \cdots : z_n]$ be the homogeneous coordinate in the complex projective space $\mathbb{C}P^n$. We define the complex manifold $X \subset \mathbb{C}P^n$ by

$$X := \{[1 : z_1 : \cdots : z_n] \in \mathbb{C}P^n |\, z_i \neq 0,\ (1 \leq i \leq n)\} \cong (\mathbb{C}^*)^n.$$

X is a natural projective embedding of $(\mathbb{C}^*)^n$. We use the restriction of the Fubini-Study metric as the metric on X. (Note that this metric is different from the natural flat metric on $(\mathbb{C}^*)^n$.)

For a holomorphic map $f : \mathbb{C} \to X$, we define its norm $|df|(z)$ by setting

$$|df|(z) := \sqrt{2}\,|df(\partial/\partial z)| \quad \text{for all } z \in \mathbb{C}. \tag{1}$$

Here $\partial/\partial z = \frac{1}{2}\left(\partial/\partial x - \sqrt{-1}\partial/\partial y\right)$, and the normalization factor $\sqrt{2}$ comes from $|\partial/\partial z| = 1/\sqrt{2}$.

The main result of this paper is the following.

Theorem 1.1. *Let $f : \mathbb{C} \to X$ be a holomorphic map. Suppose there are a non-negative integer m and a positive constant C such that*

$$|df|(z) \leq C|z|^m, \quad (|z| \geq 1). \tag{2}$$

2000 *Mathematics Subject Classification.* 32H30.

Key words and phrases. entire holomorphic curve in $(\mathbb{C}^*)^n$, polynomial growth.

Then there are polynomials $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ *with* $\deg g_i(z) \leq m+1$, $(1 \leq i \leq n)$, *such that*

$$f(z) = [1 : e^{g_1(z)} : e^{g_2(z)} : \cdots : e^{g_n(z)}]. \tag{3}$$

Conversely, if a holomorphic map $f(z)$ *is expressed by* (3) *with polynomials* $g_i(z)$ *of degree at most* $m+1$, $f(z)$ *satisfies the "polynomial growth condition"* (2).

The direction (3) $\Rightarrow$ (2) is easier, and the substantial part of the argument is the direction (2) $\Rightarrow$ (3).

If we set $m = 0$ in the above, we get the following corollary.

COROLLARY 1.2. *Let* $f : \mathbb{C} \to X$ *be a holomorphic map with bounded derivative, i.e.,* $|df|(z) \leq C$ *for some positive constant* C. *Then there are complex numbers* a_i *and* b_i, $(1 \leq i \leq n)$, *such that*

$$f(z) = [\,1 : e^{a_1 z + b_1} : e^{a_2 z + b_2} : \cdots : e^{a_n z + b_n}\,].$$

This is the theorem of [BD, Appendice]. The author also proves this in [T, Section 6].

REMARK 1.3. The essential point of Theorem 1.1 is the statement that the degrees of the polynomials $g_i(z)$ are at most $m+1$. Actually, it is easy to prove that if $f(z)$ satisfies the condition (2) then $f(z)$ can be expressed by (3) with polynomials $g_i(z)$ of degree at most $2m+2$. (See Section 4.)

Theorem 1.1 states that holomorphic curves in X can be characterized by the growth rate of their derivatives. We can formulate this fact more clearly as follows;

Let $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ be polynomials, and define $f : \mathbb{C} \to X$ by (3). We define the integer $m \geq -1$ by setting

$$m + 1 := \max(\deg g_1(z), \deg g_2(z), \cdots, \deg g_n(z)). \tag{4}$$

We have $m = -1$ if and only if f is a constant map. m can be obtained as the growth rate of $|df|$:

THEOREM 1.4. *If* $m \geq 0$, *we have*

$$\limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} = m.$$

COROLLARY 1.5. *Let* λ *be a non-negative real number, and let* $[\lambda]$ *be the maximum integer not greater than* λ. *Let* $f : \mathbb{C} \to X$ *be a holomorphic map, and suppose that there is a positive constant* C *such that*

$$|df|(z) \leq C|z|^{\lambda}, \quad (|z| \geq 1). \tag{5}$$

Then we have a positive constant C' *such that*

$$|df|(z) \leq C'|z|^{[\lambda]}, \quad (|z| \geq 1).$$

PROOF. If f is a constant map, the statement is trivial. Hence we can suppose f is not constant. From Theorem 1.1, f can be expressed by (3) with polynomials $g_i(z)$ of degree at most $[\lambda]+2$. Since f satisfies (5), we have

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq \lambda.$$

From Theorem 1.4, this shows $\deg g_i(z) \leq [\lambda]+1$ for all $g_i(z)$. Then, Theorem 1.1 gives the conclusion. □

2. Proof of (3) ⇒ (2)

Let $f : \mathbb{C} \to X$ be a holomorphic map. From the definition of X, we have holomorphic maps $f_i : \mathbb{C} \to \mathbb{C}^*$, $(1 \leq i \leq n)$, such that $f(z) = [1 : f_1(z) : \cdots : f_n(z)]$. The norm $|df|(z)$ in (1) is given by

$$(6) \qquad |df|^2(z) = \frac{1}{4\pi} \Delta \log \left(1 + \sum_{i=1}^{n} |f_i(z)|^2\right), \quad (\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}).$$

Suppose that f is expressed by (3), i.e., $f_i(z) = \exp(g_i(z))$ with a polynomial $g_i(z)$ of degree $\leq m+1$. We will repeatedly use the following calculation in this paper.

$$(7) \qquad \begin{aligned} |df|^2 &= \frac{1}{\pi}\left[\frac{\sum_i |f_i'|^2}{(1+\sum_i |f_i|^2)^2} + \frac{\sum_{i<j} |g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(1+\sum_i |f_i|^2)^2}\right], \\ &\leq \frac{1}{\pi}\left[\sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \sum_{i<j} \frac{|g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(|f_i|^2 + |f_j|^2)^2}\right], \\ &= \frac{1}{\pi}\left[\sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \sum_{i<j} \frac{|(f_i/f_j)'|^2}{(1+|f_i/f_j|^2)^2}\right], \\ &= \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2. \end{aligned}$$

Here we set

$$|df_i| := \frac{1}{\sqrt{\pi}} \frac{|f_i'|}{1+|f_i|^2} \quad \text{and} \quad |d(f_i/f_j)| := \frac{1}{\sqrt{\pi}} \frac{|(f_i/f_j)'|}{1+|f_i/f_j|^2}.$$

These are the norms of the differentials of the maps f_i, $f_i/f_j : \mathbb{C} \to \mathbb{C}P^1$.

We have $f_i(z) = \exp(g_i(z))$ and $f_i(z)/f_j(z) = \exp(g_i(z) - g_j(z))$, and the degrees of the polynomials $g_i(z)$ and $g_i(z) - g_j(z)$ are at most $m+1$. Then, the next Lemma gives the desired conclusion:

$$|df|(z) \leq C|z|^m, \quad (|z| \geq 1),$$

for some positive constant C.

LEMMA 2.1. *Let $g(z)$ be a polynomial of degree $\leq m+1$, and set $h(z) := e^{g(z)}$. Then we have a positive constant C such that*

$$|dh|(z) = \frac{1}{\sqrt{\pi}}\frac{|h'(z)|}{1+|h(z)|^2} \leq C|z|^m, \quad (|z| \geq 1).$$

PROOF. We have

$$\sqrt{\pi}\,|dh| = \frac{|g'|}{|h|+|h|^{-1}} \leq |g'| \min(|h|, |h|^{-1}) \leq |g'|.$$

Since the degree of $g'(z)$ is at most m, we easily get the conclusion. □

3. Preliminary estimates

In this section, k is a fixed positive integer.

The following is a standard fact in the Nevanlinna theory.

LEMMA 3.1. *Let $g(z)$ be a polynomial of degree k, and set $h(z) = e^{g(z)}$. Then we have a positive constant C such that*

$$\int_1^r \frac{dt}{t}\int_{|z|\leq t} |dh|^2(z)\, dxdy \leq Cr^k, \quad (r \geq 1).$$

PROOF. Since $|dh|^2 = \frac{1}{4\pi}\Delta \log(1+|h|^2)$, Jensen's formula gives

$$\int_1^r \frac{dt}{t}\int_{|z|\leq t} |dh|^2\, dxdy = \frac{1}{4\pi}\int_{|z|=r} \log(1+|h|^2)\, d\theta - \frac{1}{4\pi}\int_{|z|=1} \log(1+|h|^2)\, d\theta.$$

Here (r,θ) is the polar coordinate in the complex plane. We have

$$\log(1+|h|^2) \leq 2\,|\mathrm{Re}\, g(z)| + \log 2 \leq Cr^k, \quad (r := |z| \geq 1).$$

Thus we get the conclusion. □

Let I be a closed interval in $\mathbb{R}$ and let $u(x)$ be a real valued function defined on I. We define its $\mathcal{C}^1$-norm $\|u\|_{\mathcal{C}^1(I)}$ by setting

$$\|u\|_{\mathcal{C}^1(I)} := \sup_{x\in I} |u(x)| + \sup_{x\in I} |u'(x)|.$$

For a Lebesgue measurable set E in $\mathbb{R}$, we denote its Lebesgue measure by $|E|$.

LEMMA 3.2. *There is a positive number ε satisfying the following: If a real valued function $u(x) \in \mathcal{C}^1[0,\pi]$ satisfies*

$$\|u(x) - \cos x\|_{\mathcal{C}^1[0,\,\pi]} \leq \varepsilon,$$

then we have

$$|u^{-1}([-t,t])| \leq 4t \quad \textit{for any } t \in [0,\varepsilon].$$

PROOF. The proof is just an elementary calculus. For any small number $\delta > 0$, if we choose ε sufficiently small, we have

$$u^{-1}([-t,t]) \subset [\pi/2-\delta, \pi/2+\delta].$$

Let x_1 and x_2 be any two elements in $u^{-1}([-t,t])$. From the mean value theorem, we have $y \in [\pi/2-\delta, \pi/2+\delta]$ such that

$$u(x_1) - u(x_2) = u'(y)\,(x_1 - x_2).$$

From $\sin(\pi/2) = 1$, we can suppose that

$$|u'(y)| \geq 1/2.$$

Hence

$$|x_1 - x_2| \leq 2\,|u(x_1) - u(x_2)| \leq 4t.$$

Thus we get

$$|u^{-1}([-t,t])| \leq 4t.$$

□

Using a scale change of the coordinate, we get the following.

LEMMA 3.3. *There is a positive number ε satisfying the following: If a real valued function $u(x) \in \mathcal{C}^1[0,2\pi]$ satisfies*

$$\|u(x) - \cos kx\|_{\mathcal{C}^1[0,\,2\pi]} \leq \varepsilon,$$

then we have

$$|u^{-1}([-t,t])| \leq 8t \quad \textit{for any } t \in [0,\varepsilon].$$

PROOF.

$$u^{-1}([-t,t]) = \bigcup_{j=0}^{2k-1} u^{-1}([-t,t]) \cap [j\pi/k,\,(j+1)\pi/k].$$

Applying Lemma 3.2 to $u(x/k)$, we have

$$|u^{-1}([-t,t]) \cap [0,\pi/k]\,| \leq 4t/k.$$

In a similar way,

$$|u^{-1}([-t,t]) \cap [j\pi/k,\,(j+1)\pi/k]\,| \leq 4t/k, \quad (j = 0, 1, \cdots, 2k-1).$$

Thus we get the conclusion. □

Let E be a subset of $\mathbb{C}$. For a positive number r, we set

$$E(r) := \{\theta \in \mathbb{R}/2\pi\mathbb{Z} \,|\, re^{i\theta} \in E\}.$$

In the rest of the section, we always assume $k \geq 2$.

LEMMA 3.4. *Let C be a positive constant, and let $g(z) = z^k + a_1 z^{k-1} + \cdots + a_k$ be a monic polynomial of degree k. Set*

$$E := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g(z)| \leq C|z|\}.$$

Then we have a positive number r_0 such that

$$|E(r)| \leq 8C/r^{k-1}, \quad (r \geq r_0).$$

PROOF. Set $v(z) := \mathrm{Re}\,(a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k)$. Then we have

$$|\mathrm{Re}\, g(re^{i\theta})| \leq Cr \quad \Longleftrightarrow \quad |\cos k\theta + v(re^{i\theta})/r^k| \leq C/r^{k-1}.$$

Set $u(\theta) := \cos k\theta + v(re^{i\theta})/r^k$. It is easy to see that

$$\|u(\theta) - \cos k\theta\|_{\mathcal{C}^1[0,2\pi]} \leq \mathrm{const}/r, \quad (r \geq 1).$$

Then we can apply Lemma 3.3 to this $u(\theta)$, and we get

$$|E(r)| = |u^{-1}([-C/r^{k-1}, C/r^{k-1}])| \leq 8C/r^{k-1}, \quad (r \gg 1).$$

□

The following is the key lemma.

LEMMA 3.5. *Let $g(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ be a polynomial of degree k, $(a_0 \neq 0)$. Set*

$$E := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g(z)| \leq |z|\}.$$

Then we have a positive number r_0 such that

$$|E(r)| \leq \frac{8}{|a_0| r^{k-1}}, \quad (r \geq r_0).$$

PROOF. Let $\arg a_0$ be the argument of a_0, and set $\alpha := \arg a_0/k$. We define the monic polynomial $g_1(z)$ by

$$g_1(z) := \frac{1}{|a_0|} g(e^{-i\alpha} z) = z^k + \cdots .$$

Then we have

$$|\mathrm{Re}\, g(re^{i\theta})| \leq r \Longleftrightarrow |\mathrm{Re}\, g_1(re^{i(\theta+\alpha)})| \leq r/|a_0|.$$

Hence the conclusion follows from Lemma 3.4. □

LEMMA 3.6. *Let $g(z)$ be a polynomial of degree k, and we define E as in Lemma 3.5. Set $h(z) := e^{g(z)}$. Then we have*

$$\int_{\mathbb{C} \setminus E} |dh|^2(z)\, dxdy < \infty.$$

Proof. Since $|h| = e^{\operatorname{Re} g}$, the argument in the proof of Lemma 2.1 gives

$$\sqrt{\pi}\,|dh| \leq |g'| \min(|h|, |h|^{-1}) \leq |g'|\, e^{-|\operatorname{Re} g|}.$$

$g'(z)$ is a polynomial of degree $k-1$, and we have $|\operatorname{Re} g| > |z|$ for $z \in \mathbb{C} \setminus E$. Hence we have a positive constant C such that

$$|dh|(z) \leq C|z|^{k-1}e^{-|z|}, \quad \text{if } z \in \mathbb{C} \setminus E \text{ and } |z| \geq 1.$$

The conclusion follows from this estimate. □

4. Proof of (2) ⇒ (3)

Let $f = [1 : f_1 : f_2 : \cdots : f_n] : \mathbb{C} \to X$ be a holomorphic map with $|df|(z) \leq C|z|^m$, ($|z| \geq 1$). Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering, we have entire holomorphic functions $g_i(z)$ such that $f_i(z) = e^{g_i(z)}$. We will prove that all $g_i(z)$ are polynomials of degree $\leq m+1$. The proof falls into two steps. In the first step, we prove all $g_i(z)$ are polynomials. In the second step, we show $\deg g_i(z) \leq m+1$. The second step is the harder part of the proof.

Schwarz's formula gives[1]

$$\pi r^k g_i^{(k)}(0) = k! \int_{|z|=r} \operatorname{Re}\left(g_i(z)\right) e^{-k\sqrt{-1}\theta} d\theta = k! \int_{|z|=r} \log|f_i(z)|\, e^{-k\sqrt{-1}\theta} d\theta, \quad (k \geq 1).$$

We have

$$|\log|f_i|| \leq \log(|f_i| + |f_i|^{-1}) = \log(1 + |f_i|^2) - \log|f_i| \leq \log(1 + \sum |f_j|^2) - \log|f_i|.$$

Hence

$$\pi r^k |g_i^{(k)}(0)| \leq k! \int_{|z|=r} \log(1 + \sum |f_j|^2)\, d\theta - k! \int_{|z|=r} \log|f_i|\, d\theta.$$

Since $\log|f_i| = \operatorname{Re} g_i(z)$ is a harmonic function, the second term in the above is equal to the constant $-2\pi k! \operatorname{Re} g_i(0)$. Since $|df|^2 = \frac{1}{4\pi}\Delta \log(1 + \sum |f_j|^2)$, Jensen's formula gives

$$\frac{1}{4\pi} \int_{|z|=r} \log(1 + \sum |f_j|^2)\, d\theta - \frac{1}{4\pi} \int_{|z|=1} \log(1 + \sum |f_j|^2)\, d\theta = \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df|^2(z)\, dxdy.$$

Thus we get

$$\frac{r^k}{4k!} |g_i^{(k)}(0)| \leq \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df|^2(z)\, dxdy + \text{const}. \tag{8}$$

Since $|df|(z) \leq C|z|^m$, ($|z| \geq 1$), this shows $g_i^{(k)}(0) = 0$ for $k \geq 2m+3$. Hence $g_i(z)$ are polynomials.

[1] The idea of using Schwarz's formula is due to [BD, Appendice]. The author gives a different approach in [T, Section 6].

Next we will prove $\deg g_i(z) \leq m+1$. We define $E_i,\ E_{ij} \subset \mathbb{C}$, $(1 \leq i \leq n, 1 \leq i < j \leq n)$, by setting

$$\begin{aligned}
&\deg g_i(z) \leq m+1 \Longrightarrow E_i := \emptyset,\\
&\deg g_i(z) \geq m+2 \Longrightarrow E_i := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g_i(z)| \leq |z|\},\\
&\deg(g_i(z) - g_j(z)) \leq m+1 \Longrightarrow E_{ij} := \emptyset,\\
&\deg(g_i(z) - g_j(z)) \geq m+2 \Longrightarrow E_{ij} := \{z \in \mathbb{C} |\, |\mathrm{Re}\,(g_i(z) - g_j(z))| \leq |z|\}.
\end{aligned}$$

We set $E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij}$. Then we have $E(r) = \bigcup_i E_i(r) \cup \bigcup_{i<j} E_{ij}(r)$ for $r > 0$. From Lemma 3.5, we have positive constants r_0 and C' such that

$$|E(r)| \leq C'/r^{m+1}, \quad (r \geq r_0). \tag{9}$$

We have

$$\begin{aligned}
&\int_1^r \frac{dt}{t} \int_{|z|\leq t} |df|^2(z)\, dxdy \\
&\qquad = \int_1^r \frac{dt}{t} \int_{E\cap\{|z|\leq t\}} |df|^2(z)\, dxdy + \int_1^r \frac{dt}{t} \int_{E^c\cap\{|z|\leq t\}} |df|^2(z)\, dxdy.
\end{aligned} \tag{10}$$

Using (9) and $|df|(z) \leq C|z|^m$, $(|z| \geq 1)$, we can estimate the first term in (10) as follows:

$$\int_{E\cap\{1\leq|z|\leq t\}} |df|^2(z)\, dxdy \leq C^2 \int_{E\cap\{1\leq|z|\leq t\}} r^{2m+1}\, drd\theta = C^2 \int_1^t r^{2m+1} |E(r)| dr.$$

If $t \geq r_0$, we have

$$\int_{r_0}^t r^{2m+1} |E(r)| dr \leq C' \int_{r_0}^t r^m dr = \frac{C'}{m+1} t^{m+1} - \frac{C'}{m+1} {r_0}^{m+1}.$$

Thus

$$\int_1^r \frac{dt}{t} \int_{E\cap\{|z|\leq t\}} |df|^2(z)\, dxdy \leq \mathrm{const} \cdot r^{m+1}, \quad (r \geq 1). \tag{11}$$

Next we will estimate the second term in (10) by using the inequality (7) given in Section 2:

$$|df|^2 \leq \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2.$$

If $\deg g_i(z) \leq m+1$, Lemma 3.1 gives

$$\int_1^r \frac{dt}{t} \int_{E^c\cap\{|z|\leq t\}} |df_i|^2(z)\, dxdy \leq \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df_i|^2(z)\, dxdy \leq \mathrm{const} \cdot r^{m+1}.$$

If $\deg g_i(z) \geq m+2$, Lemma 3.6 gives

$$\int_{E^c\cap\{|z|\leq t\}} |df_i|^2(z)\, dxdy \leq \int_{E_i^c\cap\{|z|\leq t\}} |df_i|^2(z)\, dxdy \leq \mathrm{const}.$$

The terms for $|d(f_i/f_j)|$ can be also estimated in the same way, and we get

$$\int_1^r \frac{dt}{t}\int_{E^c\cap\{|z|\le t\}} |df|^2(z)\,dxdy \le \mathrm{const}\cdot r^{m+1}, \quad (r\ge 1). \tag{12}$$

From (10), (11), (12), we get

$$\int_1^r \frac{dt}{t}\int_{|z|\le t} |df|^2(z)\,dxdy \le \mathrm{const}\cdot r^{m+1}, \quad (r\ge 1).$$

From (8), this shows $g_i^{(k)}(0) = 0$ for $k \ge m+2$. Thus $g_i(z)$ are polynomials with $\deg g_i(z) \le m+1$. This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.4 and a corollary

5.1. **Proof of Theorem 1.4.** The proof of Theorem 1.4 needs the following lemma.

LEMMA 5.1. *Let $k\ge 1$ be an integer, and let δ be a real number satisfying $0<\delta<1$. Let $g(z) = a_0z^k + a_1z^{k-1} + \cdots + a_k$ be a polynomial of degree k, $(a_0 \ne 0)$. We set $h(z) := e^{g(z)}$ and define $E\subset\mathbb{C}$ by*

$$E := \{z\in\mathbb{C} \,|\, |\mathrm{Re}\, g(z)| \le |z|^\delta\}.$$

Then we have

$$\int_{\mathbb{C}\setminus E} |dh|^2 < \infty,$$

and there is a positive number r_0 such that

$$|E(r)| \le \frac{8}{|a_0| r^{k-\delta}}, \quad (r\ge r_0).$$

PROOF. This can be proven by the methods in Section 3. We omit the detail. □

Let $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ be polynomials, and define the holomorphic map $f:\mathbb{C}\to X$ and the integer $m\ge -1$ by (3) and (4). Here we suppose $m\ge 0$, i.e., f is not a constant map. We will prove Theorem 1.4.

From Theorem 1.1, we have

$$|df|(z) \le \mathrm{const}\cdot |z|^m, \quad (|z|\ge 1).$$

It follows

$$\limsup_{r\to\infty} \frac{\max_{|z|=r}\log|df|(z)}{\log r} \le m.$$

We want to prove that this is actually an equality. Suppose

$$\limsup_{r\to\infty} \frac{\max_{|z|=r}\log|df|(z)}{\log r} \lneqq m.$$

Then, if we take $\varepsilon>0$ sufficiently small, we have a positive number r_0 such that

$$|df|(z) \le |z|^{m-\varepsilon}, \quad (|z|\ge r_0). \tag{13}$$

Schwarz's formula gives the inequality (8):

$$\frac{r^k}{4k!}|g_i^{(k)}(0)| \leq \int_1^r \frac{dt}{t}\int_{|z|\leq t} |df|^2(z)\, dxdy + \text{const}, \quad (k \geq 0). \tag{14}$$

Let δ be a positive number such that $0 < \delta < 2\varepsilon$. We define E_i and E_{ij}, $(1 \leq i \leq n,\ 1 \leq i < j \leq n)$, by setting

$$\begin{aligned}
&\deg g_i(z) \leq m \Longrightarrow E_i := \emptyset, \\
&\deg g_i(z) = m+1 \Longrightarrow E_i := \{z \in \mathbb{C} \,|\, |\mathrm{Re}\, g_i(z)| \leq |z|^\delta\}, \\
&\deg(g_i(z) - g_j(z)) \leq m \Longrightarrow E_{ij} := \emptyset, \\
&\deg(g_i(z) - g_j(z)) = m+1 \Longrightarrow E_{ij} := \{z \in \mathbb{C} \,|\, |\mathrm{Re}\,(g_i(z) - g_j(z))| \leq |z|^\delta\}.
\end{aligned}$$

We set $E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij}$. Then, if we take r_0 sufficiently large, we have

$$|E(r)| \leq \text{const}/r^{m+1-\delta}, \quad (r \geq r_0). \tag{15}$$

We have

$$\begin{aligned}
&\int_1^r \frac{dt}{t}\int_{|z|\leq t} |df|^2(z)\, dxdy \\
&\qquad = \int_1^r \frac{dt}{t}\int_{E\cap\{|z|\leq t\}} |df|^2(z)\, dxdy + \int_1^r \frac{dt}{t}\int_{E^c\cap\{|z|\leq t\}} |df|^2(z)\, dxdy.
\end{aligned}$$

From (13) and (15), the first term can be estimated as in Section 4:

$$\int_1^r \frac{dt}{t}\int_{E\cap\{|z|\leq t\}} |df|^2(z)\, dxdy \leq \text{const}\cdot r^{m+1-(2\varepsilon-\delta)}, \quad (r \geq 1).$$

Using Lemma 5.1 and the inequality $|df|^2 \leq \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2$, we can estimate the second term:

$$\int_1^r \frac{dt}{t}\int_{E^c\cap\{|z|\leq t\}} |df|^2(z)\, dxdy \leq \text{const}\cdot \log r + \text{const}\cdot r^m, \quad (r \geq 1).$$

Thus we get

$$\int_1^r \frac{dt}{t}\int_{E\cap\{|z|\leq t\}} |df|^2(z)\, dxdy \leq \text{const}\cdot r^{m+1-(2\varepsilon-\delta)}, \quad (r \geq 1).$$

Note that $2\varepsilon - \delta$ is a positive number. Using this estimate in (14), we get

$$g_i^{(k)}(0) = 0, \quad (k \geq m+1).$$

This shows $\deg g_i(z) \leq m$. This contradicts the definition of m.

Remark 5.2. The following is also true:

$$\limsup_{r\to\infty} \frac{\max_{|z|\leq r} \log |df|(z)}{\log r} = m.$$

Proof. We have

$$m = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \le \limsup_{r\to\infty} \frac{\max_{|z|\le r} \log |df|(z)}{\log r}.$$

And we have $|df|(z) \le \mathrm{const}\cdot |z|^m$, $(|z|\ge 1)$. Thus

$$\limsup_{r\to\infty} \frac{\max_{|z|\le r} \log |df|(z)}{\log r} \le m.$$

□

5.2. **Order of the Shimizu-Ahlfors characteristic function.** For a holomorphic map $f:\mathbb{C}\to X$, we define the Shimizu-Ahlfors characteristic function $T(r,f)$ by

$$T(r,f) := \int_1^r \frac{dt}{t}\int_{|z|\le t} |df|^2(z)\, dxdy, \quad (r\ge 1).$$

The order ρ_f of $T(r,f)$ is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

ρ_f can be obtained as the growth rate of $|df|$:

Corollary 5.3. *For a holomorphic map $f:\mathbb{C}\to X$, we have*

$$\rho_f < \infty \iff \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} < \infty.$$

If these values are finite and f is not a constant map, then we have

$$\rho_f = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1.$$

Proof. If $\rho_f < \infty$, the estimate (14) shows that f can be expressed by (3) with polynomials $g_1(z), \cdots, g_n(z)$. Then we have

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} < \infty.$$

The proof of the converse is trivial.

Suppose $\rho_f < \infty$. Then we can express f by $f(z) = [1 : e^{g_1(z)} : \cdots : e^{g_n(z)}]$ with polynomials $g_1(z), \cdots, g_n(z)$. We set $f_i(z) := e^{g_i(z)}$, and define the integer m by (4). Theorem 1.4 gives

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1 = m + 1.$$

The estimate (14) gives

$$m + 1 \le \rho_f.$$

Since $|df| = \frac{1}{4\pi}\Delta \log(1+\sum |f_i|^2)$, Jensen's formula gives

$$T(r,f) = \frac{1}{4\pi}\int_{|z|=r} \log(1+\sum_i |f_i|^2)\, d\theta - \frac{1}{4\pi}\int_{|z|=1} \log(1+\sum_i |f_i|^2)\, d\theta.$$

Since $\deg g_i(z) \leq m+1$, we have

$$\log(1 + \sum_i |f_i|^2) \leq \mathrm{const} \cdot r^{m+1}, \quad (r \geq 1).$$

Hence

$$\rho_f \leq m+1.$$

Thus we get

$$\rho_f = m + 1 = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1.$$

□

REMARK 5.4. Of course, the statement of Corollary 5.3 is not true for general entire holomorphic curves in the complex projective space $\mathbb{C}P^n$. For example, let $f : \mathbb{C} \to \mathbb{C}P^1$ be a non-constant elliptic function. Since $|df|$ is bounded all over the complex plane, we have

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} = 0.$$

And it is easy to see

$$\rho_f = 2 \neq \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1.$$

References

[BD] F. Berteloot, J. Duval, Sur l'hyperbolicité de certains complémentaires, Enseign. Math. **47** (2001) 253-267

[T] M. Tsukamoto, A packing problem for holomorphic curves, preprint, arXiv: math.CV/0605353

Masaki Tsukamoto
Department of Mathematics, Faculty of Science
Kyoto University
Kyoto 606-8502
Japan
E-mail address: `tukamoto@math.kyoto-u.ac.jp`