Markov type of Alexandrov spaces of nonnegative curvature*†

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Abstract

We prove that Alexandrov spaces X of nonnegative curvature have Markov type 2 in the sense of Ball. As a corollary, any Lipschitz continuous mapping from a subset of X into a 2-uniformly convex Banach space is extended as a Lipschitz continuous mapping on the entire space X.

1 Introduction

The aim of the present article is to contribute to the nonlinearization of the geometry of Banach spaces from the viewpoint of metric geometry. Among them, our main object is Markov type of metric spaces due to Ball.

Markov type is a generalization of Rademacher type of Banach spaces. Rademacher type and cotype describe the behaviour of sums of independent random variables in Banach spaces, and these properties have fruitful analytic and geometric applications (cf. [LT] and [MS]). Enflo [E] first gave a generalized notion of type of metric spaces, which is called Enflo type now, and a variant of Enflo type was studied by Bourgain, Milman and Wolfson [BMW]. After them, Ball [B] introduced the notion of Markov type of metric spaces, and showed its importance in connection with the extension problem of Lipschitz mappings. Namely, he showed that any Lipschitz continuous mapping from a subset of metric space X having Markov type 2 into a reflexive Banach space having Markov cotype 2 can be extended to a Lipschitz mapping on the entire space X. Here Markov cotype of Banach spaces is a notion also introduced by Ball. It is worthful to mention that how to formulate a notion of cotype for general metric spaces has been an important question. See [MN1] for a recent breakthrough on this topic.

Until recently, only known examples of spaces possessing Markov type 2 had been Hilbert spaces and their bi-Lipschitz equivalents. Naor, Peres, Schramm and Sheffield

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[NPSS] broke the situation and showed that 2-uniformly smooth Banach spaces and some negatively curved metric spaces have Markov type 2 (Example 2.7). They also asked whether CAT(0)-spaces have Markov type 2 or not. We will answer this question affirmatively, but *under the reverse curvature bound*.

Our main theorem asserts that Alexandrov spaces of nonnegative curvature have Markov type 2 (Theorem 4.3). As an immediate corollary by virtue of Ball's extension theorem, any Lipschitz continuous mapping from a subset of an Alexandrov space of nonnegative curvature into a reflexive Banach space having Markov cotype 2 can be extended to a Lipschitz mapping on the entire space (Corollary 4.5). Compare this with [LS], [LPS] and [LN]. Our discussion is based on a different technique from that used in [NPSS]. The key tool is the inequality (3.1) in Theorem 3.2 due to Sturm.

The article is organized as follows. We review the theories of linear and nonlinear types and Alexandrov spaces of nonnegative curvature in Sections 2 and 3, respectively. Section 4 is devoted to the proof of the main theorem. Finally, in Section 5, we give a short remark on nonlinearizations of the 2-uniform smoothness and convexity of Banach spaces in connection with the curvature bounds in metric geometry.

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2 Nonlinear types

In this section, we recall Rademacher type and cotype of Banach spaces and some extensions of Rademacher type to nonlinear spaces. We refer to [LT] and [MS] for basic facts on Rademacher type and cotype. Throughout the article, we restrict ourselves to the case of $p = 2$, i.e., we will treat only type 2 and cotype 2.

A Banach space $(V, \|\cdot\|)$ is said to have *Rademacher type* 2 if there is a constant $K \geq 1$ such that, for any $N \in \mathbb{N}$ and $\{v_i\}_{i=1}^N \subset V$, we have

$$\frac{1}{2^N} \sum_{\varepsilon \in \{-1,1\}^N} \left\| \sum_{i=1}^N \varepsilon_i v_i \right\|^2 \leq K^2 \sum_{i=1}^N \|v_i\|^2, \tag{2.1}$$

where $\varepsilon = (\varepsilon_i)_{i=1}^N$. A fundamental example of a space possessing Rademacher type 2 is a 2-uniformly smooth Banach space. A Banach space $(V, \|\cdot\|)$ is said to be *2-uniformly smooth* (or have *modulus of smoothness of power type* 2) if there is a constant $S \geq 1$ such that

$$\left\| \frac{v+w}{2} \right\|^2 \geq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - \frac{S^2}{4}\|v-w\|^2 \tag{2.2}$$

holds for all $v, w \in V$. The infimum of such a constant S is denoted by $S_2(V)$. A 2-uniformly smooth Banach space has Rademacher type 2. For instance, for $2 \leq p < \infty$, an L^p-space $L^p(Z)$ over an arbitrary measure space Z is 2-uniformly smooth, and hence it has Rademacher type 2. Note that, if V is a Hilbert space, then the parallelogram identity yields the equality in (2.2) with $S = 1$.

Rademacher cotype 2 and the 2-uniform convexity of a Banach space are defined similarly by replacing (2.1) and (2.2) with

$$\frac{1}{2^N}\sum_{\varepsilon\in\{-1,1\}^N}\left\|\sum_{i=1}^N \varepsilon_i v_i\right\|^2 \ge \frac{1}{K^2}\sum_{i=1}^N \|v_i\|^2, \tag{2.3}$$

$$\left\|\frac{v+w}{2}\right\|^2 \le \frac{1}{2}\|v\|^2+\frac{1}{2}\|w\|^2-\frac{1}{4C^2}\|v-w\|^2, \tag{2.4}$$

respectively. Denote by $C_2(V)$ the least constant $C \ge 1$ satisfying (2.4). A 2-uniformly convex Banach space has Rademacher cotype 2. In particular, for $1 < p \le 2$, $L^p(Z)$ is 2-uniformly convex, and hence it has Rademacher cotype 2.

The first nonlinear extension of Rademacher type was given by Enflo.

Definition 2.1 (Enflo type, [E]) A metric space (X, d) is said to have *Enflo type* 2 if there is a constant $K \ge 1$ such that, for any $N \in \mathbb{N}$ and $\{x_\varepsilon\}_{\varepsilon\in\{-1,1\}^N} \subset X$, it holds that

$$\sum_{\varepsilon\in\{-1,1\}^N} d(x_\varepsilon, x_{-\varepsilon})^2 \le K^2 \sum_{\varepsilon\sim\varepsilon'} d(x_\varepsilon, x_{\varepsilon'})^2, \tag{2.5}$$

where $\varepsilon = (\varepsilon_i)_{i=1}^N$ and $\varepsilon \sim \varepsilon'$ holds if $\sum_{i=1}^N |\varepsilon_i - \varepsilon'_i| = 2$ (i.e., ε and ε' are adjacent). The least such a constant $K \ge 1$ is denoted by $E_2(X)$.

By taking $x_\varepsilon = \sum_{i=1}^N \varepsilon_i v_i$, we easily see that Enflo type 2 implies Rademacher type 2 for Banach spaces. However, the converse is not known in general. See [NS] for a partial positive result and [MN2] for a related work.

We next recall Markov type introduced by Ball. As is indicated in its name, we use a Markov chain to define Markov type. For $N \in \mathbb{N}$, consider a stationary, reversible Markov chain $\{M_l\}_{l\in\mathbb{N}\cup\{0\}}$ on the state space $\{1, 2, \dots, N\}$ with transition probabilities $a_{ij} := \Pr(M_{l+1} = j \mid M_l = i)$. Namely, if we set $\pi_i := \Pr(M_0 = i)$, then $\{\pi_i\}_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfy

$$0 \le \pi_i, a_{ij} \le 1, \quad \sum_{i=1}^N \pi_i = 1, \quad \sum_{j=1}^N a_{ij} = 1, \quad \pi_i a_{ij} = \pi_j a_{ji} \tag{2.6}$$

for all $i, j = 1, 2, \dots, N$. The third and fourth inequalities guarantee the stationariness ($\sum_{i=1}^N \pi_i a_{ij} = \pi_j$) and the reversibility of the Markov chain $\{M_l\}_{l\in\mathbb{N}\cup\{0\}}$.

Definition 2.2 (Markov type, [B, Definition 1.3]) A metric space (X, d) is said to have *Markov type* 2 if there is a constant $K \ge 1$ such that, for any $\alpha \in (0, 1)$, $N \in \mathbb{N}$, $\{x_i\}_{i=1}^N \subset X$, $\{\pi_i\}_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfying (2.6), we have

$$(1-\alpha)\sum_{i,j=1}^N \pi_i c_{ij} d(x_i, x_j)^2 \le K^2\alpha \sum_{i,j=1}^N \pi_i a_{ij} d(x_i, x_j)^2, \tag{2.7}$$

where we set $C = (c_{ij})_{i,j=1}^N = (1-\alpha)(I-\alpha A)^{-1}$ and I stands for the identity matrix. The infimum of $K \ge 1$ satisfying (2.7) is denoted by $M_2(X)$.

We remark that Ball's original definition concerns only the case of $\pi_i \equiv N^{-1}$. The above slightly extended formulation can be found in [NPSS]. Note that

$$C = (1-\alpha)(I-\alpha A)^{-1} = (1-\alpha)\sum_{l=0}^{\infty} \alpha^l A^l.$$

Hence $C = (c_{ij})_{i,j=1}^N$ also satisfies

$$0 \le c_{ij} \le 1, \quad \sum_{j=1}^N c_{ij} = 1, \quad \pi_i c_{ij} = \pi_j c_{ji}$$

for all $i, j = 1, 2, \dots, N$.

We recall some important properties of Markov type. Markov type has an equivalent form which is more convenient in some circumstances. For $l \in \mathbb{N}$ and $A = (a_{ij})_{i,j=1}^N$, we set $A^l = (a_{ij}^{(l)})_{i,j=1}^N$. In particular, $a_{ij}^{(1)} = a_{ij}$.

Theorem 2.3 ([B, Theorem 1.6]) *Let (X, d) be a metric space and assume that there is a constant $K \ge 1$ such that the inequality*

$$\sum_{i,j=1}^N \pi_i a_{ij}^{(l)} d(x_i, x_j)^2 \le K^2 l \sum_{i,j=1}^N \pi_i a_{ij} d(x_i, x_j)^2 \tag{2.8}$$

holds for all $l \in \mathbb{N}$, $N \in \mathbb{N}$, $\{x_i\}_{i=1}^N \subset X$, $\{\pi_i\}_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfying (2.6). *Then (X, d) has Markov type* 2 *with $M_2(X) \le K$. Conversely, if (X, d) has Markov type* 2, *then (X, d) satisfies* (2.8) *with $K = 2\sqrt{e}M_2(X)$.*

Markov type is known to be strong enough to imply Enflo type.

Proposition 2.4 ([NS, Proposition 1]) *If a metric space (X, d) has Markov type* 2, *then it has Enflo type* 2.

To state Ball's theorem which guarantees the usefulness of Markov type, we need to define Markov cotype of Banach spaces also introduced by Ball.

Definition 2.5 (Markov cotype, [B, Definition 1.5]) A Banach space $(V, \|\cdot\|)$ is said to have *Markov cotype* 2 if there is a constant $K \ge 1$ such that, for any $\alpha \in (0, 1)$, $N \in \mathbb{N}$, $\{v_i\}_{i=1}^N \subset V$ and $A = (a_{ij})_{i,j=1}^N$ satisfying (2.6) with $\pi_i \equiv N^{-1}$, we have

$$\alpha \sum_{i,j=1}^N a_{ij} \left\| \sum_{k=1}^N (c_{ik} - c_{jk}) v_k \right\|^2 \le K^2(1-\alpha) \sum_{i,j=1}^N c_{ij} \|v_i - v_j\|^2,$$

where we set $C = (c_{ij})_{i,j=1}^N = (1-\alpha)(I-\alpha A)^{-1}$. We denote by $N_2(V)$ the infimum of such a constant $K \ge 1$.

We remark that Markov cotype is strictly stronger than Rademacher cotype, for $L^1(Z)$ has Rademacher cotype 2 and does not have Markov cotype 2 (see [B]). It is known that a 2-uniformly convex Banach space $(V, \|\cdot\|)$ has Markov cotype 2 with $N_2(V) \le 2C_2(V)$ ([B, Theorem 4.1]). For a Lipschitz continuous map $f : X \longrightarrow Y$ between metric spaces, we denote by $\mathbf{Lip}(f)$ its Lipschitz constant, that is,

$$\mathbf{Lip}(f) := \sup_{x,y \in X,\ x \ne y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

Theorem 2.6 ([B, Theorem 1.7]) *Let (X, d) be a metric space having Markov type 2 and $(V, \|\cdot\|)$ be a reflexive Banach space having Markov cotype 2. Then, for any Lipschitz continuous map $f : Z \longrightarrow V$ from a subset $Z \subset X$, there exists a Lipschitz continuous extension $\tilde{f} : X \longrightarrow V$ of f with*

$$\mathbf{Lip}(\tilde{f}) \le 3M_2(X)N_2(V)\,\mathbf{Lip}(f).$$

In particular, if $(V, \|\cdot\|)$ is a 2-uniformly convex Banach space, then we have

$$\mathbf{Lip}(\tilde{f}) \le 6M_2(X)C_2(V)\,\mathbf{Lip}(f).$$

We end this section with several examples of spaces having Markov type.

Example 2.7 (i) (Hilbert spaces, [B, Proposition 1.4]) A Hilbert space $(H, \langle\cdot,\cdot\rangle)$ has Markov type 2 with $M_2(H) = 1$.

(ii) (Products) For two metric spaces (X_1, d_1) and (X_2, d_2) having Markov type 2, let (X, d) be the l^2-product of them, that is, $X := X_1 \times X_2$ and

$$d\big((x_1, x_2), (y_1, y_2)\big) := \{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2\}^{1/2}$$

for $(x_1, x_2), (y_1, y_2) \in X$. Then (X, d) has Markov type 2 with

$$M_2(X) \le \max\{M_2(X_1), M_2(X_2)\}.$$

(iii) (The bi-Lipschitz equivalence) Given two metric spaces (X_1, d_1) and (X_2, d_2), if (X_1, d_1) has Markov type 2 and if there is a bi-Lipschitz homeomorphism $f : X_1 \longrightarrow X_2$, then (X_2, d_2) has Markov type 2 with

$$M_2(X_2) \le \mathbf{Lip}(f)\,\mathbf{Lip}(f^{-1})M_2(X_1).$$

(iv) (Gromov-Hausdorff limits) If a sequence of (pointed) metric spaces $\{(X_i, d_i)\}_{i=1}^{\infty}$ converges to a (pointed) metric space (X, d) in the sense of the (pointed) Gromov-Hausdorff convergence and if every (X_i, d_i) has Markov type 2 with $\liminf_{i\to\infty} M_2(X_i) < \infty$, then (X, d) has Markov type 2 with

$$M_2(X) \le \liminf_{i\to\infty} M_2(X_i).$$

(v) (2-uniformly smooth Banach spaces, [NPSS]) A 2-uniformly smooth Banach space $(V, \|\cdot\|)$ has Markov type 2 with $M_2(V) \le 4S_2(V)$.

(vi) (Trees and hyperbolic groups, [NPSS]) There exists a universal constant C_t for which every tree T with arbitrary positive edge lengths has Markov type 2 with $M_2(T) \leq C_t$. There also exists a universal constant C_h such that every δ-hyperbolic group has Markov type 2 with $M_2 \leq C_h(1+\delta)$. More precisely, we fix a presentation of the group and consider its Cayley graph G equipped with the word metric. If G is δ-hyperbolic as a metric space, then it has Markov type 2 with $M_2(G) \leq C_h(1+\delta)$. Naor et al. have obtained an estimate for general δ-hyperbolic metric spaces, and it implies the above results.

3 Alexandrov spaces of nonnegative curvature

In this section, we recall the definition of Alexandrov spaces of nonnegative curvature. We refer to [BGP] as a standard reference.

A metric space (X, d) is said to be *geodesic* if every two points $x, y \in X$ can be connected by a curve $\gamma : [0,1] \longrightarrow X$ from x to y with $\mathrm{length}(\gamma) = d(x,y)$. A rectifiable curve $\gamma : [0,1] \longrightarrow X$ is called a *geodesic* if it is locally minimizing and has a constant speed. A geodesic $\gamma : [0,1] \longrightarrow X$ is said to be *minimal* if it satisfies $\mathrm{length}(\gamma) = d(\gamma(0), \gamma(1))$.

Definition 3.1 ([BGP]) A geodesic metric space (X, d) is called an *Alexandrov space of nonnegative curvature* if, for all three points $x, y, z \in X$ and any minimal geodesic $\gamma : [0,1] \longrightarrow X$ between y and z, we have

$$d\left(x, \gamma\left(\frac{1}{2}\right)\right)^2 \geq \frac{1}{2} d(x,y)^2 + \frac{1}{2} d(x,z)^2 - \frac{1}{4} d(y,z)^2.$$

In particular, a complete Riemannian manifold is an Alexandrov space of nonnegative curvature if and only if its sectional curvature is nonnegative everywhere. There is a rich and deep theory on the geometry and the analysis on Alexandrov spaces, but we need only the following theorem due to Sturm which plays a key role in the next section.

Theorem 3.2 ([S, Theorem 1.4]) *A geodesic metric space (X, d) is an Alexandrov space of nonnegative curvature if and only if, for any $N \in \mathbb{N}$, $\{x_i\}_{i=1}^N \subset X$, $y \in X$ and $\{a_i\}_{i=1}^N \subset [0,1]$ with $\sum_{i=1}^N a_i = 1$, we have*

$$\sum_{i,j=1}^{N} a_i a_j \{d(x_i, x_j)^2 - d(x_i, y)^2 - d(x_j, y)^2\} \leq 0. \tag{3.1}$$

The inequality (3.1) corresponds to the following fact in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. For any $N \in \mathbb{N}$, $\{v_i\}_{i=1}^N \subset H$ and $\{a_i\}_{i=1}^N \subset [0,1]$ with $\sum_{i=1}^N a_i = 1$,

$$\begin{aligned}
&\sum_{i,j=1}^N a_i a_j \{\|v_i - v_j\|^2 - \|v_i - w\|^2 - \|v_j - w\|^2\} \\
&= 2\sum_{i,j=1}^N a_i a_j \langle v_i - w, w - v_j\rangle = 2\Big\langle \Big(\sum_{i=1}^N a_i v_i\Big) - w, w - \Big(\sum_{j=1}^N a_j v_j\Big)\Big\rangle \\
&= -2\Big\|\Big(\sum_{i=1}^N a_i v_i\Big) - w\Big\|^2 \le 0
\end{aligned}$$

holds for all $w \in H$.

4 Markov type of Alexandrov spaces

In this section, we prove our main theorem. Throughout the section, let (X, d) be an Alexandrov space of nonnegative curvature, and fix $N \in \mathbb{N}$, $\{x_i\}_{i=1}^N \subset X$, $\{\pi_i\}_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfying (2.6). For $1 \le i, j \le N$ and $l \in \mathbb{N}$, set $d_{ij} := d(x_i, x_j)$ and

$$\mathcal{E}(l) := \sum_{i,j=1}^N \pi_i a_{ij}^{(l)} d_{ij}^2.$$

Recall the notation $A^l = (a_{ij}^{(l)})_{i,j=1}^N$ and that (2.6) implies

$$0 \le a_{ij}^{(l)} \le 1, \quad \sum_{j=1}^N a_{ij}^{(l)} = 1, \quad \pi_i a_{ij}^{(l)} = \pi_j a_{ji}^{(l)}$$

for all $i, j = 1, 2, \dots, N$.

Lemma 4.1 *For any $l \in \mathbb{N}$, we have $\mathcal{E}(2l) \le 2\mathcal{E}(l)$.*

Proof. We calculate

$$\begin{aligned}
\mathcal{E}(2l) &= \sum_{i,j,k=1}^N \pi_i a_{ij}^{(l)} a_{jk}^{(l)} d_{ik}^2 \\
&= \sum_{i,j,k=1}^N \pi_j a_{ji}^{(l)} a_{jk}^{(l)} (d_{ji}^2 + d_{jk}^2 + d_{ik}^2 - d_{ji}^2 - d_{jk}^2) \\
&= \sum_{i,j,k=1}^N \pi_j a_{ji}^{(l)} a_{jk}^{(l)} (d_{ji}^2 + d_{jk}^2) + \sum_{j=1}^N \pi_j \Big\{ \sum_{i,k=1}^N a_{ji}^{(l)} a_{jk}^{(l)} (d_{ik}^2 - d_{ji}^2 - d_{jk}^2) \Big\}.
\end{aligned}$$

Since $\sum_{i=1}^N a_{ji}^{(l)} = 1$, we have

$$\sum_{i,j,k=1}^N \pi_j a_{ji}^{(l)} a_{jk}^{(l)} (d_{ji}^2 + d_{jk}^2) = \sum_{i,j=1}^N \pi_j a_{ji}^{(l)} d_{ji}^2 + \sum_{j,k=1}^N \pi_j a_{jk}^{(l)} d_{jk}^2 = 2\mathcal{E}(l).$$

Moreover, applying Theorem 3.2 with $a_i = a_{ji}^{(l)}$ and $y = x_j$, we obtain

$$\sum_{i,k=1}^{N} a_{ji}^{(l)} a_{jk}^{(l)} (d_{ik}^2 - d_{ji}^2 - d_{jk}^2) \le 0$$

for every j. This completes the proof. $\square$

Lemma 4.2 *For any $l \in \mathbb{N}$ and $\alpha \in (0,1)$, we have*

$$\alpha^{2l}\mathcal{E}(2l) + 2\alpha^{2l+1}\mathcal{E}(2l+1) + \alpha^{2l+2}\mathcal{E}(2l+2) \le 2(1+\alpha)\alpha^l\{\alpha^l\mathcal{E}(l) + \alpha^{l+1}\mathcal{E}(l+1)\}.$$

Proof. The proof is essentially similar to that of Lemma 4.1. We first observe

$$\begin{aligned}
&\alpha^{2l}\mathcal{E}(2l) + 2\alpha^{2l+1}\mathcal{E}(2l+1) + \alpha^{2l+2}\mathcal{E}(2l+2) \\
&= \alpha^{2l}\{\mathcal{E}(2l) + 2\alpha\mathcal{E}(2l+1) + \alpha^2\mathcal{E}(2l+2)\} \\
&= \alpha^{2l} \sum_{i,j,k=1}^{N} \pi_i (a_{ij}^{(l)} + \alpha a_{ij}^{(l+1)})(a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)}) d_{ik}^2 \\
&= \alpha^{2l} \sum_{i,j,k=1}^{N} \pi_j (a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)})(a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)})(d_{ji}^2 + d_{jk}^2) \\
&\qquad + \alpha^{2l} \sum_{j=1}^{N} \pi_j \left\{ \sum_{i,k=1}^{N} (a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)})(a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)})(d_{ik}^2 - d_{ji}^2 - d_{jk}^2) \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,k=1}^{N} \pi_j (a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)})(a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)})(d_{ji}^2 + d_{jk}^2) \\
&= (1+\alpha) \sum_{i,j=1}^{N} \pi_j (a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)}) d_{ji}^2 + (1+\alpha) \sum_{j,k=1}^{N} \pi_j (a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)}) d_{jk}^2 \\
&= 2(1+\alpha)\{\mathcal{E}(l) + \alpha\mathcal{E}(l+1)\}.
\end{aligned}$$

Fix j and apply Theorem 3.2 to $a_i = (1+\alpha)^{-1}(a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)})$ and $y = x_j$. Then we find

$$\sum_{i,k=1}^{N} (a_{ji}^{(l)} + \alpha a_{ji}^{(l+1)})(a_{jk}^{(l)} + \alpha a_{jk}^{(l+1)})(d_{ik}^2 - d_{ji}^2 - d_{jk}^2) \le 0.$$

Therefore we obtain

$$\begin{aligned}
&\alpha^{2l}\mathcal{E}(2l) + 2\alpha^{2l+1}\mathcal{E}(2l+1) + \alpha^{2l+2}\mathcal{E}(2l+2) \\
&\le 2(1+\alpha)\alpha^{2l}\{\mathcal{E}(l) + \alpha\mathcal{E}(l+1)\} \\
&= 2(1+\alpha)\alpha^l\{\alpha^l\mathcal{E}(l) + \alpha^{l+1}\mathcal{E}(l+1)\}.
\end{aligned}$$

$\square$

Theorem 4.3 *Let (X,d) be an Alexandrov space of nonnegative curvature. Then (X,d) has Markov type 2 with $M_2(X) \le \sqrt{6}$.*

Proof. It suffices to show

$$(1-\alpha)\sum_{i,j=1}^{N} \pi_i c_{ij} d_{ij}^2 \le 6\alpha \sum_{i,j=1}^{N} \pi_i a_{ij} d_{ij}^2 \ \big(= 6\alpha\mathcal{E}(1)\big).$$

As $C = (1-\alpha)(I-\alpha A)^{-1} = (1-\alpha)\sum_{l=0}^{\infty} \alpha^l A^l$, we observe

$$\begin{aligned}
&(1-\alpha)\sum_{i,j=1}^{N} \pi_i c_{ij} d_{ij}^2 = (1-\alpha)^2 \sum_{i,j=1}^{N}\sum_{l=1}^{\infty} \alpha^l \pi_i a_{ij}^{(l)} d_{ij}^2 \\
&= \frac{(1-\alpha)^2}{2}\Bigg[\alpha\mathcal{E}(1) + \sum_{n=1}^{\infty}\bigg\{\alpha^{2^{n-1}}\mathcal{E}(2^{n-1}) + 2\alpha^{2^{n-1}+1}\mathcal{E}(2^{n-1}+1) \\
&\qquad + \cdots + 2\alpha^{2^n-1}\mathcal{E}(2^n-1) + \alpha^{2^n}\mathcal{E}(2^n)\bigg\}\Bigg] \\
&= \frac{(1-\alpha)^2}{2}\sum_{n=0}^{\infty} \mathcal{E}_n^\alpha,
\end{aligned}$$

where we set

$$\begin{aligned}
\mathcal{E}_0^\alpha &:= \alpha\mathcal{E}(1), \\
\mathcal{E}_1^\alpha &:= \alpha\mathcal{E}(1) + \alpha^2\mathcal{E}(2), \\
&\ \ \vdots \\
\mathcal{E}_n^\alpha &:= \alpha^{2^{n-1}}\mathcal{E}(2^{n-1}) + 2\alpha^{2^{n-1}+1}\mathcal{E}(2^{n-1}+1) \\
&\qquad + \cdots + 2\alpha^{2^n-1}\mathcal{E}(2^n-1) + \alpha^{2^n}\mathcal{E}(2^n).
\end{aligned}$$

It follows from Lemma 4.2 that, for all $n \ge 2$,

$$\begin{aligned}
\mathcal{E}_n^\alpha &= \alpha^{2^{n-1}}\mathcal{E}(2^{n-1}) + 2\alpha^{2^{n-1}+1}\mathcal{E}(2^{n-1}+1) \\
&\qquad + \cdots + 2\alpha^{2^n-1}\mathcal{E}(2^n-1) + \alpha^{2^n}\mathcal{E}(2^n) \\
&= \{\alpha^{2^{n-1}}\mathcal{E}(2^{n-1}) + 2\alpha^{2^{n-1}+1}\mathcal{E}(2^{n-1}+1) + \alpha^{2^{n-1}+2}\mathcal{E}(2^{n-1}+2)\} \\
&\qquad + \{\alpha^{2^{n-1}+2}\mathcal{E}(2^{n-1}+2) + 2\alpha^{2^{n-1}+3}\mathcal{E}(2^{n-1}+3) + \alpha^{2^{n-1}+4}\mathcal{E}(2^{n-1}+4)\} \\
&\qquad + \cdots + \{\alpha^{2^n-2}\mathcal{E}(2^n-2) + 2\alpha^{2^n-1}\mathcal{E}(2^n-1) + \alpha^{2^n}\mathcal{E}(2^n)\} \\
&\le 2(1+\alpha)\alpha^{2^{n-2}}\{\alpha^{2^{n-2}}\mathcal{E}(2^{n-2}) + \alpha^{2^{n-2}+1}\mathcal{E}(2^{n-2}+1)\} \\
&\qquad + 2(1+\alpha)\alpha^{2^{n-2}+1}\{\alpha^{2^{n-2}+1}\mathcal{E}(2^{n-2}+1) + \alpha^{2^{n-2}+2}\mathcal{E}(2^{n-2}+2)\} \\
&\qquad + \cdots + 2(1+\alpha)\alpha^{2^{n-1}-1}\{\alpha^{2^{n-1}-1}\mathcal{E}(2^{n-1}-1) + \alpha^{2^{n-1}}\mathcal{E}(2^{n-1})\} \\
&\le 4\alpha^{2^{n-2}}\{\alpha^{2^{n-2}}\mathcal{E}(2^{n-2}) + 2\alpha^{2^{n-2}+1}\mathcal{E}(2^{n-2}+1) \\
&\qquad + \cdots + 2\alpha^{2^{n-1}-1}\mathcal{E}(2^{n-1}-1) + \alpha^{2^{n-1}}\mathcal{E}(2^{n-1})\} \\
&= 4\alpha^{2^{n-2}}\mathcal{E}_{n-1}^\alpha.
\end{aligned}$$

Moreover, Lemma 4.1 implies

$$\mathcal{E}_1^\alpha = \alpha\mathcal{E}(1) + \alpha^2\mathcal{E}(2) \le (1+2\alpha)\alpha\mathcal{E}(1).$$

Thus we see, for all $n \ge 2$,

$$\mathcal{E}_n^\alpha \le 4^{n-1}\alpha^{2^{n-2}}\alpha^{2^{n-3}}\cdots\alpha\mathcal{E}_1^\alpha \le 4^{n-1}\alpha^{2^{n-1}-1}(1+2\alpha)\alpha\mathcal{E}(1).$$

We estimate

$$\begin{aligned}
1+\sum_{n=2}^{\infty}4^{n-1}\alpha^{2^{n-1}-1} &= \sum_{n=0}^{\infty}2^n 2^n\alpha^{2^n-1} \\
&= 1+(2\alpha+2\alpha)+(4\alpha^3+4\alpha^3+4\alpha^3+4\alpha^3)+\cdots \\
&\le \sum_{l=1}^{\infty} l\alpha^{(l-1)/2} = \sum_{l=0}^{\infty}(l+1)\alpha^{l/2} \\
&= \frac{1}{(1-\alpha^{1/2})^2} = \frac{(1+\alpha^{1/2})^2}{(1-\alpha)^2}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
(1-\alpha)\sum_{i,j=1}^{N}\pi_i c_{ij} d_{ij}^2 &= \frac{(1-\alpha)^2}{2}\sum_{n=0}^{\infty}\mathcal{E}_n^\alpha \\
&\le \frac{(1-\alpha)^2}{2}\left\{1+(1+2\alpha)\sum_{n=1}^{\infty}4^{n-1}\alpha^{2^{n-1}-1}\right\}\alpha\mathcal{E}(1) \\
&\le \frac{(1-\alpha)^2}{2}\left\{1+\frac{(1+2\alpha)(1+\alpha^{1/2})^2}{(1-\alpha)^2}\right\}\alpha\mathcal{E}(1) \\
&= \frac{1}{2}\{(1-\alpha)^2+(1+2\alpha)(1+\alpha^{1/2})^2\}\alpha\mathcal{E}(1) \\
&\le 6\alpha\mathcal{E}(1).
\end{aligned}$$

This completes the proof. □

We have two corollaries by Proposition 2.4 and Theorem 2.6.

Corollary 4.4 *Let (X,d) be an Alexandrov space of nonnegative curvature. Then (X,d) has Enflo type 2.*

Corollary 4.5 *Let (X,d) be an Alexandrov space of nonnegative curvature and $(V,\|\cdot\|)$ be a reflexive Banach space having Markov cotype 2. Then, for any Lipschitz continuous map $f : Z \longrightarrow V$ from a subset $Z \subset X$, there exists a Lipschitz continuous extension $\tilde{f} : X \longrightarrow V$ of f with*

$$\mathbf{Lip}(\tilde{f}) \le 3\sqrt{6}N_2(V)\,\mathbf{Lip}(f).$$

In particular, if $(V,\|\cdot\|)$ is 2-uniformly convex, then we have

$$\mathbf{Lip}(\tilde{f}) \le 6\sqrt{6}C_2(V)\,\mathbf{Lip}(f).$$

We mention that our bound of the ratio of Lipschitz constants is independent of the dimension of X. Compare this with [LN, Theorem 1.6].

5 Additional remarks

This section is devoted to a short remark toward a nonlinearization of the 2-uniform smoothness (and convexity). As we have already seen in (2.2), the 2-uniform smoothness of a Banach space is defined by using the inequality

$$\left\| \frac{v+w}{2} \right\|^2 \geq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - \frac{S^2}{4}\|v-w\|^2. \tag{5.1}$$

By replacing v and w with $w+v$ and $w-v$, this inequality is rewritten as

$$\left\| \frac{v+w}{2} \right\|^2 \leq \frac{S^2}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - \frac{1}{4}\|v-w\|^2. \tag{5.2}$$

Now natural generalizations of (5.1) and (5.2) are the following. Let (X,d) be a geodesic metric space. For any three points $x,y,z \in X$ and minimal geodesic $\gamma : [0,1] \longrightarrow X$ from y to z, we have

$$d\left(x, \gamma\left(\frac{1}{2}\right)\right)^2 \geq \frac{1}{2}d(x,y)^2 + \frac{1}{2}d(x,z)^2 - \frac{S^2}{4}d(y,z)^2 \tag{5.3}$$

or

$$d\left(x, \gamma\left(\frac{1}{2}\right)\right)^2 \leq \frac{S^2}{2}d(x,y)^2 + \frac{1}{2}d(x,z)^2 - \frac{1}{4}d(y,z)^2. \tag{5.4}$$

We will say that a geodesic metric space (X,d) *satisfies* (5.3) (or (5.4)) if (5.3) (or (5.4)) holds for all $x,y,z \in X$ and all minimal geodesic $\gamma : [0,1] \longrightarrow X$ from y to z. On one hand, the inequality (5.3) generalizes the nonnegatively curved property in the sense of Alexandrov which corresponds to the case of $S=1$ (see Section 3). On the other hand, the inequality (5.4) extends the CAT(0)-inequality which amounts to the case of $S=1$ (cf. [BH]). This is a reason why both negatively and positively curved spaces have Markov type 2. Compare Example 2.7 and Theorem 4.3.

We mention that we can also regard (5.1) as an upper curvature bound of the unit sphere (see [O1]), and that the reverse inequality of (5.3) (a generalized 2-uniform convexity) has been studied in [O2].

As an application of the inequality (5.4), we give an example of a nonlinear and non-Riemannian (in other words, Finslerian) space possessing Enflo type 2. We first prove a lemma.

Lemma 5.1 *Let a geodesic metric space* (X,d) *satisfy* (5.4). *Then, for any four points* $w,x,y,z \in X$*, we have*

$$d(w,y)^2 + d(x,z)^2 \leq S^2\{d(w,x)^2 + d(y,z)^2\} + d(w,z)^2 + d(y,x)^2.$$

Proof. Take a minimal geodesic $\gamma : [0,1] \longrightarrow X$ between x and z. Then (5.4) yields that

$$d\left(w, \gamma\left(\frac{1}{2}\right)\right)^2 \leq \frac{S^2}{2}d(w,x)^2 + \frac{1}{2}d(w,z)^2 - \frac{1}{4}d(x,z)^2,$$
$$d\left(y, \gamma\left(\frac{1}{2}\right)\right)^2 \leq \frac{S^2}{2}d(y,z)^2 + \frac{1}{2}d(y,x)^2 - \frac{1}{4}d(x,z)^2.$$

Thus we see

$$
\begin{aligned}
d(w,y)^2 &\le \left\{ d\left(w, \gamma\left(\frac{1}{2}\right)\right) + d\left(\gamma\left(\frac{1}{2}\right), y\right) \right\}^2 \\
&\le 2\left\{ d\left(w, \gamma\left(\frac{1}{2}\right)\right)^2 + d\left(\gamma\left(\frac{1}{2}\right), y\right)^2 \right\} \\
&\le S^2\{d(w,x)^2 + d(y,z)^2\} + d(w,z)^2 + d(y,x)^2 - d(x,z)^2.
\end{aligned}
$$

This is the required inequality. □

Proposition 5.2 *If a geodesic metric space (X,d) satisfies (5.4), then it has Enflo type 2 with $E_2(X) \le S$. In particular, a* CAT(0)*-space (X,d) has Enflo type 2 with $E_2(X) = 1$, and a 2-uniformly smooth Banach space $(V, \|\cdot\|)$ has Enflo type 2 with $E_2(V) \le S_2(V)$.*

Proof. We shall prove by induction in $N \in \mathbb{N}$. In the case of $N = 1$, for any $\{x_1, x_{-1}\} \subset X$, we immediately see

$$
d(x_1, x_{-1})^2 + d(x_{-1}, x_1)^2 \le S^2\{d(x_1, x_{-1})^2 + d(x_{-1}, x_1)^2\}.
$$

Fix $N \ge 2$ and suppose that, for any $\{x_\delta\}_{\delta \in \{-1,1\}^{N-1}} \subset X$, it holds that

$$
\sum_{\delta \in \{-1,1\}^{N-1}} d(x_\delta, x_{-\delta})^2 \le S^2 \sum_{\delta \sim \delta'} d(x_\delta, x_{\delta'})^2,
$$

where $\delta = (\delta_i)_{i=1}^{N-1}$ and $\delta \sim \delta'$ holds if $\sum_{i=1}^{N-1} |\delta_i - \delta'_i| = 2$. Now we choose an arbitrary $\{x_\varepsilon\}_{\varepsilon \in \{-1,1\}^N} \subset X$. For each $\delta \in \{-1,1\}^{N-1}$, Lemma 5.1 implies

$$
\begin{aligned}
&d(x_{(\delta,1)}, x_{(-\delta,-1)})^2 + d(x_{(\delta,-1)}, x_{(-\delta,1)})^2 \\
&\le S^2\{d(x_{(\delta,1)}, x_{(\delta,-1)})^2 + d(x_{(-\delta,-1)}, x_{(-\delta,1)})^2\} + d(x_{(\delta,1)}, x_{(-\delta,1)})^2 + d(x_{(-\delta,-1)}, x_{(\delta,-1)})^2.
\end{aligned}
$$

Summing up this inequality in $\delta \in \{-1,1\}^{N-1}$, we have

$$
\begin{aligned}
\sum_{\varepsilon \in \{-1,1\}^N} d(x_\varepsilon, x_{-\varepsilon})^2 \le S^2 &\sum_{\delta \in \{-1,1\}^{N-1}} \{d(x_{(\delta,1)}, x_{(\delta,-1)})^2 + d(x_{(\delta,-1)}, x_{(\delta,1)})^2\} \\
&+ \sum_{\delta \in \{-1,1\}^{N-1}} \{d(x_{(\delta,1)}, x_{(-\delta,1)})^2 + d(x_{(\delta,-1)}, x_{(-\delta,-1)})^2\}.
\end{aligned}
$$

By our assumption, the second term is estimated by

$$
\begin{aligned}
&\sum_{\delta \in \{-1,1\}^{N-1}} \{d(x_{(\delta,1)}, x_{(-\delta,1)})^2 + d(x_{(\delta,-1)}, x_{(-\delta,-1)})^2\} \\
&\qquad \le S^2 \sum_{\delta \sim \delta'} \{d(x_{(\delta,1)}, x_{(\delta',1)})^2 + d(x_{(\delta,-1)}, x_{(\delta',-1)})^2\}.
\end{aligned}
$$

Therefore we obtain

$$
\sum_{\varepsilon \in \{-1,1\}^N} d(x_\varepsilon, x_{-\varepsilon})^2 \le S^2 \sum_{\varepsilon \sim \varepsilon'} d(x_\varepsilon, x_{\varepsilon'})^2.
$$

This completes the proof. □

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