On Loops in the Hyperbolic Loci of the Complex Hénon Maps

Zin ARAI
Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
(email: arai@math.kyoto-u.ac.jp)

1 Introduction: Hubbard's Conjectures

The aim of this article is to give answers to the conjectures of John Hubbard about the topology of hyperbolic horseshoe loci of the complex Hénon maps

$$H_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2 : (x, y) \mapsto (x^2 + c - ay, x).$$

Here a and c are the parameters which take values in $\mathbb{C}$. These conjectures are treated in a recent paper by Bedford and Smillie [5]. Here we will briefly describe their formulation of the problem.

Let us define

$$\begin{aligned} K^{\mathbb{C}}_{a,c} &:= \{p \in \mathbb{C}^2 : \{H^n_{a,c}(p)\}_{n\in\mathbb{Z}} \text{ is bounded}\}, \\ K^{\mathbb{R}}_{a,c} &:= K^{\mathbb{C}}_{a,c} \cap \mathbb{R}^2, \\ \mathcal{H}^{\mathbb{R}} &:= \{(a, c) \in \mathbb{R}^2 : H_{a,c}|K^{\mathbb{R}}_{a,c} \text{ is a hyperbolic full horseshoe}\}, \\ \mathcal{H}^{\mathbb{C}} &:= \{(a, c) \in \mathbb{C}^2 : H_{a,c}|K^{\mathbb{C}}_{a,c} \text{ is a hyperbolic full horseshoe}\}. \end{aligned}$$

By a hyperbolic full horseshoe we mean an uniformly hyperbolic invariant set which is topologically conjugate to the full shift map of Σ_2, the space of bi-infinite sequences of two symbols.

Define three specific regions in the parameter space of real and complex Hénon maps by

$$\begin{aligned} DN &:= \{(a, c) \in \mathbb{R}^2 : c < -(5 + 2\sqrt{5})\,(|a| + 1)^2/4\}, \\ EMP &:= \{(a, c) \in \mathbb{R}^2 : c > (|a| + 1)^2/4\}, \\ HOV &:= \{(a, c) \in \mathbb{C}^2 : |c| > 2(|a| + 1)^2\}. \end{aligned}$$

The first result about the hyperbolicity of the Hénon map was obtained by Devaney and Nitecki, which says that $DN \subset \mathcal{H}^{\mathbb{R}}$. They also proved that if we choose a parameter value from EMP, then $K^{\mathbb{R}}_{a,c}$ is empty. Later, Hubbard and Oberste-Vorth improved their result by showing that $HOV \subset \mathcal{H}^{\mathbb{C}}$.

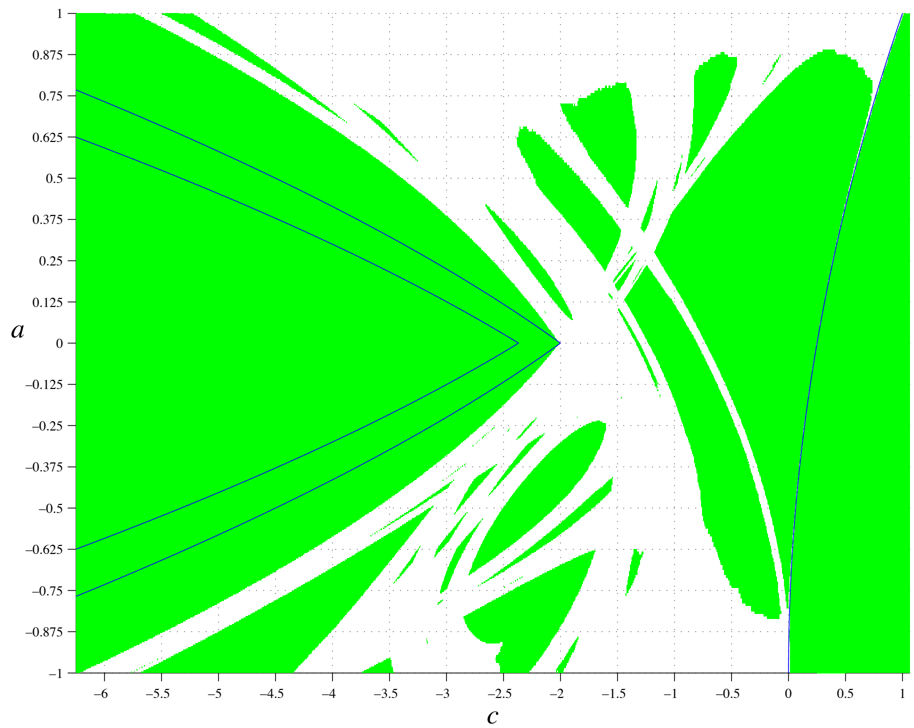

Figure 1: (a subset of) the Hyperbolic plateaus of the real Hénon map. Three solid lines describes the boundaries of DN, HOV and EMP, from left to right.

One of the Hubbard's conjectures is on the relation between $\mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^{\mathbb{C}}$. It follows from the result of Bedford, Lyubich and Smillie [3, Theorem 10.1] that $\mathcal{H}^{\mathbb{R}} \subset \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$. The problem is whether there is a real parameter value in $\mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$ which is not contained in $\mathcal{H}^{\mathbb{R}}$ but has non-trivial hyperbolic dynamics on the real plane. To be more precise, we divide $\mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$ into three sets of parameter values.

Definition 1 (Bedford and Smillie [5])**.** We call $(a, c) \in \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$ is of type 1 if $(a, c) \in \mathcal{H}^{\mathbb{R}}$, and of type 2 if $K^{\mathbb{R}}_{a,c} = \emptyset$. Otherwise we say it is of type 3.

Since $DN \subset \mathcal{H}^{\mathbb{R}} \subset \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$, the set of type 1 parameter values is non-empty. The set of type 2 parameter values is also non-empty since it contains $EMP \cap HOV$. However, the existence of a type 3 parameter value was open.

Conjecture 1 (Hubbard)**.** There exists a parameter value of type 3.

Moreover, Hubbard conjectured that there are infinitely many type 3 parameter values which have mutually different dynamics. Precisely, the conjecture was given in terms of the monodoromy representation of the fundamental group of the hyperbolic horseshoe locus. Denote by $\mathcal{H}_0^{\mathbb{C}}$ the component of $\mathcal{H}^{\mathbb{C}}$ that contains HOV. Fix a point $(a_0, c_0) \in DN$ and the canonical conjugacy $h_0 : K_{a_0,c_0}^{\mathbb{C}} \to \Sigma_2$.

Given a loop $\gamma : [0,1] \to \mathcal{H}_0^{\mathbb{C}}$ with the base point (a_0, c_0), we can construct a continuous family of conjugacies $h_t : K_{\gamma(t)}^{\mathbb{C}} \to \Sigma_2$ along γ for all $t \in [0,1]$. Then by setting $\rho(\gamma) := h_1 \circ h_0^{-1}$, we can define a homomorphism

$$\rho : \pi_1(\mathcal{H}_0^{\mathbb{C}}, (a_0, c_0)) \to \mathrm{Aut}(\Sigma_2).$$

Conjecture 2 (Hubbard)**.** The homomorphism ρ is surjective

In the rest of the paper, we will give an affirmative answer to Conjecture 1 and a partial result about Conjecture 2.

2 Main Results

Theorem 1. *There exist parameter values of type3. In fact, the parameters*

$$(a, c) \in \{-1\} \times [-5.625, -4.5625] \ \cup \ \{1\} \times [-5.4785, \ -5.3215]$$

consist of type 3 parameter values.

Let $\gamma_\emptyset$, γ_r and γ_p be loops in $\mathcal{H}_0^{\mathbb{C}}$ with the base point (a_0, c_0) that cross the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$ only once (except the base point) at $(1, 10)$, $(-1, 5)$ and $(1, 5.4)$, respectively.

Theorem 2. *The identity element and $\rho(\gamma_\emptyset)$, $\rho(\gamma_r)$ and $\rho(\gamma_p)$ are mutually distinct elements of* $\mathrm{Aut}(\Sigma_2)$*.*

The proof of these theorems are the consequence of the following two main lemmas.

Lemma 3. *On parameter regions colored in Figure 2 and 3, $H_{a,c}$ is uniformly hyperbolic on its chain recurrent set $\mathcal{R}(H_{a,c})$.*

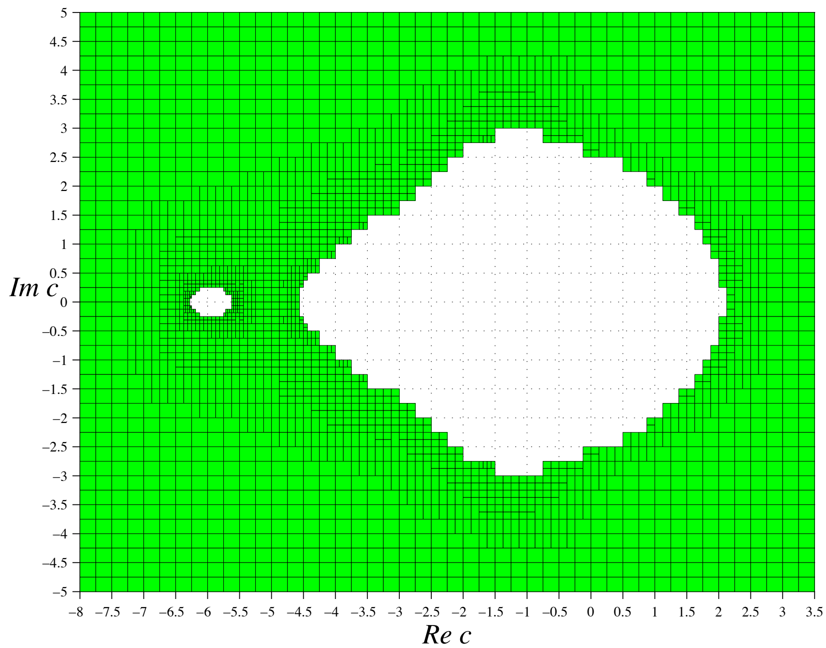

Figure 2: (a subset of) the hyperbolic full horseshoe locus of family $H_{-1,c}$

Note that Lemma 3 does not directly imply that $K^{\mathbb{C}}_{a,c}$ is uniformly hyperbolic since these two sets do not necessarily coincide in general. However, it can be shown that they do coincide in the hyperbolic horseshoe locus.

Corollary 4. *On parameter regions colored in Figure 2, 3, $H_{a,c}|K^{\mathbb{C}}_{a,c}$ is uniformly hyperbolic full horseshoe.*

Proof of Corollary 4. Choose a point (a, c) in the colored region in Figure 2 or 3, and a path γ from $(1, -10)$ to (a, c) which is contained in the region. At $(1, -10)$, $K_{a,c} = \mathcal{R}(H_{a,c})$ holds. By the semicontinuity of $K_{a,c}$ [4, Theorem 3.1] and the $\mathcal{R}$-structural stability, it follows that this equality holds all points along γ and hence at (a, c). Thus Lemma 3 implies the corollary. □

Lemma 5. *The dynamics of $H_{a,c}|K^{\mathbb{R}}_{a,c}$ are mutually not conjugate for parameter values $(a, c) = (1, -10)$, $(1, -5.4)$, $(1, +10)$ and $(-1, -5)$.*

Remark that at $(a, c) = (1, -10)$ the dynamics is a hyperbolic full horseshoe, i.e. $(1, -10)$ is of type 1, and $K^{\mathbb{R}}_{a,c} = \emptyset$ at $(a, c) = (1, +10)$, i.e. $(1, 10)$ is of

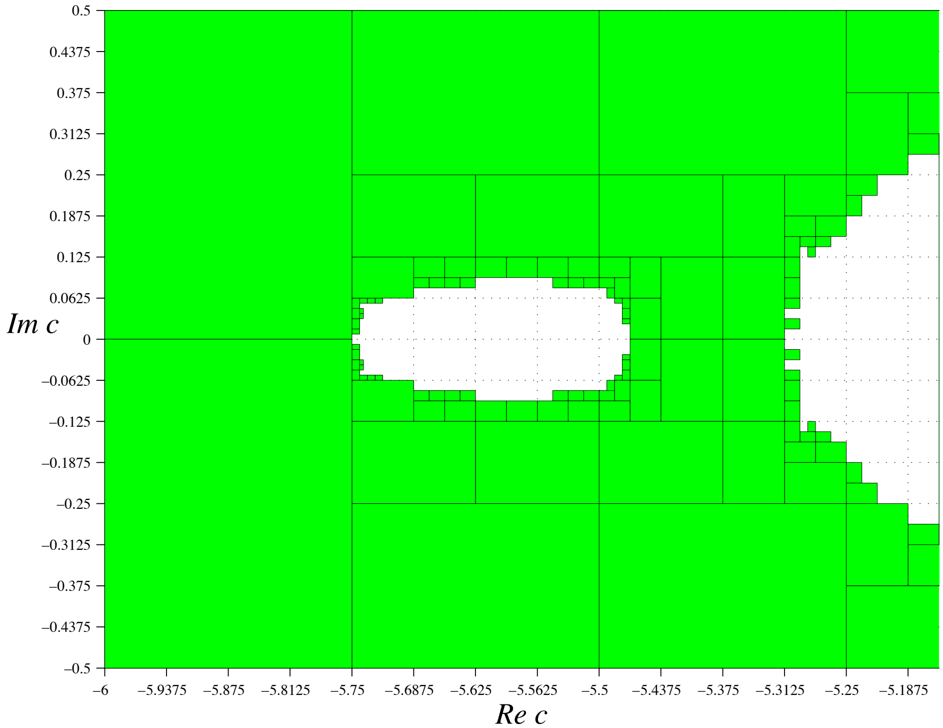

Figure 3: (a subset of) the hyperbolic full horseshoe locus of the family $H_{1,c}$

type 2.

The proof of these two lemmas are computer assisted, and the algorithms will be given in the later sections.

Proof of Theorem 1. Let R_r and R_p be the colored regions in Figure 2 and 3, respectively. We put

$$I_r = \{-1\} \times [-5.625, -4.5625], \quad I_p = \{1\} \times [-5.4785, -5.3215]$$

so that $I_r \subset R_r \cap \{\mathrm{Im}\ c = 0\}$ and $I_p \subset R_p \cap \{\mathrm{Im}\ c = 0\}$ hold. We will show that I_r and I_p consist of type 3 parameter values.

Let (a_0, c_0) be a point in $I_r \cup I_p$. Then it is easy to see that $(a_0, c_0) \in \mathcal{H}^{\mathbb{C}}$ because we can take a continuous path from DN to (a_0, c_0) within R_r or R_p. We are now left to check that $K^{\mathbb{R}}_{a_0,c_0}$ is not empty and the dynamics on it is not conjugate to the full shift of Σ_2, but this follows from Lemma 5. This proves Theorem 1. $\square$

To prove Theorem 2, we use the theorem of Bedford and Smillie. Let (a_1, c_1) and (a_2, c_2) be parameter values in $\mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$ and γ_1 and γ_2 be loops in $\mathcal{H}_0^{\mathbb{C}}$ with base point $(a_0, c_0) \in DN$ that path through (a_1, c_1) and (a_2, c_2), respectively.

Theorem 6 ([5, Theorem 5.2])**.** *Assume* $\rho(\gamma_1) = \rho(\gamma_2)$. *Then* $H_{a_1,c_1}|K^{\mathbb{R}}_{a_1,c_1}$ *and* $H_{a_2,c_2}|K^{\mathbb{R}}_{a_2,c_2}$ *are conjugate.*

Theorem 2 is a direct consequence of Theorem6 and Lemma 5.

3 Algorithm to prove the hyperbolicity

In this section, we briefly recall the algorithm developed by the author [1]. We also refer the reader to the papers of Suzanne Lynch Hruska [9, 10] for another algorithm.

Let f be a diffeomorphism on a manifold M and Λ a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

Definition 2. We say f is *uniformly hyperbolic* on Λ, or Λ is a *uniformly hyperbolic invariant set* if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n|_{E^s}\| < c\lambda^n \quad \text{and} \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n$$

hold for all $n \geq 0$. Here $\|\cdot\|$ denotes a metric on M.

We note that this definition involves two constants, c and λ. If we try to prove hyperbolicity according to this usual definition, we must control two parameters at the same time, and the algorithm would be rather complicated. Although we can omit the constant c by choosing a suitable metric on M, constructing such a metric is also a difficult problem in general. The situation is the same even if we use the standard "cone fields" argument.

To avoid this computational difficulty, we introduce the notion of quasi-hyperbolicity. Recall that the differential of f induces a dynamical system $Tf : TM \to TM$. By restricting it to the invariant set $T\Lambda$, we obtain $Tf : T\Lambda \to T\Lambda$. An orbit of Tf is called a trivial orbit if it is contained in the zero section of the bundle $T\Lambda$.

Definition 3. We say that f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

It is easy to see that hyperbolicity implies quasi-hyperbolicity. The converse is not true in general. However, when $f|_\Lambda$ is chain recurrent, these two notions coincide.

Theorem 7 ([6, 14]). *Assume that $f|_\Lambda$ is chain recurrent, that is, $\mathcal{R}(f|_\Lambda) = \Lambda$. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

Next, we rephrase the definition of quasi-hyperbolicity in terms of isolating neighborhoods. Recall that a compact set N is an isolating neighborhood (see [12]) with respect to f if the maximal invariant set

$$\mathrm{Inv}_f N := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}$$

is contained in int N, the interior of N. An invariant set S of f is said to be isolated if there is an isolating neighborhood N such that $\mathrm{Inv}_f N = S$.

Note that linearity of Tf in the fiber direction implies that if there is a non-trivial bounded orbit of $Tf : T\Lambda \to T\Lambda$, then any neighborhood N of the zero-section of $T\Lambda$ contains a non-trivial bounded orbit. Therefore, the definition of quasi-hyperbolicity is equivalent to saying that the zero section of the tangent bundle $T\Lambda$ is an isolated invariant set with respect to $Tf : T\Lambda \to T\Lambda$. Furthermore, it suffice to find an isolating neighborhood that contains the zero section.

Proposition 8. *Assume that $N \subset T\Lambda$ is a isolating neighborhood with respect to $Tf : T\Lambda \to T\Lambda$ and N contains the image of the zero-section of $T\Lambda$. Then Λ is quasi-hyperbolic.*

Now we check that the hypothesis of Theorem 7 is satisfied in our case of the complex Hénon maps.

Let us define

$$R(a,c) := \frac{1}{2}(1 + |a| + \sqrt{(1+|a|)^2 + 4c}),$$
$$S(a,c) := \{(x,y) \in \mathbb{C}^2 : |x| \le R(a,c), |y| \le R(a,c)\}.$$

Then we can prove the following.

Lemma 9. *The chain recurrent set $\mathcal{R}(H_{a,c})$ is contained in $S(a,c)$. Furthermore, $H_{a,c}$ restricted to $\mathcal{R}(H_{a,c})$ is chain recurrent.*

We fix the Jacobian parameter a to $+1$ or -1 and consider $H_{1,c}$ and $H_{-1,c}$ as one-parameter families with parameter c. In parameter c plane, we define a square C which is defined by

$$C = \{c \in \mathbb{C} : |\mathrm{Im}\ c| < 8, |\mathrm{Re}\ c| < 8\}.$$

Note that the region $(\mathbb{C}^2 \setminus HOV) \cap \{a = \pm 1\}$ is contained in $\{\pm 1\} \times C$.

To prove Lemma 3, we run the algorithm 15 of [1] with initial parameter set C and for family $H_{\pm 1,c}$. The computation was done on 2GHz PowerPC 970 CPU and computational time required to obtain Figure 2 and 3 were 530.3 hours and 654.0 hours, respectively.

4 Counting Periodic Orbits

In the last section, we give an algorithm to prove Lemma 5. This is done by showing that $H_{a,c}$ has different numbers of periodic point in $K^{\mathbb{R}}_{a,c}$ at these parameter values. In fact, we claim that the numbers of points in $\mathrm{Fix}(H^n_{a,c}) \cap \mathbb{R}^2$ is exactly as in Figure 4.

	$(1, -10)$	$(1, -5.4)$	$(-1, -5)$	$(1, +10)$
$n = 3$	8	8	2	0
$n = 4$	16	16	16	0
$n = 5$	32	22	22	0
$n = 6$	64	52	52	0
$n = 7$	128	114	72	0

Figure 4: The number of points in $\mathrm{Fix}(H^n_{a,c}) \cap \mathbb{R}^2$

We use the Conley index theory to prove the claim. The reader not familiar with the Conley index may consult [11, 12].

First we give a lower bound for the number of periodic points. Consider the dynamics restricted on the real plane $H_{a,c}|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ and numerically find the periodic points. Next we construct an index pair for each periodic points in $\mathbb{R}^2$. Then the existence of periodic points in these index pairs will be established using the following Conley index version of Lefschetz fixed point theorem.

Theorem 10 ([11, Theorem 10.102]). *Let (P_1, P_0) be an index pair for f and f_{P*} the homology index map induced by f. If*

$$\sum_k (-1)^k \operatorname{tr} f^n_{P*k} \neq 0$$

then $\mathrm{Inv}(\mathrm{cl}\,(P_1 \setminus P_0), f)$ *contains a fixed point of f^n.*

This theorem assures that there is at least one fixed point in each index pair and therefore we obtain a lower bound for the number of points in $\mathrm{Fix}(H^n_{a,c}) \cap \mathbb{R}^2$.

To obtain an upper bound, two methods can be applied.

One is to prove the uniqueness of the fixed point of $H^n_{a,c}$ in each index pair. Since all the periodic points are hyperbolic in these cases, this can be done using a Hartman-Grobman type theorem [2, Proposition 4.1].

The other one uses the fact that the number of the fixed points of $H^n_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2$ does not depends on the parameter and equals to 2^n, counted with multiplicity [8, Theorem 3.1]. Note that in our case, the uniform hyperbolicity implies that the multiplicity is always 1 and there are exactly 2^n distinct fixed points. Therefore, if there are k distinct fixed points of $H^n_{a,c}$ in $\mathbb{C}^2 \setminus \mathbb{R}^2$, then the number of the fixed points of $H^n_{a,c}$ in $K^{\mathbb{R}}_{a,c}$ is $2^n - k$. Now we can use Theorem 10 to prove the existence of fixed points in $\mathbb{C}^2 \setminus \mathbb{R}^2$, and this gives an upper bound for the number of points in $\mathrm{Fix}(H^n_{a,c}) \cap \mathbb{R}^2$.

In fact, in all four parameter values of Lemma 5, the lower bound and upper bounded obtained by this method coincide. This proves our claim.

Acknowledgments. First of all, the author would like to thank J. Hubbard, whose enthusiasm for the complex Hénon maps drove him to this study. He also thank E. Bedford, J. Smillie and S. Huruska for many valuable suggestions and their hospitality. This work is partially supported by JSPF.

References

[1] Z. Arai, On Hyperbolic Plateaus of the Hénon Maps, preprint.

[2] Z. Arai and K. Mischaikow, Rigorous computations of homoclinic tangencies, preprint.

[3] E. Bedford, M. Lyubich and J. Smillie, Polynomial diffeomorphisms of C^2. IV: The measure of maximal entropy and laminar currents, *Invent. math.*, **112** (1993), 77–125.

[4] E. Bedford and J. Smillie, Polynomial diffeomorphisms of C^2: currents, equilibrium measure and hyperbolicity, *Invent. math.*, **103** (1991), 69–99.

[5] E. Bedford and J. Smillie, The Hénon family: The complex horseshoe locus and real parameter values, preprint.

[6] R. C. Churchill, J. Franke and J. Selgrade, A geometric criterion for hyperbolicity of flows, *Proc. Amer. Math. Soc.*, **62** (1977), 137–143.

[7] R. Devaney and Z. Nitecki, Shift automorphisms in the Hénon mapping, *Commun. Math. Phys.*, **67** (1979), 137–146.

[8] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dynam. Systems* **9** (1989), 67–99.

[9] S. L. Hruska, A numerical method for proving hyperbolicity of complex Hénon mappings, preprint, `arXiv:math.DS/0406004`.

[10] S. L. Hruska, Rigorous numerical studies of the dynamics of polynomial skew products of $\mathbb{C}^2$, preprint, `arXiv:math.DS/0502038`.

[11] T. Kaczynski, K. Mischaikow and M. Mrozek, *Computational Homology*, Applied Mathematical Sciences, **157**, Springer-Verlag, 2004.

[12] K. Mischaikow and M. Mrozek, The Conley index theory, *Handbook of Dynamical Systems II*, North-Holland, 2002, 393–460.

[13] C. Robinson, *Dynamical systems; stability, symbolic dynamics, and chaos*, 2nd ed., CRC Press, Boca Raton, FL, 1999.

[14] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splitting for linear differential systems I, *J. Differential Equations*, **27** (1974) 429–458.