Ricci curvature and almost spherical multi-suspension

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1 Introduction

In this paper, we give a generalization of suspension theorem for almost maximal diameter that is proved by J.Cheeger and T.H.Colding. As a consequence, we can get some sphere theorems. And we introduce the relation to first eigenvalue of Laplacian and to the structure of tangent cone of non-collapsing limit spaces.

One of the main results of this paper is the following theorem;

Main Theorem 1
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold $(n \geq 2)$ with $Ric_M \geq n-1$, we assume there exists $p_i, q_i \in M$ $(i = 1, 2, \cdots, k)$ such that

$$\text{for each } i,\ |\overline{p_i, q_i} - \pi| < \epsilon \text{ holds, for } i \neq j,\ \ |\overline{p_i, p_j} - \tfrac{\pi}{2}| < \epsilon \text{ holds.}$$

Here $\epsilon < \epsilon_n$ is sufficiently small positive number. Then we have

1. $k \leq n+1$.
2. *If $1 \leq k \leq n-1$, then there exist a compact length space Z $(diam_Z \leq \pi)$ such that*
$$d_{GH}(M, \mathbf{S}^{k-1} * Z) < \Psi(\epsilon|n).$$
3. *If $k = n, or\ n+1$, then*
$$d_{GH}(M, \mathbf{S}^n) < \Psi(\epsilon|n).$$
Especially, M is diffeomorphic to $\mathbf{S}^n$.

Here, $\Psi(\epsilon|n)$ is a function from $\mathbf{R}_{>0} \times \mathbf{N}$ to $\mathbf{R}_{>0}$ such that for each $n \in \mathbf{N}$

$$\lim_{\epsilon \to 0} \Psi(\epsilon|n) = 0.$$

And $\mathbf{S}^{k-1} * Z$ is k-fold spherical suspension of Z.
d_{GH} is the Gromov-Hausdorff distance between compact metric spaces.
Main Theorem 1 gives some sphere theorem in case $k = n$.

Let us review some related result.

Let M be an n dimensional complete Riemannian manifold with $Ric_M \geq n-1$.

Then, we have

$$diam_M \leq \pi, \quad rad_M \leq \pi, \quad vol(M) \leq vol(\mathbf{S}^n).$$

Especially, M is compact.

Here, $diam_m$, rad_M, $vol(M)$ are diameter, radius, volume of M each, and $\mathbf{S}^n$ is n dimensional standard unit sphere in $n+1$ dimensional Eucildean space.

Of couse, if M is isometric to $\mathbf{S}^n$, then above inequality are equality. Conversely, *If above some inequality satisfies equality, then M is isometric to $\mathbf{S}^n$.*

Now, we consider perturbation version of this.

The perturbation version for volume and radius is the following that is proved by T.H.Colding.

Theorem 1.1 (T.H.Colding [10, 11])
With notation as above, we assume

$$vol(M) \geq vol(\mathbf{S}^n) - \epsilon \quad (\text{ or } rad_M \geq \pi - \epsilon).$$

Here $\epsilon < \epsilon_n$ is sufficiently small positive number. Then we have

$$d_{GH}(M, \ \mathbf{S}^n) < \Psi(\epsilon|n).$$

Especially, M is diffeomorphic to $\mathbf{S}^n$.

Here, last stetement, "diffeomorphic" is a result of stability theorem that is proved by J.Cheeger and T.H.Colding. (See Theorem 2.26 in section 2)

But a statement corresponding to a diameter is not true. But the following result is proved by J.Cheeger and T.H.Colding.

Theorem 1.2 (J.Cheeger, T.H.Colding [4])
With notation as above, we assume

$$diam_M \geq \pi - \epsilon.$$

Here $\epsilon < \epsilon_n$ is sufficiently small positive number. Then there exist a compact length space Z with $diam_Z \leq \pi$ such that

$$d_{GH}(M, \ \mathbf{S}^0 * Z) < \Psi(\epsilon|n).$$

Theorem 1.2 corresponds to case $k = 1$ of Main Theorem 1.

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

denotes eigenvalues of Laplacian on M.

Main Theorem 2
With notation as above,

$$\textit{Assumption of Main Theorem 1 holds} \iff \lambda_k \le n + \epsilon.$$

We will explain this theorem in section 3.

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2 Proof of Main Theorem 1

2.1 Preliminaries

Notation A function $\psi : \mathbf{R}^k_{>0} \times \mathbf{R}^l \to \mathbf{R}_{>0}$ such that

$$\lim_{\epsilon_1,\epsilon_2,\cdots\epsilon_k \to 0} \psi(\epsilon_1, \epsilon_2, \cdots \epsilon_k | c_1, c_2, \cdots, c_k) = 0,$$

it is denoted by

$$\Psi(\epsilon_1, \epsilon_2, \cdots \epsilon_k | c_1, c_2, \cdots, c_k). \quad \text{(or simply, } \Psi)$$

Therefore, for example,

$$2\Psi(\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_k | c_1, c_2, \cdots, c_l) = \Psi(\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_k | c_1, c_2, \cdots, c_l).$$

And we use the following notation;

$$a = b \pm \Psi \iff |a - b| < \Psi.$$

Z is a metric space, Z and $z \in Z, r > 0$, we put

$$\mathbf{B}_r(z) := \{w \in Z | \overline{z,w} < r\}, \quad \overline{\mathbf{B}}_r(z) := \{w \in Z | \overline{z,w} \le r\}, \quad \partial \mathbf{B}_r(z) := \{w \in Z | \overline{z,w} = r\}.$$

Here, $\overline{z,w}$ is a distance between z and w.

Definition 2.1 (length space)
We say that Z is a length space if for each $z_1, z_2 \in Z$, there exist a continuous map $c : [0,1] \to Z$ such that

$$\textit{length}(c) = \overline{z_1, z_2}$$

Remark 2.2
We skip the definition of the length for above continuous map. See [2] for details.

Definition 2.3 (spherical susupension)
We define a metric on $[0,\pi] \times Z/_\sim$ (here, $\sim$ is an equivalent relation such that $\{0\} \times Z$ and $\{\pi\} \times Z$ goes to point each.) as

$$\overline{(t_1,z_1),(t_2,z_2)} \stackrel{\text{def}}{=} \arccos(\cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos \min\{\overline{z_1,z_2},\pi\}).$$

Then, this metric space is denoted by

$$\mathbf{S}^0 * Z$$

and we call spherical suspension of Z.
And we define

$$\mathbf{S}^k * Z := \overbrace{\mathrm{S}^0 * (\mathrm{S}^0 * (\cdots * (\mathrm{S}^0}^{k+1} * Z))\cdots).$$

Remark 2.4
If Z is compact, then $\mathbf{S}^0 * Z$ is compact. And, if Z is length space, then $\mathbf{S}^0 * Z$ is also a length space.

We put $\mathcal{M} = \{$isometry class of compact matric space$\}$, then,

$$\mathbf{S}^0 * : \mathcal{M} \longrightarrow \mathcal{M}$$

is uniformly continuous map for d_{GH}.
Namely, for W, Z : are compact metric space,

$$d_{GH}(Z,W) < \epsilon \Longrightarrow d_{GH}(\mathbf{S}^0 * Z, \mathbf{S}^0 * W) < \Psi(\epsilon)$$

holds.

Now we introduce a segment inequarity.

For an n dimensional complete Riemannian manifold M ($n \geq 2$) with $Ric_M \geq n-1$, $g : M \longrightarrow \mathbf{R}_{\geq 0}$, we put $\mathcal{F}_g : M \times M \longrightarrow \mathbf{R}_{\geq 0}$ into

$$\mathcal{F}_g(x,y) := \inf_{\gamma} \int_{\gamma} g(\gamma(t))dt.$$

Here, infinimum runs all normal geodesic γ from x to y.

Theorem 2.5 (J.Cheeger, T.H.Colding [3])
With notation as above,

$$\int_{M \times M} \mathcal{F}_g(x,y)dxdy \leq C(n) vol M \int_M g(x)dx.$$

Here, $C(n)$ is a positive constant depending only on n.

Remark 2.6
In fact, above theorem is a special case of segment inequality that is proved by J.Cheeger and T.H.Colding. They prove the statement under $Ric_M \geq -(n-1)$. But, in this situation, it is sufficient to prove main result.

2.2 Proof of Almost cosine formura (Analytic part)

From now on, fix an integer $n \geq 2$, a positive number ϵ, and M always denotes an n dimensional complete Riemannian manifold with $Ric_M \geq n-1$ and $p, q \in M$ such that $\overline{p,q} \geq \pi - \epsilon$ holds. We put $f(x) := \cos \overline{p,x}$.

Lemma 2.7 (T.H.Colding [10])
With notation as above, there exist $\tilde{f} \in C^\infty(M)$ such that

$$\frac{1}{vol(M)} \int_M |f(x) - \tilde{f}(x)|^2 dx < \Psi(\epsilon|n),$$

$$\frac{1}{vol(M)} \int_M |\nabla f - \nabla \tilde{f}|^2 dx < \Psi(\epsilon|n),$$

$$\frac{1}{vol(M)} \int_M |\mathrm{Hess}_{\tilde{f}} + \tilde{f} g_M|^2 dx < \Psi(\epsilon|n).$$

Here g_M is Riemannian metric on M.

Lemma 2.8 (K.Grove, P.Petersen [17])
For each $x \in M$,

$$\overline{p,x} + \overline{q,x} - \overline{p,q} < \Psi(\epsilon|n).$$

Lemma 2.9
For each $x \in M$, $t \in [-1,1]$ $\quad (f^{-1}(t) \neq \phi)$,

1. *If $f(x) \leq t$, then*

$$\overline{x, f^{-1}(t)} + \overline{p, f^{-1}(t)} - \overline{x,p} = 0.$$

2. *If $f(x) > t$, then*

$$\overline{p,x} + \overline{x, f^{-1}(t)} - \overline{p, f^{-1}(t)} < \Psi(\epsilon|n).$$

Proof.

1. It is easy to see that there exist $y \in f^{-1}(t)$ such that

$$\overline{p,y} + \overline{x,y} = \overline{p,x}$$

On the other hand, for each $z \in f^{-1}(t)$,

$$\begin{aligned} \overline{x,y} &= \overline{p,x} - \overline{p,y} \\ &= \overline{p,x} - \overline{p,z} \\ &\leq \overline{x,z} \end{aligned}$$

So,

$$\overline{x,y} = \overline{x, f^{-1}(t)}.$$

This gives the claim.

2. We can assume $f(q) \leq t$ without loss of generality. Similarly above argument, there exist $y \in f^{-1}(t)$ such that

$$\overline{x,y} + \overline{y,q} = \overline{x,q}$$

From Lemma 2.8,

$$\overline{p,x} + \overline{x,y} - \overline{p,y} < \Psi.$$

So, for each $z \in f^{-1}(t)$,

$$\begin{aligned} \overline{x,y} &\leq \overline{p,y} - \overline{p,x} + \Psi \\ &= \overline{p,z} - \overline{p,x} + \Psi \\ &\leq \overline{z,x} + \Psi. \end{aligned}$$

Therefore,

$$\left| \overline{x,y} - \overline{x, f^{-1}(t)} \right| < \Psi$$

This gives the claim. $\square$

Lemma 2.10
We take a $\tilde{f} \in C^\infty(M)$ as in Lemma 2.7.

Then, for each $x, y, z \in M$, there exists $\hat{x}, \hat{y}, \hat{z} \in M$ with the following properties.

1. $\overline{x,\hat{x}} < \Psi(\epsilon|n)$, $\quad \overline{y,\hat{y}} < \Psi(\epsilon|n)$, $\quad \overline{z,\hat{z}} < \Psi(\epsilon|n)$,
$|f(\hat{x}) - \tilde{f}(\hat{x})| < \Psi(\epsilon|n)$, $\quad |f(\hat{y}) - \tilde{f}(\hat{y})| < \Psi(\epsilon|n)$, $\quad |f(\hat{z}) - \tilde{f}(\hat{z})| < \Psi(\epsilon|n)$.

2. *For each two elements in $\hat{x}, \hat{y}, \hat{z}$, one is not contained cut locus of the other.*

3. *There exist a unique normal geodesic from $\hat{x}$ to $\hat{y}$;*

$$\sigma : [0, \overline{\hat{x},\hat{y}}] \to M$$

and $U \subset [0, \overline{\hat{x},\hat{y}}]$ such that

   (a) *U is open and has full volume. And for each $u \in U$, there exist a unique normal geodesic from $\hat{z}$ to $\sigma(u)$,*

$$\tau_u : [0, l(u)] \to M \quad (l(u) := \overline{\hat{z}, \sigma(u)}).$$

   (b) *It has following property;*

$$\int_U \left|f(\sigma(u)) - \tilde{f}(\sigma(u))\right|^2 du < \Psi(\epsilon|n),$$

$$\int_U \left||\nabla \tilde{f}|(\sigma(u)) - \sin \overline{p, \sigma(u)}\right|^2 du < \Psi(\epsilon|n),$$

$$\int_U \int_0^{l(u)} \left|\mathrm{Hess}_{\tilde{f}} + \tilde{f} g_M\right|(\tau_u(s)) ds du < \Psi(\epsilon|n).$$

Proof. From Theorem 2.5,

$$\frac{1}{\left(vol(M)\right)^3} \int_{M^3} \mathcal{F}_{\mathcal{F}_{|\mathbf{Hess}_{\tilde{f}} + \tilde{f} g_M|}(c,\ )}(a,b) dadbdc < \Psi(\epsilon|n) \qquad (\#)$$

We take specific $\Psi(\epsilon|n)$ such that above inequality (#) and Lemma 2.7 holds, and denotes by $\psi_0(\epsilon|n)$.

i.e

1. $\psi_0 : \mathbf{R}_{>0} \times \mathbf{N} \to \mathbf{R}_{>0}$
2. For each $n \in \mathbf{N}$,
$$\lim_{\epsilon \to 0} \psi_0(\epsilon|n) = 0$$
holds.
3. Above inequality (#) and Lemma 2.7 holds if we replace the $\Psi(\epsilon|n)$ by $\psi_0(\epsilon|n)$.

And, we take $\psi_i : \mathbf{R}_{>0} \times \mathbf{N} \to \mathbf{R}_{>0}$ $(i = 1, 2)$ with the following properties;

I. For each $n \in \mathbf{N}$,
$$\lim_{\epsilon \to 0} \psi_i(\epsilon|n) = 0$$
holds.

II.
$$\lim_{\epsilon \to 0} \frac{\psi_{i-1}(\epsilon|n)}{\psi_i(\epsilon|n)} = 0. \qquad (i = 1, 2)$$

$\tilde{M} \subset M^3$ is a subset of M^3 consists of element $(a, b, c) \in M^3$ with the following properties;

- For each two elements in a, b, c, one is not contained cut locus of the other, and $\overrightarrow{a, b} \cap C_c$ is 0-set in $\overrightarrow{a, b}$. Here, C_c is cut locus of c.
- $|f(a) - \tilde{f}(a)| \leq \psi_1(\epsilon|n)$, $|f(b) - \tilde{f}(b)| \leq \psi_1(\epsilon|n)$, $|f(c) - \tilde{f}(c)| \leq \psi_1(\epsilon|n)$.

- $$\int_{\overrightarrow{a,b}} |f-\tilde{f}|^2 \leq \psi_1(\epsilon|n), \qquad \int_{\overrightarrow{a,b}} \big||\nabla \tilde{f}| - \sin \overline{p,}\ \big|^2 \leq \psi_1(\epsilon|n).$$

- $$\mathcal{F}_{\mathcal{F}_{|\mathbf{Hess}_{\tilde{f}} + \tilde{f} g_M|^2}(c,\ )}(a,b) \leq \psi_1(\epsilon|n).$$

Then, by using segment inequality, we have

$$vol(\tilde{M}) \geq (1-\psi_2(\epsilon|n))\big(vol(M)\big)^3.$$

From this and Bishop-Gromov's volume comparison theorem, we have the claim. □

Lemma 2.11
For each $x \in M$, $t \in [-1,1]$ and $z \in f^{-1}(t)$, $y \in f^{-1}(t)$ such that

$$\overline{x,y} = \overline{x, f^{-1}(t)},$$

we take $\hat{x}, \hat{y}, \hat{z}$ as in Lemma 2.10. (We use same notation in Lemma 2.10 below.)

1. *If $f(x) \leq t$, then*

$$\int_U \big|\nabla \tilde{f}(\sigma(u)) - \sin(\overline{p,x} - u)\sigma'(u)\big|^2 du < \Psi(\epsilon|n).$$

2. *If $f(x) > t$, then*

$$\int_U \big|\nabla \tilde{f}(\sigma(u)) + \sin(\overline{p,x} + u)\sigma'(u)\big|^2 du < \Psi(\epsilon|n).$$

Proof. First, Note that we have the following;

1. For each $u \in U$,
$$|\overline{p,\sigma(u)} - (\overline{p,x} - u)| < \Psi(\epsilon|n).$$

2. For each $u \in U$,
$$|\overline{p,\sigma(u)} - (\overline{p,x} + u)| < \Psi(\epsilon|n).$$

We skip the proof of this result because it is easy to prove by Lemma 2.8.

We will give only the proof of case 1 by using this result. (Case 2 is also similally argument.)

$$\int_U \left|\nabla \tilde{f}(\sigma(u)) - \sin(\overline{p,x} - u)\sigma'(u)\right|^2 du$$

$$= \int_U \big(|\nabla \tilde{f}|^2(\sigma(u)) - 2\sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u) + \sin^2(\overline{p,x} - u)\big)du$$

$$= \int_U \big(\sin^2(\overline{p,x} - u) - 2\sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u) + \sin^2(\overline{p,x} - u)\big)du \pm \Psi$$

$$= 2\int_U \big(\sin^2(\overline{p,x} - u) - \sin(\overline{p,x} - u)(\tilde{f}\circ\sigma)'(u)\big)du \pm \Psi$$

$$= 2\int_U \sin^2(\overline{p,x} - u)du - 2\big[\sin(\overline{p,x} - u)\tilde{f}\circ\sigma(u)\big]_0^{\overline{\hat{x},\hat{y}}}$$

$$+ 2\int_U - \cos(\overline{p,x} - u)\tilde{f}\circ\sigma(u)du \pm \Psi$$

$$= 2\int_U \sin^2(\overline{p,x} - u)du - 2\big(\sin(\overline{p,\hat{x}} - \overline{\hat{x},\hat{y}})\tilde{f}(\hat{y}) - \sin\overline{p,x}\ \tilde{f}(\hat{x})\big)$$

$$+ 2\int_U - \cos^2(\overline{p,x} - u)du \pm \Psi$$

$$= 2\int_U \big(\sin^2(\overline{p,x} - u) - \cos^2(\overline{p,x} - u)\big)du$$

$$- 2\big(\sin\overline{p,\hat{y}}\cos\overline{p,y} - \sin\overline{p,x}\cos\overline{p,x}\big) \pm \Psi$$

$$= -2\int_U \cos(2\overline{p.x} - 2u)du - \sin 2\overline{p,y} + \sin 2\overline{p,x} \pm \Psi$$

$$= \big[\sin(2\overline{p,x} - 2u)\big]_0^{\overline{\hat{x},\hat{y}}} - \sin 2\overline{p,y} + \sin 2\overline{p,x} \pm \Psi$$

$$= \sin 2(\overline{p,\hat{x}} - \overline{\hat{x},\hat{y}}) - \sin 2\overline{p,x} - \sin 2\overline{p,y} + \sin 2\overline{p,x} \pm \Psi$$

$$= \Psi \qquad \square$$

Lemma 2.12
Under same assumption as in Lemma 2.11,

$$\left|\frac{\cos\overline{\hat{z},\hat{x}}-\cos\overline{p,\hat{z}}\cos\overline{p,\hat{x}}}{\sin\overline{p,\hat{x}}}-\frac{\cos\overline{\hat{y},\hat{z}}-\cos\overline{p,\hat{y}}\cos\overline{p,\hat{z}}}{\sin\overline{p,\hat{y}}}\right|\times\min\{\sin^2\overline{p,\hat{x}},\ \sin^2\overline{p,\hat{y}}\}<\Psi(\epsilon|n)$$

Proof. We will give only the proof under assumption of case 1 in Lemma 2.11.

$$\left|\frac{\cos\overline{\hat{z},\hat{x}}-\cos\overline{p,\hat{z}}\cos\overline{p,\hat{x}}}{\sin\overline{p,\hat{x}}}-\frac{\cos\overline{\hat{y},\hat{z}}-\cos\overline{p,\hat{y}}\cos\overline{p,\hat{z}}}{\sin\overline{p,\hat{y}}}\right|=\left|\int_U\left(\frac{\cos l(u)-\cos\overline{p,\hat{z}}\cos(\overline{p,\hat{x}}-u)}{\sin(\overline{p,\hat{x}}-u)}\right)'du\right|$$

$$=\left|\int_U\left\{\frac{\big(-\sin l(u)\ l'(u)-\cos\overline{p,\hat{z}}\sin(\overline{p,\hat{x}}-u)\big)\sin(\overline{p,\hat{x}}-u)}{\sin^2(\overline{p,\hat{x}}-u)}\right.\right.$$

$$\left.\left.+\frac{\big(\cos l(u)-\cos\overline{p,\hat{z}}\cos(\overline{p,\hat{x}}-u)\big)\cos(\overline{p,\hat{x}}-u)}{\sin^2(\overline{p,\hat{x}}-u)}\right\}du\right|$$

$$\leq\frac{1}{\min\{\sin^2\overline{p,\hat{x}},\sin^2\overline{p,\hat{y}}\}}\left\{\int_U\left|-\sin l(u)<\tau_u'(l(u)),\sigma'(u)>\sin(\overline{p,\hat{x}}-u)\right.\right.$$

$$\left.\left.+\cos l(u)f(\sigma(u))-\cos\overline{p,\hat{z}}\ \right|du\pm\Psi\right\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat{x}},\sin^2\overline{p,\hat{y}}\}}\left\{\int_U\left|-\frac{d\tilde{f}\circ\tau_u(s)}{ds}\Big|_{s=l(u)}\sin l(u)+\cos l(u)\tilde{f}(\tau_u(l(u)))-\tilde{f}(\tau_u(0))\right|du\pm\Psi\right\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat{x}},\sin^2\overline{p,\hat{y}}\}}\left\{\int_U\left|\int_0^{l(u)}\frac{d}{ds}\Big(-\frac{d\tilde{f}\circ\tau_u(s)}{ds}\sin s+\cos s\ \tilde{f}(\tau_u(s))\Big)ds\right|du\pm\Psi\right\}$$

$$\leq\frac{1}{\min\{\sin^2\overline{p,\hat{x}},\sin^2\overline{p,\hat{y}}\}}\left\{\int_U\int_0^{l(u)}\left|\mathbf{Hess}_{\tilde{f}}+\tilde{f}g_M\right|(\tau_u(s))dsdu\pm\Psi\right\}$$

$$=\frac{1}{\min\{\sin^2\overline{p,\hat{x}},\sin^2\overline{p,\hat{y}}\}}\ \Psi \qquad \square$$

Proposition 2.13 (Almost cosine formura)
There exist $\delta=\delta(\epsilon,n)>0$, ($\lim_{\epsilon\to0}\delta(\epsilon,n)=0$) *with the following property;*
For each $x\in M$*, we take* $z_x\in\partial\mathbf{B}_{\frac{\pi}{2}}(p)$ *such that*

$$\overline{x, z_x} = \overline{x, \partial \mathrm{B}_{\frac{\pi}{2}}(p)}.$$

Then, for each $x, x' \in M \setminus (\mathrm{B}_\delta(p) \cup \mathrm{B}_\delta(q))$,

$$\cos \overline{x, x'} = \cos \overline{p, x} \cos \overline{p, x'} + \sin \overline{p, x} \sin \overline{p, x'} \cos \overline{z_x, z_{x'}} \pm \Psi(\epsilon|n)$$

holds.

Proof. This is clear by Lemma 2.12. □

2.3 Proof of Main Theorem 1 (Geometric part)

In this section, We will estimate several Gromov-Hausdorff distance.

Lemma 2.14
Let $\epsilon > 0$, M *be an* n *dimensional complete Riemannian manifold* ($n \geq 2$) *with* $Ric_M \geq n-1$, *we assume there exists* $p, q \in M$ *such that* $\overline{p, q} \geq \pi - \epsilon$ *holds.*
Then,

$$d_{GH}\big(M, \mathbf{S}^0 * \partial \mathbf{B}_{\frac{\pi}{2}}(p)\big) < \Psi(\epsilon|n).$$

Here the metric on $\partial \mathbf{B}_{\frac{\pi}{2}}(p)$ *is the restriction of* M.

Proof. Under same notation in Lemma 2.13, we define

$$\phi : M \setminus \big(\mathbf{B}_\delta(p) \cup \mathbf{B}_\delta(q)\big) \rightarrow \mathbf{S}^0 * \partial \mathbf{B}_{\frac{\pi}{2}}(p) = [0, \pi] \times \partial \mathbf{B}_{\frac{\pi}{2}}(p) / \{0, \pi\} \times \partial \mathbf{B}_{\frac{\pi}{2}}(p)$$

as

$$\phi(x) = (\overline{p, x}, z_x).$$

This gives $\Psi(\epsilon|n)$-Hausdorff approximation by Proposition 2.13.
Therefore, from this, the claim is clear. □

Now, We take specific $\Psi(\epsilon|n)$ such that Lemma 2.8 and Proposition 2.13 holds and denotes by $\psi_3(\epsilon|n)$.
Namely,

1. $\psi_3 : \mathbf{R}_{>0} \times \mathbf{N} \rightarrow \mathbf{R}_{>0}$
2. For each n,
$$\lim_{\epsilon \to 0} \psi_3(\epsilon|n) = 0$$
holds.
3. Lemma 2.8 and Propositon 2.13 holds if we replace the $\Psi(\epsilon|n)$ in the conclusion by $\psi_3(\epsilon|n)$.

Besides, we take a function $\psi_i (i = 4, 5, 6, 7, 8, 9, 10)$ with the following properties;

I. $\psi_i : \mathbf{R}_{>0} \times \mathbf{N} \to \mathbf{R}_{>0}$

II. For each n,

$$\lim_{\epsilon \to 0} \psi_i(\epsilon|n) = 0, \qquad \lim_{\epsilon \to 0} \frac{\psi_{i-1}(\epsilon|n)}{\psi_i(\epsilon|n)} = 0 \qquad (i = 4, 5, 6, 7, 8, 9, 10)$$

holds.

Remark 2.15
More exactly, we take ψ_i to justify the following statement.

Lemma 2.16
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$, we assume there exists $p_1, q_1, p_2, q_2 \in M$ with the following properties.

1. $|\overline{p_1, q_1} - \pi| < \epsilon$, $|\overline{p_2, q_2} - \pi| < \epsilon$, $|\overline{p_1, p_2} - \frac{\pi}{2}| < \epsilon$.
2. $\partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big) = \phi$.

Then,

$$d_{GH}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1), \mathbf{S}^0\big) < \Psi(\epsilon|n).$$

Especially,

$$d_{GH}(M, \mathbf{S}^1) < \Psi(\epsilon|n).$$

Proof. This is clear. □

Lemma 2.17
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$, we assume there exists $p_1, q_1, p_2, q_2 \in M$ with the following properties.

1. $|\overline{p_1, q_1} - \pi| < \epsilon$, $|\overline{p_2, q_2} - \pi| < \epsilon$, $|\overline{p_1, p_2} - \frac{\pi}{2}| < \epsilon$.
2. $\partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big) \neq \phi$.

Then, for each $x \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big)$,

$$\overrightarrow{p_2, x} \subset \mathbf{B}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \text{ and } \overrightarrow{q_2, x} \subset \mathbf{B}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big).$$

Here, for $x, y \in M$,

$\overrightarrow{x, y} := \{z \in M |$ *There exist γ : normal geodesic from x to y, such that* $z \in Im(\gamma)\}$.

Proof. We will give only the proof of the statement for $\overrightarrow{p_2, x}$.
(The proof of the other is similar.)
First, we remark that

- for each $t \in [0, \overline{p_2, x}]$, $x \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big)$, $\sigma : [0, \overline{p_2, x}] \to M$ is normal geodesic from p_2 to x, we have

$$\sigma(t) \in M \setminus \big(\mathbf{B}_{\delta(\epsilon|n)}(p_1) \cup \mathbf{B}_{\delta(\epsilon|n)}(q_1)\big).$$

Because, from Lemma 2.8 and triangle inequality,

$$\begin{aligned}
\overline{p_1, \sigma(t)} &\geq \frac{1}{2}(\overline{p_1, p_2} + \overline{p_1, x} - \overline{p_2, x}) \\
&\geq \frac{1}{2}\big(\frac{\pi}{2} - \epsilon + \frac{\pi}{2} - (\pi + \psi_3(\epsilon|n) - \psi_7(\epsilon|n)\big) \\
&= \frac{1}{2}\big(\psi_7(\epsilon|n) - \psi_3(\epsilon|n) - \epsilon\big) \\
&\geq \delta(\epsilon|n)
\end{aligned}$$

and

$$\begin{aligned}
\overline{p_1, \sigma(t)} &\leq \frac{1}{2}(\overline{p_1, p_2} + \overline{p_1, x} + \overline{p_2, x}) \\
&\leq \frac{1}{2}\big(\frac{\pi}{2} + \frac{\pi}{2} + \epsilon + \pi + \psi_3(\epsilon|n) - \psi_7(\epsilon|n)\big) \\
&= \pi - \frac{1}{2}\big(\psi_7(\epsilon|n) - \epsilon - \psi_3(\epsilon|n)\big).
\end{aligned}$$

So

$$\begin{aligned}
\overline{q_1, \sigma(t)} &\geq \pi - \epsilon - \overline{p_1, \sigma(t)} \\
&\geq \pi - \epsilon - \big(\pi - \frac{1}{2}(\psi_7(\epsilon|n) - \epsilon - \psi_3(\epsilon|n))\big) \\
&= \frac{1}{2}\big(\psi_7(\epsilon|n) - 3\epsilon - \psi_3(\epsilon|n)\big) \\
&\geq \delta(\epsilon|n).
\end{aligned}$$

So, we can use Proposition 2.13, for $z_t \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_1)$ is an element such that $\overline{\sigma(t), z_t} = \overline{\sigma(t), \partial \mathbf{B}_{\frac{\pi}{2}}(p_1)}$ holds,

$$\cos \overline{p_2, \sigma(t)} = \cos \overline{p_1, p_2} \cos \overline{p_1, \sigma(t)} + \sin \overline{p_1, p_2} \sin \overline{p_1, \sigma(t)} \cos \overline{p_2, z_t} \pm 10\psi_3(\epsilon|n),$$

$$\cos \overline{\sigma(t), x} = \cos \overline{p_1, \sigma(t)} \cos \overline{p_1, x} + \sin \overline{p_1, \sigma(t)} \sin \overline{p_1, x} \cos \overline{x, z_t} \pm 10\psi_3(\epsilon|n).$$

From this, we have

$$\begin{aligned}
\cos \overline{p_2, \sigma(t)} &= \sin \overline{p_1, \sigma(t)} \cos \overline{p_2, z_t} \pm 20\psi_3(\epsilon|n), \\
\cos \overline{\sigma(t), x} &= \sin \overline{p_1, \sigma(t)} \cos \overline{x, z_t} \pm 20\psi_3(\epsilon|n).
\end{aligned}$$

1. *The case* $\overline{p_2, \sigma(t)} \le \frac{\pi}{2}, \overline{\sigma(t), x} \le \frac{\pi}{2}$.

   In this case
$$\cos \overline{p_2, \sigma(t)} \le \cos \overline{p_2, z_t} + \psi_4(\epsilon|n),$$
$$\cos \overline{\sigma(t), x} \le \cos \overline{x, z_t} + \psi_4(\epsilon|n).$$

   So
$$\overline{p_2, \sigma(t)} \ge \overline{p_2, z_t} - \frac{1}{100}\psi_5(\epsilon|n),$$
$$\overline{\sigma(t), x} \ge \overline{x, z_t} - \frac{1}{100}\psi_5(\epsilon|n),$$

   and from triangle inequality, we have
$$\left|\overline{p_2, \sigma(t)} - \overline{p_2, z_t}\right| \le \frac{1}{10}\psi_5(\epsilon|n),$$
$$\left|\overline{\sigma(t), x} - \overline{x, z_t}\right| \le \frac{1}{10}\psi_5(\epsilon|n).$$

   So
$$\cos \overline{p_2, \sigma(t)} = \sin \overline{p_1, \sigma(t)} \cos \overline{p_2, \sigma(t)} \pm \frac{1}{2}\psi_5(\epsilon|n),$$
$$\cos \overline{\sigma(t), x} = \sin \overline{p_1, \sigma(t)} \cos \overline{\sigma(t), x} \pm \frac{1}{2}\psi_5(\epsilon|n).$$

   Thus, we have
$$\left|\overline{p_1, \sigma(t)} - \frac{\pi}{2}\right| < \psi_6(\epsilon|n).$$

   Therefore, in this case, the claim is true.

2. *The case* $\overline{p_2, \sigma(t)} > \frac{\pi}{2}$.

   In this case, from the result of case 1,
$$\cos \overline{\sigma(\frac{\overline{p_2, x}}{2}), \sigma(t)} = \sin \overline{p_1, \sigma(t)} \cos \overline{\sigma(\frac{\overline{p_2, x}}{2}), z_t} \pm 10\psi_6(\epsilon|n),$$
$$\cos \overline{\sigma(t), x} = \sin \overline{p_1, \sigma(t)} \cos \overline{x, z_t} \pm 20\psi_3(\epsilon|n).$$

   An argument after this is similar to case 1.

3. *The case* $\overline{\sigma(t), x} > \frac{\pi}{2}$.

   This case is also similar to an argument in case 2. □

Lemma 2.18
Under same assumption as in Lemma 2.17,
$\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \ne \phi$ *and,*
$$d_{GH}\left(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1), \mathbf{S}^0 * \left(\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)\right)\right) < \Psi(\epsilon|n).$$

Proof. For $x \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus (\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2))$,

we take $z \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2)$ such that

$$\overline{x,z} = \overline{x, \partial \mathbf{B}_{\frac{\pi}{2}}(p_2)}.$$

From Lemma 2.17,

$$z \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big).$$

Then, we define

$$\phi : \partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus (\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)) \to \mathbf{S}^0 * \Big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)\Big)$$

as

$$\phi(x) = (\overline{p_2, x}, z).$$

By Proposition 2.13, this gives $\Psi(\epsilon|n)$-Hausdorff approximation. □

Lemma 2.19
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold $(n \geq 2)$ with $Ric_M \geq n-1$, we assume there exists $p_1, q_1, p_2, q_2 \in M$ with the following properties;

1. $|\overline{p_1, q_1} - \pi| < \epsilon$, $|\overline{p_2, q_2} - \pi| < \epsilon$, $|\overline{p_1, p_2} - \frac{\pi}{2}| < \epsilon$.
2. $\partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big) \neq \phi$.
3. *There exists* $x, y \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$ *with* $\overline{x,y} \geq \pi - \psi_9(\epsilon|n)$ *such that*

$$\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \ \setminus \ \big(\mathbf{B}_{\psi_{10}(\epsilon|n)}(x) \cup \mathbf{B}_{\psi_{10}(\epsilon|n)}(y)\big) = \phi.$$

Then,

$$d_{GH}\Big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big), \ \ \mathbf{S}^0\Big) < \Psi(\epsilon|n).$$

Especially,

$$d_{GH}(M, \ \mathbf{S}^2) < \Psi(\epsilon|n).$$

Proof. This is clear. □

Lemma 2.20
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold $(n \geq 2)$ with $Ric_M \geq n-1$, we assume there exists $p_1, q_1, p_2, q_2 \in M$ with the following properties;

1. $|\overline{p_1, q_1} - \pi| < \epsilon$, $|\overline{p_2, q_2} - \pi| < \epsilon$, $|\overline{p_1, p_2} - \frac{\pi}{2}| < \epsilon$.

2. $\partial \mathbf{B}_{\frac{\pi}{2}}(p_1) \setminus \big(\mathbf{B}_{\psi_7(\epsilon|n)}(p_2) \cup \mathbf{B}_{\psi_7(\epsilon|n)}(q_2)\big) \neq \phi$.

3. For each $x, y \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$ *with* $\overline{x, y} \geq \pi - \psi_9(\epsilon|n)$,

$$\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \setminus \big(\mathbf{B}_{\psi_{10}(\epsilon|n)}(x) \cup \mathbf{B}_{\psi_{10}(\epsilon|n)}(y)\big) \neq \phi$$

Then, for each $x, y \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$, *there exist* $z \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$ *such that*

$$\big|\overline{z, x} - \frac{1}{2}\overline{x, y}\big| < \Psi(\epsilon|n), \qquad \big|\overline{z, y} - \frac{1}{2}\overline{x, y}\big| < \Psi(\epsilon|n).$$

Especially, there exist a compact length space Z *with* $diam_Z \leq \pi$ *such that*

$$d_{GH}(\partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big),\ Z) < \Psi(\epsilon|n).$$

Therefore,

$$d_{GH}(M, \mathbf{S}^1 * Z) < \Psi(\epsilon|n).$$

Proof.

1. *The case* $\psi_9(\epsilon|n) \leq \overline{x, y} \leq \pi - \psi_9(\epsilon|n)$.

   By Lemma 2.17, (or similarly argument of the proof) there exist $w \in \overrightarrow{x, y}$ such that

   $$\overline{x, w} = \frac{1}{2}\overline{x, y}, \qquad \overline{y, w} = \frac{1}{2}\overline{x, y}$$

   and

   $$w \in \mathbf{B}_{\psi_{10}(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \cap \mathbf{B}_{\psi_{10}(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_2)\big).$$

   We take $\hat{w} \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_1)$ such that $\overline{w, \hat{w}} < \psi_{10}(\epsilon|n)$ holds.

   In addition, we take $z \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$ such that

   $$\overline{\hat{w}, z} = \overline{\hat{w}, \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)},$$

   then,

   $$\overline{z, w} < \Psi(\epsilon|n).$$

   Therefore, in this case, the claim is true.

2. *The case* $\overline{x, y} > \pi - \psi_9(\epsilon|n)$.

   By the assumption, there exist

   $$w \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big) \setminus \big(\mathbf{B}_{\psi_{10}(\epsilon|n)}(x) \cup \mathbf{B}_{\psi_{10}(\epsilon|n)}(y)\big).$$

   Since

$$\overline{x,w} + \overline{w,y} \geq \overline{x,y} \geq \pi - \psi_9(\epsilon|n),$$

we have

$$\max\{\overline{x,w}, \overline{w,y}\} \geq \tfrac{1}{2}\big(\pi - \psi_9(\epsilon|n)\big).$$

So, we may assume

$$\overline{x,w} \geq \tfrac{1}{2}\big(\pi - \psi_9(\epsilon|n)\big).$$

We take $\hat{w} \in \overrightarrow{x,w}$ such that

$$\overline{x,\hat{w}} = \frac{1}{2}\big(\pi - \psi_9(\epsilon|n)\big)$$

and $z \in \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)$ such that

$$\overline{\hat{w},z} = \overline{\hat{w}, \partial \mathbf{B}_{\frac{\pi}{2}}(p_2) \cap \overline{\mathbf{B}}_{\psi_8(\epsilon|n)}\big(\partial \mathbf{B}_{\frac{\pi}{2}}(p_1)\big)}.$$

Then,

$$\Big|\overline{x,z} - \frac{\pi}{2}\Big| < \Psi(\epsilon|n).$$

From this and Lemma 2.8, we have the claim.

3. *The case* $\overline{x,y} < \psi_9(\epsilon|n)$

   In this case, we take $z = y$.

We can prove the last claim by using Gromov's pre-compactness theorem. □

From above results, we have next proposition.

Proposition 2.21
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$, and $p_1, q_1, p_2, q_2 \in M$ such that

$$|\overline{p_1,q_1} - \pi| < \epsilon, \quad |\overline{p_2,q_2} - \pi| < \epsilon, \quad |\overline{p_1,p_2} - \tfrac{\pi}{2}| < \epsilon.$$

Then, one of the following 1,2,3 holds.

1. *There exist a compact length space Z with $diam_Z \leq \pi$ such that*

$$d_{GH}(M, \mathbf{S}^1 * Z) < \Psi(\epsilon|n).$$

2. $$d_{GH}(M, \mathbf{S}^2) < \Psi(\epsilon|n).$$

3. $$d_{GH}(M, \mathbf{S}^1) < \Psi(\epsilon|n).$$

From similarly argument, we can show the next proposition.

Proposition 2.22
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$ and $p_i, q_i \in M$ $(i = 1, 2, \cdots, k)$ such that

for each i, $|\overline{p_i, q_i} - \pi| < \epsilon$ holds, and for $i \neq j$, $|\overline{p_i, p_j} - \frac{\pi}{2}| < \epsilon$ holds.

Then, one of the following 1,2,3 holds.

1. *There exist a compact length space Z with $diam_Z \leq \pi$ such that*
$$d_{GH}(M, \mathbf{S}^{k-1} * Z) < \Psi(\epsilon|n).$$
2. $$d_{GH}(M, \mathbf{S}^k) < \Psi(\epsilon|n).$$
3. $$d_{GH}(M, \mathbf{S}^{k-1}) < \Psi(\epsilon|n).$$

Now, we give next lemma without the proof.

Lemma 2.23 (T.H.Colding [11])
For each $n \in \mathbf{N}$ ($n \geq 2$), there exist $C(n) > 0$ with the following property. If an integer k satisfies $0 \leq k < n$, and an n dimensional complete Riemannian manifold M satisfies $Ric_M \geq n-1$,
$$d_{GH}(M, \mathbf{S}^k) \geq C(n)$$
holds.

Proposition 2.24
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$, and $p_i, q_i \in M$ $(i = 1, 2, \cdots, k)$ such that

for each i, $|\overline{p_i, q_i} - \pi| < \epsilon$ holds, and for $i \neq j$, $|\overline{p_i, p_j} - \frac{\pi}{2}| < \epsilon$ holds.

Then, we have the following .

1. *If $1 \leq k \leq n-1$, then there exist a compact length space Z with $diam_Z \leq \pi$ such that*
$$d_{GH}(M, \mathbf{S}^{k-1} * Z) < \Psi(\epsilon|n).$$
2. *If $k = n$, then*
$$d_{GH}(M, \mathbf{S}^n_+) < \Psi(\epsilon|n),$$
or,
$$d_{GH}(M, \mathbf{S}^n) < \Psi(\epsilon|n).$$

Here,
$\mathbf{S}^n_+ := \{\mathbf{x} = (x_1, x_2, \cdots, x_{n+1}) \in \mathbf{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1, x_{n+1} \geq 0\}$, *and the metric is the restriction of* $\mathbf{S}^n$.

3. *If* $k = n+1$, *then*

$$d_{GH}(M, \mathbf{S}^n) < \Psi(\epsilon|n).$$

Proof. This is a consequence of Proposition 2.22 and Lemma 2.23. □

Finally, we recall the following.

Theorem 2.25 (J.Cheeger, T.H.Colding [3, 5])
If $(M_i, p_i)_{i \in \mathbf{N}}$ *is sequence such that* M_i *are* n *dimensional complete Riemannian manifolds with* $Ric_{M_i} \geq -(n-1)$, $p_i \in M_i$ *and* Z *is proper length space (i.e length space and its bounded closed subsets are compact),* $z \in Z$

$(M_i, p_i) \to (Z, z) \quad (i \to \infty)$ *: non-collapsing, pointed Gromov-Hausdorff convergence*

then, for each tangent cone at z *in* Z, $T_z Z$

$$T_z Z \ncong \mathbf{R}^{n-1} \times \mathbf{R}_{\geq 0}.$$

Here,
$\mathbf{R}^{n-1} \times \mathbf{R}_{\geq 0} := \{\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n | x_n \geq 0\}$, *the metric is the restriction of* $\mathbf{R}^n$.

Theorem 2.26 (J.Cheeger, T.H.Colding [3, 5, 16])
For M*:* n *dimensional compact Riemannian manifold with* $Ric_M \geq -(n-1)$, *there exist* $\delta = \delta(M)$ *with the following property. If* N*:* n *dimensional compact Riemannian manifold,* $Ric_N \geq -(n-1)$ *such that*

$$d_{GH}(M, N) < \delta$$

then M *is diffeomorphic to* N*.*

Proof of Main Theorem 1.
Proposition 2.24 and Theorem 2.25, 2.26 implies Main Theorem 1. □

Remark 2.27
Theorem 1.1 follows from Main Theorem 1.

Let M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n-1$.

From Bishop-Gromov's volume comparison theorem, we have

$$vol(M) \geq vol(\mathbf{S}^n) - \epsilon \quad \Longrightarrow \quad rad_M \geq \pi - \Psi(\epsilon|n)$$

Now, we consider the situation with $rad_M \geq \pi - \epsilon$.

Then, we have

for each $p \in M$, there exist $q \in M$ such that $\overline{p,q} \geq \pi - \epsilon$ holds.

First, we take arbitrary $p_1 \in M$.
Then, from above, there exist $q_1 \in M$ such that

$$\overline{p_1, q_1} \geq \pi - \Psi(\epsilon|n).$$

Thus, from Main Theorem 1, M is close to the space of 1-fold suspension of some compact length space.
Especially, there exist $p_2 \in M$ such that

$$|\overline{p_1, p_2} - \frac{\pi}{2}| < \Psi(\epsilon|n).$$

Similarly, there exist $q_2 \in M$ such that

$$\overline{p_2, q_2} \geq \pi - \Psi(\epsilon|n).$$

Thus, M is close to the space of 2-fold suspension of some compact length space.
If we repeat this argument, then the assumption of Main Theorem 1 for case $k = n + 1$ holds. It implies Theorem 1.1.

3 First eigenvalue of Laplacian

In this section, we give the relation between Main Theorem 1 and first eigenvalue of Laplacian. Let M be an n dimensional complete Riemannian manifold with $Ric_M \geq n - 1$.

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots .$$

denotes eigenvalues of Laplacian on M.

Theorem 3.1 (A.Lichnerowicz, M.Obata [20, 21])
With notation as above,

$$\lambda_1 \geq n.$$

And the inequality is equality if and only if M is isometric to $\mathbf{S}^n$.

Now, we consider perturbation version of this statement.

Theorem 3.2 (S.Y.Cheng, T.H.Colding, C.B.Croke [8, 10, 13])
1. *If $diam_M \geq \pi - \epsilon$, then $\lambda_1 \leq n + \Psi(\epsilon|n)$ holds.*
2. *If $\lambda_1 \leq n + \epsilon$, then $diam_M \geq \pi - \Psi(\epsilon|n)$ holds.*

If we consider simillaly statement for λ_{n+1}, then we have the following;

Theorem 3.3 (P.Petersen [26])

1. *If* $rad_M \geq \pi - \epsilon$, *then* $\lambda_{n+1} \leq n + \Psi(\epsilon|n)$ *holds.*
2. *If* $\lambda_{n+1} \leq n - \epsilon$, *then* $rad_M \geq \pi - \Psi(\epsilon|n)$ *holds.*

These means the following;

$$\lambda_1 \leq n + \epsilon \Longleftrightarrow \text{Assumption of Main Theorem 1 for } k = 1 \text{ holds.}$$

$$\lambda_{n+1} \leq n + \epsilon \Longleftrightarrow \text{Assumption of Main Theorem 1 for } k = n + 1 \text{ holds.}$$

We would like to consider whether a statement corresponding to λ_k is right.

Theorem 3.4 (P.Petersen [26])
We have,

$$\lambda_k \leq n + \epsilon \Longrightarrow \textit{Assumption of Main Theorem 1 holds.}$$

Remark 3.5
This is stated in [26] introduction. We will give the proof later.

We have a converse of it. They together imply

Main Theorem 2
We have,

$$\textit{Assumption of Main Theorem 1 holds} \Longleftrightarrow \lambda_k \leq n + \Psi(\epsilon|n).$$

The rest of this papers devoted by the proof of Main Theorem 2.

First, we consider the case $k = 2$. i.e,
Let $\epsilon > 0$, M be an n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_M \geq n - 1$, and $p_1, q_1, p_2, q_2 \in M$ such that

$$|\overline{p_1, q_1} - \pi| < \epsilon, \quad |\overline{p_2, q_2} - \pi| < \epsilon, \quad |\overline{p_1, p_2} - \tfrac{\pi}{2}| < \epsilon.$$

In this situation, we put $f_i(x) = \cos \overline{p_i, x}$ $(i = 1, 2)$ and take $\tilde{f}_i \in C^\infty(M)$ as in Lemma 2.7. Then we have the following

$$\frac{1}{vol(M)} \int_M f_i^2 dx = \frac{1}{n+1} \pm \Psi(\epsilon|n) \qquad (3.1)$$

$$\frac{1}{vol(M)} \int_M |\nabla f_i|^2 dx = \frac{n}{n+1} \pm \Psi(\epsilon|n) \qquad (3.2)$$

$$\frac{1}{vol(M)}\int_M |\Delta \tilde{f}_i(x) + n\tilde{f}_i(x)|^2 dx < \Psi(\epsilon|n). \qquad (3.3)$$

(See Lemma 1.10 in [10])

Remark 3.6
Here, $\Delta = tr(\text{Hess})$. So, eigenvalues of Laplacian that we are considering now is one for $-\Delta = d^*d$.

Lemma 3.7
With notation as above,

$$\frac{1}{vol(M)}\int_M \tilde{f}_i^2 dx = \frac{1}{n+1} \pm \Psi(\epsilon|n),$$

$$\frac{1}{vol(M)}\int_M |\nabla \tilde{f}_i|^2 dx = \frac{n}{n+1} \pm \Psi(\epsilon|n).$$

Especially,

$$\frac{1}{vol(M)}\int_M \tilde{f}_1\tilde{f}_2 dx = \frac{1}{vol(M)}\int_M f_1 f_2 dx \pm \Psi(\epsilon|n),$$

$$\frac{1}{vol(M)}\int_M g_M(\nabla \tilde{f}_1, \nabla \tilde{f}_2) dx = \frac{1}{vol(M)}\int_M g_M(\nabla f_1, \nabla f_2) dx \pm \Psi(\epsilon|n).$$

Proof.

$$\frac{1}{vol(M)}\int_M \tilde{f}_i^2 dx = \frac{1}{vol(M)}\int_M (\tilde{f}_i - f_i + f_i)^2 dx$$

$$= \frac{1}{vol(M)}\int_M (\tilde{f}_i - f_i)^2 dx + \frac{2}{vol(M)}\int_M f_i(\tilde{f}_i - f_i) dx + \frac{1}{vol(M)}\int_M f_i^2 dx$$

$$= \frac{1}{n+1} \pm \Psi(\epsilon|n) \quad (\because \text{Cauchy-Schwartz inequality})$$

The proof of other equality is similar. □

Lemma 3.8
We have the following;

$$\frac{1}{vol(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx = -\frac{1}{vol(M)}\int_M f_1 f_2 dx \pm \Psi(\epsilon|n).$$

Especially, from Lemma 3.7,

$$\frac{1}{vol(M)}\int_M g_M(\nabla \tilde{f}_1, \nabla \tilde{f}_2)dx = -\frac{1}{vol(M)}\int_M \tilde{f}_1 \tilde{f}_2 dx \pm \Psi(\epsilon|n).$$

Proof. First, we take specific $\Psi(\epsilon|n)$ satisfies the conclusion of all statement in section 2 and denotes by $\psi_{11}(\epsilon|n)$. And we take a function ψ_{12} with the following properties;

1. $\psi_{12} : \mathbf{R}_{>0} \times \mathbf{N} \to \mathbf{R}_{>0}$.
2. For $\delta = \delta(\epsilon|n)$ is in Proposition 2.13,

$$\frac{\psi_{11}(\epsilon|n)^{\frac{1}{100}}}{\delta(\epsilon|n)} < \psi_{12}(\epsilon|n)$$

   holds.

We put

$$A_{p_1} := \mathrm{B}_{3\delta}(p_1) \cup \mathrm{B}_{3\delta}(q_1) \cup C_{p_1}.$$

For each $x \in M \setminus A_{p_1}, s \in [0, \overline{p_1, x}]$, we define $c_x(s) \in M$ as

$c_x(s)$ is a point on segment $\overrightarrow{p_1, x}$ such that $\overline{x, c_x(s)} = s$ holds.

Then,

$$\frac{1}{vol(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx = \frac{1}{vol(M)}\int_{M\setminus A_{p_1}} g_M(\nabla f_1, \nabla f_2)dx \pm \psi_{11}(\epsilon|n)$$

$$\left(\because \frac{vol(A_{p_1})}{vol(M)} < \psi_{11}(\epsilon|n)\right)$$

$$= \frac{1}{vol(M)}\int_{M\setminus A_{p_1}} g_M(\nabla f_1, \nabla \tilde{f}_2)dx \pm 2\psi_{11}(\epsilon|n)$$

$$= \frac{1}{vol(M)}\int_{M\setminus A_{p_1}} \sin \overline{p_1, x}\frac{d\tilde{f}_2 \circ c_x(s)}{ds}\Big|_{s=0} dx \pm 2\psi_{11}(\epsilon|n)$$

$$= \frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \Big\{ \sin \overline{p_1, x} \Big(\frac{\tilde{f}_2 \circ c_x(\delta) - \tilde{f}_2 \circ c_x(0)}{\delta}$$

$$- \frac{1}{\delta} \int_0^\delta (\delta - s) \frac{d^2 \tilde{f}_2 \circ c_x(s)}{ds^2} ds \Big) \Big\} dx \pm 2\psi_{11}(\epsilon|n)$$

$$= \frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \sin \overline{p_1, x} \Big(\frac{f_2 \circ c_x(\delta) - f_2 \circ c_x(0)}{\delta} \Big) dx$$

$$+ \frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \sin \overline{p_1, x} \Big(\frac{\tilde{f}_2 \circ c_x(\delta) - f_2 \circ c_x(\delta)}{\delta} \Big) dx \tag{1}$$

$$- \frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \sin \overline{p_1, x} \Big(\frac{\tilde{f}_2 \circ c_x(0) - f_2 \circ c_x(0)}{\delta} \Big) dx \tag{2}$$

$$- \frac{1}{\delta vol(M)} \int_{M\setminus A_{p_1}} \sin \overline{p_1, x} \int_0^\delta (\delta - s) \Big(\frac{d^2 \tilde{f}_2 \circ c_x(s)}{ds^2} + \tilde{f}_2 \circ c_x(s) \Big) ds dx \tag{3}$$

$$+ \frac{1}{\delta vol(M)} \int_{M\setminus A_{p_1}} \sin \overline{p_1, x} \int_0^\delta (\delta - s) \tilde{f}_2 \circ c_x(s) ds dx \pm 2\psi_{11}(\epsilon|n). \tag{4}$$

Now, we will prove the following;

Claim

$$|(1)| < \psi_{12}(\epsilon|n) \tag{3.4}$$

$$|(2)| < \psi_{12}(\epsilon|n) \tag{3.5}$$

$$|(3)| < \psi_{12}(\epsilon|n) \tag{3.6}$$

$$|(4)| < \psi_{12}(\epsilon|n) \tag{3.7}$$

Proof of claim.

1. *Proof of (3.4).*

$$|(1)| \leq \frac{1}{\delta vol(M)} \int_{M \setminus A_{p_1}} \left| \tilde{f}_2 \circ c_x(\delta) - f_2 \circ c_x(\delta) \right| dx \qquad (5)$$

We use next estimate;

Estimate 1 There exist $C(n) > 0$ such that for each integrable function $h : M \to \mathbf{R}_{\geq 0}$,

$$\frac{1}{vol(M)} \int_{M \setminus A_{p_1}} h \circ c_x(\delta) dx \leq \frac{C(n)}{vol(M)} \int_M h(x) dx.$$

Proof of estimate 1. We put

$$S_{p_1}(1) \subset T_{p_1} M : \text{ unit sphere}$$

and for $u \in S_{p_1}(1)$,

$$t(u) := \text{distance from } p_1 \text{ to cut locus of direction of } u \ > 0$$

$$\hat{S}_{p_1}(1) := \{u \in S_{p_1}(1) | t(u) > 3\delta\}$$

$$\theta(t, u) := t^{n-1} \big(\det(g_{ij}|_{\exp_{p_1}(tu)})\big)^{\frac{1}{2}} \qquad \Big(g_{ij} := g_M\big(\frac{\partial}{\partial x_i}, \ \frac{\partial}{\partial x_j}\big) \Big).$$

Then,

$$\begin{aligned}
\int_{M \setminus A_{p_1}} h \circ c_x(\delta) dx &\leq \int_{\hat{S}_{p_1}(1)} \int_{3\delta}^{t(u)} h \circ c_{\exp_{p_1}(tu)}(\delta) \theta(t, u) dt du \\
&= \int_{\hat{S}_{p_1}(1)} \int_{3\delta}^{t(u)} h(\exp_{p_1}((t - \delta)u) \theta(t, u) dt du \\
&= \int_{\hat{S}_{p_1}(1)} \int_{2\delta}^{t(u) - \delta} h(\exp_{p_1}(\hat{t} u)) \theta(\hat{t} + \delta, u) d\hat{t} du \qquad (6)
\end{aligned}$$

From Laplacian comparison theorem, there exist $C(n) > 0$ such that

$$\theta(\hat{t}+\delta, u) \leq \frac{\sin^{n-1}(\hat{t}+\delta)}{\sin^{n-1}\hat{t}}\, \theta(\hat{t}, u) \leq C(n)\, \theta(\hat{t}, u) \quad (u \in \hat{S}_{p_1}(1),\ \hat{t} \in [2\delta,\ t(u)]\ ).$$

So,

$$(6) \leq C(n) \int_{\hat{S}_{p_1}(1)} \int_{2\delta}^{t(u)-\delta} h(\exp_{p_1}(\hat{t}u))\theta(\hat{t}, u) d\hat{t} du$$

$$\leq C(n) \int_{S_{p_1}(1)} \int_{0}^{t(u)} h(\exp_{p_1}(\hat{t}u))\theta(\hat{t}, u) d\hat{t} du$$

$$= C(n) \int_M h(x) dx$$

Therefore, we divide this by $vol(M)$, we have **estimate 1**. $\square$

From this estimate,

$$(5) \leq \frac{C(n)}{\delta vol(M)} \int_M |\tilde{f}_2 - f_2| dx$$

$$\leq \frac{C(n)}{\delta} \Big(\frac{1}{vol(M)} \int_M |\tilde{f}_2 - f_2|^2 dx \Big)^{\frac{1}{2}}$$

$$\leq \frac{C(n)}{\delta} (\psi_{11}(\epsilon|n))^{\frac{1}{2}}$$

$$< \psi_{12}(\epsilon|n)$$

Therefore, We have (3.4). $\square$

2. *Proof of (3.5).*

$$|(2)| \leq \frac{1}{\delta vol(M)} \int_M |\tilde{f}_2 - f_2| dx$$

$$\leq \frac{1}{\delta}\Big(\int_M |\tilde{f}_2 - f_2|^2 dx\Big)^{\frac{1}{2}}$$

$$\leq \frac{1}{\delta}(\psi_{11}(\epsilon|n))^{\frac{1}{2}}$$

$$< \psi_{12}(\epsilon|n)$$

Therefore, We have (3.5). □

3. *Proof of (3.6).*

$$|(3)| \leq \frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \int_0^\delta \big|\mathbf{Hess}_{\tilde{f}_2} + \tilde{f}_2 g_M\big|(c_x(s))dsdx \qquad \cdots (7)$$

We use next estimate;

Estimate 2 There exist $C(n) > 0$ such that for each integrable function $h : M \to \mathbf{R}_{\geq 0}$,

$$\frac{1}{vol(M)} \int_{M\setminus A_{p_1}} \int_0^\delta h \circ c_x(s)dsdx \leq \frac{C(n)\delta}{vol(M)} \int_M h(x)dx.$$

We skip this proof because it is similar to estimate 1.

Then,

$$(7) \leq \frac{C(n)\delta}{vol(M)} \int_M \big|\mathbf{Hess}_{\tilde{f}_2} + \tilde{f}_2 g_M\big|dx$$

$$\leq C(n)\delta\Big(\frac{1}{vol(M)} \int_M \big|\mathbf{Hess}_{\tilde{f}_2} + \tilde{f}_2 g_M\big|^2 dx\Big)^{\frac{1}{2}}$$

$$< \psi_{12}(\epsilon|n)$$

Therefore, We have (3.6). □

4. *Proof of (3.7).*

From Lemma 3.7 and **estimate 2**, the proof is similar to |(3)|. $\square$

So, we have **claim**. $\square$

From this claim, we have the following;

$$\frac{1}{vol(M)}\int_M g_M(\nabla f_1, \nabla f_2)dx = \frac{1}{vol(M)}\int_{M\setminus A_{p_1}} \sin\overline{p_1, x}\Big(\frac{f_2\circ c_x(\delta) - f_2\circ c_x(0)}{\delta}\Big)dx$$

$$\pm\, 4\psi_{12}(\epsilon|n) \qquad (8)$$

By Almost cosine formura,

$$\sin\overline{p_1, x}(f_2\circ c_x(\delta) - f_2\circ c_x(0)) = \sin\overline{p_1, x}\ \big(\cos\overline{p_1, p_2}\ \cos\overline{p_1, c_x(\delta)}$$

$$+\sin\overline{p_1, p_2}\ \sin\overline{p_1, c_x(\delta)}\ \frac{\cos\overline{p_2, x} - \cos\overline{p_1, p_2}\cos\overline{p_1, x}}{\sin\overline{p_1, p_2}\ \sin\overline{p_1, x}}\big)$$

$$-\sin\overline{p_1, x}\ \cos\overline{p_2, x} \pm \psi_{11}(\epsilon|n)$$

$$= \big(\sin(\overline{p_1, x} - \delta) - \sin\overline{p_1, x}\big)\cos\overline{p_2, x} \pm 3\psi_{11}(\epsilon|n)$$

We use mean value theorem,

$$(8) = \frac{1}{vol(M)}\int_{M\setminus A_{p_1}} -\cos\overline{p_2, x}\cos\overline{p_1, x}dx \pm 6\psi_{12}(\epsilon|n)$$

$$= -\frac{1}{vol(M)}\int_M f_1 f_2 dx \pm 10\psi_{12}(\epsilon|n).$$

So, we have Lemma 3.8. $\square$

Lemma 3.9
We have

$$\Big|\frac{1}{vol(M)}\int_M g_M(\nabla\tilde{f}_1, \nabla\tilde{f}_2)dx\Big|,\ \ \Big|\frac{1}{vol(M)}\int_M \tilde{f}_1\tilde{f}_2 dx\Big| < \ \Psi(\epsilon|n).$$

Proof. From (3.1), (3.2) and (3.3), we have

$$\frac{1}{vol(M)}\int_M g_M(\nabla\tilde{f}_1,\nabla\tilde{f}_2)dx=\frac{n}{vol(M)}\int_M \tilde{f}_1\tilde{f}_2dx\pm\Psi(\epsilon|n).$$

From this and Lemma 3.8, we have the statement. □

Theorem 3.10
We have

$$\lambda_2\le n+\Psi(\epsilon|n).$$

Proof. From Lemma 3.9, we have

$$\tilde{f}_1,\ \tilde{f}_2 \text{ are linearly independent in } L^2_1(M).$$

So, from min-max principle, we have

$$\lambda_2\le\sup\Big\{\int_M|\nabla(a_1\tilde{f}_1+a_2\tilde{f}_2)|^2dx\Big/\int_M(a_1\tilde{f}_1+a_2\tilde{f}_2)^2dx\Big|\,a_1,\ a_2\in\mathbf{R},\ a_1^2+a_2^2\neq 0\Big\}.$$

And from Lemma 3.9, for $a_1^2+a_2^2\neq 0$, we have

$$\int_M|\nabla(a_1\tilde{f}_1+a_2\tilde{f}_2)|^2dx\Big/\int_M(a_1\tilde{f}_1+a_2\tilde{f}_2)^2dx\le n+\Psi(\epsilon|n).$$

Theorem 3.10 holds. □

The proof of general case of Main Theorem 2 is similar.

Proof of Theorem 3.4.
Let us prove Theorem 3.4. We first recall some inequalities proved in [26].
Let $\tilde{f}_i\in C^\infty(M)$ $(i=1,2)$ be eigenfunctions with

$$-\Delta\tilde{f}_i=\lambda_i\tilde{f}_i,\quad |\lambda_i-n|<\epsilon,\quad \int_M\tilde{f}_1\tilde{f}_2dx=0.$$

Then we may assume that

$$\tilde{f}_i^2+|\nabla\tilde{f}_i|^2\le 1,$$

$$\frac{1}{vol(M)}\int_M\tilde{f}_i^2dx=\frac{1}{n+1}\pm\Psi(\epsilon|n),$$

$$\frac{1}{vol(M)}\int_M |\nabla \tilde{f}_i|^2 dx = \frac{n}{n+1} \pm \Psi(\epsilon|n),$$

$$\frac{1}{vol(M)}\int_M |\tilde{f}_i^2 + |\nabla \tilde{f}_i|^2 - 1| dx < \Psi(\epsilon|n).$$

holds. (See Lemma 3.1 in [26].)

So, for each $p \in M$, there exist $\tilde{p} \in M$ such that

$$\overline{p,\tilde{p}} < \Psi(\epsilon|n) \text{ and } \tilde{f}_i^2(\tilde{p}) + |\nabla \tilde{f}_i|^2(\tilde{p}) = 1 \pm \Psi(\epsilon|n).$$

Now, we take specific $\Psi(\epsilon|n)$ satisfies the above inequalities, and denotes by $\psi_{13}(\epsilon|n)$.

And, we take $p_i, q_i \in M$ with

$$\tilde{f}_i(p_i) = \max \tilde{f}_i, \quad \tilde{f}_i(q_i) = \min \tilde{f}_i.$$

For $g_i(x) := \tilde{f}_i(p_i) - \tilde{f}_i(x) + \psi_{13}(\epsilon|n)$, $h_i(x) := \tilde{f}_i(x) - \tilde{f}_i(q_i) + \psi_{13}(\epsilon|n) \in C^\infty(M)$, by using Cheng-Yau's gradient estimate, we have

$$\frac{|\nabla g_i|^2}{g_i^2}, \quad \frac{|\nabla h_i|^2}{h_i^2} < \frac{C(n)}{\psi_{13}(\epsilon|n)}.$$

Here, $C(n)$ is a positive constant depending only on n. (See [3, 9].)

Thus, If we take $\tilde{p}_i, \tilde{q}_i \in M$ as above, then

$$|\nabla \tilde{f}_i|^2(\tilde{p}_i), \quad |\nabla \tilde{f}_i|^2(\tilde{q}_i) < \Psi(\epsilon|n).$$

Especially, we have

$$|\tilde{f}_i(p_i) - 1|, \quad |\tilde{f}_i(q_i) + 1| < \Psi(\epsilon|n).$$

Now, we put $f_i(x) := \cos \overline{p_i, x}$, by $|\nabla \arccos \tilde{f}_i| \leq 1$, we have

$$\tilde{f}_i > f_i - \Psi(\epsilon|n).$$

So, in the barrier sense,

$$\Delta(\tilde{f}_i - f_i) < \Psi(\epsilon|n).$$

From Theorem 7.2 in [26], we have

$$|\tilde{f}_i - f_i| < \Psi(\epsilon|n)$$

Especially,

$$\overline{p_i, q_i} \geq \pi - \Psi(\epsilon|n).$$

So, by (3.1), (3.2) we have

$$\frac{1}{vol(M)}\int_M |\nabla f_i - \nabla \tilde{f}_i|^2 dx < \Psi(\epsilon|n).$$

From a calculation similar to the proof of Lemma 3.8 and Lemma 3.9, we have

$$\frac{1}{vol(M)}\int_M \tilde{f}_1\tilde{f}_2 dx = \frac{\cos\overline{p_1,p_2}}{n+1} \pm \Psi(\epsilon|n). \cdots (3.8)$$

Since left hand side is equal to 0, we have

$$|\overline{p_1,p_2} - \frac{\pi}{2}| < \Psi(\epsilon|n).$$

Therefore, we have Theorem 3.4. □

We remark that the above argument also gives an alternative proof of Theorem 3.3 in [26].

Corollary 3.11
There exist a positive constant $C(n)$ depending only on n such that for M : n dimensional complete Riemannian manifold with $Ric_M \geq n-1$,

$$\lambda_{n+2} \geq C(n) > n.$$

Proof. If the assertion is false, then there exists a compact length space Y and for each $k \in \mathbf{N}$, complete Riemannian manifold M_k with $Ric_{M_k} \geq n-1$ such that the $(n+2)$-th eigenvalue λ^k_{n+2} satisfies

$$\lim_{k\to\infty} \lambda^k_{n+2} = n,$$

$$M_k \longrightarrow Y : \text{Gromov Hausdorff convergence}.$$

From (3.8), there exists $p_i, q_i \in Y (i = 1, 2, \cdots, n+2)$ such that

for each i, $\overline{p_i,q_i} = \pi$ holds, and for $i \neq j, \overline{p_i,p_j} = \frac{\pi}{2}$ holds.

This is contradiction by Main Theorem 1. □

Corollary 3.12
For M : n dimensional complete Riemannian manifold with $Ric_M \geq n-1$,

$$|\lambda_n - n| < \epsilon \Longrightarrow |\lambda_{n+1} - n| < \Psi(\epsilon|n).$$

Proof. This is clear by case $k = n$ of Main Theorem 1,2. □

4 A note on the relation to the structure of tangent cone of non-collapsing limit spaces

In this section, We remark that on Main Theorem 1 is similar to some results of the structure of tangent cone of limit space due to J.Cheeger, T.H.Colding.

Definition 4.1 (metric cone)
For Z: metric space, we define a metric on $[0, \infty) \times Z / \{0\} \times Z$ as

$$\overline{(t_1, z_1), (t_2, z_2)} \overset{\text{def}}{=} (t_1^2 + t_2^2 - 2t_1 t_2 \cos \min\{\overline{z_1, z_2}, \pi\})^{\frac{1}{2}}.$$

This metric spaces is denoted by

$$C(Z) \qquad (z^* := [(0, z)])$$

and is called by metric cone of Z.

Now, we consider following situation; $\{M_i\}_{i \in \mathbf{N}}$: n dimensional complete Riemannian manifolds ($n \geq 2$) with $Ric_{M_i} \geq -(n-1)$, $m_i \in M_i$, and Y : proper metric space with $y \in Y$,

- $(M_i, m_i) \to (Y, y) \quad (i \to \infty)$: pointed Gromov-Hausdorff convergence
- There exist $v > 0$ such that for each i

$$vol(\mathbf{B}_1(m_i)) \geq v > 0.$$

First, we review a result about the tangent cone $T_y Y$ at y in Y.

Theorem 4.2 (J.Cheeger, T.H.Colding [3, 5])
There exist a compact length space Z with $diam_Z \leq \pi$ such that

$$C(Z) \cong T_y Y.$$

Next, we would like to introduce the suspension structure for Z in Theorem 4.2. The following results also follows from results in [3, 5].

Theorem 4.3
If there exists $p_i, q_i \in Z$ $(i = 1, 2, \cdots, k)$ such that

for each i, $\overline{p_i, q_i} = \pi$ holds, and for $i \neq j$, $\overline{p_i, p_j} = \frac{\pi}{2}$ holds.

then,

1. *$k \leq n$.*
2. *If $1 \leq k \leq n-2$, then there exist a compact length space X with $diam_X \leq \pi$ such that*

$$Z \cong \mathbf{S}^{k-1} * X.$$

3. *If* $k = n-1$, *or* n, *then*

$$Z \cong \mathbf{S}^{n-1}.$$

Proof. First we remark that

1. Generally, for a metric space X, there exist a natural isomorphism

$$C(\mathbf{S}^{k-1} * X) \cong \mathbf{R}^k \times C(X).$$

   We next remark the equality below follows from splitting theorem by J.Cheeger, T.H.Colding.

2. If there exists $z_1, z_2 \in Z$ such that $\overline{z_1, z_2} = \pi$ holds, then for each $z \in Z$,

$$\overline{z_1, z} + \overline{z, z_2} = \pi.$$

   Compare Lemma 2.8.

3. Since $dim_H Z = n-1$ by the assumption of Theorem 4.3

$$Z \not\cong \mathbf{S}^k.$$

   for $1 \leq k \leq n-2$.

   Here $dim_H Z$ is Hausdorff dimension of Z. Compare Lemma 2.23.

Theorem 4.3 follows from these and an argument is similar to section 2.3. □

We will introduce to relation between Theorem 4.3 and some property of singular set of Y. we put

$$\mathcal{R} := \{y_1 \in Y | \text{ For any tangent cone } T_{y_1}Y \text{ at } y_1,\ T_{y_1}Y \ \cong \mathbf{R}^n.\}$$

$$\mathcal{S} := Y \setminus \mathcal{R}$$

$$\mathcal{S}_k := \{y_1 \in Y | \text{Any tangent cone } T_{y_1}Y \text{ does not have splitting factor } \mathbf{R}^{k+1}\}$$

Here, k is a non-negative integer.
Then, known result for $dim_H \mathcal{S}_k$ is the following;

Theorem 4.4 (J.Cheeger, T.H.Colding [3, 5])
With notation as above,

$$dim_H \mathcal{S}_k \leq k.$$

Theorem 4.5 (J.Cheeger, T.H.Colding [3, 5])
With notation as above,

$$\mathcal{S} = \mathcal{S}_{n-2}.$$

Especially,

$$dim_H \mathcal{S} \leq n-2.$$

Remark 4.6
Let us explain relation of Main Theorem 1 to the splitting theorem of the limit space. We consider the following situation ; $\{M_i\}_{i\in\mathbf{N}}$: n dimensional complete Riemannian manifold ($n \geq 2$) with $Ric_{M_i} \geq n-1$, Z: compact metric space,

$$M_i \to Z \quad (i \to \infty)\text{: Gromov-Hausdorff convergence.}$$

We consider metric cone of Z, $C(Z)$, Almost cosine formura implies

$$\text{Splitting theorem holds for } (C(Z), z^*).$$

i.e.

If $C(Z)$ has a line passing z^*, then there exist a compact metric space X with $diam_X \leq \pi$ such that

$$C(Z) \cong \mathbf{R} \times C(X).$$

We can apply splitting theorem also to $C(X)$.

Main theorem 1 is proved by applying to this argument iteratively.

And, the statement for $k = n$ of Main Theorem 1 is

$$C(Z) \not\cong \mathbf{R}^n \times \mathbf{R}_{\geq 0}.$$

Compare Theorem 2.25.

Note that these things, we have, Main Theorem 1 is equivalent to the following;

Main Theorem 1' For above Z, if there exists $p_i, q_i \in Z$ $(i = 1, 2, \cdots, k)$ such that

$$\overline{p_i, q_i} = \pi, \text{ and } \det\big((\cos \overline{p_i, p_j})_{i,j}\big) \neq 0 \qquad (*)$$

then

1. $k \leq n+1$.

2. If $1 \leq k \leq n-1$, then there exist a compact length space X with $diam_X \leq \pi$ such that

$$Z \cong \mathbf{S}^{k-1} * X.$$

3. If $k = n, n+1$, then

$$Z \cong \mathbf{S}^n.$$

We can replace the assumption of Theorem 4.3 by above $(*)$.

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