A NOTE ON THE BERGMAN METRIC OF BOUNDED HOMOGENEOUS DOMAINS

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Let D be a bounded domain in $\mathbb{C}^n$ and let ds^2_D be the Bergman metric on D. In function theory of several variables, the Bergman metric has played an important role as a canonically defined invariant metric.

Recently it was shown that ds^2_D is complete if D is hyperconvex, i.e. if D admits a bounded plurisubharmonic exhaustion function (cf. [1], [3], [8]). This achievement reflects the developments of pluripotential theory on hyperconvex domains (cf. [2]).

On the other hand, it has long been known that some of the hyperconvex domains, strongly pseudoconvex domains and bounded symmetric domains for instance, share a property stronger than the Bergman completeness.

Namely, the Bergman metric on such domains is not only complete but also admits a potential function whose gradient is bounded, when it is measured with respect to ds^2_D. A remarkable consequence of this property is, according to a recent work of B.-Y. Chen [4], that the L^2 $\overline{\partial}$-cohomology groups of type (p, q) vanish if $p + q \neq n$ and that they are infinite dimensional if $p + q = n$. (See also [6] and [7].)

Therefore, given any domain D with a complete Bergman metric, it is interesting to decide whether or not D enjoys this property. Homogeneous domains are particularly interesting because the description of the actions of the group of biholomorphic automorphisms on the middle-degree L^2 $\overline{\partial}$-cohomology groups may then become a significant project which is very likely to be profitable.

The purpose of this note is to report the following simple but indispensable observation towards this direction of research.

Theorem 1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$. Then there exists a real analytic function φ on D satisfying*

$$ds^2_D = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta} dz_\alpha d\overline{z}_\beta$$

and

$$\left| \sum_{\alpha=1}^{n} \frac{\partial \varphi}{\partial z_\alpha} dz_\alpha \right|_{ds^2_D} = const.$$

Here $|\cdot|_{ds^2_D}$ stands for the length with respect to ds^2_D.

Proof. As was shown in [11], every bounded homogeneous domain say D is biholomorphically equivalent to a domain $\Omega \subset \mathbb{C}^n$ such that the group

$$\{u \in \mathrm{Aut}(\Omega) \mid u \text{ extends to an affine transformation on } \mathbb{C}^n\}$$

acts transitively on Ω.

Let $\sigma : D \to \Omega$ be a biholomorphism, and let $K_D(z)$ (resp. $K_\Omega(w)$) be the Bergman kernel function on D (resp. on Ω). Then we have

$$\begin{aligned} K_D(z) &= K_\Omega(\sigma(z))\big|\det \partial\sigma(z)/\partial z\big|^2, \\ ds_D^2 &= \sigma^* ds_\Omega^2 \end{aligned}$$

and

$$ds_D^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K_\Omega(\sigma(z))}{\partial z_\alpha \partial \overline{z}_\beta} dz_\alpha d\overline{z}_\beta.$$

What we want to show is that

$$\big|\partial \log K_\Omega(\sigma(z))\big|_{ds_D^2} = \text{const.}$$

To see this, take any two points $\zeta, \xi \in D$ and let α be an affine transformation such that $\alpha(\Omega) = \Omega$ and $\sigma(\zeta) = \alpha(\sigma(\xi))$.

We note that

$$\partial \log K_\Omega(\alpha(w)) = \partial \log K_\Omega(w)$$

since α is affine.

Let $\tau = \sigma^{-1} \circ \alpha^{-1} \circ \sigma$. Then we have

$$\begin{aligned} \big|\big(\partial \log K_\Omega(\sigma(z))\big)\big|_{z=\zeta}\big|_{ds_D^2} &= \big|\big(\partial \log K_\Omega(\alpha(\sigma(\tau(z)))))\big|_{z=\zeta}\big|_{ds_D^2} \\ &= \big|\big(\partial \log K_\Omega(\sigma(\tau(z)))\big)\big|_{z=\zeta}\big|_{ds_D^2} \\ &= \big|\tau^*\big(\big(\partial \log K_\Omega(\sigma(z))\big)\big|_{z=\xi}\big)\big|_{ds_D^2} \\ &= \big|\big(\partial \log K_\Omega(\sigma(z))\big)\big|_{z=\xi}\big|_{ds_D^2}. \end{aligned}$$

The last equality holds because τ is an isometry with respect to the Bergman metric. □

As is naturally expected, the following is true.

Proposition. *For any bounded homogeneous domain D, the constant arising as the length of $\partial\varphi$ in Theorem 1 does not depend on the choice of φ.*

Proof. Clearly it suffices to consider Ω instead of D. Let ψ be a C^2 function on Ω satisfying $\partial\overline{\partial}\psi = \partial\overline{\partial} \log K_\Omega$ and $|\partial\psi| = \text{const}$. We put $\omega = \partial\psi - \partial \log K_\Omega$. Then ω is a holomorphic 1-form on Ω satisfying

$$d|\partial \log K_\Omega + \omega|^2 = 0.$$

Let $w_0 \in \Omega$ be any point, let ξ be any tangent vector of Ω at w_0, and let α_t be a 1-parameter group of affine transformations of Ω generated by a vector field X such that $X(w_0) = \xi$.

Since α_t are all isometries with respect to ds_Ω^2, the Lie derivatives of ds_Ω^2 vanish with respect to X. Since α_t are all affine, the Lie derivatives of $\partial \log K_\Omega$ also vanish.

Therefore we obtain

$$\begin{aligned}0 &= \xi|\partial \log K_\Omega + \omega|^2 \\ &= \left(X|\partial \log K_\Omega + \omega|^2\right)_{w_0} \\ &= \langle (X\omega)_{w_0}, \omega_{w_0}\rangle + \langle \omega_{w_0}, (X\omega)_{w_0}\rangle.\end{aligned} \tag{1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product.

For the simplicity of notation we write $\xi\omega$ for $(X\omega)_{w_0}$ and write the result of (1) as

$$\langle \xi\omega, \omega\rangle + \langle \omega, \xi\omega\rangle = 0.$$

Similarly, denoting by J the complex structure of Ω, we have

$$\langle (J\xi)\omega, \omega\rangle + \langle \omega, (J\xi)\omega\rangle = 0.$$

Hence

$$\langle (\xi - iJ\xi)\omega, \omega\rangle + \langle \omega, (\xi + iJ\xi)\omega\rangle = 0.$$

Since ω is holomorphic we have

$$(\xi - iJ\xi)\omega = 0.$$

Therefore

$$\langle \omega, (\xi + iJ\xi)\omega\rangle = 0,$$

and hence

$$\langle \omega, \xi\omega\rangle = 0.$$

Since ξ was arbitrary, for any holomorphic vector field $\widetilde{\xi}$ on a neighbourhood of w_0 satisfying $\widetilde{\xi}(w_0) = \xi$ one has

$$\left\langle \omega, \widetilde{\xi}\omega \right\rangle = 0$$

near w_0.

Therefore, similarly as above we obtain

$$\begin{aligned}&\left\langle \xi\omega, \xi\omega \right\rangle + \left\langle \omega, \xi\left(\widetilde{\xi}\omega\right) \right\rangle = 0, \\ &\left\langle (J\xi)\omega, \xi\omega \right\rangle + \left\langle \omega, (J\xi)\left(\widetilde{\xi}\omega\right) \right\rangle = 0, \\ &\langle (\xi + iJ\xi)\omega, \xi\omega\rangle = 0,\end{aligned}$$

so that $\langle \xi\omega, \xi\omega\rangle = 0$.

Hence ω is holomorphic and parallel, which is impossible unless $\omega = 0$. □

That every bounded homogeneous domain D is equivalent to an affinely homogeneous domain Ω is a consequence of the fact that every bounded homogeneous domain is analytically equivalent to a Siegel domain of the 2nd kind. (See also [11].) Since K_Ω is not exhaustive on Ω (e.g. $K_\Omega(w) = \pi^{-1}(\operatorname{Re} w)^{-2}$ if $\Omega = \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$), it is not clear whether or not Ω (or D) is hyperconvex. Nevertheless, combining Theorem 1 with completeness of the Bergman metrics on (not necessarily homogeneous) Siegel domains of the 2nd kind as was proved by Nakajima [10], together with B.-Y. Chen's theorem, we obtain the following.

Theorem 2. *Let (D, ds_D^2) be a bounded homogeneous domain equipped with the Bergman metric and let $H^{p,q}_{(2)}(D)$ be the L^2 $\overline{\partial}$-cohomology group of D of type (p,q). Then*

$$\dim H^{p,q}_{(2)}(D) = \begin{cases} \infty & \text{if } p+q=n \\ 0 & \text{if } p+q \neq n \end{cases}$$

Moreover $H^{p,q}_{(2)}(D)$ are Hausdorff.

Note. B.-Y. Chen kindly gave us the following comments after we finished writing the manuscript.
1. Completeness of the Bergman metric of D follows from the homogeneity of D (an exercise!).
2. H. Donnelly [5] has proved that $\partial \log K_D$ is bounded with respect to ds_D^2, which is already sufficient to conclude Theorem 2.
3. Any bounded homogeneous domain D is hyperconvex. In fact, as was shown by Mok and Yau [9], D admits an Einstein-Kähler metric whose volume form $V(z) = \bigwedge_{\alpha=1}^n (\sqrt{-1} dz_\alpha \wedge d\overline{z}_\alpha)$ satisfy $V(z) > d(z)^{-2}(\log d(z))^{-2}$ near the boundary of D. Here $d(z)$ denotes the euclidean distance to the boundary. Since D is homogeneous, the volume form of ds_D^2 and that of the Einstein-Kähler metric are different only by a constant multiple. Hence $\lim_{z \to \partial D} K_D(z) = \infty$. Combining this with Donnelly's result on the boundedness of $\partial \log K_D$, we can conclude that $-(\log K_D + M)^{-1}$ is a plurisubharmonic bounded exhaustion function of D for sufficiently large M.

In spite of all this, Theorem 1 might still be of independent interest because of the

Question: How does the length of $\partial\varphi$ depend on D ?

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