Entropy comparisons and codings on interacting maps

Tsuyoshi Kato

Introduction

Families of maps provide us with dynamical systems by compositions of them. Usually they will show very complicated structures, and direct observations of them seem difficult to understand their behaviours. By extracting some rough values of them by use of projections, it becomes possible to look at them geometrically. They are given by families of symbolic dynamics. This is a basic idea of realizing geometric pattern formation from random dynamics. The original detailed dynamics will be referred as micro and the latter as macro.

Dynamics by families of maps admit several algebraic operations between themselves. Of particular interests for us are given by contracting compositions of maps or codings of symbols. Thus starting from a finite family of maps, one can produce infinite hierarchies of dynamics and operations between them.

One merit to produce such hierarchies is to compare with dynamical invariants on them, which measure degree of pattern formation. In this paper, we study such dynamical systems from entropy view points. Essentially we will introduce two types of them, *topological* and *informative entropies*. Our main interests here will be to compare the entropy values between these dynamical systems.

Let us take two continuous maps on the interval:

$$f, g : [0,1] \to [0,1]$$

and consider their iterations:

$$O_1(x) = \{f^k(x)\}_{k=0,1,\dots}, \quad O_2(x) = \{g^k(x)\}_{k=0,1,\dots}.$$

We call them the *oscillatins* ([K2]).

Let us define *interaction* of these orbits below. For this, let X_2 be the one sided full shift with two alphabets $\{0,1\}$:

$$X_2 = \{(k_0, k_1, \dots) : k_i \in \{0,1\}\}.$$

Then for each element $\bar{k} = (k_0, k_1, \dots) \in X_2$, we will associate a family of maps:

$$\{h^m(x)\}_{k=0,1,\dots}, \quad h^m : [0,1] \to [0,1]$$

as follows. Let us put:

$$d_i(x) = \begin{cases} f(x) & i = 0, \\ g(x) & i = 1. \end{cases}$$

Then we define the family of maps $\{h^m\}_{m=0}^{\infty}$ by:

$$h^m(x) \equiv d_{k_m} \circ d_{k_{m-1}} \circ \cdots \circ d_{k_0}(x).$$

Let:

$$\pi : [0,1] \to \{0,1\}$$

be a measurable map given by $\pi([0, \frac{1}{2})) \equiv 0$ and $\pi((\frac{1}{2}, 1]) \equiv 1$.

For a.e. $x \in [0,1]$, one can compose $\{h^m(x)\}_m$ with π and obtains another element:

$$\bar{k}' \equiv \pi((h^0(x), h^1(x), \dots)) \equiv (\pi \circ h^0(x), \pi \circ h^1(x), \dots) \in X_2.$$

Thus for each element $\bar{k} \in X_2$, one has assigned another element $\bar{k}'$. We denote this assignment:

$$\Phi(f,g)(x) : X_2 \to X_2$$

by $\Phi(f,g)(x)(\bar{k}) \equiv \pi((h^0(x), h^1(x), \dots))$. It is expressed as a famliy of symbolic dynamics:

$$\Phi(f,g) : [0,1] \times X_2 \to [0,1] \times X_2$$

by $\Phi(f,g)(x, \bar{k}) = (x, \Phi(f,g)(x)(\bar{k}))$. This is the most basic dynamical systems in this paper, and we call it the *interaction map*.

By increasing the number of maps $\{f_1, \dots, f_a\}$, one obtains parallely corresponding interaction maps $\Phi : [0,1] \times X_a \to [0,1] \times X_a$ by use of projections $\pi : [0,1] \to \{1, \dots, a\}$.

Let $\mathbf{I} = [0,1] \times [0,1] \times \dots$ be the infinite product of the interval with the product topology. For a.e. $\bar{x} = (x_0, x_1, \dots) \in \mathbf{I}$, the projection $\pi(\bar{x}) = (\pi(x_0), \pi(x_1), \dots)$ gives an element in X_2. So it determines a family of maps $\{h^k\}_k$ as above. Let us fix $x \in [0,1]$. Then the family of maps above gives a map:

$$\tilde{\Phi}(x) : \mathbf{I} \to \mathbf{I}, \quad (x_0, x_1, \dots) \to (h^0(x), h^1(x), \dots).$$

In this dynamics, the flow $\{\tilde{\Phi}(x)^t(\bar{x})\}_{t=0}^{\infty} \subset \mathbf{I}$ will behave quite comlicated manner under the very random compositions of maps. This motivates our formulation of interaction maps which wastes very detailed information and traces only rough values of them.

Even though interaction maps will be much more simplified compared with the above one on the infintie product of the intervals, at the same time they contain rich geometric structures. For example one can realize flows of some infinite integrable systems which possess solitons by such interaction maps ([K3], [K4]). So one has a chance to create some patterns by such rough maps. Such principle is called *pattern formation* and in this paper we study such micro-macro comparisons from entropy view points.

For $\bar{k} \in X_a$, let us write $\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots, \bar{k}_n^t, \dots) \in X_a$. From dynamical view point, the interaction map has two directions. One is n-direction which we call holizontal, and the other is t-direction and we call it vertical, or time direction.

The interaction map decomposes into smaller pieces as dynamical systems. In 1.B, we study strurcutre of such spaces from topological entropy view points.

For holizontal direction, there is a natural fibration of interaction maps. Let us put

$$\bar{X}(\{f_i\}_i) = \{(\bar{k}, \Phi(x)(\bar{k}), x) : x, \bar{k}\}, \bar{X}(\{f_i\}_i; \bar{k}) = \{(\Phi(x)(\bar{k}), x) : x\}.$$

They are subsets of $X_a \times X_a \times [0,1]$ and $X_a \times [0,1]$ respectively. X_a has the standard shift σ, and using it, the above spaces admit canonical shifts by $\bar{\sigma}(\bar{k}, \bar{l}, x) = (\sigma(\bar{k}), \sigma(\bar{l}), f_{k_0}(x))$. Then one obtains the shift commuting Lipschitz fibration:

$$\bar{X}(\{f_i\}_i; \bar{k}) \hookrightarrow \bar{X}(\{f_i\}_i) \mapsto X_a.$$

From the well known Bowen's fibration theorem, one obtains estimates between values of topological entorpy (**cor 1.2**):

$$h_t(X_a) = \log a \leq h_t(\bar{X}(\{f_i\}_i)) \leq \log a + \sup_{\bar{k} \in X_a} h_t(\bar{X}(\{f_i\}_i; \bar{k})).$$

Thus behaviour of $h_t(\bar{X}(\{f_i\}_i))$ depends on that of $h_t(\bar{X}(\{f_i\}_i; \bar{k}))$. For the latter space, again one obtains another fibration. Let us put:

$$\bar{Y}(\{f_i\}_i, \bar{k}) = \{\Phi(\bar{k})(x); x\}, \quad \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l}) = \{x; \Phi(\bar{k})(x) = \bar{l}\}.$$

These are subsets of X_a and $[0, 1]$ respectively and admit similar shifts. Then one obtains the shift commuting Lipschitz surjection:

$$\bar{Y}(\{f_i\}_i, \bar{k}, \bar{l}) \hookrightarrow \bar{X}(\{f_i\}_i, \bar{k}) \to \bar{Y}(\{f_i\}_i, \bar{k}).$$

Under some conditions, this admits a structure of continuous fibration.

Unlike to the above case, this fibration changes under the action of shifts. Thus in order to obtain estimates of values of topological entorpy, we show an equivariant version of the Bowen's theorem.

Let X, Y, Z be shifted spaces by σ, and $E_z \subset X$ and $F_z \subset Y$ be a parametrized families by $z \in Z$ so that $\sigma(E_z) \subset E_{\sigma(z)}$ hold for all $z \in Z$, and similar for F_z. Let $T_z : E_z \to F_z$ be an equivariant Lipschitz fibration. Then the inequalities hold (**thm 1.2**):

$$h_t(F_z) \leq h_t(E_z) \leq h_t(F_z) + \sup_{y \in F_z} h_t(T_z^{-1}(y))$$

for each $z \in Z$.

By applying thorem 1.2, one obtains the estimates on toological entropy under some condition as above (**cor 1.3**):

$$\begin{aligned} h_t(\bar{Y}(\sigma; \{f_i\}_i, \bar{k})) &\leq h_t(\sigma; \bar{X}(\{f_i\}_i, \bar{k})) \\ &\leq h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k})) + \sup_{\bar{l} \in X_a} h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})). \end{aligned}$$

When dynamics shows some monotonicity on each $\bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})$, then the entropy on it vanishes. Under such conditions, one obtains the equality (**prop 1.2**):

$$h_t(\bar{Y}(\sigma; \{f_i\}_i, \bar{k})) = h_t(\sigma; \bar{X}(\{f_i\}_i, \bar{k})).$$

In particular one obtains the upper bound of the values (**cor 1.4**):

$$h_t(\bar{X}(\{f_i\}_i)) \leq 2 \log a.$$

In 1.C and 3.B, we study cell automata from the entropy view point. By modyfying composition way of maps, one can realize Lotka Volterra cell automaton and box and ball system (BBS) by interaction maps ([K3],[K4]).

For the LV case, the most different point from the ordinary interactions is that in order to determine k_i^{t+1}, one is required to know the values of k_j^t until $j = i+1$. It turns out that this phenomena is reflected to values of topological entropy. In fact for non cell type interaction, the topological entropy of the pointwise interaction, $\Phi(x) : X_a \to X_a$ is always trivial (**lem 1.3**). On the other hand for cell type one, one can immediately construct interactions so that the pointwise interactions have value $\log a$ of topological entropy (**lem 1.8**).

In general interactions can be changed to cell type ones. One can see that the values of the topological entropy of families of interactions also drastically change. By use of the tent map f and $g = 1 - f$, the corresponding interaction map $\Phi(f, g)$ and its cell type one $\Phi(f, g, J)$ have their values respectively as (**lem 1.10**):

$$h_t(\Phi(f, g)) = \log 2, \quad h_t(\Phi(f, g, J)) = \infty.$$

In 2.B we study dynamics of interaction maps for vertical direction. By use of restrictions, one obtains vertical slices of interaction maps as an infinite inclusions of dynamical systems. There are canonical factor maps between them. From such structure, we study amalgamation of these dynamics which give topological conjugacies, and so the same values of topolgical entropy.

In the case of BBS, such restriction above gives a surjection to a subshift of finite type. In particular one obtains a rough estimate from below (**lem 3.3**):

$$h_t(BBS) \geq \frac{1+\sqrt{5}}{2} > 0.$$

The interaction maps $\Phi(\{f_i\}_i) : [0,1] \times X_a \to [0,1] \times X_a$ can admit several operations so that one can obtain other maps of the same type as above. From the micro to macro principle view point, one wanders which will admit more structures of patterns. So it would be natural to compare with the values of the corresponding entropies which will reflect such phenomena.

From the information theory view points, there are algebraic operations on the set of words which we call the *codings* (section 4). There are quite various types of codings. Let $\{f_1, \dots, f_a\}$ be a family of maps. For each $\bar{k}_n \in (X_a)_n$, one can associate another map $g(\bar{k}_n)$ by composition:

$$g(\bar{k}_n) = f_{k_n} \circ f_{k_{n-1}} \circ \cdots \circ f_{k_1}.$$

By this way one obtains another families of maps

$$\mathbf{C}_n = \cup_{\bar{k}_n \in (X_a)_n} g(\bar{k}_n) \equiv \{g_1, \dots, g_{a^n}\}.$$

We call it *contraction* of $\{f_i\}_i$. Using this new family of maps, one obtains another interaction maps $\Phi(\{g_j\}_j) : [0,1] \times X_{a^n} \to [0,1] \times X_{a^n}$ by using the projection $\pi : [0,1] \to \{1, \dots, a^n\}$, $\pi((\frac{k}{a^n}, \frac{k+1}{a^n})) = k+1$.

By letting $n \to \infty$, the division of the interval becomes more elaborate, on the other hand by the effect of many times compositions, some informations on the dynamics of maps will be lost. So it will be a natural question which n will be the best in order to induce the richest information on the dynamics of the interacting maps. In particular one can ask whether it might be possible to recover the dynamical information of the original maps $\{f_i\}_i$ from such interaction maps as $n \to \infty$.

The lexicographic order gives a map $\psi_0 : (X_a)_n \to \{1, \dots, a^n\}$ and it induces a map $\psi : X_a \to X_a a^n$. Thus a contraction gives a diagram as below, which is non commutative diagram in general.

Of particular interest for us is *compression* which is given by decreasing number of contracted maps g_j $(4.B)$. It gives a map φ_0 : $(X_a)_m \to \{1, \ldots, t\}$, $t \leq a^n$, and induces $\varphi : X_a \to X_t$. So a compression from $\{f_1, \ldots, f_a\}$ to $\{g_1, \ldots, g_t\}$ also gives a non commutative diagram in general:

$$\begin{array}{ccc} [0,1] \times X_a & \xrightarrow{\varphi} & [0,1] \times X_t \\ \Phi(\{f_i\}_i) \downarrow & & \Phi(\{g_j\}_j) \downarrow \\ [0,1] \times X_a & \xrightarrow{\varphi} & [0,1] \times X_t \end{array}$$

For a particular case which we call the compression by an n-th projection, one obtains commutative diagrams between the interaction maps on the original family $\{f_i\}_i$ and on the compressed one $\{g_i\}_i$ above (**prop 4.1**).

In principle codings give smaller values of entropies, since it eliminates detailed information of the original dynamics. The compression by an n-th projection gives the optimal case. In fact using commutativity, one obtains the equality of the topological entropy (**cor 4.1**):

$$h_t(f_1, \ldots, f_a) = h_t(g_1, \ldots, g_a)$$

From measure theory view points, one can induce a canonical measure Q on each $(X_a)_n$, which is determined only by a family of maps $\{f_1, \ldots, f_a\}$. Once one has a measure, at least two entropies, informative h_s and conditional h_c ones are defined, which work well for degree zero and one Markov processes respectively. In general dynamics of interacting maps shows behaviours of higher degree Markov. In this paper in order to treat such case, we introduce the *interacting entropy* h_i which measures how maps are well interacting. All these entropies are given by determining the initial value $\bar{x} \in X_a$.

When the interaction is Markov, then it coincides with the conditional one (**cor 5.2**):

$$h_i(\bar{x}) = h_c(\bar{x}).$$

Our formulation will reflect much more interacting situations. In general the interacting entropy show smaller values than the topological one (**lem 5.5**, **thm 5.1**):

$$h_t(\bar{X}(\{f_i\}_i, \bar{x})) \geq h_i(\bar{x}), \quad h_t(f_1 \dots, f_a) \geq h_i(f_1, \dots, f_a).$$

When the measure Q satisfies some homogeneity, then h_i coincides with h_t, and so the above inequality is optimal (**cor 5.3**).

A commutative compression $\varphi_0 : (X_a)_m \to \{1, \dots, t\}$ from $\{f_i\}_{i=1}^a$ to $\{g_j\}_{j=1}^t$ gives a priori estimates (**thm 5.2**):

$$(n+1)h_i(\bar{x}_n;\{f_i\}_i) \\ \leq (l+1)h_i(\varphi(\bar{x}_n); \{g_i\}_i) + \sup_{\bar{y}_l \in (X_t)_l} \log \sharp \varphi^{-1}(\bar{y}_l)$$

where $(n+1) = (l+1)(m+1)$.

The interacting entropy tends to decrease their values when the length of words go to infinity. In order to study degree of such decay, we introduce *divergence*

$$D(P||Q)$$

between two measures (5.C). By use of D, one introduces the minimum exponent $m_n = m_n(Q, \bar{x}_n)$ (5.C.2).

In the case when Q is memeoryless in vertical direction and Markov in holizontal one, one obtains a decay estimate of h_i (**cor 5.5**):

$$h_i(\bar{x}_n) \leq p(n) \exp(-nm_n)$$

where p is a polynomial. In particular when $m_n \geq \lambda > 0$ satisfies a lower bound by a positive number, then $h_i(\bar{x}_n)$ decays expenentially

One of main aim to introduce h_i is to study some qualitative properties of codings. From informative view points, one of the most important interests of codings is when they can be decodable, in other words when they are injections.

In this paper we study such property when the induced measure Q is memoryless in time and Markov for $\bar{x}$. Passing through ergodicity and low of large numbers, one obtains a result which states that for small coding rate R with respect to the value of interacting entropy, almost all long random words cannot be decodable (**thm 6.3**).

In precise suppose Q is moreover ergodic. Let $\varphi_0 : (X_a)_n \to (X_t)_m$ be a coding, and put $R = m/n$. When the inequality:

$$R \log t < h_i(\bar{x})$$

is satisfied, then for any other map $\psi_0 : (X_t)_m \to (X_a)_n$, any small $\lambda > 0$ and for all large n, the estimate:

$$P_e \equiv \Pr\ (\psi \circ \varphi(\bar{Y}_n) \neq \bar{Y}_n) \geq 1 - \lambda$$

holds.

The conclusion can be stated for non Markov cases. By introducing ergodicity for such cases in $5.D$, finally we would like to propose:

Conjecture: When the induced measure is ergodic, then the above statement still holds.

1 Topological entropy

1.A Topological entropy of interacting maps: Let $a \geq 1$ be a positive integer, and $f_1, \dots, f_a : [0,1] \to [0,1]$ be a famliy of maps. We define *interaction* of these famillies of maps below. Let:

$$X_a = \{(k_0, k_1, \dots) : k_i \in \{1, \dots, a\}\}$$

be the one sided full shift. It is compact and admits a metric structure by:

$$d(\{x_i\}_i, \{y_j\}_j) = \Sigma_{i=0}^{\infty} \frac{1 - \delta_{x_i, y_i}}{2^i}.$$

We equip with the standard metric on $[0,1]$ and with the product metric on $[0,1] \times X_a$.

For each element $\bar{k} = (k_0, k_1, \dots) \in X_a$, we will associate a family of maps:

$$\{h^m(x)\}_{k=0,1,\dots}, \quad h^m : [0,1] \to [0,1]$$

by:

$$h^m(x) \equiv f_{k_m} \circ f_{k_{m-1}} \circ \dots \circ f_{k_0}(x).$$

We call the famliy as the *interaction maps.*

Let us put a subset $S(f_1, \dots, f_a; \bar{k}) = \{x \in [0,1] : h^m(x) \in \{\frac{i}{a}\}_{i=1}^{a-1}$ for some $m\}$ in $[0,1]$. We call it the *singular set.* The regular set with respect to $\bar{k}$ is given by $R(f_1, \dots, f_a; \bar{k}) \equiv [0,1] \backslash S(f_1, \dots, f_a; \bar{k})$.

The *regular set* of the family of maps $\{f_1, \dots, f_a\}$ is defined by:

$$R(f_1, \dots, f_a) \equiv \cap_{\bar{k} \in X_a} R(f_1, \dots, f_a; \bar{k}) \subset [0,1].$$

Example: Let f be the tent map, $f|[0, \frac{1}{2}](x) = 2x$ and $f|[\frac{1}{2}, 1](x) = 2 - 2x$, and g be its reverse, $g(x) = 1 - f(x)$. The regular set is:

$$R(f, g) = [0,1] \backslash \{\frac{k}{2^n}; n = 1, 2, \dots, 1 \leq k \leq 2^n - 1\}.$$

By definition, the following hold:

Lemma 1.1 *Let* $R(f_1, \dots, f_a) \subset [0,1]$ *be the regular set. Then:*

$$f_i(R(f_1, \dots, f_a)) \subset R(f_1, \dots, f_a)$$

are satisfied for all $i = 1, \dots, a$.

Let:

$$\pi : [0,1] \backslash \{\frac{1}{a}, \frac{2}{a}, \dots, \frac{a-1}{a}\} \to \{0, 1, \dots, a\}$$

be a measurable map given by $\pi((\frac{i-1}{a}, \frac{i}{a})) \equiv i$ for $i = 1, \dots, a$.

Let $\bar{k} \in X_a$ and $\{h^m\}_m$ be the corresponding famliy of maps. For each $x \in R(f_1, \dots, f_a)$, one can compose $\{h^m(x)\}_m$ with π and obtains another element:

$$\bar{k}' \equiv \pi((h^0(x), h^1(x), \dots)) \equiv (\pi \circ h^0(x), \pi \circ h^1(x), \dots) \in X_a.$$

Thus for each element $\bar{k} \in X_a$, one has assigned $\bar{k}' \in X_a$. We denote it as $\Phi(\{f_i\}_i)(x) : X_a \to X_a$ by $\Phi(\{f_i\}_i)(x)(\bar{k}) \equiv \pi((h^0(x), h^1(x), \dots))$. It gives a family of symbolic dynamics:

$$\Phi(f_1, \dots, f_a) : [0,1] \times X_a \to [0,1] \times X_a$$

with domain $R(f_1, \dots, f_a)$, by $\Phi(\{f_i\}_i)(x, \bar{k}) = (x, \Phi(\{f_i\}_i)(x)(\bar{k}))$. This is the most basic dynamics in this paper. We call it the *interaction map*.

1.A.2 Topological entropy of interacting maps: Below we define the *topological entropy* for maps on metric spaces into itselves. For our purpose we will extend the underlying spaces for families of maps $\{g^k : Y \to Y\}_{k=0}^{\infty}$ which generalizes the single map h case by assigning g^k corresponding to the i-th iterates of h.

Let (Y, d) be a compact metric space, and $Y^{\mathbf{N}}$ be the product space. It is also compact and admits an induced metric by $\bar{d}(\{x_i\}_i, \{y_j\}_j) = \Sigma_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$.

A subset $E \subset Y^{\mathbf{N}}$ is called (n, ϵ) *separated*, if for any $\{x_i\}_i \neq \{y_i\}_i \in E$, there is some $0 \leq j \leq n$ so that the inequality holds:

$$d(x_j, y_j) > \epsilon.$$

A subset $F \subset Y^{\mathbf{N}}$ is (n, ϵ) *net* if for any $\{x_i\}_i \in Y^{\mathbf{N}}$, there is some $\{y_i\}_i \in F$ so that the inequalities hold:

$$d(x_j, y_j) \leq \epsilon$$

for all $0 \leq j \leq n$.

For a subset $K \subset Y^{\mathbf{N}}$, let $r_n(\epsilon, K) \in \mathbf{N}$ be the smallest cardinarity among (n, ϵ) nets in K. Similarly let $s_n(\epsilon, K)$ be the largest cardinarity of (n, ϵ) separated sets in K. Then we define:

$$r(\epsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K),$$
$$s(\epsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K).$$

The following holds (see [MS] $p164$):

Lemma 1.2 *(1) Both $r(\epsilon, K)$ and $s(\epsilon, K)$ are decreasing with respect to ϵ.*

(2) The inequalities hold:

$$r_n(\epsilon, K) \leq s_n(\epsilon, K) \leq r_n(\frac{1}{2}\epsilon, K).$$

For any subset $K \subset Y^{\mathbf{N}}$, the *topological entropy* of K is given by:

$$h_t(K) = \lim_{\epsilon \to 0} r(\epsilon, K) = \lim_{\epsilon \to 0} s(\epsilon, K).$$

Let $\{g^k : Y \to Y\}_{k=1}^{\infty}$ be a family of maps, and $S \subset Y$ be a subset. Then we put the set:

$$K = \{(x, g^1(x), g^2(x), \dots); x \in S\} \subset Y^{\mathbf{N}}.$$

The topological entropy with respect to $(\{g^k\}_k, S)$ is given by:

$$h_t(\{g^k\}_k; S) = \lim_{\epsilon \to 0} r(\epsilon, K) = \lim_{\epsilon \to 0} s(\epsilon, K).$$

For a single map $f : Y \to Y$, the topological entropy $h_t(f; S$ is also given by using its iterations $\{f^k\}_k$.

Let $f_1, \dots, f_a : [0,1] \to [0,1]$ be a family of maps on the interval, and $\Phi : [0,1] \times X_a \to [0,1] \times X_a$ be the corresponding interaction map with the regular set $R = R(f_1, \dots, f_a)$.

Definition 1.1 *The topological entropy of the interacting map is given by:*

$$h_t(f_1, \dots, f_a) = h_t(\Phi; R \times X_a).$$

This is the most basic topological entropy in this paper. From measure theoretic view points, we will introduce other entropies in later sections.

Let $X_a^{\mathbf{N}}$ be the infinite product of X_a with the product topology. This is a compact metrizable space. We denote its elements as $(\bar{k}_0, \bar{k}_1, \dots) = \{k_i^s\}_{i,s=0}^{\infty}$ There is a canonical shift on $X_a^{\mathbf{N}}$ by:

$$\sigma(\bar{k}_0, \bar{k}_1, \dots) = (\bar{k}_1, \bar{k}_2, \dots).$$

Let Φ be the interaction map corresponding to $f_1, \dots, f_a$ and choose $x \in R(f_1, \dots, f_a)$. Then one obtains a map:

$$\Psi(x) : X_a \to X_a^{\mathbf{N}}, \quad \Psi(x)(\bar{k}) = (\bar{k}, \Phi(\bar{k}, x), \Phi(\bar{k}, x)^2, \Phi(\bar{k}, x)^3, \dots).$$

This describes the iteration dynamics of $\Phi(x)$. We also denote $\Psi : [0,1] \times X_a \to X_a^{\mathbf{N}}$ by $\Psi(x, \bar{k}) = \Psi(x)(\bar{k})$.

Now we put:

$$\mathbf{L}(f_1,\dots,f_a) = \cup_{x\in R(f_1,\dots,f_a),\bar{k}\in X_a}\Psi(x)(\bar{k}) \subset X_a^{\mathbf{N}}$$

and call it the *orbit space of the interaction map*. This is the shift invariant subset.

For positive integers $n,t \geq 0$, let us put the set $(X_a)_n^{t+1}$ and the projections:

$$(X_a)_n^{t+1} = \{\{k_i^s\}_{0\leq i\leq n}^{0\leq s\leq t}; k_i^s \in \{1,\dots,a\}\},$$
$$\bar{\pi}_{n,t} : X_a^{\mathbf{N}} \to (X_a)_n^{t+1}, \quad \{k_i^s\}_{i,s\in\mathbf{N}} \to \{k_i^s\}_{0\leq i\leq n}^{0\leq s\leq t}.$$

We denote $(X_a)_n = (X_a)_n^1$ as the set of all words of length $n+1$.

Now we define the *topogical entropy of the orbit spaces* by:

$$h_t(\mathbf{L}(f_1,\dots,f_a)) = \lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t+1}\log\sharp\bar{\pi}_{n,t}(\mathbf{L}(f_1,\dots,f_a)).$$

Proposition 1.1 *The equality holds:*

$$h_t(f_1,\dots,f_a) = h_t(\mathbf{L}(f_1,\dots,f_a)).$$

Proof: Let $\Phi : [0,1]\times X_a \to [0,1]\times X_a$ be the interacting maps and $R = R(f_1,\dots,f_a)$ be the regular set. Let $S(\epsilon,t) \subset R\times X_a$ be an (ϵ,t) net for Φ. Then for each large n, there is a small $\epsilon > 0$ so that $\bar{k}_n \neq \bar{k}'_n \in (X_a)_n$ implies $d(\bar{k}_n,\bar{k}'_n) \geq \epsilon$. So $\sharp S(\epsilon,t) \geq \sharp\bar{\pi}_{n,t}(\mathbf{L}(\{f_i\}_i))$ holds. Thus:

$$\limsup_{t\to\infty}\frac{1}{t+1}\log\sharp S(\epsilon,t) \geq \limsup_{t\to\infty}\frac{1}{t+1}\log\sharp\bar{\pi}_{n,t}(\mathbf{L}(\{f_i\}_i))$$

holds, and one obtains the inequality $h_t(\Phi) \geq h_t(\mathbf{L}(\{f_i\}_i))$.

Conversely, if two elements $(x,\bar{k}),(x',\bar{k}) \in S(\epsilon,t)$ satisfy the equality $\pi_{n,t}(\{\Phi(x)^i(\bar{k})\}_i) = \pi_{n,t}(\{\Phi(x')^i(\bar{k})\}_i)$, then by ϵ separation, $|x - x'| \geq \epsilon$ must be satisfied. So $\frac{1}{\epsilon}\sharp\bar{\pi}_{n,t}(\mathbf{L}(f,g)) \geq \sharp S(\epsilon,t)$ holds. Thus:

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t+1}\log\sharp\bar{\pi}_{n,t}(\mathbf{L}(\{f_i\}_i))\\ &= \limsup_{t\to\infty}\frac{1}{t+1}\log[\frac{1}{\epsilon}\sharp\bar{\pi}_{n,t}(\mathbf{L}(\{f_i\}_i))] \geq \limsup_{t\to\infty}\frac{1}{t+1}\log\sharp S(\epsilon,t)\end{aligned}$$

holds, and one obtains the other inequality $h_t(\Phi) \leq h_t(\mathbf{L}(\{f_i\}_i))$.

This completes the proof.

Example: Let f be the tent map and g be its reverse. Consider the vertical projection $\bar{\pi}_{0,\infty} : X_a^{\mathbf{N}} \to X_a$ by $\{k_i^s\}_{i,s} \to \{k_0^s\}_s$. Since $\bar{\pi}_{0,\infty} : \mathbf{L}(f,g) \cong X_2$ gives a homeomorphism, the topological entropy is give by:

$$h_t(f,g) = \log 2.$$

Let us choose any $x \in R(f_1, \dots, f_a)$, and consider the pointwise dynamics:

$$\Phi(x) : X_a \to X_a.$$

We see that the entropy of $\Phi(x)$ is trivial.

Lemma 1.3 *The topological entropy $h_t(\Phi(x))$ is equal to 0 for any $x \in R(f_1, \dots, f_a)$.*

Proof: Let us put $\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots)$, $t = 0, 1, \dots$ Then k_m^t is determined by $(k_0, \dots, k_m)$ for any t.

Let $C_1, \dots, C_{a^N} \subset X_a$ be all the set of cylinders of length N. If one chooses elements $\bar{k}_i$ arbitrarily from C_i, then the set $\{\bar{k}_1, \dots, \bar{k}_{2^N}\}$ gives the largest $(t, \frac{1}{a^N})$ separated net for all t. Thus the topological entropy is:

$$h_t(\Phi(x)) = \lim_{n \to \infty} \frac{1}{t+1} \log a^N = 0.$$

This completes the proof.

Later when one considers cell type interactions in $1.C.3$, the pointwise topological entropy becomes non trivial.

1.A.3 Topological entropy in micro scale: So far we have considered symbolic dynamics arising from families of maps. This is obtained by wasting some detailed information on their dynamics by use of projections. Here we will consider dynamics which will contain all informations arising from families of maps.

Let $\mathbf{I} = [0,1] \times [0,1] \times \dots$ be the infinite product of the interval with the product topology. So a sequence $\{\{x_i^t\}_{i=0}^{\infty}\}_t \subset \mathbf{I}$ converges when every restriction to finite length words $\{\{x_i^t\}_{i=0}^N\}_t \subset \mathbf{I}$ converges. $\mathbf{I}$ is a compact metric space.

For $\bar{x} = (x_0, x_1, \dots) \in \mathbf{I}$, when any of x_i does not hit the set $\{\frac{i}{a}\}_{i=1}^{a-1}$, then the projection

$$\pi(\bar{x}) = (\pi(x_0), \pi(x_1), \dots)$$

gives an element in X_a. So it determines a family of maps $\{h^k\}_k$ as above.

Let $x \in R(f_1, \dots, f_a)$. Then the family of maps above gives a map:

$$\tilde{\Phi}(x) : \mathbf{I} \to \mathbf{I}, \quad (x_0, x_1, \dots) \to (h^0(x), h^1(x), \dots).$$

By this way one obtains the interaction map:

$$\tilde{\Phi} : [0,1] \times \mathbf{I} \to [0,1] \times \mathbf{I}, \quad \tilde{\Phi}(x, \mathbf{x}) = (x, \tilde{\Phi}(x)(\mathbf{x})).$$

The interaction maps we have considered so far are the reduction of $\tilde{\Phi}$ as:

$$\Phi : [0,1] \times X_2 \to [0,1] \times X_2.$$

Let $R = R(f_1, \dots, f_a)$ be the regular set.

Definition 1.2 *The topological entropy in micro scale is given by:*

$$h_t^m(f_1, \dots, f_a) = h_t(\tilde{\Phi}) = h_t(\tilde{\Phi}; R \times \mathbf{I}).$$

Clearly the inequality holds:

$$h_t^m(f_1, \dots, f_a) \geq h_t(f_1, \dots, f_a).$$

1.B Entropies for holizontal direction: A continuous map between metric spaces $T : E \to F$ is called a *fibration*, if for each small $\epsilon > 0$, there is $\delta > 0$ so that for any $y, y' \in F$ with $d(y, y') \leq \delta$ and $x \in E$ with $T(x) = y$, the inequality:

$$d(x, T^{-1}(y')) \leq \epsilon$$

holds.

Let $\sigma : E \to E$ and $\sigma' : F \to F$ be two metric spaces equipped with continuous maps between themselves. We call these maps the *shifts.* A continuous map $T : E \to F$ is a *factor map*, if it is surjective fibration which commutes with the shifts.

In $1.B$ we study structure of dynamics for holizontal direction from topological entropy view point by use of fibration structure of the dynamics.

Let $\{f_1, \dots, f_a\}$ be a family of maps, and consider the corresponding interaction map $\Phi : [0,1] \times X_a \to [0,1] \times X_a$. Φ has two directions, holizontal and vertical ones. Here one will consider entropies of the dynamics of maps for holizontal directions.

Let $R = R(f_1, \dots, f_a)$ be the regular set, and put the total shift dynamics:

$$\bar{X}(\{f_i\}_i) = \{(\bar{k}, \Phi(x)(\bar{k}), x) : x \in R, \bar{k} \in X_a\} \subset X_a \times X_a \times [0,1].$$

Let $\sigma : X_a \to X_a$ be the shift given by $\sigma(k_0, k_1, \dots) = (k_1, k_2, \dots)$. Then there is the induced one by:

$$\bar{\sigma} : X_a^2 \times [0,1] \to X_a^2 \times [0,1]$$

where $\bar{\sigma}(\bar{k}, \bar{l}, x) = (\sigma(\bar{k}), \sigma(\bar{l}), f_{k_0}(x))$, where $\bar{k} = (k_0, k_1, \dots)$. Then by lemma 1.1, $\bar{X}(\{f_i\}_i)$ is a $\bar{\sigma}$ invariant subset in $X_a^2 \times [0,1]$.

The *shift entropy* for $\{f_1, \dots, f_a\}$ is given by:

$$h_t(\bar{\sigma}, \bar{X}(f_1, \dots, f_a)).$$

Let us put:

$$\bar{X}(\{f_i\}_i; \bar{k}) = \{(\Phi(x)(\bar{k}), x) : x \in R\} \subset X_a \times [0,1].$$

This also admits a canonical shift by:

$$\bar{\sigma}(\Phi(x)(\bar{k}), x)) = (\sigma(\Phi(x)(\bar{k})), f_{k_0}(x))$$

where $\bar{k} = (k_0, k_1, \dots)$.

$\bar{X}(\{f_i\}_i)$ admits the shift commuting fibration:

$$\bar{X}(\{f_i\}_i; \bar{k}) \hookrightarrow \bar{X}(\{f_i\}_i) \mapsto X_a.$$

The map $\bar{X}(\{f_i\}_i) \to X_a$, $(\bar{k}, \Phi(x)(\bar{k}), x) \to \bar{k}$ is clearly a *Lipschitz map.*

Theorem 1.1 *Let (X, d) and (Y, d') be compact metric spaces, and $E \subset X^{\mathbf{N}}$ and $F \subset Y^{\mathbf{N}}$ be shift invariant subsets. If $T : E \to F$ is a Lipschitz factor map, then the inequalities hold:*

$$h_t(F) \leq h_t(E) \leq h_t(F) + \sup_{y \in F} h_t(T^{-1}(y)).$$

Proof: When both E and F are compact, then the conclusion is known by Bowen's theorem. In general let $\bar{E} \subset X$ be the closure of E. Thus both $\bar{E}$ and $\bar{F}$ are compact. Since T is Lipschitz, it extends to a shift commuting Lipschitz map $T : \bar{E} \to \bar{F}$. Thus the conclusion follows, since the topological entropy gives the same values on the closure of the spaces (cf. lem 1.4). This completes the proof.

We will verify such fibration theorem for more general cases in the next section.

In particular one obtains the following:

Corollary 1.1 *Let $\{h^k : X \to X\}_k$ and $\{g^k : Y \to Y\}_k$ be two families of maps on relatively compact metric spaces.*

Suppose there is a Lipschitz factor map $T : X \to Y$ with $T \circ h^k = g^k \circ T$. Then the inequalities hold:

$$h_t(\{g^k\}) \leq h_t(\{h^k\}) \leq h_t(\{g^k\}) + \sup_{y \in Y} h_t(\{h^k\}; T^{-1}(y)).$$

Corollary 1.2 *We have the inequalities:*

$$h_t(X_a) = \log a \leq h_t(\bar{X}(\{f_i\}_i)) \leq \log a + \sup_{\bar{k} \in X_a} h_t(\bar{X}(\{f_i\}_i; \bar{k})).$$

Thus in order to know about the value $h_t(\bar{X}(\{f_i\}_i))$, one needs to study behaviour of $h_t(\bar{X}(\{f_i\}_i; \bar{k}))$.

1.B.2 Equivariant Bowen's theorem for Lipschitz fibration: Let (X, d) and (Y, d') be two compact metric spaces equipped with the shifts $\sigma : X \to X$ and $\sigma' : Y \to Y$. Suppose there is another compact metric space with a shift (Z, σ'') so that families of subsets $E_z \subset X$ and $F_z \subset Y$ are equivariantly assigned for each $z \in Z$:

$$\sigma(E_z) \subset E_{\sigma''(z)}, \quad \sigma'(F_z) \subset F_{\sigma''(z)}.$$

An *equivariant map* is a continuously parametrized map:

$$T_z : E_z \to F_z, \quad z \in Z$$

so that

$$\sigma'(T_z(m)) = T_{\sigma''(z)}(\sigma(m))$$

holds.

An equivariant map $T_z : E_z \to F_z$ is an *equivariant Lipschitz fibration*, if $T : E \to F$ is a fibration such that the family $\{T_z\}_{z\in Z}$ has a uniform Lipschitz constant with respect to z (the Lipschitz constants for T_z are uniformly bounded by a constant independently of z).

Lemma 1.4 *Let $T_z : E_z \to F_z$ be an equivariant Lipschitz fibration.*

Then for any small $\mu, \epsilon > 0$, there is some $\delta > 0$ so that for any $y \in F_z$ and $y' \in F_{z'}$ with $d(y, y') \leq \delta$, and any μ net $R_\mu \subset T_z^{-1}(y)$, there is an $\mu + \epsilon$ net $R_{\mu+\epsilon} \subset T_{z'}^{-1}(y')$ with the same cardinality:

$$\sharp R_\mu = \sharp R_{\mu+\epsilon}.$$

Our aim in this section is to verify the following:

Theorem 1.2 *Let $T_z : E_z \to F_z$ be an equivariant Lipschitz fibration. Then for each $z \in Z$, the inequalities hold:*

$$h_t(F_z) \leq h_t(E_z) \leq h_t(F_z) + \sup_{y\in F_z} h_t(T_z^{-1}(y)).$$

Proof: The proof is given by a modification of the one in [MS] $p165$.

We put $E = \cup_{z\in Z} E_z$, and $\bar{E} \subset X$ be its closure. F and $\bar{F} \subset Y$ are similar. For $x \in E_z$, we denote $x_i = \sigma^i(x) \in E_{(\sigma'')^i(z)}$.

Let $S \subset F_z$ be a maximal (n, ϵ) separated set, and choose $x \in T_z^{-1}(y) \subset E_z$ from each point $y \in S$. We denote the set of such points by $S' \subset E_z$. Because T_z is Lipschitz, S' is also $(n, C\epsilon)$ separated net for some C. Thus the first inequality $h_t(F_z) \leq h_t(E_z)$ holds.

Let $R_y = R_y(n, \epsilon)$ be a minimal (n, ϵ) net in $T_z^{-1}(y) \subset E_z$, and $r(n, \epsilon)$ be its cardinarity. Let us put $a = \sup_{y\in F_z} h_t(T_z^{-1}(y))$, and fix small $\epsilon, \alpha > 0$. Then there is a large $m(y)$ so that the inequalities hold:

$$a + \alpha \geq h_t(T_z^{-1}(y)) + \alpha \geq \frac{1}{m(y)} \log r(m(y), \epsilon).$$

Put $D_n(x, 2\epsilon) = \{x' \in \bar{E}_z : d(x_i, x'_i) < 2\epsilon, 0 \leq i \leq n\}$, and $U_y = \cup_{x \in R_y} D_{m(y)}(x, 2\epsilon)$. U_y contains $T_z^{-1}(y)$.

Let $\{W_{y_1}, \ldots, W_{y_q}\}$ be a finite cover of $\bar{F}_z$ so that $T^{-1}(W_{y_i}) \subset U_{y_i}$.

Let us put $M_z = \sup_i \{m(y_1), \ldots, m(y_q)\}$. Let $\delta_z > 0$ be a Lebesgue number, $B(y, \delta_z) \subset W_{y_i}$ for some $i = i(y)$.

We show that for each $z \in Z$, one can choose these points so that $\delta_z \geq \mu > 0$ and $M_z \leq C$ are uniformly bounded from below and above respectively, independently of $z \in Z$. Then the rest of the proof is parallel to [MS].

Let us put:

$$G_n(x, 2\epsilon) = \{x' \in \bar{E} : d(x_i, x'_i) < 2\epsilon, 0 \leq i \leq n\}$$

and $V_y = \cup_{x \in R_y} G_{m(y)}(x, 2\epsilon)$. V_y contains $T_z^{-1}(y)$.

Let $\{Y_{y_1}, \ldots, Y_{y_p}\}$ be a finite cover of $\bar{F}$ so that $T^{-1}(Y_{y_i}) \subset V_{y_i}$ for $y_i \in E_{z_i}$, where $T : \bar{E} \to \bar{F}$. Let $\delta > 0$ be a Lebesgue number, $B(y, \delta) \subset Y_{y_i}$ for some $i = i(y)$.

Firstly for $i = 1, \ldots, p$, we choose $R_{y_i}(m(y_i), \epsilon) \subset T^{-1}(y_i)$. For each $y' \in F$, choose i so that $y' \in Y_{y_i} \cap F_{z'}$ holds. Then using lemma 1.4, let us choose $R_{y'}(m(y_i), \epsilon') \subset T^{-1}(y')$ with $\sharp R_{y_i}(m(y_i), \epsilon) = \sharp R_{y'}(m(y_i), \epsilon')$. We choose all Y_{y_i} so small that $\epsilon' \leq 1.5\epsilon$ hold.

By this way for each $y \in F$, one has chosen $(m(y), 1.5\epsilon)$ net $R_y \subset T^{-1}(y)$ satisfying $a + \alpha \geq \frac{1}{m(y)} \log \sharp R_y$, where $\sup_y m(y) < \infty$. Re-choose a minimal $(m(y), 1.5\epsilon)$ net $R'_y \subset T^{-1}(y)$. Since $\sharp R'_y \leq \sharp R_y$ holds, still the inequality $a + \alpha \geq \frac{1}{m(y)} \log \sharp R'_y$ hold. Thus for each $y \in F$, we have chosen $m(y)$ among the finite set $\{m(y_1), \ldots, m(y_p)\}$. In particular their values are uniformly bounded.

For each $z \in Z$, we take finite sets of points $\{y_1^z, \ldots, y_p^z\} \subset F_z$ so that $y_i^z \in V_{y_i} \cap F_z \equiv W_i^z$. By the construction, still the Lebesgue number of the cover $\{W_1^z, \ldots, W_p^z\} \subset F_z$ is δ. This finishes the proof.

1.B.3 Slice entropy: Let us choose a family of maps $(f_1, \dots, f_a)$, $\bar{k} \in X_a$ and the corresponding family of maps $\{h^m\}_m$. Let us denote the regular set $R = R(f_1, \dots, f_a)$.

Recall $\bar{X}(\{f_i\}_i, \bar{k}) \subset X_a \times [0, 1]$ in $1.B$. This again admits another fibration as follows. Let us fix $\bar{l} \in X_a$. Then we put:

$$\bar{Y}(\{f_i\}_i, \bar{k}) = \{\Phi(\bar{k})(x); x \in R(\{f_i\}_i)\} \subset X_a,$$
$$\bar{Y}(\{f_i\}_i, \bar{k}, \bar{l}) = \{x \in R(\{f_i\}_i); \Phi(\bar{k})(x) = \bar{l}\} \subset [0, 1].$$

There are shift σ on both spaces as:

$$\sigma : \bar{Y}(\{f_i\}_i, \bar{k}) \to \bar{Y}(\{f_i\}_i, \sigma(\bar{k})),$$
$$\sigma : \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l}) \to \bar{Y}(\{f_i\}_i, \sigma(\bar{k}), \sigma(\bar{l}))$$

where the former is induced from the one on X_a, and the latter is given by $\sigma(x) = f_{k_0}(x)$.

Using this, one obtains the shift commuting Lipschitz surjection:

$$\bar{Y}(\{f_i\}_i, \bar{k}, \bar{l}) \hookrightarrow \bar{X}(\{f_i\}_i, \bar{k}) \to \bar{Y}(\{f_i\}_i, \bar{k}).$$

Lemma 1.5 *The inequality holds:*

$$h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k})) \leq h_t(\sigma; \bar{X}(\{f_i\}_i, \bar{k})).$$

Proof: This follows from the first part of the proof of theorem 1.2.

Definition 1.3 *A strange sequence with respect to $\bar{k}$ is $\{x_i\}_{i=0}^{\infty} \subset R(\{f_i\}_i)$ such that there is a positive $\epsilon > 0$ so that $\Phi(\bar{k})(x_i)$ converges to some $\bar{l}$ and a uniform estimate holds:*

$$d(x_i, \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})) \geq \epsilon$$

for all large i.

Remark: (1) In particular $\bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})$ is non empty.

(2) Let $\{h^m\}_m$ be the family of maps corresponding to $\bar{k}$. If $\{x_i\}_i$ is a strange sequence and a subsequence converges to some $x \in [0, 1]$, then $h^m(x) = \frac{i}{a}$ must be satisfied for some m and $i = 1, \dots, a - 1$.

Lemma 1.6 *Suppose there are no strange sequences for $\bar{k}$. Then*

$$T_{\bar{k}} : \bar{X}(\{f_i\}_i, \bar{k}) \to \bar{Y}(\{f_i\}_i, \bar{k})$$

$(\Phi(x)(\bar{k}), x) \to \Phi(x)(\bar{k})$ *is a fibration.*

Proof: By the above remark (1), $\bar{Y}(\{f_i\}_i, \bar{k})$ is closed and so compact. Then the conclusion follows from the definition of fibration $(1.B)$. This completes the proof.

By applying theorem 1.2, one obtains the following:

Corollary 1.3 *Suppose there are no strange sequences for $\bar{k}$. Then the inequalities hold:*

$$\begin{aligned} h_t(\bar{Y}(\sigma; \{f_i\}_i, \bar{k})) &\leq h_t(\sigma; \bar{X}(\{f_i\}_i, \bar{k})) \\ &\leq h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k})) + \sup_{\bar{l} \in X_a} h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})). \end{aligned}$$

Let $\{h^m\}_m$ be the family of maps corresponding to $\bar{k}$. $\bar{k}$ is called *monotone* with respect to $\{f_i\}_{i=1}^a$, if for each $\bar{l} \in X_a$, $h^m | \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})$ are monotone for all $m = 0, 1, \ldots$

Example: Let us put $I_1 = [0, \frac{1}{2})$ and $I_2 = (\frac{1}{2}, 1]$. We say that a map $f : [0,1] \to [0,1]$ is *half dividing*, if:

$$f(I_i) \subset I_j$$

hold for $i = 1, 2$ and $j = j(i)$.

Let f and g be both half dividing, piecewise monotone with their turning points only at most $\frac{1}{2}$.

Then each $\bar{k}$ is monotone with respect to (f, g).

Proposition 1.2 *Suppose $\bar{k}$ is monotone and there are no strange sequences for $\bar{k}$. Then the equality holds:*

$$h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k})) = h_t(\sigma; \bar{X}(\{f_i\}_i, \bar{k})).$$

Proof: It is enough to see $h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})) = 0$. Let $\{h^m\}_m$ be the family of maps corresponding to $\bar{k}$, and put $K = \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})$. $h^m | K$

are all monotone. Let $S_m(\epsilon) \subset h^m(K) \subset [0,1]$ be an ϵ net. Then

$$T_n = \cup_{m=0}^n (h^m)^{-1}(S_m(\epsilon))$$

is an (n, ϵ) net, and by monotonicity, its cardinarity is less that $C_\epsilon n$. So the result follows from the estimate:

$$h_t(\sigma; \bar{Y}(\{f_i\}_i, \bar{k}, \bar{l})) \leq \lim \sup_{n \to \infty} \frac{1}{n} \log(C_\epsilon n) = 0.$$

This completes the proof.

Corollary 1.4 *Suppose all $\bar{k}$ satisfy the above two conditions. Then the shift entropy satisfies an upper bound:*

$$h_t(\bar{\sigma}; \bar{X}(f_1, \dots, f_a)) \leq 2 \log a.$$

Proof: Since $\bar{Y}(\{f_i\}_i, \bar{k}) \subset X_a$, the topological entropy satisfies the estimate $h_t(\bar{Y}(\{f_i\}_i, \bar{k})) \leq h_t(X_a) = \log a$. Thus combining with cor 1.2, one obtains the desired estimate. This completes the proof.

1.C Cell type equations: So far we have considered dynamics of the form:

$$\Phi : [0,1] \times Y^{\mathbf{N}} \to [0,1] \times Y^{\mathbf{N}} \quad (*)$$

for some topological spaces Y, which induces families of maps by restriction:

$$\Phi_M : [0,1] \times Y^M \to [0,1] \times Y^M \quad (**)$$

where $M = 1, 2, \dots$

Cell type equations below can be also expressed as $(*)$ above, but they cannot admit such restriction as $(**)$. Such phenomena becomes important in $1.C.3$ where one considers pointwise topological entropy.

The discrete Lotka Volterra equation is give by the following ([HT]):

$$\frac{V_n^{t+1}}{V_n^t} = \frac{1 + \delta V_{n+1}^t}{1 + \delta V_{n-1}^{t+1}}.$$

Let us take a large $L > 0$, and put:

$$S_L = \{\{V_n^t\}_{n,t} : \text{ solutions of } (*), 0 \leq V_n^t \leq L\}.$$

It is a compact subset in $[0, L]^{\mathbf{N}}$.

Recall that the topological entropy of orbit spaces is defined on subsets in $[0, L]^{\mathbf{N}}$ in $1.A$. The *topological entropy of the discrete Lotka Volterra* is given by:

$$h_t(dLV) = \lim \sup_{L \to \infty} h_t(S_L).$$

1.C.2 From discrete to ultra discrete Lotka Volterra: By change of variables as $V_n^t = \exp(\frac{v_n^t}{\epsilon})$ and $\delta = \exp(-\frac{L_0}{\epsilon})$, one can rewrite it as:

$$v_n^{t+1} - v_n^t = \epsilon \log(1 + \exp(\frac{v_{n+1}^t - L_0}{\epsilon})) - \epsilon \log(1 + \exp(\frac{v_{n-1}^{t+1} - L_0}{\epsilon})) \quad (*).$$

By letting $\epsilon \to 0$, one obtains the Lotka Volterra cell automaton:

$$v_n^{t+1} - v_n^t = \max\{0, v_{n+1}^t - L_0\} - \max\{0, v_{n-1}^{t+1} - L_0\}.$$

For a large $L >> 0$, we put:

$$T_L = \{\{v_n^t\}_{n,t} : \text{ solutions of } (*), v_n^t \in \{0, \dots, L\}\}.$$

It is a compact subset in $X_{L+1}^{\mathbf{N}}$.

The *topological entropy of the Lotka Volterra cell automaton* is given by:

$$h_t(udLV) = \lim \sup_{L \to \infty} h_t(T_L).$$

Question: What are the relations between these entropies:

$$h_t(dLV), \quad h_t(udLV).$$

Recall that the discrete Lotka Volterra map is parametrized by ϵ. Does it follow:

$$\lim_{\epsilon \to 0} h_t(dLV) = h_t(udLV).$$

1.C.3 Pointwise topological entropies: In order to express geometric cell automata including Lotka-Volterra cell automaton and box and ball system by use of families of maps, one has generalized interaction of maps in [K3], [K4]. Here we study topological entropies for the dynamics of the corresponding interaction pointwisely.

The LV cell automaton admits the following type of the structure. For simplicity we only use two maps. General cases are similar.

Let:

$$J : \{0,1\}^2 \to \{0,1\}$$

be a map, and take two maps f and g on $[0,1]$. Then one obtains a famliy of maps $\{h^m\}_m$ inductively by:

$$h^{m+1} = d_{J(k_{m+1},k_{m+2})} \circ h^m$$

where $d_i = f$ or g as in the introduction. We also call the family as the interaction with respect to (f, g, J, x).

Then by the same way as before one obtains a dynamics:

$$\Phi(x) : X_2 \to X_2, \quad \Phi(x)(\bar{k}) = (\pi(x), \pi(h^1(x)), \dots).$$

We call this interaction map *cell type*.

Let us take $\bar{k} = (k_0, k_1, \dots) \in X_2$ and put $\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots)$, $t = 0, 1, \dots$ Notice the following:

Lemma 1.7 *k_i^t is determined by $(k_0, \dots, k_{i+t})$.*

Now we put the *pointwise topological entropy* for (f, g, J, x) by:

$$h_t(f, g, J, x) \equiv h_t(\Phi(x)).$$

When $J(k_m, k_{m+1}) = k_m$ holds, this is the interaction we have considered in $1.A$, and $h_t(f, g, J, x) = 0$ holds by lemma 1.3.

Let us consider the cell type interaction.

Lemma 1.8 *The estimate holds:*

$$h_t(f, g, J, x) \leq \log 2.$$

Proof: Let us put $\Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots)$, $t = 0, 1, \dots$ Then k_m^t is determined by $(k_0, \dots, k_{m+t})$ for any t. Thus as in the proof of lemma 1.3, at most 2^{m+t} elements spans the largest $(t, \frac{1}{2^m})$ separated net for all t. Thus the topological entropy satisfies the estimate:

$$h_t(\Phi(x)) \leq \lim_{t\to\infty} \frac{1}{t} \log 2^{t+m} = \log 2.$$

This completes the proof.

The above estimate is optimal.

Let us say that a pair of maps (f, g) is *opposite*, if for a.e. $x \in [0, 1]$,

$$\pi(f(x)) \neq \pi(g(x)) \in \{0, 1\}$$

hold.

For example, $f(x) = 1 - x$ and $g(x) = x$ is an opposite pair.

Lemma 1.9 *Let f and g be opposite, and J satisfies $J(a, 0) = a$ and $J(a, 1) = a + 1$ for $a = 0, 1$ mod 2.*

Then for the corresponding interaction Φ and any $x \in R(f, g) \subset [0, 1]$,

$$h_t(f, g, J, x) = \log 2$$

hold.

Proof: Recall the projections in $1.A$:

$$\bar{\pi}_{n,t} : X_2^{\mathbf{N}} \to (X_2)_n^{t+1}, \quad \{a_i^s\}_{i,s \in \mathbf{N}} \to \{a_i^s\}_{0 \leq i \leq n}^{0 \leq s \leq t}$$

By the condition, it is easy to check that any element $\{a_i^s\}_{0 \leq i \leq n}^{0 \leq s \leq t} \in (X_2)_{n+1}^{t+1}$ is uniquely determined by the initial sequence $(k_0, \ldots, k_{n+t})$.

The number of $\{(k_0, \ldots, k_{n+t})\}$ is 2^{n+t+1}. Thus as in the proof of lemma 1.4, $h_t(\Phi(x)) = \lim_{t \to \infty} \frac{1}{t} \log 2^{n+t+1} = \log 2$ holds.

This completes the proof.

Example: Let f be the tent map and g be its reverse. Recall that for the non cell type interaction, the topological entropy is equal to $h_t(f, g) = \log 2$ $(1.A.2)$.

The pair (f, g) is opposite. Let us choose J as above. Then one has the generalized interaction map:

$$\Phi(f, g, J) : [0, 1] \times X_2 \to [0, 1] \times X_2.$$

Lemma 1.10 *The topological entropy satisfies:*

$$h_t(f, g, J) \equiv h_t(\Phi(f, g, J)) = \infty.$$

Proof: Let $\mathbf{L}(f,g,J) \subset X_2^{\mathbf{N}}$ be the orbit space of the interaction. Then the above proof shows that it coincides with $X_2^{\mathbf{N}}$. Thus by prop 1.1,

$$\begin{aligned} h_t(f,g,J) &= \lim_n \limsup_t \frac{1}{t+1} \log \sharp \bar{\pi}_{n,t} \mathbf{L}(f,g,J) \\ &= \lim_n \limsup_t \frac{1}{t+1} \log 2^{(n+1)(t+1)} = \infty. \end{aligned}$$

This completes the proof.

Let us fix $x \in [0,1]$ and L_0, L. In [K3], we have seen that the Lotka Volterra cell automaton:

$$v_n^{t+1} - v_n^t = \max(L_0, v_{n+1}^t) - \max(L_0, v_{n-1}^{t+1})$$

can be represented by a cell type interaction from a family of piecewise linear maps. In general the images of the flow are not bounded, and so one has to denote it as:

$$\Phi(x) : X_{L+1} \to X_\infty, \quad \Phi(x)^t(\bar{k}) = (k_0^t, k_1^t, \dots).$$

By a parallel argument as above, one may obtain some estimate of the value of the pointwise topological entropy of the LV cell automaton from below.

2 Vertical slices and amalgamation

Let us take a family of maps $\{f_1, \dots, f_a\}$ and consider the corresponding interaction map $\Phi : X_a \times [0,1] \to X_a \times [0,1]$ with domain $R = R(f_1, \dots, f_a)$. In this section we study its holizontalwise restrictions as:

$$\Phi_n : [0,1] \times (X_a)_n \to [0,1] \times (X_a)_n$$

for $n = 1, 2, \dots$ where:

$$(X_a)_n = \{0, \dots, a\} \times \dots \times \{0, \dots, a\} = \{(k_0, \dots, k_n) : k_i \in \{0, \dots, a\}\}$$

is all the set of words of length $n+1$.

Such restriction is impossible for cell type interactions $(1.C)$, and here we will only consider non-cell type.

The *topological entropy* at n *step* is given by:

$$h_t(\Phi_n) \equiv h_t(\Phi_n; R \times X_a).$$

Proposition 2.1 *Suppose the value of the topological entropy of the interaction by $\{f_1, \dots, f_a\}$ is finite. Then the following holds:*

$$h_t(f_1, \dots, f_a) \equiv h_t(\Phi) = \lim_{n \to \infty} \sup h_t(\Phi_n).$$

Proof: Let us denote elements $\bar{a} = (a_0, \dots, a_n)$. We restrict the metric on $(X_a)_n$ as $|\bar{a} - \bar{b}| = \Sigma_{i=0}^n \frac{|a_i - b_i|}{2^i}$.

Let $S(\Phi)(n, \epsilon) = \{(x_1, \bar{k}^1), \dots, (x_l, \bar{k}^l)\}$ be a largest (n, ϵ) separated net of Φ. We denote $\bar{k}^i = (k_0^i, k_1^i, \dots)$, and its restriction by $\bar{k}_n^i = (k_0^i, k_1^i, \dots, k_n^i) \in (X_a)_n$ for $i = 1, \dots, l$.

Clearly the estimate

$$\sharp S(\Phi)(n, \epsilon) \geq \sharp S(\Phi_m)(n, \epsilon)$$

holds. Thus $h_t(\Phi) \geq h_t(\Phi_m)$ hold for all $m = 1, 2, \dots$

Conversely let $S(\Phi)(n, \epsilon) = \{(x_1, \bar{k}^1), \dots, (x_l, \bar{k}^l)\}$ be a largest (n, ϵ) separated net of Φ. Then for large $m = m(\epsilon)$, $\{(x_1, \bar{k}_m^1), \dots, (x_l, \bar{k}_m^l)\}$ is an $(n, \frac{\epsilon}{2})$ separated net. Thus one obtains the inequality:

$$\sharp S(\Phi)(n, \epsilon) \leq \sharp S(\Phi_m)(n, \frac{\epsilon}{2}).$$

Let us denote $s(\Phi)(n, \epsilon) = \sharp S(\Phi)(n, \epsilon)$. By definition, this number is independent of chioice of $S(\Phi)(n, \epsilon)$. Moreover $s(\Phi_m)(n, \epsilon)$ increase with respect to m. At the same time it is decreasing with respect to ϵ. Thus from the estimate:

$$\begin{aligned} h_t(\Phi)(\epsilon) &\equiv \limsup_n \frac{1}{n} \log s(\Phi)(n, \epsilon) \\ &\leq h_t(\Phi_m)(\frac{\epsilon}{2}) \leq \lim_{m \to \infty} h_t(\Phi_m)(\frac{\epsilon}{2}) \equiv \tilde{h}_t(\Phi)(\frac{\epsilon}{2}) \end{aligned}$$

one obtains the inequality:

$$h_t(\Phi) \leq \lim_{\epsilon \to 0} \tilde{h}_t(\Phi)(\epsilon) \equiv \tilde{h}_t(\Phi).$$

Since $h_t(\Phi_m)(\epsilon)$ is decreasing for ϵ and increasing for m, the limits exchange:

$$\tilde{h}_t(\Phi) = \lim_{\epsilon \to 0} \lim_{m \to \infty} h_t(\Phi_m)(\epsilon) = \lim_{m \to \infty} \lim_{\epsilon \to 0} h_t(\Phi_m)(\epsilon) = \lim \sup_{m \to \infty} h_t(\Phi_m).$$

Thus the reverse estimate also holds. This completes the proof.

Definition 2.1 *The interaction by $\{f_1, \dots, f_a\}$ is of finite type, if the equality holds for all large $m \geq m_0$:*

$$h_t(f_1, \dots, f_a) \equiv h_t(\Phi) = h_t(\Phi_m).$$

In general $h_t(\Phi) \geq h_t(\Phi_m)$ holds.

Examples: (1) Let $I_0 = [0, \frac{1}{2})$ and $I_1 = (\frac{1}{2}, 1]$, and suppose f and g satisfy both $f(I_i) \subset I_j$ and $g(I_k) \subset I_m$ for some $j = j(i), m = m(k)$ (e.g., $f(x) = 1 - x$ and $g(x) = x$). Then (f, g) is of finite type. In fact $h_t(\Phi_n) = h_t(\Phi) = 0$ hold.

(2) Let f be the tent map and $g = 1 - f$ be its reverse. Then (f, g) is of finite type. In fact $h_t(\Phi_n) = h_t(\Phi) = \log 2$ hold for $n = 0, 1, 2, \dots$

2.B Factor maps: Let us take a family of maps $\{f_1, \dots, f_a\}$, and $R(f_1, \dots, f_a) \subset [0, 1]$ be the regular set. Let us put:

$$O_n = \{(\bar{x}_n^l)_{l \geq 0} : \Phi(x)^l(\bar{x}_n^0) = \bar{x}_n^l, x \in R(\{f_i\}_i)\} \subset (X_a)_n^{\mathbf{N}}$$

and its *shift* σ by:

$$\sigma((\bar{x}_n^0, \bar{x}_n^1, \bar{x}_n^2, \dots)) = (\bar{x}_n^1, \bar{x}_n^2, \dots).$$

O_n is a σ invariant subset.

The *total subshifts* Y_n is defined by:

$$Y_n \equiv \{(\bar{x}_n^l)_{l \geq 0} : \Phi(x_l)(\bar{x}_n^l) = \bar{x}_n^{l+1}, x_l \in R(\{f_i\}_i), l = 0, 1, \dots\}.$$

It is a subshift of finite type, and (O_n, σ) is a subshift of (Y_n, σ).

Question: Suppose O_n is closed in $(X_a)_n^{\mathbf{N}}$. When (O_n, σ) is a subshift of finite type ?

Clearly for the above example (1), (O_n, σ) is a subshift of finite type.

Let us consider the family of subshifts:

$$\{(O_n, \sigma)\}_{n=1,2,\dots}$$

as above. By the construction, one obtains a family of *factor maps* (shift commuting surjection):

$$\pi_n; (O_n, \sigma) \to (O_{n-1}, \sigma), \quad \bar{x}_n^l = (x_0^l, \dots, x_n^l) \to \bar{x}_{n-1}^l = (x_0^l, \dots, x_{n-1}^l)$$

by projection. It also induces the factor maps $\pi_n : (Y_n, \sigma) \to (Y_{n-1}, \sigma)$.

We say that $(\bar{x}_n^l)_{l\geq 0}$ and $(\bar{y}_n^l)_{l\geq 0}$ form a *diamond* of length $k > 0$, if there is some i so that
(1) $\pi_n(\{x_0^l, \dots, x_n^l\}_{l=i}^{i+k}) = \pi_n(\{y_0^l, \dots, y_n^l\}_{l=i}^{i+k})$,
(2) $\{x_0^l, \dots, x_n^l\}_{l=i}^{i+k} \neq \{y_0^l, \dots, y_n^l\}_{l=i}^{i+k}$, and
(3) $\{x_0^i, \dots, x_n^i\} = \{y_0^i, \dots, y_n^i\}$ and $\{x_0^{i+k}, \dots, x_n^{i+k}\} = \{y_0^{i+k}, \dots y_n^{i+k}\}$
hold.

Proposition 2.2 *(1) Let W_n be O_n or Y_n. If W_n does not contain any diamonds, then π_n is exactly a to* 1 *map. In particular $h_t(W_n, \sigma) = h_t(W_{n-1}, \sigma)$ holds.*
(2) If Y_n has a diamond, then π_n is uncountable to one at some point. Moreover if Y_n is irreducible, then the strict inequality $h_t(Y_n, \sigma) > h_t(Y_{n-1}, \sigma)$ holds.

The proof is given in [Ki] $p98$.

Example: Let f be the tent map and g be its reverse. Then for $n \geq 2$, O_n does not contain any diamonds, π_n is two to one map and $h_t(O_n) = \log 2$.
Y_n is the full shift $(X_2)_n$. Thus $h_t(Y_n) = (n+1)\log 2$.

2.C Amalgamation: In general it will not so easy to compare values of topological entropies for two symbolic dynamics with different alphabets. An amalgamation is an operation to decrease the numbers of alphabets, which gives topologically conjugate pairs. In particular they give the same values of the topological entropy.

Below we study the amalgamation of total subshifts. For any word $w \in (X_a)_n$, let $f(w) \subset (X_a)_n$ be a subset so that any $v \in f(w)$ satisfies $\Phi(x)(w) = v$ for some $x \in R(\{f_i\}_i)$. Similarly $p(w) \subset (X_a)_n$ be a subset so that any $u \in p(w)$ satisfies $\Phi(x)(u) = w$ for some $x \in R(\{f_i\}_i)$. $p(w)$ is called the *predecessor set* and $f(w)$ the *follower set*.

Let us consider Y_{n+1}. For a pair $\{w, w'\} \subset (X_a)_{n+1}$ and $v \in (X_a)_n$, we construct another symbolic dynamics $Y(w, w')$, whose alphabets are $[(X_a)_{n+1} \backslash \{w, w'\}] \cup \{v\}$.

Suppose that $w, w' \in (X_a)_{n+1}$ satisfies the followings:

$$p([w]) = p([w']), \quad f([w]) \cap f([w']) = \phi.$$

Then we define a new shift $Y(w, w')$ by the followings. Let us denote by f and f' the follower sets for Y_{n+1} and $Y(w, w')$ respectively. Then:

(1) for $x \neq v, w, w'$, $f(x) = f'(x)$,
(2) for $x \neq v$ and $w, w' \in f(x)$, then $f'(x) = [f(x) \backslash \{w, w'\}] \cup \{v\}$,
(3) if $w, w' \neq f(w) \cup f(w')$, then $f'(v) = f(w) \cup f(w')$, and
(4) if $w, w' \in f(w) \cup f(w')$, then $f'(v) = [f(w) \cup f(w') \backslash \{w, w'\}] \cup \{v\}$.

Thus by this way one has obtained a new symbolic dynamics $Y(w, w')$ which is called the *amalgamation*. We say that (w, w') is an amalgamated pair.

These are topologically conjugate:

$$(Y_{n+1}, \sigma) \cong (Y(w, w'), \sigma).$$

Suppose there is an involution $\tau : (X_a)_{n+1} \to (X_a)_{n+1}$ without any fixed points. Then it gives a division of the set into pairs:

$$\{(w_1, \tau(w^1)), \dots, (w_{2^n}, \tau(w_{2^n}))\}.$$

We say that the involution induces an amalgamation, if each $(w_i, \tau(w_i))$ is an amalgamated pair.

When an involution induces an amalgamation, then the succesive procedure above gives a symbolic dynamics Y' with a^n number of alphabets, and one can assign each alphabet by an element in $(X_a)_n$.

If one can choose an assignment of the alphabets so that the corresponding Y' is the same as Y_n, then we say that it is a *reduction* at $n+1$-stage. So when one obtains reduction at $n+1$ stage, then Y_{n+1} and Y_n are topologically conjugate. Thus:

Lemma 2.1 *If there is a reduction at $n+1$ stage, then (Y_{n+1}, σ) and (Y_n, σ) are mutually topologically conjugate.*

In particular the topological entropies are mutually equal:

$$h_t(\sigma; Y_{n+1}) = h_t(\sigma; Y_n).$$

Example: Let f and g satisfy the followings:

$$\frac{1}{2} < f(x) < 1, \quad 0 < x < \frac{1}{2}, \quad 0 < f(x) < \frac{1}{2}, \quad \frac{1}{2} < x < 1,$$
$$0 < g(x) < \frac{1}{2}, \quad 0 < x < \frac{1}{2}, \quad \frac{1}{2} < g(x) < 1, \quad \frac{1}{2} < x < 1.$$

Then each Y_{n+1} admits an reduction for $n \geq 1$. For example, Y_2 can be described as:

$$(00, 10) \to (01, 10), \quad (01, 11) \to (00, 11).$$

Thus $(X_2)_2$ has amalgamated pairs

$$(01, 10), \quad (00, 11).$$

Then letting:

$$\{10, 01\} \to 0, \quad \{00, 11\} \to 1$$

one obtains $Y' = Y_1$.

For Y_3, one has the following table:

	000	001	010	011	100	101	10	111
$(0, \frac{1}{2})$	101	100	110	111	010	011	001	000
$(\frac{1}{2}, 1)$	010	011	001	000	101	100	110	111
	01	00	11	10	01	00	11	10

where each vertical pair at the second and the third line is an amalgamated pair. The last line gives an assignment to $(X_2)_2$. For example, $\{101, 010\}$ is assigned by 01, and so on. By this way one obtains $Y' = Y_2$.

2.C.2 Amalgamation of maps: The symbolic dynamics Y_{n+1} are obtained from famililies of maps. Here we will consider deformations of families of maps which induce amalgamation of the symbolic dynamics. In the case the families pass through some critial point at which the dynamics change. We call the family at the point *threshold*.

Let us take a family of maps $\{f_1, \dots, f_l\}$ and denote the projection $\pi : [0,1] \to \{1, \dots, l\}$ by $\pi(I_i) = i$ where $I_{i+1} = (\frac{i}{l}, \frac{i+1}{l})$, $i = 0, \dots, l-1$. One obtains the total shifts of finite type Y_n and the subshifts $O_n \subset Y_n$ of orbit spaces.

Let $\{f_{1_1}, f_{1_2}, f_2, \dots, f_l\}$ be a family of $l+1$ maps and $\Phi(x) : X_{l+1} \to X_{l+1}$ be the corresponding interaction maps, where we have division of the intervals as $\{I_{1_1}, I_{1_2}, I_2, \dots, I_l\} \subset [0,1]$ with $I_{i_1} \cup I_{i_2} \cup p_1 = I_1$, $p_1 = \frac{1}{l+1}$.

We will consider to make amalgamation by contracting two maps into one. Here we will only consider the case $n = 0$. In this case amalgamation has rather combinatorial nature. If one considers higher steps $n \geq 1$, then subtle analytic properties of maps will reflect.

Notice the inclusions:

$$O_1 \subset Y_1 \subset X_{l+1}.$$

We denote $p(j) = \{k : f_k(x) \in I_j \text{ for some } x \in R(f_1, \dots, f_l)\}$ and $f(j) = \{l : f_j(x) \in I_l \text{ for some } x \in R(f_1, \dots, f_l)\}$.

Suppose the following two conditions:

(1) $p(1_1) = p(1_2)$ and (2) $f(1_1) \cap f(1_2) = \phi$.

Then by amalgamation, there is $Y_1' \subset X_l$ with the alphabets $\{1, 2, \dots, l\}$ by changing both 1_j, $j = 1, 2$ to 1. This gives a topological conjugate:

$$(Y_1, \sigma) \cong (Y_1', \sigma).$$

When there is another map f_1 so that Y_1 corresponding to $\{f_1, \dots, f_l\}$

is the same as Y_1', then we say that $\bar{f}_2 \equiv \{f_1, f_2, \ldots, f_l\}$ is an *amalgam* of $\bar{f}_1 \equiv \{f_{1_1}, f_{1_2}, f_2, \ldots, f_l\}$.

Let us say that an amalgam $\bar{f}_2$ of $\bar{f}_1$ as above have a *threshold*, if there are continuous families of maps $(f_{1_1}^t, f_{1_2}^t)$ and f_i^t which satisfy $f_j^0 = f_j$ for $0 \leq t \leq 1$, $1 \leq i \leq l$, and some $0 < s < 1$ so that

$$(1) \quad f_{1_1}^1|[0,s] = f_1^1|[0,s], \quad f_{1_2}^1|[s,1] = f_1^1|[s,1],$$

(2) the interaction maps for both $\{f_{1_1}^t, f_{1_2}^t, f_2^t, \ldots, f_l^t\}$ and $\{f_1^t, f_2^t, \ldots, f_l^t\})$ are independent of t respectively for all $0 \leq t < 1$, and

(3) $\{f_1^t, f_2^t, \ldots, f_l^t\})$ is an amalgam of $\{f_{1_1}^t, f_{1_2}^t, f_2^t, \ldots, f_l^t\}$ for each $0 \leq t < 1$.

Example: Let us consider two diagrams:

$$A_1: \begin{array}{ccccc} (1_1) & \longleftrightarrow & (3) & & \\ & \swarrow & & \nwarrow & \\ (1_2) & \longrightarrow & (4) & \longleftrightarrow & (2) \end{array} \qquad A_2: \begin{array}{ccc} (1) & \longleftrightarrow & (3) \\ \downarrow & & \uparrow \\ (4) & \longleftrightarrow & (2) \end{array}$$

Each A_i corresponds to a subshift of finite type Y_1 and Y_1', where $Y_1 \subset X_5$ and $Y_1' \subset X_4$. Y_1' can be amalgamated to Y_1 by contracting $\{1_1, 1_2\}$ to 1.

These are constructed by interaction maps using families of maps $\{f_{1_1}, f_{1_2}, f_2, f_3, f_4\}$ and $\{f_1, f_2, f_3, f_4\}$ respectively, where we will use two projections $\pi_1 : [0,1] \to \{1, \ldots, 5\}$ and $\pi_2 : [0,1] \to \{1, \ldots, 4\}$ by:

$$\pi_j((\frac{i}{5}, \frac{i+1}{5})) = i, \quad i = 2,3,4, j = 1,2,$$
$$\pi_1(\frac{i-1}{5}, \frac{i}{5})) = 1_i, \quad i = 1,2, \quad \pi_2(0, \frac{2}{5})) = 1.$$

Then we have families of maps satisfying:

$$\frac{3}{5} < f_{1_1} < \frac{4}{5}, \quad \frac{4}{5} < f_{1_2} < 1,$$
$$\frac{3}{5} < f_1 < 1, \quad f_1(0) < \frac{4}{5} < f_1(1), \quad \frac{3}{5} < f_2 < 1, \quad f_2(0) < \frac{4}{5} < f_2(1),$$
$$0 < f_3 < \frac{2}{5}, \quad f_3(0) < \frac{1}{5} < f_3(1), \quad \frac{2}{5} < f_4 < \frac{3}{5}.$$

These give interaction maps whose total shifts at the stage 1 are Y_1 and Y_1' respectively.

It is easy to obtain parametrizations of maps $\{f_{1_1}^t, f_{1_2}^t, f_2^t, f_3^t, f_4^t\}$ and $\{f_1^t, f_2^t, f_3^t, f_4^t\}$ respectively, so that for all $0 \leq t < 1$, they satisfy the above properties, where at $t = 1$, $\{f_{1_1}^1|[0, \frac{1}{2}], f_{1_2}^1|[\frac{1}{2}, 1]\}$ and f_1^1 coincide mutually. Thus $\{f_{1_1}, f_{1_2}, f_2, f_3, f_4\}$ and $\{f_1, f_2, f_3, f_4\}$ have threshold mutually.

3 Block maps

Let $t = 1, 2, 3, \dots$, X_t be the one sided full shift, and X_t^0 be all the set of words of finite lengths.

Let $s \geq 1, m, r \geq 0$ be integers, and $\varphi_0 : (X_a)_{m+r+1} \to X_t^0$ be a map. The corresponding (r, m, s)-*block map*:

$$\varphi : X_a \to X_t, (k_0, k_1, \dots) \to (k_0', k_1', \dots)$$

is given by:

$$k_i' = \varphi_0(k_{si-r}, k_{si-r+1}, \dots, k_{si+m}).$$

In this section we will consider the following cases of maps of the form:

(1) $(r, m, 1)$-block maps are called just block maps in symbolic dynamics.

(2) $(0, m-1, m)$-block maps are called *codings*, and we will study such maps in later sections. In particular we study *contractions* which will be given by contractions of families of maps by codings.

(3) Let Σ_a be the both sided full shift with the alphabets $\{1, \dots, a\}$. Then by the same way one can obtain (r, m, s)-block maps $\varphi : \Sigma_a \to \Sigma_t$ from φ_0.

(4) Let us denote:

$$X_a^\vee = \{(\dots, k_{-n}, k_{-n+1}, \dots, k_0) : k_j \in \{1, \dots, a\}\}$$

and consider a continuous map:

$$\varphi_0 : X_a^\vee \to \{1, \dots, t\}.$$

Then one can also obtain the corresponding $(-\infty, 0, s)$-block map $\varphi : \Sigma_a \to \Sigma_t$. A $(-\infty, 0, 1)$-block map will be used below in order to treat some classes of infinite integrable systems.

Block maps are general in the sense of the following:

Lemma 3.1 *Let $\varphi : X_a \to X_b$ be a continuous map which satisfies quasi-commututativity with the shifts as:*

$$\varphi(\sigma^n(\bar{k})) = \sigma(\varphi(\bar{k})).$$

Then it is an $(0, m, n)$ block map for some m.

See [Ki] $p26$.

3.B Transformations on interaction of maps: Let $\{f_1, \dots, f_l\}$ and $\{g_1, \dots, g_m\}$ be two families of maps, and $\Phi(\{f_i\}_i) : [0,1] \times X_l \to [0,1] \times X_l$ and $\Phi(\{g_j\}_j) : [0,1] \times X_m \to [0,1] \times X_m$ be the corresponding interaction maps respectively.

Let $Y_l \subset X_l$ and $Y_m \subset X_m$ be subsets which are both invariant under the iterations of the interaction maps respectively. For example one may take $Y_l = \cup_{x \in R(f_1,\dots,f_l)} \Phi(x)(X_l)$.

Let us take another family of maps $\{\alpha_a\}_{a=1}^l$. Then using the projection $\pi : [0,1] \to \{1, \dots, m\}$, one obtains the corresponding interaction maps $\Phi(\{\alpha_a\}_a) : [0,1] \times X_l \to [0,1] \times X_m$.

We say that $\Phi(\{f_i\}_i) : [0,1] \times Y_l \to [0,1] \times Y_l$ and $\Phi(\{g_j\}_j) : [0,1] \times Y_m \to [0,1] \times Y_m$ are mutually *equivalent*, if there is a family of maps $\{\alpha_a\}_{a=1}^l$ so that on $R \equiv R(f_a, \dots, f_l) \cap R(g_1, \dots, g_m) \subset [0,1]$,

(1) the corresponding interaction maps admits the isomorphic restriction as $\Phi(\{\alpha_a\}_a) : R \times Y_l \cong R \times Y_m$ and

(2) the following diagram commutes for all $x \in R$:

$$\begin{array}{ccc} Y_l & \xrightarrow{\Phi(\{\alpha_a\}_a(x))} & Y_m \\ \Phi(\{f_i\}_i)(x) \downarrow & & \Phi(\{g_j\}_j)(x) \downarrow \\ Y_l & \xrightarrow{\Phi(\{\alpha_a\}_a(x))} & Y_m \end{array}$$

The following is clear:

Lemma 3.2 *If $\Phi(\{f_i\}_i)|[0,1] \times Y_l$ and $\Phi(\{g_j\}_j) : [0,1] \times Y_m$ are mutually equivalent, then they have the same topological entropy:*

$$h_t(\Phi(\{f_i\}_i)|R \times Y_l) = h_t(\Phi(\{g_j\}_j) : R \times Y_m).$$

3.B.2 Box and ball system: The box and ball system is a dynamics on Σ_2 which has arose in mathematical physics. It can be expressed as a modified interaction of maps ([K4]).

We recall the BBS as follows.

Let $\sigma = (\dots, v_{-n}, \dots, v_0, v_1, \dots) \in \Sigma_2$ be a bi-infinite sequence by $\{0,1\}$ such that for all sufficiently large $n >> 0$, $v_n = v_{-n} = 0$. Let us denote the set of such sequences by $\Sigma_2^0 \subset \Sigma_2$. It is shift invariant.

Let us choose an element $\sigma \in \Sigma_2^0$ and $(i_1 < i_2 < \cdots < i_m)$ be all the indices with $v_{i_l} = 1$ in σ. Notice that $\sigma \in |Sigma_2^0$ is uniquely determined by the index sets of 1 above. Let $T(\sigma)_1 = (\dots, v_{-m}^1, \dots, v_0^1, v_1^1, \dots) \in \Sigma_2^0$ be another element defined as follows; let $j_1 \geq i_1$ be the smallest index with the property that it is larger than i_1 and $v_{j_1} = 0$. Then $v_l^1 = v_l$ except $l = i_1$ and j_1, and we put $v_{i_1}^1 = 0$ and $v_{j_1}^1 = 1$.

Next we do the same thing for $v_{i_2}^1 = v_{i_2}$ in $T(\sigma)_1$, and find another smallest index $j_2 \geq i_2$ with $v_{j_2}^1 = 0$. Then we exchange 0 and 1 in $v_{i_2}^1$ and $v_{j_2}^1$ as above. The result is denoted by $T(\sigma)_2$.

We continue this process for $i_3, i_4, \dots$ until i_m, and finally one obtains the desired $T(\sigma) \equiv T(\sigma)_m \in \Sigma_2^0$.

Thus one has obtained a continuous bijective map:

$$T : \Sigma_2^0 \cong \Sigma_2^0$$

which is calld the *box and ball system* (BBS).

The topological entropy of BBS is given by:

$$h_t(\text{BBS}) = h_t(T; \Sigma_2^0).$$

Let $\Sigma_2^0(N) \subset \Sigma_2^0$ be the subset whose elements are consisted by sequences with exactly N number of 1. Then the BBS system induces a bijection $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$.

Let $\pi_0 : \Sigma^2 \to \{0,1\}$, $\pi_0(\{k_i\}_{i\in\mathbf{Z}}) = k_0$ be the projection. It induces:

$$\pi_0 : (\Sigma^2)^{\mathbf{N}} \to X_2$$

by $\pi_0(\{k_i^t\}_{i\in\mathbf{Z},t=0,1,\dots}) = \{k_0^t\}_{t=0,1,\dots}$.

Let $\mathbf{L}(BBS) = \{(\bar{k}, T(\bar{k}), T^2(\bar{k}), \dots); \bar{k} \in \Sigma^2\}$ be the orbit spaces of BBS, and denote $\mathbf{L}(BBS)_0 \equiv \pi_0(\mathbf{L}(BBS)_0) \subset X_2$.

Lemma 3.3 $\mathbf{L}(BBS)_0$ *is the subshift of finite type with the transition matrix*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

In particular one obtains a rough estimate of the topological entropy:

$$h_t(BBS) \geq \log \frac{1+\sqrt{5}}{2}.$$

The latter follows from the Perron Frobenius theory.

3.B.3 Transformation from BBS to LV: The BBS can be transformed to Lotka Volterra cell automaton by change of variables as follows.

The BBS and the Lotka Volterra cell automaton are given by the equations respectively:

$$B_n^{t+1} = \min\{1 - B_n^t, \Sigma_{i=-\infty}^{n-1}(B_i^t - B_i^{t+1})\}$$
$$V_n^{t+1} - V_n^t = \max\{L, V_{n+1}^t\} - \max\{L, V_{n-1}^{t+1}\}$$

A transformation between them is given by the succesive three procedures:

$S_{n+1}^{t+1} - S_n^t = \min\{0, 1 - S_{n+1}^t + S_n^{t+1}\}, \quad S_n^t = \Sigma_{i=-\infty}^n B_i^t,$
$U_{n+1}^{t+1} - U_n^t = \max\{0, U_n^{t+1} - 1\} - \max\{0, U_{n+1}^t - 1\}, \quad U_n^t = S_{n+1}^t - S_n^{t+1},$
$V_n^{t+1} - V_n^t = \max\{1, V_{n+1}^t\} - \max\{1, V_{n-1}^{t+1}\}, \quad V_{t-n}^n = U_n^t.$

From block maps view points, the first is $(-\infty, 0, 1)$-block map.

Let us consider the second step:

$$(\mathrm{BBS}_2) : S_{n+1}^{t+1} - S_n^t = \min\{0, 1 - S_{n+1}^t + S_n^{t+1}\}$$

and denote its transformation by T_2. We denote $\Sigma(\mathrm{BBS}_2)_L \subset \Sigma_L$ all the set of sequences $\{S_i\}_{i \in \mathbf{Z}} \subset \{0, \dots, L-1\}$ such that the corresponding orbits $\{S_i^t\}_i = T_2^t(\{S_i\}_i)\}_{t=0,1,\dots} \subset \{0, \dots, L-1\}$ are all bounded by L. Thus T_2 gives a dynamics as:

$$T_2 : \Sigma(\mathrm{BBS}_2)_L \to \Sigma(\mathrm{BBS}_2)_L.$$

Let $g : \{0, \dots, L-1\}^2 \times [0,1] \to \mathbf{R}$ be a family of maps on the interval, and $\pi : [0,1] \to \{0, \dots, L-1\}$ be the projection. Then for each $\bar{k} = (k_0, k_1, \dots)$, one obtains an infinite family of maps $\{h^m\}_m$ by:

$$h^{n+1}(x) = g_{k_n, k_{n+1}} \circ h^n(x).$$

Let us say that g is *marked at* $-\infty$, if $g_{0,0} \equiv 0$ holds.

Lemma 3.4 *The above automaton BBS_2 can be expressed as an interaction of maps which is marked at $-\infty$:*

$$g : \{1, \dots, L\}^2 \times [0,1] \to \mathbf{R}, \quad g_{a,b}(x) = \frac{a-b}{L} + \min\{\frac{b}{L}, (\frac{1}{L} + x)\}.$$

We have a map from BBS to BBS_2:

$$\Phi_2 : \Sigma_2^0(L) \to \Sigma_L^0.$$

Let us denote its image as:

$$Y_L = \Phi_2(\Sigma_2^0(L)) \subset \Sigma_L^0.$$

Proposition 3.1 *The BBS and BBS_2 are mutually equivalent by*

$$\Phi_2 : \Sigma_2^0(L) \cong Y_L.$$

Now put:

$$\Sigma(\mathrm{BBS}_2) = \cup_{L \in \mathbf{N}} \Sigma(\mathrm{BBS}_2)_L$$

and $Y \equiv \cup_{L \in \mathbf{N}} Y_L$. The topological entropy of BBS_2 is given by:

$$h_t(\mathrm{BBS}_2) = h_t(T_2; \Sigma(\mathrm{BBS}_2)).$$

The above equivalence induces $\Phi_2 : \Sigma_2^0 \cong Y \subset \Sigma(\mathrm{BBS}_2)$.

Corollary 3.1 *The estimates hold:*

$$\log \frac{1+\sqrt{5}}{2} \leq h_t(BBS) \leq h_t(BBS_2).$$

Proof of proposition: We have described BBS as an interaction map using four maps $\{f_1, f_2, f_3, f_4\}$ and some permutation ([K4]).

Let $\Sigma_2^0(L) \subset \Sigma_2^0$ be as above. Then we will construct a family of maps $\Phi(\{\alpha_a\}_a) : \Sigma_2^0(L) \to Y_L \subset \Sigma_L$ which induce the desired equivalence.

Notice that BBS and BBS$_2$ are related by the equations:

$$S_n^t = \Sigma_{i=-\infty}^n B_i^t, \quad S_n^t - S_{n-1}^t = B_n^t.$$

Thus one chooses a family of maps satisfying:

$$\alpha : \{0,1\} \times [0,1] \to [0,1], \quad \alpha(0,x) = x, \alpha(1,x) = x+1.$$

The corresponding interacting map is a $(-\infty, 0, 1)$ block map.

The converse direction is given by an automaton:

$$\beta : \{0, \dots, L-1\} \times [0,1] \to \mathbf{R}, \quad \beta(k,y) = \frac{k}{L} - y.$$

It represents a $(-1, 0, 1)$ block map.

This completes the proof.

Remarks: (1) Even though the number of maps in BBS is four, which is much smaller compared with BBS$_2$ (L^2), the former has to modify the interaction by use of permutation groups ([K4]).

(2) For the third and the final steps also, they are invertible transformations, but they do not preserve the time. We will treat these cases below.

3.B.4 Block maps by line segments: Let $\mathbf{Z} \times \mathbf{N} \subset \mathbf{R} \times \mathbf{R}_+$ be the integer lattice in the upper half plane, and let $l \subset \mathbf{R} \times \mathbf{R}_+$ be a line which may be unbounded. We put $l_0 = l \cap \mathbf{Z} \times \mathbf{N}$ and $m = \sharp l_0 \in \mathbf{N} \cup \{\infty\}$.

Let $F : \mathbf{Z} \times \mathbf{N} \to \mathbf{Z} \times \mathbf{N}$ be a function. Then we denote:

$$l_0(n,t) = F(n,t) + l_0 \subset \mathbf{Z} \times \mathbf{N},$$
$$L_0 = \{l_0(n,t) : (n,t) \in \mathbf{Z} \times \mathbf{N}\}.$$

L_0 is all the set of parallel line segments.

Let $\{(k_{-i}^t, \dots, k_0^t, k_1^t, \dots)\}_{t \geq 0, i \in \mathbf{Z}}$ be a flow. One may regard that it is a function on $\mathbf{Z} \times \mathbf{N}$ by $(n,t) \to k_n^t$. We denote $k(l_0(n,t)) = \{k_i^s\}_{(i,s) \in l_0(n,t)}$. We identify it as $k(l_0(n,t)) \in (X_a)_m$.

Let $\varphi_0 : (X_a)_m \to \{1, \dots, t\}$ be a map. The corresponding (l_0, F)-*block map*:

$$\varphi : \Sigma_a^{\mathbf{N}} \to \Sigma_t^{\mathbf{N}}, \{(\dots, k_0^t, k_1^t, \dots)\}_t \to \{(\dots, l_0^t, l_1^t, \dots)\}_t$$

is given by:

$$l_i^t = \varphi_0(k(l_0(i,t))).$$

We say that it is a *block map by the line segments* (l_0, F).

Lemma 3.5 *The third and final transformations from BBS to LV are both invertible, and they are all given by block maps by line segments.*

Proof: From $\{S_n^t\}$ to $\{U_n^t\}$, one has $l_0 = \{(0,1), (-1,0)\}$ and $F = \mathrm{id}$. From $\{U_n^t\}$ to $\{V_n^t\}$, one has $l_0 = (0,0)$ and $F(n,t) = (t-n, n)$.

Finally using the equality $\Sigma_{x=0}^{\infty} U_{n-x}^{t+x} = S_{n+1}^t$, one sees that from $\{U_n^t\}$ to $\{S_n^t\}$, $l_0 = \{(-s,s) : s \in \mathbf{N}\}$ and $F(n,t) = (n-1,t)$.

This completes the proof.

By the same way, one can generalize the notion of equivalence so that two interaction of maps are equivalent, if they are transformed by invertible block maps by line segments which are all expressible as interactions of maps.

Corollary 3.2 *The third and final transformations from BBS to LV are both equivalent by line segments.*

4 Contraction

Recall that $(X_a)_{n-1}$ is all the set of words of length n with the alphabets $\{1, \dots, a\}$. Let $\varphi_0 : (X_a)_{n-1} \to \{1, \dots, b\}$ be a map. Then φ_0

determines a continuous map:

$$\varphi : X_a \to X_b, \quad k'_i = \varphi(k_{ni}, \dots, x_{(i+1)n-1}).$$

It is an $(0, n-1, n)$ block map. Let σ be the shift. Then it satisfies commutativity:

$$\varphi(\sigma^n(\bar{k})) = \sigma(\varphi(\bar{k}))$$

for all $\bar{k} \in X_a$.

The lexicographic order gives the corresponding map:

$$\psi_0 : (X_a)_{n-1} \to \{1, \dots, a^n\}$$

which is called the *contraction*. So it induces the continuous map:

$$\psi : X_a \to X_{a^n}$$

for all $n = 1, 2, \dots$

Let $\{f_1, \dots, f_a\}$ be a family of maps. For each $\bar{k}_n \in (X_a)_{n-1}$, one can associate another map $g(\bar{k}_n)$ by composition:

$$g(\bar{k}_n) = f_{k_{n-1}} \circ f_{k_{n-2}} \circ \dots \circ f_{k_0}$$

where $\bar{k}_n = (k_0, \dots, k_{n-1})$. By this way one obtains families of maps of cardinarity a^n:

$$\mathbf{C}_n = \cup_{\bar{k}_n \in (X_a)_n} g(\bar{k}_n) \equiv \{g_1, \dots, g_{a^n}\}$$

where the indices of the set is given by the lexicographic way. For example when $n = 2$ and $a = 2$, $g_1 = f_1 \circ f_1$, $g_2 = f_2 \circ f_1$, $g_3 = f_1 \circ f_2$ and $g_4 = f_2 \circ f_2$. It is called the n-th *contracted family of maps.*

Let $\pi_a : [0,1] \to \{1, \dots, a\}$ be the projection. Then using π_a, one has the interaction map $\Phi(\{f_i\}_{i=1}^a) : [0,1] \times X_a \to [0,1] \times X_a$. Similarly using $\pi_{a^n} : [0,1] \to \{1, \dots, a^n\}$, one obtains another interaction map:

$$\Phi(\{g_j\}_j) : [0,1] \times X_{a^n} \to [0,1] \times X_{a^n}.$$

using the famliy $\{g_j\}_j$. So one obtains a diagram:

$$\begin{array}{ccc} [0,1] \times X_a & \xrightarrow{\psi} & [0,1] \times X_{a^n} \\ \Phi(\{f_i\}_i) \downarrow & & \Phi(\{g_j\}_j) \downarrow \\ [0,1] \times X_a & \xrightarrow{\psi} & [0,1] \times X_{a^n} \end{array}$$

which is not commutative in general.

Below we study entropy hehaviours of both sides restricted on the spaces where the diagram above commute.

4.B Compression: Let us take a family of maps $\{f_1, \dots, f_a\}$ and choose $n-1$. Then as above one obtains another family of maps $\mathbf{C} = \{g_1, \dots, g_{a^n}\}$.

Let us put $\mathbf{C}_i = \{g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{a^n}\}$ whose cardinarity is $a^n - 1$, and fix another projection $\pi' : [0,1] \to \{1, \dots, a^n - 1\}$.

We say that $\mathbf{C}_i$ and $\mathbf{C}_j$ are *equivalent* with respect to π', if the corresponding interacting maps:

$$\Phi(\mathbf{C}_i), \Phi(\mathbf{C}_j) : [0,1] \times X_{a^n-1} \to [0,1] \times X_{a^n-1}$$

give the same map mutually, after reordering the indices of them. We say that $\mathbf{C}_i$ or $\mathbf{C}_j$ are *compression* of $\mathbf{C}$ with respect to π'.

By identifying g_i with g_j, one obtains another map $\varphi_0 : (X_a)_{n-1} \to \{1, \dots, a^n - 1\}$ which gives $\varphi : X_a \to X_{a^n-1}$.

Example: Let $\{f_1, f_2\}$ be two maps. Then the second contracted family is denoted as $\mathbf{C}_2 = \{g_1, g_2, g_3, g_4\}$.

Let f_1 be the tent map and f_2 be its reverse, $f_2(x) = 1 - f_1(x)$. Consider the set:

$$\mathbf{C} = \{g_1, g_2, g_3, g_4\} = \{f_1^2, f_2 f_1, f_1 f_2, f_2^2\}.$$

Since $f_1^2 = f_1 f_2$ holds, $\mathbf{C}_1 = \{f_2 f_1, f_1 f_2, f_2^2\}$ and $\mathbf{C}_3 = \{f_2 f_1, f_1^2, f_2^2\}$ are mutually equivalent.

One has the corresponding block map as:

$$\varphi_0 : (X_2)_2 \to \{1, 2, 3\}, \quad (1,1), (2,1) \to 1, (1,2) \to 2, (2,2) \to 3.$$

For any subset $J \subset \{1, \dots, a\}$, let us denote by $\mathbf{C}_J = \{g_j : j \notin J\}$. Suppose $(\mathbf{C}_i, \mathbf{C}_j)$ and $(\mathbf{C}_k, \mathbf{C}_l)$ are equivalent pairs where $\{i,j\} \cap \{k,l\} = \phi$. Then one can again compress $\mathbf{C}_i$ to obtain another $\mathbf{C}_{i,k}$. By continuing this process, one obtains a compressed family of maps $\mathbf{C}_J \subset \mathbf{C}$. Then finally one obtains a *minimal element*:

$$\mathbf{C}' = \{g_{i_1}, \dots, g_{i_t}\}$$

which is no longer compressible.

Let us choose a compression $\mathbf{C}$. Thus one obtains a map:

$$C : (X_a)_{n-1} \to \{1, \ldots, t\}$$

and we denote the corresponding map by:

$$\varphi : X_a \to X_t.$$

We call it the *associated compression map.*

Example: Let us consider the above example. They also satisfy more equalities $f_2^2 = f_2 f_1$. So one obtains more equivalent pairs $\mathbf{C}_{1,3}$ and $\mathbf{C}_{2,4}$. Thus one obtains a minimal element $\mathbf{C} = \{f_1 f_2, f_2 f_1\} \equiv \{g_1, g_2\}$.

Thus the associated compression map is:

$$\psi_0 : (X_2)_2 \to X_2, \quad (1,1), (2,1) \to 1, (1,2), (2,2) \to 2$$

and the diagram:

$$\begin{array}{ccc} X_2 & \stackrel{\psi}{\longrightarrow} & X_2 \\ \Phi(\{f_1, f_2\})(x) \downarrow & & \Phi(\{g_1, g_2\})(x) \downarrow \\ X_2 & \stackrel{\psi}{\longrightarrow} & X_2 \end{array}$$

which in fact commutes (prop 4.1).

Remark: Notice that in order to obtain equivalent pairs, one doesnot necessarily require equalities of the maps themselves as above. For example when $f_1(x) = 1 - x$ and g be the identity, $\mathbf{C}_1$and $\mathbf{C}_4$, $\mathbf{C}_2$ and $\mathbf{C}_3$ are mutually equivalent. Still one obtains the same equivalences when f_1 and f_2 are slightly perturbed away from $\frac{1}{2}$.

Let $\{f_1, \ldots, f_a\}$ be a family of maps and $\mathbf{C}$ be a compression which induces a map $(X_a)_{n-1} \to \{1, \ldots, t\}$. Then corresponding interaction map:

$$\Phi(\mathbf{C}) : [0,1] \times X_t \to [0,1] \times X_t$$

is called the *associated interaction map* with $\mathbf{C}$. The following is clear:

Lemma 4.1 *Let $R(f_1, \ldots, f_a) \subset [0,1]$ be the regular set. Then the the inclusion $R(f_1, \ldots, f_a) \subset R(g_1, \ldots, g_t)$ holds.*

Definition 4.1 *Let $\mathbf{C}_I = \{g_1, \dots, g_t\}$ be a compression. We say that $x \in R(f_1, \dots, f_a)$ is a compressible point, if the following diagram commutes:*

$$\begin{array}{ccc} X_a & \xrightarrow{\varphi} & X_t \\ \Phi(\{f_i\}_i)(x) \downarrow & & \Phi(\{g_j\}_j)(x) \downarrow \\ X_a & \xrightarrow{\varphi} & X_t \end{array}$$

where φ is the associated block map.

In that case it is an *infinitely compressible point* which saids commutativities of the diagrams:

$$\begin{array}{ccc} X_a & \xrightarrow{\varphi} & X_t \\ \Phi^s(\{f_i\}_i)(x) \downarrow & & \Phi^s(\{g_j\}_j)(x) \downarrow \\ X_a & \xrightarrow{\varphi} & X_t \end{array}$$

for all $s = 0, 1, \dots$ We denote all the set of compressible points by:

$$J(\mathbf{C}) \subset [0, 1].$$

Let $\mathbf{C} : (X_a)_{n-1} \to \{1, \dots, a\}$ be a compression. Thus by definition one has assigned some element $\bar{k}(i) = (k_0(i), \dots, k_{n-1}(i)) \in (X_a)_{n-1}$ for each $i = 1, \dots, a$.

We say that $\mathbf{C}$ *depends only on the last factor*, if for each $\mathbf{C}(l_0, \dots, l_{n-1}) \in \{1, \dots, a\}$, it satisfies the equalities:

$$k_n(\mathbf{C}(l_0, \dots, l_{n-1})) = l_{n-1}.$$

Proposition 4.1 *Let $C : (X_a)_{n-1} \to \{1, \dots, a\}$ be a compression so that it depends only on the last factor.*

Then the compressible points satisfy the equality:

$$J(\mathbf{C}) = R(f_1, \dots, f_a).$$

We say that such compression is the n-th *projection*.

Proof: Let $\{f_1, \dots, f_a\}$ be a family of maps and denote the compression between maps by $\mathbf{C} : \{f_1, \dots, f_a\} \to \{g_1, \dots, g_a\}$.

Recall the notation $g(\bar{k}_n) = f_{k_{n-1}} \circ f_{k_{n-2}} \circ \cdots \circ f_{k_0}$ for $\bar{k}_n \in (X_a)_n$. For any element $\bar{k}_{n-1} = (k_0, \dots, k_{n-2}) \in (X_a)_{n-1}$, we put $\bar{k}_{n-1} i = (k_0, \dots, k_{n-2}, i) \in (X_a)_n$ for $i = 1, \dots, a$.

Then by definition, $\mathbf{C}(g(\bar{k}_{n-1}i)) = f_i \circ g(\bar{k}'_{n-1})$ are satisfied for some $\bar{k}'_{n-1} \in (X_a)_{n-1}$. Thus the result follows. This completes the proof.

Examples: Let f be the tent map and g be its reverse. Then the compression satisfies the above condition. Thus $J(\mathbf{C}) = [0,1] \backslash \{\frac{k}{2^n}; 1 \leq k \leq 2^n - 1, n = 1, 2, \dots\}$.

Since a compression by an n-th projection induces a surjection $\varphi : X_a \to X_a$, it follows from the above proposition that one obtains the equality for the topological entropies of the corresponding interaction maps:

Corollary 4.1 *Suppose* $\mathbf{C} : \{f_1, \dots, f_a\} \to \{g_1, \dots, g_a\}$ *is a compression by an n-th projection. Then the topological entropies satisfy the equality:*

$$h_t(f_1, \dots, f_a; R) = h_t(g_1, \dots, g_a; R)$$

where $R = R(f_1, \dots, f_a) = J(\mathbf{C})$.

In the above example, f be the tent map and g be its reverse, then the estimate holds:

$$h_t(f, g) = h_t(f^2, g^2) = \log 2.$$

Proof: Recall the orbit space:

$$\mathbf{L}(f_1, \dots, f_a) = \{\{\Phi(x)(\bar{k}); x \in R(f_1, \dots, f_a), \bar{k} \in X_a\} \subset X_a^{\mathbf{N}}$$

in 1.A. For positive integers $m, t \geq 0$, let $\bar{\pi}_{m,t} : X_2^{\mathbf{N}} \to (X_2)_m^{t+1}$, $\{a_i^s\}_{i,s \in \mathbf{N}} \to \{a_i^s\}_{0 \leq i \leq m}^{0 \leq s \leq t}$ be the projections.

Then by the assumption, the equalities hold for all m, t:

$$\sharp \bar{\pi}_{nm,t}(\mathbf{L}(f_1, \dots, f_a)) = \sharp \bar{\pi}_{m,t}(\mathbf{L}(g_1, \dots, g_a)).$$

By proposition 1.1, the topological entropy satisfies the equality $h_t(f_1, \dots, f_a) = h_t(\mathbf{L}(f_1, \dots, f_a))$, where the latter is given by

$$\lim_{m \to \infty} \limsup_{t \to \infty} \frac{1}{t+1} \log \sharp \bar{\pi}_{m,t}(\mathbf{L}(f_1, \dots, f_a)).$$

Thus the equality holds. This completes the proof.

5 Interaction entropy

In this section, we study several properties of interaction of maps from measure theoretic view points. In particular we introduce measure theoretic entropies which we call the *interacting entropy.* When the interaction satisfies the Markov property, then it coincides with the conditional entropy.

Later on we always assume that the regular set $R(f_1, \dots, f_a) \subset [0,1]$ has positive measure with respect to the standard one μ_0. For simplicity, we assume $R(f_1, \dots, f_a)$ has full measure, $\mu_0(R(f_1, \dots, f_a)) = 1$.

5.A Induced measure: Let $\{f_1, \dots, f_a\}$ be a family of maps on the interval, and consider the corresponding interacting map $\Phi(x) : X_a \to X_a$. We equip with the standard measure μ_0 on $[0,1]$.

For each pair $\bar{x}_n, \bar{y}_n \in (X_a)_n$, let

$$P(\bar{x}_n : \bar{y}_n) \subset [0,1]$$

be the set of regular points $x \in R(f_1, \dots, f_a)$ satisfying $\Phi(x)(\bar{x}_n) = \bar{y}_n$.

Definition 5.1 *Let us fix $\bar{x}_n \in (X_a)_n$, and let $\bar{Y}_n \in (X_a)_n$ be a random variable. The induced probability from $\{f_1, \dots, f_a\}$ is the one on $(X_a)_n$ given by:*

$$Q_n(\bar{x}_n : \bar{Y}_n) \equiv \mu_0(P(\bar{x}_n : \bar{Y}_n)).$$

The *informative entropy* was introduced by Shannon ([S]). Let us fix $x, y \in \{1, \dots, a\}$. In our formulation it is by the following:

$$h_s(x) = -\Sigma_{w \in \{1,\dots,a\}} Q_1(x, w) \log Q_1(x, w).$$

When the probability is memoryless, then this will be the suitable one to analyze. In our case, interacting maps usually possess memory, and so we introduce relative version of the informative entropy. This plays an important role when the interaction is Markov. Let:

$$Q((x,y)|(x',y')) = \frac{Q((x',x);(y',y))}{Q(x',y')}$$

be the conditional probability.

The *conditional entropy* $h_c(x,x')$ is given by:

$$h_c(x,x') = -\Sigma_{(z,w)\in\{1,\dots,a\}^2} \\ Q(x,z)Q((x',w)|(x,z))\log Q((x',w)|(x,z)).$$

Let us fix $\bar{x}_n \in (X_a)_n$. The conditional entropy with respect to this word is given by:

$$\begin{aligned} h_c(\bar{x}_n) &= -\frac{1}{n+1}\Sigma_{(z,w)\in\{1,\dots,a\}^2}\Sigma_{i=1}^n \qquad Q(x_{i-1},z)Q((x_i,w)|(x_{i-1},z)) \\ &\quad \log Q((x_i,w)|(x_{i-1},z)) \\ &= \frac{1}{n+1}\Sigma_{i=1}^n h_c(x_{i-1},x_i). \end{aligned}$$

Let $\bar{x} \in X_a$. The conditional entropy of $\bar{x}$ is given by:

$$h_c(\bar{x}) = \lim\sup_{n\to\infty} h_c(\bar{x}_n)$$

where $\bar{x}_n \in (X_a)_n$ is the restriction of $\bar{x}$.

5.A.2 Markov property: Let us choose $\bar{x}_n \in (X_a)_n$.

Definition 5.2 *We say that Q is Markov with respect to $\bar{x}_n$, if for all $\bar{y}_n \in (X_a)_n$, the following holds:*

$$\begin{aligned} &Q_n((x_0,\dots,x_n):(y_1,\dots,y_n)) \\ &\qquad = Q(x_0,y_0)\Pi_{i=1}^n Q((x_i,y_i)|(x_{i-1},y_{i-1})). \end{aligned}$$

We say that Q is *Markov up to* n, if it is so for any $\bar{x}_n$. We also say that it is Markov for $\bar{x} \in X_a$, if for its restrictions $\bar{x}_n \in (X_a)_n$, it is so with respect to $\bar{x}_n$.

Example: A continuous map f on the interval is *half dividing*, if the followings hold:

$$f(I_i) \subset I_j, \quad I_1 = [0,\frac{1}{2}), I_2 = (\frac{1}{2},1], \quad j = j(i) \in \{1,2\}.$$

Lemma 5.1 *If two maps f and g are both half dividing, then the measure Q is Markov up to infinity.*

In fact,

$$Q(x,y) = 1 \text{ or } \frac{1}{2}, \text{ or } 0, \quad Q((x,y)|(x',y')) = 1 \text{ or } 0$$

hold.

Let us consider the tent map, $f|[0,\frac{1}{2}](x) = 2x$, $f|[\frac{1}{2},1](x) = 2-2x$, and $g(x) = -f(x)+1$ be its reverse.

Lemma 5.2 *For f and g be as above, the corresponding Q is Markov up to infinity, where the equality holds:*

$$Q_n(\bar{x}_n : \bar{y}_n) = \frac{1}{2^{n+1}}.$$

Proof: Since $g \circ f = g^2$ and $f \circ g = f^2$ hold, it follows $h^n \equiv d_{k_{n-1}} \circ \cdots \circ d_{k_0} = f^n$ or g^n. Thus in any case they are all PL maps with the slopes $\pm 2^{n+1}$. All turning values are 0 or 1.

So it is immediate to see that $h^n|[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}]$ are monotone with the range $[0,1]$ for $k = 0, \dots, 2^{n+1}-1$. Then by induction, one can check that for each pair $(\bar{x}_n, \bar{y}_n)$, $P(\bar{x}_n, \bar{y}_n) \subset [0,1]$ is a union of $(\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}})$ for some $k \in \{0, \dots, 2^{n+2}-1\}$.

Then again by induction, one can see:

$$Q_{n+1}(\bar{x}_{n+1} : \bar{y}_{n+1}) = \frac{1}{2} Q_n(\bar{x}_n, \bar{y}_n) = \cdots = \frac{1}{2^{n+1}}.$$

In particular $Q_1((x,y)|(x',y')) = \frac{1}{2}$ hold.

This completes the proof.

From the proof, it follows the following:

Corollary 5.1 *Let $\tilde{f}$ and $\tilde{g}$ be sufficiently small perturbations of the above f and g on intervals $(\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}})$, $k = 0, 1, \dots, 2^{n+1}-1$.*

Then the corresponding Q_n is Markov up to n.

5.B Interacting entropy: Let $\{f_1, \dots, f_a\}$ be a family of maps, and $Q_n(\bar{x}_n, \quad)$ be the induced probability with respect to the standard measure μ_0 on $[0,1]$.

The *interacting entropy* is defined by:

$$h_i(\bar{x}_n) = -\frac{1}{n+1}\Sigma_{\bar{y}_n \in (X_a)_n} Q_n(\bar{x}_n, \bar{y}_n) \log Q_n(\bar{x}_n, \bar{y}_n).$$

Let $\bar{x} \in X_a$ and $\bar{x}_n \in (X_a)_n$ be its restriction. Then the interacting entropy for $\bar{x}$ is given by:

$$h_i(\bar{x}) = \limsup_{n\to\infty} h_i(\bar{x}_n).$$

Example: Let f be the tent map and g be its reverse as before. Then:

$$(n+1)h_i(\bar{x}_n) = \Sigma_{\bar{y}_n \in (X_2)_n} 2^{-(n+1)} \log 2^{n+1} = (n+1)\log 2.$$

In particular for any $\bar{x} \in X_2$ and its restriction $\bar{x}_n$,

$$h_i(\bar{x}) = h_i(\bar{x}_n) = \log 2$$

holds.

We will often use the following general estimate. It is called the *log-sum inequality*:

Lemma 5.3 *Let $\{a_i\}_i$ and $\{b_j\}_j$ be two families by non negeative numbers with $a = \Sigma_i a_i, b = \Sigma_j b_j < \infty$. Then the estimate holds:*

$$\Sigma_{i=0}^{\infty} a_i \log \frac{a_i}{b_i} \geq a \log \frac{a}{b}.$$

5.B.2 Basic properties of interacting entropy: Below we have basic estimates and relations with other entropies.

Notice that $\sharp(X_a)_n = a^{n+1}$. So from the log-sum inequality, one obtains the a priori estimate. The above example is the optimal case.

Lemma 5.4

$$h_i(\bar{x}_n) \leq \log a.$$

Proof: By the log-sum inequality,

$$\begin{aligned}
&- \Sigma_{\bar{y}_n \in (X_a)_n} Q_n(\bar{x}_n, \bar{y}_n) \log Q_n(\bar{x}_n, \bar{y}_n) \\
&\leq -(\Sigma_{\bar{y}_n \in (X_a)_n} Q_n(\bar{x}_n, \bar{y}_n)) \log \frac{\Sigma Q_n(\bar{x}_n, \bar{y}_n)}{a^{n+1}} \\
&= \log a^{n+1} = (n+1)\log a.
\end{aligned}$$

This completes the proof.

Proposition 5.1 *Suppose Q is Markov up to n. Then the interacting entropy satisfies the equality:*

$$h_i(\bar{x}_n) = h_c(\bar{x}_n) + \frac{1}{n+1} h_s(x_0).$$

Proof: By the assumption, one obtains the equalities:

$$\begin{aligned}
-\Sigma_{\bar{y}_n \in (X_a)_n} & Q_n(\bar{x}_n, \bar{y}_n) \log Q_n(\bar{x}_n, \bar{y}_n) \\
&= -\Sigma Q(x_0, y_0) \Pi_{i=1}^n Q((x_i, y_i)|(x_{i-1}, y_{i-1})) \\
&\quad \log Q(x_0, y_0) \Pi_{i=1}^n Q((x_i, y_i)|(x_{i-1}, y_{i-1})) \\
&= -\Sigma_{y_0} Q(x_0, y_0) \log Q(x_0, y_0) \\
&- \Sigma_{i=1}^n Q(x_{i-1}, y_{i-1}) Q((x_i, y_i)|(x_{i-1}, y_{i-1})) \\
&\quad \log Q((x_i, y_i)|(x_{i-1}, y_{i-1})) \\
&= h_s(x_0) + \Sigma_{i=1}^n h_c(x_{i-1}, x_i) = h_s(x_0) + (n+1) h_c(\bar{x}_n).
\end{aligned}$$

This completes the proof.

Corollary 5.2 *Let $\bar{x} \in X_a$ and Q be Markov with respect to $\bar{x}$. Then the equality holds:*

$$h_i(\bar{x}) = h_c(\bar{x}).$$

Recall $\bar{X}(\{f_i\}_i, \bar{k}) \subset X_a \times [0,1]$ in $1.B$.

Lemma 5.5

$$h_t(\bar{X}(\{f_i\}_i, \bar{k})) \geq h_i(\bar{k})$$

holds.

Proof: By lemma 1.5, the estimate $h_t(\bar{Y}(\{f_i\}_i, \bar{k})) \leq h_t(\bar{X}(\{f_i\}_i, \bar{k}))$ holds. Thus it is enough to verify the inequality $h_t(\bar{Y}(\{f_i\}_i, \bar{k})) \geq h_i(\bar{k})$.

Recall $\bar{Y}(\{f_i\}_i, \bar{k}_n) \subset (X_a)_n$ and $\bar{Y}(f, g, \bar{k}_n, \bar{l}_n) \subset [0,1]$ in $1.B.2$. It satisfies the equality $Q_n(\bar{x}_n, \bar{y}_n) = \mu_0(\bar{Y}(\{f_i\}_i, \bar{x}_n, \bar{y}_n))$, where μ_0 is the standard metsure on $[0,1]$.

By the log-sum inequality, the estimate holds:

$$\begin{aligned}
(n+1) h_i(\bar{x}_n) &= -\Sigma_{\bar{y}_n \in (X_a)_n} Q_n(\bar{x}_n, \bar{y}_n) \log Q_n(\bar{x}_n, \bar{y}_n) \\
&\leq -\log \frac{1}{\sharp \bar{Y}(\{f_i\}_i, \bar{k}_n)} = \log \sharp \bar{Y}(\{f_i\}_i, \bar{k}_n)
\end{aligned}$$

Thus by letting $n \to \infty$, one obtains the desired estimate. This completes the proof.

We say that Q_n is *homogeneous*, if

$$Q_n(\bar{x}_n, \bar{y}_n) = C$$

are the same for all $\bar{y}_n \in (X_a)_n$. We say that Q is homogeneous, if Q_n is so for all $n = 1, 2, \ldots$

It follows from the above proof, the equality $h_t(\bar{Y}(\{f_i\}_i, \bar{k})) = h_i(\bar{k})$ holds, if and only if Q_n is homogeneous.

Example: Let f be the tent map and g be its reverse. Then Q_n is homogeneous for any $\bar{x}_n$.

Corollary 5.3 *Suppose there are no strange sequences for the family $\{f_i\}_i$, and $\bar{k}$ is monotone. Then the equality holds:*

$$h_t(\bar{X}(\{f_i\}_i, \bar{k})) = h_i(\bar{k})$$

if and only if Q is homogeneous.

Proof: This follows from combination of proposition 1.2 and the above remark. This completes the proof.

Example: Let f be the tent map and g be its reverse. Then this pair satisfies all the conditions above. Thus the equalities hold:

$$h_t(\bar{X}(f, g, \bar{k})) = h_i(\bar{k}) = \log 2.$$

5.B.3 Interacting entropy for the iteration: Let $\{f_1, \ldots, f_a\}$ be a family of maps and $\Phi : [0,1] \times X_a \to [0,1] \times X_a$ be the coresponding interaction map.

Let us choose two integers $t, n \geq 1$, and take a subset:

$$\{y_i^s\}_{0 \leq i \leq n}^{1 \leq s \leq t} \subset (X_a)_n^t \equiv (X_a)_n \times \cdots \times (X_a)_n.$$

Let us fix $\bar{x}_n \in (X_a)_n$ and put subsets in $[0,1]$:

$$P(\bar{x}_n, \{y_i^s\}_{0 \leq i \leq n}^{1 \leq s \leq t}) = \{x \in R(f_1, \ldots, f_a) : \\ \Phi(x)^s(\bar{x}_n) = (y_1^s, \ldots, y_n^s), 1 \leq s \leq t\}.$$

Let μ_0 be the standard measure on $[0,1]$. Then the induced measure on $(X_a)^t_n$ is given by:

$$Q(\bar{x}_n; \{y_i^s\}_{0\leq i\leq n}^{1\leq s\leq t}) = \mu_0(P(\bar{x}_n, \{y_i^s\}_{0\leq i\leq n}^{1\leq s\leq t})).$$

The *interacting entropies* with the intial data $\bar{x}_n$ are defined by:

$$h_i(\bar{x}_n, t) = -\frac{1}{t}\Sigma_{\{y_i^s\}_{i,s}\in (X_a)^t_n} Q(\bar{x}_n; \{y_i^s\}_{i,s}) \log Q(\bar{x}_n; \{y_i^s\}_{i,s}),$$
$$h_i(\Phi; \bar{x}_n) = \limsup_{t\to\infty} h_i(\bar{x}_n, t).$$

Definition 5.3 *The interacting entropy of the interaction map Φ is given by:*

$$h_i(f_1, \ldots, f_a) = \limsup_{n\to\infty} \Sigma_{\bar{x}_n\in(X_a)_n} a^{-(n+1)} h_i(\Phi; \bar{x}_n).$$

Example: Let f be the tent map and g be its reverse. Then:

$$Q(\bar{x}_n, \{y_i^s\}_{i,s}) = \frac{1}{2^{n+t}}$$

hold for all $\bar{x}_n$ and t. Thus

$$h_i(\bar{x}_n, t) = (n+t)\log 2, \quad h_i(\Phi; \bar{x}_n) = \log 2.$$

hold respectively. In particular the interacting entropy for $\{f, g\}$ is obtained as:

$$h_i(f, g) = \Sigma_{\bar{x}_n\in(X_2)_n} \frac{1}{2^{n+1}} \log 2 = \log 2.$$

We have the basic inequality:

Theorem 5.1 *The topological entropy is larger than the interacting entropy:*

$$h_i(f_1, \ldots, f_a) \leq h_t(f_1, \ldots, f_a).$$

Proof: Let $\Phi : [0,1]\times X_a \to [0,1]\times X_a$ be the interacting map, and denote its orbit spaces by $\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{k}) = \cup_{x\in R(\{f_i\}_i)}\Psi(x)(\bar{k}) \subset X_a^{\mathbf{N}}$ and $\mathbf{L}(\{f_i\}_{i=1}^a) = \cup_{\bar{k}\in X_a}\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{k})$. By proposition 1.1, the topological entropy $h_t(\Phi)$ is equal to the one of its orbit space $h_t(\mathbf{L}(\{f_i\}_{i=1}^a))$.

For positive integers $n, t \geq 0$, let us put the projections:

$$\bar{\pi}_{n,t} : X_a^{\mathbf{N}} \to (X_a)_n^{t+1}, \quad \{a_i^s\}_{i,s \in \mathbf{N}} \to \{a_i^s\}_{0 \leq i \leq n}^{0 \leq s \leq t}.$$

Notice the inclusion:

$$\{(\bar{x}_n; \{y_i^s\}_{i=0,s=1}^{i=n,s=t}) : Q(\bar{x}_n; \{y_i^s\}_{i,s}) \neq 0\} \subset \pi_{n,t}(\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{x}_n)).$$

Since $\Sigma_{\{y_i^s\}_{i,s}} Q(\bar{x}_n; \{y_i^s\}_{i,s}) = 1$ holds, it follows by the log-sum inequality (lemma 5.3), one obtains the estimate:

$$\begin{aligned} & - \Sigma_{\{y_i^s\}_{i,s}} Q(\bar{x}_n; \{y_i^s\}_{i,s}) \log Q(\bar{x}_n; \{y_i^s\}_{i,s}) \\ & \leq - \log \frac{1}{\sharp \pi_{n,t}(\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{x}_n))} = \log \sharp \pi_{n,t}(\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{x}_n)). \end{aligned}$$

By concavity of $\log x$, one obtains the estimate:

$$\Sigma_{\bar{x}_n \in (X_a)_n} \frac{1}{a^{n+1}} \log \sharp \pi_{n,t}(\mathbf{L}(\{f_i\}_{i=1}^a)(\bar{x}_n)) \leq \log \frac{1}{a^{n+1}} \sharp \pi_{n,t}(\mathbf{L}(\{f_i\}_{i=1}^a)).$$

Thus by dividing by $\frac{1}{t+1}$ on the both sides, and letting firstly $t \to \infty$ and then $n \to \infty$, one obtains the desired estimate.

This completes the proof.

For the above case $\{f, g\}$ where f is the tent map and g be its reverse, the equality holds $h_i(f, g) = h_t(f, g)$.

5.B.4 Compression and interacting entropy: Let us take a family of maps $\{f_1, \dots, f_a\}$ and $\mathbf{C} = \{g_1, \dots, g_t\}$ be a compressed family with $\varphi_0 : (X_a)_m \to \{1, \dots, t\}$.

Let $J(\mathbf{C}) \subset R(f_1, \dots, f_a)$ be the set of compressible points $(4.B)$.

Suppose $J(\mathbf{C})$ has positive measure, and let us equip with the nomalized measure $\mu_0' = \frac{1}{|J(\mathbf{C})|} \mu_0$ on $J(\mathbf{C})$ so that $\mu_0'(J(\mathbf{C})) = 1$. Correspondingly let $Q(\bar{x}_n, \bar{Y}_n)$ be the induced measure on $(X_a)_n$ with respect to $(J(\mathbf{C}), \mu_0')$.

Theorem 5.2 *Let $n + 1 = (l + 1)(m + 1)$. Then the interacting entropies with respect to μ_0' satisfy the inequality:*

$$\begin{aligned} & (n+1) h_i(\bar{x}_n; \{f_i\}_i; \mu_0') \\ & \qquad \leq (l+1) h_i(\varphi(\bar{x}_n); \{g_i\}_i; \mu_0') + \sup_{\bar{y}_l \in (X_t)_l} \log \sharp \varphi^{-1}(\bar{y}_l). \end{aligned}$$

Proof: By definition, the interacting entropy is given by:

$$(n+1)h_i(\bar{x}_n) = -\Sigma_{\bar{y}_n \in (X_a)_n} Q_n(\bar{x}_n, \bar{y}_n) \log Q_n(\bar{x}_n, \bar{y}_n).$$

By commutativity of the diagram, one obtains the equality:

$$Q(\varphi(\bar{x}_n), \bar{y}_l) = \Sigma_{\bar{z}_n \in \varphi^{-1}(\bar{y}_l)} Q(\bar{x}_n, \bar{z}_n).$$

So by the log-sum inequality, one obtains the estimates:

$$-\Sigma_{\bar{z}_n} Q(\bar{x}_n, \bar{z}_n) \log Q(\bar{x}_n, \bar{z}_n) \leq -Q(\varphi(\bar{x}_n), \bar{y}_l) \log \frac{Q(\varphi(\bar{x}_n), \bar{y}_l)}{|\varphi^{-1}(\bar{y}_l)|}.$$

By summing both sides with respect to $\bar{y}_l$, one obtains the desired estimate.

Example: Let f_1 be the tent map and f_2 be its reverse. Then in $4.B$, one has obtained a compression $\mathbf{C}$ from $\{f_1, f_2\}$ to $\{g_1, g_2\}$ with $m = 1$. In that case $J(\mathbf{C}) = R(f_1, f_2) = [0,1] \backslash \{\frac{k}{2^n}; k = 1, 2, \ldots, n = 1, 2, \ldots\}$, and so $\mu'_0 = \mu_0$. $|\varphi^{-1}(\bar{y}_l)| = 2^{l+1}$ are all constant.

$h_i(\bar{x}_n; \{f_i\}_i) = \log 2$, $h_i(\varphi(\bar{x}_n); \{g_i\}_i) = \log 2$, and so in this case the equality holds $(n+1)h_i(\bar{x}_n; \{f_i\}_i) = (l+1)(h_i(\varphi(\bar{x}_n); \{g_i\}_i) + \log 2)$.

5.C Divergence: Let $\{f_1, \ldots, f_a\}$ be a family of maps, and Q be the corresponding measure.

Definition 5.4 *$\bar{x}_n$ is memoryless in time, if the two step probability:*

$$Q((x_{i-1}, x_i), (y_2, y_1)) \equiv Q(y_2, y_1)$$

is independent of choice of (x_{i-1}, x_i), $i = 1, \ldots, n$.

Lemma 5.6 *If Q is memoryless in time, then both one step probability $Q(x_i, y) \equiv Q(y)$ and the conditional one $Q((x_i, y_1)|(x_{i-1}, y_2)) \equiv Q(y_1|y_2)$ are also independent of i respectively.*

Proof: By summing up with y_2 on both sides of $Q((x_{i-1}, x_i), (y_2, y_1)) \equiv Q(y_2, y_1)$, one can see that $Q(x_i, y_1) \equiv Q(y_1)$ does not depend on x_{i-1}.

Then from the formula $Q((x_i, y_1)|(x_{i-1}, y_2)) = Q(y_2, y_1)/Q(y_2)$, the result follows. This completes the proof.

Recall that for interaction maps $\Phi(x)^t(\bar{k}) = \{k_n^t\}_{n,t}$, there are two directions. One is n-direction and the other is t-one, time direction. When Q is memoryless in time, and Markov with respect to $\bar{x} \in X_a$, then by the above lemma, the conditional entropy $h_c(x_{i-1}, x_i)$ is independent of choice of i.

Lemma 5.7 *Suppose Q is memoryless in time and Markov with respect to $\bar{x}$. Then the equalities hold:*

$$h_c(x_{i-1}, x_i) = h_c(\bar{x}) = h_i(\bar{x})$$

for any $i = 1, 2, \ldots$

Proof: By definition, $(n+1)h_c(\bar{x}_n) = \Sigma_{i=1}^n h_c(x_{i-1}, x_i) = nh_c(x_{i-1}, x_i)$. So letting $n \to \infty$, the first equality holds. The second is corollary 5.2. This completes the proof.

When Q is memoryless in time and Markov with respect to $\bar{x}$, then we will denote:

$$h_*(Q) \equiv h_*(x_{i-1}, x_i), \quad * = s, c, i.$$

For any word $\bar{y}_n = y_0 y_1 \ldots y_{n-1} \in (X_a)_n$ and $m \leq n$, we denote its prefixes as $\bar{x}_m = y_0 \ldots y_m$.

Let us put the cardinarities by:

$$N(b|\bar{y}_n) = \sharp\{i; y_i = b; i = 0, \ldots, n-1\}$$

and put the set of the succesive pairs of elements by:

$$S(\bar{y}_n) = \{(y_{i-1}, y_i)\}_{i=1}^n \subset X_a^2.$$

Similarly we put the number of $(z, w) \in \{1, \ldots, a\}^2$ in $S(\bar{y}_n)$ by:

$$0 \leq N((z, w)|\bar{y}_n) \leq n.$$

The *type* of $\bar{y}_n$ is the conditional probability on $\{1, \ldots, a\}^2$:

$$P(w|z) = \frac{N((z, w)|\bar{y}_n)}{N(z|\bar{y}_n)}.$$

We put $P(z) = \frac{N(z|\bar{y}_n)}{n}$. This gives a Markov process on$(X_a)_n$ by $P_n(\bar{y}_n) = P(y_0)P(y_1|y_0)\dots P(y_n|y_{n-1})$.

We say that two words $\bar{y}_n, \bar{y}'_n \in (X_a)_n$ give the *same type*, if the equalities hold:

$$N(z|\bar{y}_n) = N(z|\bar{y}'_n), \quad N((z,w)|\bar{y}_n) = N((z,w)|\bar{y}'_n)$$

for all $w, z \in \{1, \dots, a\}$.

Let us fix n and a type P on $\{1, \dots, a\}^2$. We put the set of words with the same type P by:

$$T^n(P) \subset (X_a)_n.$$

Thus for any element $\bar{y}_n \in T^n(P)$, the corresponding type $P(\bar{y}_n)$ is equal to P.

We also put the set of types of length n by:

$$\mathfrak{P}_n = \{P(\bar{z}_n) : \bar{z}_n \in (X_a)_n\}.$$

Notice the following:

Lemma 5.8 *The number of the set of types of length n, $\sharp\mathfrak{P}_n$ grows polynomially with respect to n.*

The conditional entropy of the type P is given by:

$$h_c(P) = -\Sigma_{(z,w)\in\{1,\dots,a\}^2} P(z)P(w|z) \log P(w|z).$$

Let us fix $x, x' \in \{1, \dots, a\}$ and put $Q(w|z) \equiv Q((x,w)|(x',z))$.

Definition 5.5 *The divergence of a type P is defined by:*

$$D(P||Q) = \Sigma_{(z,w)\in\{1,\dots,a\}^2} P(z)P(w|z) \log \frac{P(w|z)}{Q(w|z)}.$$

Proposition 5.2 *(1) If $P(z|w) = Q(z|w)$ hold, then the divergence vanishes $D(P||Q) = 0$.*

(2) $D(P||Q) \geq 0$ holds.

Proof: (1) is clear.

For (2), notice the equalities $\Sigma_{w\in\{1,\dots,a\}}P(w|z) = \Sigma_{w\in\{1,\dots,a\}}Q(w|z) = 1$. Thus from the lemma 5.3, the estimate holds:

$$D(P||Q) = \Sigma_{z,P(z)\neq 0}P(z)\Sigma_w P(w|z)\log\frac{P(w|z)}{Q(w|z)} \geq 0.$$

This compeletes the proof.

5.C.2 Measure estimates: Let us put:

$$\tilde{Q}_n(\bar{x}_n, \bar{y}_n) = \frac{Q_n(\bar{x}_n, \bar{y}_n)}{Q(x_0, y_0)}.$$

Proposition 5.3 *Let P be the type of $\bar{y}_n$.*

Suppose Q is memeryless in time, and Markov with respect to $\bar{x}_n$. Then the equality holds:

$$\tilde{Q}_n(\bar{x}_n : \bar{y}_n) = \exp(-n(h_c(P) + D(P||Q))).$$

Proof: By the assumption, the equalities hold:

$$\begin{aligned}
&Q_n(\bar{x}_n : \bar{y}_n) = Q(x_0, y_0)\Pi_{i=1}^n Q((x_i, y_i)|(x_{i-1}, y_{i-1})) \\
&\quad = Q(y_0)\Pi_{i=1}^n Q((y_i|y_{i-1})) \\
&\quad = Q(y_0)\Pi_{(w,z)\in\{1,\dots,a\}^2} Q(w|z)^{N((z,w)|\bar{y}_n)} \\
&\quad = Q(y_0)\Pi Q(w|z)^{nP(z)P(w|z)} \\
&\quad = Q(y_0)\Pi\exp(nP(z)P(w|z)\log Q(w|z)) \\
&\quad = Q(y_0)\Pi\exp(-n(-P(z)P(w|z)\log P(w|z) \\
&\qquad + P(z)P(w|z)\log\frac{P(w|z)}{Q(w|z)})) \\
&\quad = Q(y_0)\exp(-n(h_c(P) + D(P||Q))).
\end{aligned}$$

This completes the proof.

Corollary 5.4 *(1) $\tilde{P}_n(\bar{y}_n) \equiv P_n(\bar{y}_n)/P(y_0) = \exp(-nh_c(P))$ holds. In particular the estimate $P_n(\bar{y}_n) \geq \frac{1}{n}\exp(-nh_c(P))$ holds.*

(2) $-\frac{1}{n}\log\tilde{Q}_n(\bar{y}_n) \geq h_c(P)$ holds.

(1) follows by applying $P(w|z) = Q(w|z)$. (2) follows from positivity of the divergence, proposition 5.2(2).

Let us take two types $P, P' \in \mathfrak{P}_n$. We say that P' *is associated with* P, if the following holds:

$$P'_n(T^n(P)) = \sup_{Q \in \mathfrak{P}_n} P'_n(T^n(Q)).$$

P' may not be unique with respect to P.

Proposition 5.4 *(1) For any type $P \in \mathfrak{P}_n$, the inequality holds:*

$$|T^n(P)| \leq n \exp(nh_c(P)).$$

(2) If P' is associated with P, then the inequality holds:

$$\frac{1}{Cn^\alpha} \exp(n(h_c(P) + D(P||P'))) \leq |T^n(P)| \leq n \exp(nh_c(P))$$

where C, α are both universal constants.

Proof: (1) Notice the estimate $P_n(\bar{y}_n) \geq \frac{1}{n}\tilde{P}_n(\bar{y}_n)$ for any $\bar{y}_n \in T^n(P)$. Moreover for $\bar{y}_n, \bar{y}'_n \in T^n(P)$, one obtains the formula:

$$\begin{aligned} \tilde{P}_n(\bar{y}_n) &= \Pi_{N((z,w)|\bar{y}_n)>0} \frac{N((z,w)|\bar{y}_n)}{N(z|\bar{y}_n)} \\ &= \Pi_{N((z,w)|\bar{y}'_n)>0} \frac{N((z,w)|\bar{y}'_n)}{N(z|\bar{y}'_n)} = \tilde{P}_n(\bar{y}'_n). \end{aligned}$$

So $\tilde{P}_n(\bar{y}_n)$ are constant for any $\bar{y}_n \in T^n(P)$. Thus the estimates:

$$1 \geq P^n(T^n(P)) \geq \frac{1}{n}|T^n(P)|\tilde{P}_n(\bar{y}_n) \geq |T^n(P)|\frac{1}{n} \exp(-nh_c(P))$$

hold from cor 5.4 (1).

(2)

$$\begin{aligned} &1 = \Sigma_Q P'_n(T^n(Q)) \leq \Sigma_Q P'_n(T^n(P)) \leq Cn^\alpha P'_n(T^n(P)) \\ &\leq Cn^\alpha \tilde{P}'_n(T^n(P)) \leq Cn^\alpha |T^n(P)| \exp(-n(h_c(P) + D(P||P'))) \end{aligned}$$

where we used prop 5.3 and lem 5.8. This completes the proof.

Theorem 5.3 *Suppose Q is memoryless in time, and Markov with respect to $\bar{x}_n$.*

(1) Let P be any type. Then the inequality holds:

$$Q(T^n(P)) \leq n \exp(-nD(P||Q)).$$

(2) Suppose moreover $\inf_{z \in \{1,\dots,a\}} Q(z) \geq C > 0$.
If P' is associated with P, then the inequality holds:

$$\frac{C}{n^\alpha} \exp(-n(D(P||Q) + D(P||P'))) \\ \leq Q(T^n(P)) \leq n \exp(-nD(P||Q)).$$

Proof:

$$\begin{aligned} Q^n(T^n(P)) &= \Sigma_{\bar{y}_n \in T^n(P)} Q^n(\bar{x}_n, \bar{y}_n) \\ &= \Sigma_{\bar{y}_n \in T^n(P)} Q_1(y_0) \exp(-n(h_c(P) + D(P||Q)) \\ &\leq (\geq)(C)|T^n(P)| \exp(-n(h_c(P) + D(P||Q)). \end{aligned}$$

Thus combining with the prop 5.4, one obtains the desired result. This completes the proof.

Let us take any $\bar{x} \in X_a$ so that the corresponding Q is memoryless in time and Markov with respect to $\bar{x}$. Let us denote its restriction by $\bar{x}_n \in (X_a)_n$.

Let P be types of degree n. We define the *minimum decay exponent* by:

$$m_n = m(\bar{x}_n) \equiv \inf_P D(P||Q).$$

Corollary 5.5 *Let $\{f_1, \dots, f_a\}$ be a family of maps, and take $\bar{x} \in X_a$. Suppose Q is memoryless in time and Markov with respect to $\bar{x}$. Let $h_i(\bar{x}_n)$ be the interacting entropy.*

Then there is a polynomial p such that:

$$h_i(\bar{x}_n) \leq p(n) \exp(-nm_n)$$

holds. In particular when $m_n \geq \lambda > 0$ satisfies a lower bound by a positive number, then $h_i(\bar{x}_n) \to 0$.

Proof: Notice that when $Q(\bar{y}_n) > 0$ is positive, then the estimates hold:

$$Q(\bar{x}_n, \bar{y}_n) \geq \alpha\beta^n$$

where $\alpha = \min_y Q(y) > 0$ and $\beta = \min_{y,y'} Q(y,y')$.

By the log-sum inequality,

$$
\begin{aligned}
& - \Sigma_{\bar{y}_n \in T^n(P)} Q(\bar{y}_n) \log Q(\bar{y}_n) \leq -Q(T^n(P)) \log \frac{Q(T^n(P))}{|T^n(P)|} \\
& \leq n \exp(-nm_n) \log(\alpha^{-1}\beta^{-n}|T^n(P)|) \\
& \leq p'(n) \exp(-nm_n)
\end{aligned}
$$

hold. Since the number of the set of types $\mathfrak{P}_n$ is at most polynomial, the estimate $h_i(\bar{x}_n) \leq p(n)\exp(-nm_n)$ holds. Moreover when $m_n \geq \lambda > 0$ satisfies a lower bound by a positive number, then $h_i(\bar{x}_n) \to 0$. This completes the proof.

One may expect more precise bound by use of divergences and combinatorics of types. Let us put:

$$D(P) = \sup_{P''} D(P||P'')$$

where P'' are all types.

Question: Does the estimate holds:

$$\frac{C}{n^{\alpha}} \exp(-n(D(P||Q) + D(P))) \leq Q(T^n(P))$$

This will depend completely on combinatorics of types, whether there could exist some P for each P' so that P' is associated with P.

Let us put the *maximum decay exponent*:

$$l_n = l(\bar{x}_n) \equiv \sup_P D(P||Q) + D(P).$$

Conjecture: There exists a universal polynomial q of types so that the estimate

$$h_i(\bar{x}_n) \leq (q(n) + n(l_n + \log a)) \exp(-nm_n)$$

holds.

This will follow by assuming the existence of the associated type P' for each P.

5.D Stationary distribution and low of large numbers: Let Q be an $a \times a$ *probability matrix*. We say that Q is *ergodic*, if there is some $n_0 \geq 1$ so that for n_0-times multiplication of the matrix Q^{n_0}, the following positivity hold:

$$q_{i,j}^{n_0} > 0, \quad i,j \in \{1,\ldots,a\}, \quad Q^{n_0} = (q_{i,j}^{n_0}).$$

Let $\mu = (\mu_1, \ldots, \mu_a)$ be a distribution of probability. Thus each $\mu_i \geq 0$ and $\Sigma_i \mu_i = 1$ hold.

The next is well known ergodic theorem for Markov chain:

Theorem 5.4 *Let us choose $\bar{x}$ so that the corresponding Q is memoryless in time and Markov up to infinity.*

Suppose Q is ergodic. Then there is a unique distribution of probability $\mu^0 = (\mu_1^0, \ldots, \mu_a^0)$ so that the followings hold:

(1) $\mu^0 Q = \mu^0$, and

(2) $\lim_{n\to\infty} q_{i,j}^n = \mu_j^0$ for $j = 1, \ldots, a$, where the convergence is of exponential order.

We say that μ_0 is the *stationary distribution*.

Let us choose a family of maps $\{f_1, \ldots, f_a\}$ and $\bar{x} \in X_a$ so that the corresponding Q is memoryless in time and Markov up to infinity. Let $Q((x_i, w)|(x_{i-1}, z)) = Q(w|z)$ be the conditional probability for Q, and regard $Q = \{Q(w|z)\}_{z,w \in X_a}$ as an $a \times a$ matrix.

Definition 5.6 *Q is ergodic with respect to $\bar{x} \in X_a$, if it is memoryless and Markov up to infinity and $Q_1 \equiv (Q(1), \ldots, Q(a))$ is the stationary distribution.*

Example: Let f and g be tent map and its reverse correspondingly. Then $Q = (\frac{1}{2})$ which is ergodic, and the stationary distribution is given by $\mu^0 = (\frac{1}{2}, \frac{1}{2}) = (Q(1), Q(2))$.

Now we have the low of large numbers obtained by Kolmogorov:

Theorem 5.5 *Suppose Q is ergodic with respect to $\bar{x}$. Then for any positive $\epsilon > 0$ and $w, z \in \{1, \ldots, a\}$,*

$$Pr\ \{|\frac{N((z,w)|\bar{y}_n)}{n} - Q(z)Q((w|z))| > \epsilon\} \to 0$$

as $n \to \infty$.

Remark 5.1*:* Let us put:

$$Q(\bar{x}_n : (y_0, y_n)) \equiv \Sigma_{y_1,\dots,y_{n-1} \in \{1,\dots,a\}} Q(\bar{x}_n : (y_0, y_1, \dots, y_n)).$$

For non Markov case, still one can define a notion of ergodicity, by requiring that for $\bar{x} \in X_a$ and its restrictions $\bar{x}_n$,

$$\lim_{n \to \infty} Q(\bar{x}_n : (b, a)) = Q(a)$$

converges exponentially which is independent of choice of b. In this case also one may expect to hold low of large numbers type statement.

5.D.2 Neighbourhoolds of the standard measure: Let us take a famliy of maps $\{f_1, \dots, f_a\}$, and choose $\bar{x}$ so that the corresponding Q is memoryless in time and Markov up to infinity.

We say that $\bar{y}_n$ is ϵ-*typical*, if for any pair (z, w) with $Q(w|z) \neq 0$, the inequalities:

$$|N((z, w)|(\bar{x}_n, \bar{y}_n)) - nQ(z)Q(w|z)| \leq \frac{\epsilon n Q(z) Q(w|z)}{\log a}$$

hold. We denote the set of ϵ-typical words by:

$$T^n_\epsilon(Q) \subset (X_a)_n.$$

In the memoryless in time case, the conditional entropy $h_c(x_1, x_2)$ does not depend on $(x_1, x_2) \in \{1, \dots a\}^2$, and so we have denoted it as $h_c(Q)$ (5.C).

Proposition 5.5 *(1) For any $\bar{y}_n \in T^n_\epsilon(Q)$, the inequality holds:*

$$|\frac{1}{n} \log Q(\bar{x}_n, \bar{y}_n) + h_c(Q)| \leq \epsilon' = \frac{2\epsilon h_c(Q)}{\log a}$$

for all large $n = n(\epsilon)$.

(2) If Q is ergodic, then for any small $\lambda > 0$, the estimates:

$$Pr\ (Y^n \in T^n_\epsilon(Q)) \geq 1 - \lambda$$

holds for all large $n = n(\lambda)$.

(3) Under the assumption, the inequalities hold:

$$(1-\lambda)\exp(n(h_c(Q)-\epsilon)) \le |T_\epsilon^n(Q)| \le \exp(n(h_c(Q)+\epsilon)).$$

Proof: (1) The estimates hold:

$$\begin{aligned}
&|\frac{1}{n}\log Q(\bar{x}_n,\bar{y}_n) + h_c(Q)| \le -\frac{1}{n}\log Q(x_1,y_1) \\
&\quad + |\Sigma_{(z,w)}\frac{N((z,w)|\bar{y}_n)}{n}\log Q(w|z) - \Sigma_{(z,w)}Q(z)Q(w|z)\log Q(w|z)| \\
&\quad \le |\Sigma_{(z,w)}\{\frac{N((z,w)|\bar{y}_n)}{n} - Q(z)Q(w|z)\}\log Q(w|z)| + \frac{C}{n} \\
&\quad \le \Sigma_{(z,w)} - \frac{\epsilon Q(z)Q(w|z)\log Q(z,w)}{\log a} + \frac{C}{n} \\
&\quad = \frac{\epsilon h_c(Q)}{\log a} + \frac{C}{n} \le \epsilon.
\end{aligned}$$

(2) By theorem 5.4 (1) and the definition of ergodicity, $Q(z) \neq 0$ for all z. Then by the assumption and theorem 5.4,

$$\begin{aligned}
\Pr\,(\bar{Y}_n \in T_\epsilon^n(Q)^c) \le \Sigma_{(z,w)} \Pr\,\{&|N((z,w)|\bar{y}_n) - nQ(z)Q(w|z)| \\
&> \frac{\epsilon n Q(z)Q(z,w)}{\log a})\} \to 0
\end{aligned}$$

as $n \to \infty$. So (2) follows.

(3) Since the inequalities hold:

$$1 = \Sigma_{\bar{y}_n}Q(\bar{y}_n) \ge \Sigma_{\bar{y}_n \in T_\epsilon^n(Q)}Q(\bar{y}_n) \ge |T_\epsilon^n(Q)|\exp(-n(h_c(Q)+\epsilon))$$

one obtains the estimate:

$$|T_\epsilon^n(Q)| \le \exp(n(h_c(Q)+\epsilon)).$$

Conversely, one obtains the another estimate:

$$(1-\lambda) \le \Sigma_{\bar{y}_n \in T_\epsilon^n(Q)}Q(\bar{y}_n) \le |T_\epsilon^n(Q)|\exp(-n(h_c(Q)-\epsilon))$$

where we have applied (2) above. Thus one obtains the desired result.

This completes the proof.

5.E Complexity: Let $X_a^0 = \{x_0 \ldots, x_k; x_i \in \{1, \ldots, a\}, k = 1, 2, \ldots\}$ be all the set of words of finite length.

Let us decompose any word $\bar{x}_n = x_0 \ldots x_n \in (X_a)_n$ into mutually different subwords as:

$$\bar{x}_n = \bar{x}^1 \bar{x}^2 \ldots \bar{x}^s$$

where $\bar{x}^i, \bar{x}^j \in X_a^0$ are mutually different.

The largest numger s is called the *Lempel-Ziv complexity* (LV complexity):

$$c(\bar{x}_n) = \sup\{s : \bar{x}_n = \bar{x}^1 \bar{x}^2 \ldots \bar{x}^s; \bar{x}^i \neq \bar{x}^j, i \neq j\}.$$

For example, for $\bar{x}_4 = 0010 = (00)(1)(0)$ and $c(0010) = 3$.

The following estimate is known:

Lemma 5.9 (LZ) *There are constants $\epsilon_n \to 0$ so that the inequalities hold:*

$$\sqrt{2(n+1)} - 1.5 < c(\bar{x}_n) < \frac{n+1}{(1 - \epsilon_{n+1}) \log_a (n+1)}.$$

Let $\bar{x}_n = \bar{x}^1 \bar{x}^2 \ldots \bar{x}^c$ be a decomposition. Then correspondingly one defines a family of numbers $c_{l,s}$ as the number of j such that $|\bar{x}^j_{n_{j-1}+1}| = l$ and $\bar{x}^{j-1}_{n_{j-1}}$ ends with $s \in \{1, \ldots, a\}$.

So $\Sigma_{l,s} c_{l,s} = c$ and $\Sigma_{l,s} l c_{l,s} = n$ hold.

Lemma 5.10 (Ziv's inequality) *Let us choose $\bar{x}_n$ so that the corresponding Q is memoryless in time and Markov up to n. Let us decompose any $\bar{y}_n$ into mutually disjoint subwords. Then the inequalities hold:*

$$\log Q(\bar{x}_n, \bar{y}_n) \leq -\Sigma_{l,s} c_{l,s} \log c_{l,s}.$$

Theorem 5.6 (LZ) *Let us choose $\bar{x}$ so that the corresponding Q is memoryless and Markov. Then ther is a bound by the interacting entropy:*

$$\limsup_{n \to \infty} E_Q\left(\frac{c(\bar{x}_n, \bar{Y}_n) \log c(\bar{x}_n, \bar{Y}_n)}{n}\right) \leq h_i(\bar{x}).$$

Proof: For convenience we give an outline of the proof.

By the Ziv's inequality,

$$\log Q(\bar{x}_n, \bar{y}_n) \leq -c \log c - c\Sigma_{l,s} \frac{c_{l,s}}{c} \log \frac{c_{l,s}}{c}$$

hold. Thus:

$$-\frac{1}{n} \log Q(\bar{x}_n, \bar{y}_n) \geq \frac{c \log c}{n} - \frac{c}{n} \Sigma_{l,s} \frac{c_{l,s}}{c} \log \frac{c_{l,s}}{c}$$

holds. Using lemma 5.9, one can verify the second term converges to zero as $n \to \infty$. So:

$$-\frac{1}{n} \log Q(\bar{x}_n, \bar{y}_n) \geq \frac{c \log c}{n} + \delta(n) \geq \frac{c \log c}{n} - \delta(n)$$

where $\delta(n) \to 0$ as $n \to \infty$.

By taking the expectation of the both sides, one obtains:

$$\frac{1}{n} E(c(\bar{x}_n, \bar{Y}_n) \log c(\bar{x}_n, \bar{Y}_n)) \leq h_i(\bar{x}_n, \bar{Y}_n) + \delta(n)$$

holds. Thus letting $n \to \infty$, one obtains the desired estimate.

This completes the proof.

6 Codings

6.A Coding: A *coding* is a reverse operation to contraction. Below we will define a *regular coding* from $\{g_j\}_j$ to $\{f_i\}_i$ given by a contraction from $\{f_i\}_i$ to $\{g_j\}_j$.

We denote by X_t^0 the set of all words of finite length by $\{1, \ldots, t\}$. Let $\{f_i\}_{i=1}^t$ be a family of maps on the interval, and $E \subset X_t^0$ be a finite subset. An *inhomogeneous contraction* is a possibly multi-valued surjection:

$$\psi_0 : E \subset X_t^0 \to (X_a)_n.$$

A *coding* $\varphi_0 : (X_a)_n \to E \subset X_t^0$ is given by a map $\varphi_0 = \psi_0^{-1}$ for an injective multi-valued inhomogeneous contraction ψ_0.

A coding induces a continuous map:

$$\varphi : X_a \to X_t.$$

φ_0 is called *regular* if φ is an isomorphism.

Definition 6.1 *A regular coding φ_0 from $\{g_j\}_{j=1}^a$ to $\{f_i\}_{i=1}^t$ is given by $\varphi_0 = \psi_0^{-1}$, where $\{g_j\}_j$ is obtained from $\{f_i\}_i$ by the regular contraction $\psi_0 : E \cong (X_a)_n$.*

Example: Let f be $f|[0, \frac{1}{2}](x) = -x + 1$ and $f|[\frac{1}{2}, 1](x) = x$, $g(x) = -x + 1$ and $h(x) = -f(x) + 1$. Then one has the equalities:

$$gf = h, \quad fg = f, \quad g^2 = id.$$

Thus one obtains a contraction from $\{f, g\}$ to $\{f, h, id\}$ as:

$$\psi_0 : \{1 \to 1, \quad (1,2) \to 2, \quad (2,2) \to 3\}.$$

It is a regular contraction, and so $\varphi_0 = \psi_0^{-1}$ gives a regular coding from $\{f, h, id\}$ to $\{f, g\}$.

Let φ_0 be a regular coding.

Theorem 6.1 (Kraft's inequality) *A regular coding $\varphi_0 : (X_a)_n \to E \subset X_t^0$ satisfies the inequality:*

$$\Sigma_{\bar{x}_n \in (X_a)_n} t^{-|\varphi_0(\bar{x}_n)|} \leq 1$$

where $|\varphi_0(\bar{x}_n)|$ is the length of the word.

Example: Let a be a power of 2 and consider a map $\varphi_0 : \{1, \dots, a\} \to X_2^0$. For simplicity suppose $a = 2$. One can naturally associate $0 \to (0,0), 1 \to (0,1), 2 \to (1,0), 3 \to (1,1)$. This assignment has a canonical extension to the general case of a. They are all regular.

Definition 6.2 *Let $\{f_0, \dots, f_{a-1}\}$ be a family of maps with the interaction map $\Phi(\{f_i\}_i) : [0,1] \times X_a \to [0,1] \times X_a$.*

A commutative coding at $x \in [0,1]$ consistes of a coding $\varphi_0 : (X_a)_n \to X_t^0$ from $\{g_i\}_i$ to $\{f_j\}_j$ so that the following is a commutative coding:

$$\begin{array}{ccc} X_a & \xrightarrow{\varphi} & X_t \\ \Phi(\{g_i\}_i)(x) \downarrow & & \Phi(\{f_j\}_j)(x) \downarrow \\ X_a & \xrightarrow{\varphi} & X_t \end{array}$$

We denote the set of the commuting points above by $I \subset [0,1]$.

Example: Let f_1 be the tent map and f_2 be its reverse. Let us put $g_1 = f_1 f_2$ and $g_2 = f_2 f_1$. Then $\varphi_0 : \{1 \to (2,1), 2 \to (1,2)\}$ is a regular and commutative coding on $I = R(f_1, f_2) = [0,1] \backslash \{\frac{k}{2^n}; k, n = 1, 2, \dots\}$ $(4.B)$.

6.B The average code length: Let $\varphi_0 : (X_a)_n \to E \subset X_t^0$ be a coding.

We will estimate the *average code length* of φ_0 by the interacting entropy. Let $n+1 = |\bar{y}_n|$ be the length of the word.

Definition 6.3 *Let $\bar{x} \in X_a$, $\bar{x}_n \in (X_a)_n$ be its restriction, and Q be the corresponding measure. The average code length is given by:*

$$\bar{n}(\varphi_0, \bar{x}_n) = \frac{1}{n+1} \Sigma_{\bar{y} \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n) |\varphi_0(\bar{y}_n)|,$$
$$\bar{n}(\varphi_0, \bar{x}) = \limsup_{n \to \infty} \bar{n}(\varphi_0, \bar{x}_n).$$

Proposition 6.1 *Suppose a coding φ_0 is regular. Then the average coding length has a bound from below by the interacting entropy:*

$$\bar{n}(\varphi_0, \bar{x}_n) \log t \geq h_i(\bar{x}_n).$$

Proof: Since a regular coding satisfies the Kraft's inequality, it follows:

$$0 \leq -\log \Sigma t^{-|\varphi_0(\bar{y}_n)|} = \Sigma_{\bar{y}_n \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n) \log \frac{\Sigma_{\bar{y}_n \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n)}{\Sigma t^{-|\varphi_0(\bar{y}_n)|}}$$
$$\leq \Sigma_{\bar{y}_n \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n) \log \frac{Q(\bar{x}_n, \bar{y}_n)}{t^{-|\varphi_0(\bar{y}_n)|}}$$
$$= \Sigma_{\bar{y}_n \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n) \log Q(\bar{x}_n, \bar{y}_n) + \log t \Sigma_{\bar{y}_n \in (X_a)_n} Q(\bar{x}_n, \bar{y}_n) |\varphi_0(\bar{y}_n)|$$
$$= -(n+1) h_i(\bar{x}_n) + (n+1) \bar{n}(\varphi_0, \bar{x}_n) \log t.$$

This completes the proof.

In particular one has the estimates:

$$\bar{n}(\varphi_0, \bar{x}) \log t \geq h_c(\bar{x}) = h_i(\bar{x}).$$

Shannon constructed an algorithm of regular codings which satisfy small avarage code lengths. In our setting, it may depend on the initial data $\bar{x}_n$:

Theorem 6.2 (S) *There exists a regular coding $\varphi_0 : (X_a)_n \to X_t^0$ which may depend on $\bar{x}_n$ so that the following inequalities hold:*

$$h_i(\bar{x}_n) \leq \bar{n}(\varphi_0, \bar{x}_n) \log t \leq h_i(\bar{x}_n) + \log t.$$

6.C Undecodability: Let $\varphi_0 : (X_a)_n \to (X_t)_m$ be a coding. The rational number:

$$R = \frac{m}{n}$$

is called the *coding rate*.

Theorem 6.3 *Let us choose $\bar{x} \in X_a$ so that the corresponding Q is ergodic.*

If a coding $\varphi_0 : (X_a)_n \to (X_t)_m$ satisfies the inequality:

$$R \log t < h_i(\bar{x})$$

then for any other map $\psi_0 : (X_t)_m \to (X_a)_n$, $\lambda > 0$ and for all large n, the estimate:

$$P_e \equiv Pr\ (\psi \circ \varphi(\bar{Y}_n) \neq \bar{Y}_n) \geq 1 - \lambda$$

holds.

Proof: By lemma 5.7, the equalities $h_c(Q) = h_c(\bar{x}) = h_i(\bar{x})$ hold.

Let us put $\mathbf{C} = \{\bar{y}_n : \psi \circ \varphi(\bar{y}_n) = \bar{y}_n)\} \subset (X_a)_n$. One can choose a positive $0 < \lambda$ and $\delta > 0$ so that the inequalities $\lambda < \delta < h_c(Q) - R \log t$ hold. Since φ is an injection on $\mathbf{C}$, the estimates hold:

$$|\mathbf{C}| \leq t^m = \exp(m \log t) < \exp(n(h_c(Q) - \delta)).$$

Recall $T_\epsilon^n(Q)$ in $5.D.2$.

By the assumption and proposition 5.5, for any small $\lambda > 0$, there is $\epsilon > 0$ so that the estimates:

$$\begin{aligned} &\Pr\ (Y^n \in T_\epsilon^n(Q)) \geq 1 - \lambda, \\ &|\frac{1}{n} \log Q(\bar{x}_n, \bar{y}_n) + h_c(Q)| \leq \lambda, \quad \bar{y}_n \in T_\epsilon^n(Q). \end{aligned}$$

holds for all large $n = n(\lambda)$.

Thus the inequalities hold:

$$\begin{aligned} 1 - P_e = & \ \Pr \ (\bar{Y}_n \in \mathbf{C}) \\ = & \ \Pr \ (\bar{Y}_n \in \mathbf{C} \cap T_\epsilon^n(Q)^c) + \ \Pr \ (\bar{Y}_n \in \mathbf{C} \cap T_\epsilon^n(Q)) \\ \leq & \ \lambda + |\mathbf{C} \cap T_\epsilon^n(Q)| \sup_{\bar{y}_n \in T_\epsilon^n(Q)} Q(\bar{x}_n, \bar{y}_n) \\ \leq & \ \lambda + \exp(n(h_c(Q) - \delta)) \exp(-n(h_c(Q) - \lambda)) \\ = & \ \lambda + \exp(-(\delta - \lambda)n) \leq 2\lambda. \end{aligned}$$

This completes the proof.

Example: Let f_1 be the tent map and f_2 be its reverse. In $4.B$, one has obtained a compression $\mathbf{C} : \{f_1, f_2\} \to \{g_1, g_2\}$. This is a critical case in the sense that the equality holds:

$$R \log 2 = \log 2 = h_i(\bar{x}) = \log 2.$$

In fact this coding is invertible.

7 Interacting bonds

In [K4], we have constructed dynamics of weighted graphs from a family of maps. Here by using algebraic structures of interaction like contraction, we will construct *multi graphs* which is given by an inclusion of a pair of graphs. We would interpret it as a state of interacting molecules, where each component of the smaller graph will be a molecule by covalent bond, and larger one will represent an interacting state of them by hydrogen bond.

7.B Interaction graphs: Let $f, g : [0,1] \to [0,1]$ be two interval maps and $\Phi(x, f, g) : X_2 \to X_2$ be the interaction map. Let us choose another map $d : [0,1] \to [0,1]$.

Suppose for a point $z \in [0,1]$ and some $\bar{k} \in X_2$, the following equality holds:

$$\Phi(x, f, g)(\bar{k}) = \pi((d(z), d^2(z), \dots)) \equiv (\pi(d(z)), \pi(d^2(z)), \dots).$$

Then we express this by a marked oriented edge as:

$$(f, x) \xrightarrow{(g, \bar{k})} (d, z).$$

Let us choose families of maps $\{f_0, \dots, f_k\}$ and points $\{x_0, \dots, x_l\}$. For each $(i, j, x) \in \{0, \dots, k\}^2 \times \{x_0, \dots, x_l\}$, let us assign an element $\bar{k}(i, j, x) \in X_2$. By this way we have chosen a family of elements $\{\bar{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l} \subset X_2$. Then we put two sets:

$$V = \{(f_i, x_j) : 0 \le i \le k, 0 \le j \le l\} \text{ (the set of vertices)},$$
$$E = \{e_{i,j,k} : (f_i, x_h) \xrightarrow{(f_j, \bar{k}(i,j,x_h))} (f_k, x_v) :\} \text{ (the set of edges)}.$$

The *interaction graph* is a marked oriented graph, where the set of vertices V and edges E are given as above. We denote it by:

$$G(\{f_i\}_i^k; \{x_j\}_j^l; \{\bar{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l})$$

7.B.2 Multi interaction graphs: Let us fix a triple of families, $\{x_0, \dots, x_l\}$, $\{\bar{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l}$ and $\{\bar{l}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l}$ Suppose that for each pair (f_i, f_j), a compression:

$$\{f_i, f_j\} \to \{g_i, g_j\}, \quad \varphi : X_2 \to X_2$$

is assigned.

Let us have two different sets:

$$E_h = \{e_{i,j,k} : (f_i, x_h) \xrightarrow{(f_j, \bar{k}(i,j,x_h))} (f_k, x_v)\} \text{ (the set of weak edges)},$$
$$E_c = \{e_{i,j,k} : (f_i, x_h) \xrightarrow{(f_j, \bar{k}(i,j,x_h))} (f_k, x_v), (g_i, x_h) \xrightarrow{(g_j, \bar{l}(i,j,x_h))} (g_k, x_v)\}$$
$$\text{(the set of strong edges)}.$$

The set of vertices $V = \{(f_i, x_j) : 0 \le i \le k, 0 \le j \le l\}$ are the same as before.

Definition 7.1 *The bi-interaction graph is a pair of the weighted graphs:*

$$G(\{f_i\}_i; \{g_i\}_i) \equiv (G_c \subset G_h)$$

where the set of edges are E_c and E_h for G_c and G_h respectively. The set of vertices are both the same V above.

We will say that G_c is the c-graph, and G_h is the h-graph.

Proposition 7.1 *Suppose all x_h are compressible points. If one chooses $\bar{l}(i,j,x_h) = \varphi(\bar{k}(i,j,x_h))$, then the c-graph and h-graph coincide each other:*

$$G_c = G_h.$$

Notice that the bi-interaction graphs are determined by the initial values $\{\bar{k}(i,j,x_h)\}_{i,j,h=0}^{i,j=k,h=l}$ and $\{\bar{l}(i,j,x_h)\}_{i,j,h=0}^{i,j=k,h=l}$. We will denote the set of bi-interaction graphs arising from the families $\{f_0,\dots,f_k\}$, $\{g_1,\dots,g_k\}$ and $\{x_0,\dots,x_l\}$ by:

$$\mathfrak{G}(\{f_i\}_i,\{g_i\}_i;\{x_j\}_{j=0}^l).$$

Let us put:

$$B_2^{k,l} \equiv X_2^{2((k+1)^2+l+1)} = X_2 \times X_2 \times \cdots \times X_2.$$

Then the family of the interaction map gives a continuous map:

$$\Phi : B_2^{k,l} \to B_2^{k,l}$$

where:

$$\begin{aligned}
&\Phi(\{\bar{k}(i,j,x)\},\{\bar{l}(i,j,x)\}) = (\{\bar{k}'(i,j,x)\},\{\bar{l}'(i,j,x)\}),\\
&\bar{k}'(i,j,x) \equiv \Phi(f_i,f_j,x)(\bar{k}(i,j,x)),\\
&\bar{l}'(i,j,x) \equiv \Phi(g_i,g_j,x)(\bar{l}(i,j,x)).
\end{aligned}$$

This induces a dynamics on the set of the bi-interaction graphs as:

$$\Phi_* : \mathfrak{G}(\{f_i\}_{i=0}^k,\{g_i\}_i;\{x_j\}_{j=0}^l) \to \mathfrak{G}(\{f_i\}_{i=0}^k,\{g_i\}_i;\{x_j\}_{j=0}^l)$$

by

$$\begin{aligned}
\Phi_*(G(\{f_i\}_i^k,&\{g_i\}_i;\{x_j\}_j^l;\{\bar{k}(i,j,x_h)\},\{\bar{l}(i,j,x_h\}))\\
&= G(\{f_i\}_i^k,\{g_i\}_i;\{x_j\}_j^l;\Phi(\{\bar{k}(i,j,x_h)\},\{\bar{l}(i,j,x)\})).
\end{aligned}$$

Let us choose an infinite set of points $\{x_j\}_{j=0}^\infty$, and equip with a topology on:

$$\mathfrak{G}(\{f_i\}_{i=0}^k,\{g_i\}_i;\{x_j\}_{j=0}^\infty) = \cup_l \mathfrak{G}(\{f_i\}_{i=0}^k,\{g_i\}_i;\{x_j\}_{j=0}^l)$$

induced from its finite subsets. Now one obtains the *topological entropy of the interaction graphs* by:

$$h_t(\Phi_*; \mathfrak{G}(\{f_i\}_{i=0}^k, \{g_i\}_i; \{x_j\}_{j=0}^\infty)).$$

One can choose a canonically $\bar{l}(i,j,x_h) = \varphi(\bar{k}(i,j,x_h))$. Thus one obtains a sequence of the bi-interaction graphs as:

$$\begin{aligned}
&((G_0^c, G_0^h), (G_1^c, G_1^h), \dots), \\
&(G_i^c, G_i^h) = G(\{f_i\}_i^k, \{g_i\}_i; \{x_j\}_j^l; \\
&\qquad\qquad \Phi^i((\{\bar{k}(i,j,x_h)\}, \{\varphi(\bar{k}(i,j,x_h))\})_{i,j,h=0}^{i,j=k,h=l}))).
\end{aligned}$$

This gives the *dynamics of the interaction graphs.*

Suppose all x_h are compressible points. Then the equalities hold:

$$G_i^c = G_i^h, \quad i = 0, 1, \dots$$

Thus the above sequence gives a dynamics of the single interaction graphs.

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Tsuyoshi Kato,
Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606-8502, Japan