Alcove path and Nichols-Woronowicz model of K-theory on flag varieties

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Dedicated to Professor Kenji Ueno on the occasion of his sixtieth birthday

Abstract

We give a model of the K-ring of the flag varieties in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. Our model is based on the construction of the path operators by C. Lenart and A. Postnikov.

Introduction

One of the main interests in the study of the Schubert calculus on the flag varieties is a combinatorial description of the cohomology ring of the flag varieties. Fomin and Kirillov [5] constructed a combinatorial model of the cohomology of the flag variety $\mathcal{F}l_n$ of type A_{n-1} as a subalgebra in a non-commutative quadratic algebra $\mathcal{E}_n$, which is an algebra generated over $\mathbf{Z}$ by the generators $[i,j]$, $1 \leq i,j \leq n$, $i \neq j$, and subject to the relations:
(0) $[i,j] = -[j,i]$,
(1) $[i,j]^2 = 0$,
(2) $[i,j][k,l] = [k,l][i,j]$, if $\{i,j\} \cap \{k,l\} = \emptyset$,
(3) $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

They introduced the Dunkl elements $\theta_1, \ldots, \theta_n$ in $\mathcal{E}_n$ by

$$\theta_i := \sum_{j \neq i} [i,j],$$

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which commute each other, and proved that the subalgebra generated by the Dunkl elements θ_i, $i = 1, \ldots, n$, in $\mathcal{E}_n$ is isomorphic to the cohomology ring of the flag variety $\mathcal{F}l_n$. It is remarkable that the algebra $\mathcal{E}_n$ has a natural quantum deformation. The deformed algebra $\tilde{\mathcal{E}}_n$ is an algebra over the polynomial ring $\mathbf{Z}[q_1, \ldots, q_{n-1}]$ obtained by replacing the relation (1) for the algebra $\mathcal{E}_n$ by the relation

$(1)'$ $[i,j]^2 = 0$, for $j > i+1$, and $[i, i+1]^2 = q_i$.

It was conjectured in [5] that the subalgebra of $\tilde{\mathcal{E}}_n$ generated by the Dunkl elements is isomorphic to the quantum cohomology ring $QH^*(\mathcal{F}l_n)$. This conjecture has been proved by Postnikov [19]. A generalization of their construction to other root systems is given by [9].

The Hopf algebra structure related to the algebra $\mathcal{E}_n$ has been studied by Fomin and Procesi [6]. The relationship between the algebra $\mathcal{E}_n$ and a braided Hopf algebra called the Nichols algebra was pointed out by Milinski and Schneider [18]. Conjecturally, the algebra $\mathcal{E}_n$ is isomorphic to the Nichols-Woronowicz algebra $\mathcal{B}(V_{S_n})$ associated to a Yetter-Drinfeld module V_{S_n} over the symmetric group S_n. Bazlov [2] constructed a model of the coinvariant algebra of the finite Coxeter group W as a subalgebra in the corresponding Nichols-Woronowicz algebra $\mathcal{B}(V_W)$. The natural braided differential operators acting on the Nichols-Woronowicz algebra play the key role in his argument.

The purpose of this paper is to construct the model of the K-ring of the flag variety G/B as a subalgebra in the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ associated to the Yetter-Drinfeld module V_W over the corresponding Weyl group W. We introduce the multiplicative analogue of the Dunkl elements and show that the subalgebra generated by the multiplicative Dunkl elements in $\mathcal{B}(V_W)$ is isomorphic to the K-ring $K(G/B)$. Kirillov and the author [11] constructed a model of $K(G/B)$ in $\mathcal{B}(V_W)$ when the Lie group G is of classical type or of type G_2. The main result in this paper is a generalization of the result in [11, Section 5] for an arbitrary simple Lie group G.

Let P be the weight lattice in the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra of the simple Lie group G. Denote by T the maximal torus in G and $R(T)$ the representation ring of T. The K-ring $K(G/B)$ has a presentation as a quotient of the group algebra of the weight lattice P. Let us consider an algebra isomorphism $\iota : \mathbf{Z}[P] = \bigoplus_{\lambda \in P} \mathbf{Z} \cdot e^{\lambda} \to R(T)$ such that $\iota(e^{\lambda})$ is the character χ^{λ} corresponding to the weight λ. Then the T-equivariant K-algebra $K_T(G/B)$ is isomorphic to the quotient algebra $R(T) \otimes \mathbf{Z}[P]/J$, where

$J := (1 \otimes f - \iota(f) \otimes 1 \mid f \in \mathbf{Z}[P]^W)$. Lenart and Postnikov [15] introduced the path operator $R^{[\lambda]}$ acting on $K_T(G/B)$ in order to give the Chevalley-type formula in $K_T(G/B)$ which describes the multiplication by the class of the line bundle L_λ on G/B associated to the weight λ. The path operator $R^{[\lambda]}$ is defined by using the alcove path $A_0 \stackrel{-\beta_1}{\rightarrow} \cdots \stackrel{-\beta_l}{\rightarrow} A_l$ which connects the fundamental alcove A° and its translation $A^{-\lambda} := A^\circ - \lambda$. The operator $R^{[\lambda]}$ is by definition a composite of the operators

$$R^{[\lambda]} = (1 + B_{\beta_l}) \cdots (1 + B_{\beta_1}),$$

where B_β is the Bruhat operator studied by Brenti, Fomin and Postnikov [3]. Our construction of the model of $K(G/B)$ in $\mathcal{B}(V_W)$ is based on Lenart and Postnikov's definition of the operators $R^{[\lambda]}$ in [15].

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1 Nichols-Woronowicz algebra

The Nichols-Woronowicz algebra associated to a braided vector space is an analogue of the polynomial ring in a braided setting. For details on the Nichols-Woronowicz algebra, see [1], [2] and [17].

Let (V, ψ) be a braided vector space, i.e. a vector space equipped with a linear isomorphism $\psi : V \otimes V \to V \otimes V$ such that the braid relation $\psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1}$ is satisfied on $V^{\otimes n}$, where ψ_i is the linear endomorphism on $V^{\otimes n}$ obtained by applying ψ on the i-th and $(i+1)$-st components. In the following, we assume that V is a finite dimensional $\mathbf{Q}$-vector space. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced decomposition of an element $w \in S_n$, where $s_i = (i\ i{+}1)$ is the simple transposition. Then the linear map $\Psi_w := \psi_{i_1} \cdots \psi_{i_l}$ on $V^{\otimes n}$ is independent of the choice of reduced decompositions of w because of the braid relation. We define the Woronowicz symmetrizer on $V^{\otimes n}$ by $\sigma_n(\psi) := \sum_{w \in S_n} \Psi_w$. Such a definition of the braided analogue of the symmetrizer (or anti-symmetrizer) is due to Woronowicz [20].

Definition 1.1 (see [2] and [17]) *The Nichols-Woronowicz algebra $\mathcal{B}(V)$ associated to the braided vector space (V, ψ) is defined as the quotient of the tensor algebra $T(V)$ by the ideal $\bigoplus_{n \geq 0} \mathrm{Ker}(\sigma_n(\psi))$.*

Remark 1.1 For a more systematic treatment, we should work in a fixed braided category $\mathcal{C}$ of vector spaces. If the braided vector space (V, ψ) is an object in the braided vactor space, the tensor algebra $T(V)$ has a natural braided Hopf algebra structure in $\mathcal{C}$. It is known that the kernel $\bigoplus_{n\geq 0} \mathrm{Ker}(\sigma_n(\psi))$ is a Hopf ideal of $T(V)$. Hence, $\mathcal{B}(V)$ is also a braided Hopf algebra in $\mathcal{C}$.

The following is the alternative definition of the Nichols-Woronowicz algebra due to Andruskiewitsch and Schneider [1]. In [1], the algebra $\mathcal{B}(V)$ is called the Nichols algebra.

Definition 1.2 ([1]) *The graded braided Hopf algebra $\mathcal{B}(V)$ is called the Nichols-Woronowicz algebra if it satisfies the following conditions:*
(1) $\mathcal{B}^0(V) = \mathbf{Q}$,
(2) $V = \mathcal{B}^1(V) = \{x \in \mathcal{B}(V) \mid \triangle(x) = x \otimes 1 + 1 \otimes x\}$,
(3) $\mathcal{B}(V)$ *is generated by* $\mathcal{B}^1(V)$ *as a* $\mathbf{Q}$*-algebra.*

We use a particular braided vector space called the Yetter-Drinfeld module in the subsequent construction. Let Γ be a finite group.

Definition 1.3 *A* $\mathbf{Q}$*-vector space* V *is called a Yetter-Drinfeld module over* Γ, *if*
(1) V *is a* Γ*-module,*
(2) V *is* Γ*-graded, i.e.* $V = \bigoplus_{g\in\Gamma} V_g$, *where* V_g *is a linear subspace of* V,
(3) *for* $h \in \Gamma$ *and* $v \in V_g$, $h(v) \in V_{hgh^{-1}}$.

The category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of the Yetter-Drinfeld modules over a fixed group Γ is naturally braided. The tensor product of the objects V and V' of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is again a Yetter-Drinfeld module with the Γ-action $g(v \otimes w) = g(v) \otimes g(w)$ and the Γ-grading $(V \otimes V')_g = \bigoplus_{h,h'\in\Gamma,\, hh'=g} V_h \otimes V'_{h'}$. The braiding between V and V' is defined by $\psi_{V,V'}(v \otimes w) = g(w) \otimes v$, for $v \in V_g$ and $w \in V'$.

Fix a Borel subgroup B in a simple Lie group G. Let Δ be the set of the roots and Δ_+ the set of the positive roots corresponding to B. We define a Yetter-Drinfeld module $V_W := \bigoplus_{\alpha\in\Delta} \mathbf{Q}\cdot[\alpha]/([\alpha]+[-\alpha])$ over the Weyl group W. The W-action on V_W is given by $w([\alpha]) = [w(\alpha)]$. The W-degree of the symbol $[\alpha]$ is the reflection s_α. Note that $[\alpha]^2 = 0$, $\forall\alpha \in \Delta$, in the associated Nichols-Woronowicz algebra $\mathcal{B}(V_W)$. It is also easy to see $[\alpha][\beta] = [\beta][\alpha]$ when $s_\alpha s_\beta = s_\beta s_\alpha$. The following proposition can be shown by checking the quadratic relations in $\mathcal{B}(V_{S_n})$ via direct computations of the symmetrizer.

Proposition 1.1 *Fix the standard orthonormal basis* $\{e_1, \ldots, e_n\}$ *of* $\mathbf{Q}^n$*. Let* $\Delta = \{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\}$ *be the root system of type* A_{n-1}*. Then there exists a surjective algebra homomorphism*

$$\begin{array}{rccc} \eta: & \mathcal{E}_n & \to & \mathcal{B}(V_{S_n}) \\ & [i,j] & \mapsto & [e_i - e_j]. \end{array}$$

Conjecture 1.1 The algebra homomorphism η is an isomorphism.

This conjecture is now confirmed up to $n = 6$.

Take the standard realization of the root system of rank two with respect to an orthonormal basis $(e_i)_i$ as follows:
$(A_1 \times A_1) : \Delta_+^{A_1 \times A_1} = \{a_1 = e_1,\ a_2 = e_2\}$,
$(A_2) : \Delta_+^{A_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1 - e_3,\ a_3 = e_2 - e_3\}$,
$(B_2) : \Delta_+^{B_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1,\ a_3 = e_1 + e_2,\ a_4 = e_2\}$,
$(C_2) : \Delta_+^{C_2} = \{a_1 = e_1 - e_2,\ a_2 = 2e_1,\ a_3 = e_1 + e_2,\ a_4 = 2e_2\}$,
$(G_2) : \Delta_+^{G_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1 - 2e_2 + e_3,\ a_3 = -e_2 + e_3,\ a_4 = -e_1 - e_2 + 2e_3,\ a_5 = -e_1 + e_3,\ a_6 = -2e_1 + e_2 + e_3\}$.

If the set Δ of the roots contains a subset of form Δ_+^X, $X = A_1 \times A_1$, A_2, B_2, C_2 or G_2, then one can check the following relations are satisfied in the algebra $\mathcal{B}(V_W)$ (see also [11]).
$(A_1 \times A_1) : [a_1][a_2] = [a_2][a_1]$,
$(A_2) : [a_1][a_2] + [a_2][a_3] = [a_3][a_1]$,
$(B_2, C_2) : [a_1][a_2] + [a_2][a_3] + [a_3][a_4] = [a_4][a_1]$,

$$[a_1][a_2][a_3][a_2] + [a_2][a_3][a_2][a_1] + [a_3][a_2][a_1][a_2] + [a_2][a_1][a_2][a_3] = 0,$$
$$[a_2][a_3][a_4][a_3] + [a_3][a_4][a_3][a_2] + [a_4][a_3][a_2][a_3] + [a_3][a_2][a_3][a_4] = 0,$$
$$[a_1][a_2][a_3][a_4] = [a_4][a_3][a_2][a_1],$$

$(G_2) : [a_1][a_2] + [a_2][a_3] + [a_3][a_4] + [a_4][a_5] + [a_5][a_6] = [a_6][a_1]$,

$$[a_1][a_2][a_1][a_3] + [a_1][a_3][a_1][a_2] + [a_1][a_3][a_2][a_3]$$
$$= [a_2][a_1][a_3][a_1] + [a_3][a_2][a_3][a_1] + [a_3][a_1][a_2][a_1],$$
$$[a_6][a_5][a_6][a_4] + [a_6][a_4][a_6][a_5] + [a_6][a_4][a_5][a_4]$$
$$= [a_5][a_6][a_4][a_6] + [a_4][a_5][a_4][a_6] + [a_4][a_6][a_5][a_6],$$

$$
\begin{aligned}
&[a_1][a_2][a_4][a_5]+[a_2][a_3][a_4][a_5]+[a_2][a_3][a_5][a_6]+[a_5][a_3][a_4][a_2]\\
&=[a_3][a_4][a_2][a_3]+[a_3][a_4][a_3][a_4]+[a_4][a_5][a_3][a_4]+[a_6][a_4][a_3][a_1],\\
&[a_5][a_4][a_2][a_1]+[a_5][a_4][a_3][a_2]+[a_6][a_5][a_3][a_2]+[a_2][a_4][a_3][a_5]\\
&=[a_3][a_2][a_4][a_3]+[a_4][a_3][a_4][a_3]+[a_4][a_3][a_5][a_4]+[a_1][a_3][a_4][a_6],\\
&[a_1][a_2][a_3][a_4][a_5][a_6]=[a_6][a_5][a_4][a_3][a_2][a_1].
\end{aligned}
$$

The complete set of the independent defining relations for the algebra $\mathcal{B}(V_W)$ has not yet been determined in general. The relations listed above imply the following.

Proposition 1.2 *The elements $h_\alpha := 1+[\alpha]$, $\alpha \in \Delta$, satisfy the Yang-Baxter equations, i.e., if Δ contains a subset Δ' of form Δ_+^X, $X = A_1 \times A_1$, A_2, B_2, C_2 or G_2, then the elements h_α, $\alpha \in \Delta'$, satisfy the equations as follows:*
$(A_1 \times A_1) : h_{a_1}h_{a_2} = h_{a_2}h_{a_1}$,
$(A_2) : h_{a_1}h_{a_2}h_{a_3} = h_{a_3}h_{a_2}h_{a_1}$,
$(B_2, C_2) : h_{a_1}h_{a_2}h_{a_3}h_{a_4} = h_{a_4}h_{a_3}h_{a_2}h_{a_1}$,
$(G_2) : h_{a_1}h_{a_2}h_{a_3}h_{a_4}h_{a_5}h_{a_6} = h_{a_6}h_{a_5}h_{a_4}h_{a_3}h_{a_2}h_{a_1}$.

For a general braided vector space V, the elements $v \in V$ act on the algebra $\mathcal{B}(V^*)$ as braided differential operators. In the subsequent construction, we use the braided differential operator D_α for a positive root α acting on the algebra $\mathcal{B}(V_W)$ determined by the conditions:
(0) $D_\alpha(t) = 0$, for $t \in \mathcal{B}^0(V_W) = \mathbf{Q}$,
(1) $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(2) $D_\alpha(FF') = D_\alpha(F)F' + s_\alpha(F)D_\alpha(F')$ for $F, F' \in \mathcal{B}(V_W)$.
We set $D_\alpha := -D_{-\alpha}$ if α is a negative root. The following lemma is a key in the proof of the main theorem.

Lemma 1.1 (See [18, Proposition 2.4] and [2, Criterion 3.2].)

$$\cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha) = \mathcal{B}^0(V_W) = \mathbf{Q}.$$

2 Alcove path and multiplicative Chevalley elements

In this section, we define a family of the elements Ξ^λ, $\lambda \in P$, in the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ following the construction of the path operators due to Lenart and Postnikov [15].

Let W_{aff} be the affine Weyl group of the dual root system $\Delta^\vee := \{\alpha^\vee = 2\alpha/(\alpha,\alpha) \mid \alpha \in \Delta\}$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbf{Z}$, with respect to the affine hyperplanes $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^* \mid (\lambda, \alpha^\vee) = k\}$. The connected components of $\mathfrak{h}^* \setminus \cup_{\alpha\in\Delta, k\in\mathbf{Z}} H_{\alpha,k}$ are called alcoves. The fundamental alcove A° is the alcove defined by the inequalities $0 < (\lambda, \alpha^\vee) < 1$, $\forall \alpha \in \Delta_+$.

Definition 2.1 (1) *A sequence $(A_0, \ldots, A_l)$ of alcoves A_i is called an alcove path if A_i and A_{i+1} have a common wall and $A_i \neq A_{i+1}$.*
(2) *An alcove path $(A_0, \ldots, A_l)$ is called reduced if the length l of the path is minimal among those of alcove paths connecting A_0 and A_l.*
(3) *We use the symbol $A_i \xrightarrow{\beta} A_{i+1}$ when A_i and A_{i+1} have a common wall of form $H_{\beta,k}$ and the direction of the root β is from A_i to A_{i+1}.*

Let $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta_+$ be the set of the simple roots. Denote by ω_i the fundamental dominant weight corresponding to a simple root α_i, i.e. $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$. Take an alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ connecting $A_0 = A^\circ$ and $A_l = A^{-\lambda} := A^\circ - \lambda$. The sequence $(\beta_1, \ldots, \beta_l)$ appearing here is called a λ-chain.

Definition 2.2 *We define the elements $\Xi^{[\lambda]}$ in $\mathcal{B}(V_W)$ for $\lambda \in P$ by the formula*

$$\Xi^{[\lambda]} = h_{\beta_l} \cdots h_{\beta_1}.$$

We call the elements $\Xi_i := \Xi^{\omega_i}$ the multiplicative Chevalley elements.

The element $\Xi^{[\lambda]}$ is independent of the choice of an alcove path from A° to $A^{-\lambda}$. Because the elements h_α satisfy the Yang-Baxter equations and $h_\alpha h_{-\alpha} = 1$, the argument of [15, Lemma 9.3], which is implicitly given by Cherednik [4], is applicable to our case.

We use the following results from [15].

Lemma 2.1 ([15, Lemmas 12.3 and 12.4]) *Let $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ be an alcove path from $A_0 = A^\circ$ to $A_l = A^{-\lambda}$.*
(1) *The sequence $(\alpha_i, s_{\alpha_i}(\beta_1), \ldots, s_{\alpha_i}(\beta_l), -\alpha_i)$ is an $s_{\alpha_i}(\lambda)$-chain for $i = 1, \ldots, r$.*
(2) *Asume that $\beta_j = \pm\alpha_i$ for some $1 \leq j \leq l$ and $1 \leq i \leq r$. Denote by s the reflection with respect to the common wall of A_{l-j} and A_{l-j+1}. Then the sequence $(\alpha_i, s_{\alpha_i}(\beta_1), \ldots, s_{\alpha_i}(\beta_{j-1}), \beta_{j+1}, \ldots, \beta_l)$ is an $s(\lambda)$-chain.*

Proposition 2.1 ([15, Proposition 12.2]) *We have $\Xi^{[\lambda]} \cdot \Xi^{[\lambda']} = \Xi^{[\lambda'+\lambda]}$, for $\lambda, \lambda' \in P$. In particular, they commute each other.*

3 Main result

We define the operator $\pi_i : \mathbf{Z}[P] \to \mathbf{Z}[P]$ by the formula

$$\pi_i(f) := \frac{f - s_{\alpha_i}(f)}{e^{\alpha_i} - 1}$$

for the simple root α_i. The operator π_i is characterized by the conditions:
(1) $\pi_i(e^{\omega_j}) = \delta_{i,j} e^{\omega_i - \alpha_i}$,
(2) $\pi_i(fg) = \pi_i(f)g + s_{\alpha_i}(f)\pi_i(g)$.
We have an algebra homomorphism

$$\begin{array}{cccc} \varphi : & \mathbf{Z}[P] & \to & \mathbf{Z}[\Xi_1, \ldots, \Xi_r] \\ & e^{\omega_i} & \mapsto & \Xi_i. \end{array}$$

Proposition 3.1 *Let f be an element in $\mathbf{Z}[P]$. We have*

$$\Pi_i(\varphi(f)) = \varphi(\pi_i f),$$

where $\Pi_i = h_{\alpha_i}^{-1} \circ D_{[\alpha_i]}$.

Proof. It is enough to check that the operator Π_i satisfies the conditions:
(1) $\Pi_i(\Xi_j) = \delta_{i,j} \Xi^{[\omega_i - \alpha_i]}$,
(2) $\Pi_i(FF') = \Pi_i(F)F' + s_{\alpha_i}(F)\Pi_i(F')$, for $F, F' \in \mathbf{Z}[\Xi_1, \ldots, \Xi_r]$.
Let $t_i = t_{-\omega_j} \in W_{\mathrm{aff}}$ be the translation by $-\omega_i$. Since the hyperplane of form $H_{\alpha_j,k}$, $j \neq i$, does not separate the alcoves A° and $t_i^{-1}(A^\circ)$, the root $\pm\alpha_j$, $j \neq i$, can not appear as a component of the ω_i-chain $(\beta_1, \ldots, \beta_l)$ corresponding to a reduced path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ from $A_0 = A^\circ$ to $A_l = t_i(A^\circ)$ (see [8, Chapter 4]). Hence, we have $\Pi_i(\Xi_j) = 0$ if $j \neq i$. From Lemma 2.1 (2), we also have $\Pi_i(\Xi_i) = \Xi^{[\omega_i - \alpha_i]}$, so the condition (1) follows.

Take an element $f \in \mathbf{Z}[P]$ such that $\varphi(f) = F$. We obtain

$$\Pi_i(FF') = h_{\alpha_i}^{-1} D_{[\alpha]}(F)F' + h_{\alpha_i}^{-1} \cdot \varphi(s_{\alpha_i}(f)) \cdot h_{\alpha_i} \cdot h_{\alpha_i}^{-1} D_{[\alpha]}(F')$$

by applying the twisted Leibniz rule for $D_{[\alpha]}$. From Lemma 2.1 (1), one can see that $h_{\alpha_i}^{-1} \cdot \varphi(s_{\alpha_i}(f)) \cdot h_{\alpha_i} = s_{\alpha_i}(F)$. So the condition (2) is satisfied.

Theorem 3.1 *The subalgebra $\mathbf{Z}[\Xi_1, \ldots, \Xi_r]$ generated by the multiplicative Chevalley elements in the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ is isomorphic to the K-ring $K(G/B)$.*

Proof. Define the algebra homomorphism $\varepsilon : \mathbf{Z}[P] \to \mathbf{Z}$ by the assignment $e^\lambda \mapsto 1$, $\forall \lambda \in P$. Let $I \subset \mathbf{Z}[P]$ be the ideal generated by the elements of form $f - \varepsilon(f)$, $f \in \mathbf{Z}[P]^W$. Then the K-ring $K(G/B)$ is isomorphic to the quotient algebra $\mathbf{Z}[P]/I$. Take an element $g \in I$. We have $\Pi_i(\varphi(g)) = \varphi(\pi_i(g)) = 0$ for $i = 1, \ldots, r$ from Proposition 3.1. Since $w \circ D_{[\alpha]} \circ w^{-1} = D_{[w(\alpha)]}$, we obtain $D_{[\alpha]}(\varphi(g)) = 0$, $\forall \alpha \in \Delta_+$, and therefore $\varphi(g) = 0$ by Lemma 1.1. If $g \notin I$, there exists an operator ϖ on $\mathbf{Z}[P]$ obtained as a linear combination of the composites of the multiplication operators and the operators π_i such that the constant term of $\varpi(g)$ is nonzero. Hence, we can conclude that $\mathrm{Im}(\varphi) \cong \mathbf{Z}[P]/I \cong K(G/B)$.

Remark 3.1 (1) The idea of the proof of the above theorem is used in [11, Sections 5 and 6] for the root systems of classical type and of type G_2. The multiplicative Dunkl elements $\Theta_i := \Xi^{[e_i]}$ corresponding to the components of the orthonormal basis $(e_i)_i$ are used in [11]. The multiplicative Dunkl elements in the Fomin-Kirillov quadratic algebra $\mathcal{E}_n$ are introduced by Lenart and Yong [14], [16].
(2) For an arbitrary parabolic subgroup $P \supset B$, the K-ring $K(G/P)$ of the homogeneous space G/P is a subalgebra of $K(G/B)$. Hence, the algebra $\mathcal{B}(V_W)$ also contains $K(G/P)$ as a commutative subalgebra.

Bazlov [2] has proved that the subalgebra in $\mathcal{B}(V_W)$ generated by the elements $[\alpha]$ corresponding to the simple roots α is isomorphic to the nil-Coxeter algebra

$$NC_W := \mathbf{Z}\langle u_1, \ldots, u_r \rangle / (u_i^2,\ (u_i u_j)^{[m_{ij}/2]} u_i^{\nu_{ij}} - (u_j u_i)^{[m_{ij}/2]} u_j^{\nu_{ij}},\ i = 1, \ldots, r),$$

where m_{ij} is the order of $s_{\alpha_i} s_{\alpha_j}$ in W, $[m_{ij}/2]$ stands for the integer part of $m_{ij}/2$, and $\nu_{ij} := m_{ij} - 2[m_{ij}/2]$. In our case, we can show the following.

Corollary 3.1 *The subalgebra in* $\mathrm{End}(\mathcal{B}(V_W))$ *generated by the operators* $\Pi_1, \ldots, \Pi_r$ *is isomorphic to the nil-Hecke algebra*

$$NH_W := \mathbf{Z}\langle T_1, \ldots, T_r \rangle / (T_i^2 + T_i,\ (T_i T_j)^{[m_{ij}/2]} T_i^{\nu_{ij}} - (T_j T_i)^{[m_{ij}/2]} T_j^{\nu_{ij}},\ i = 1, \ldots, r)$$

via the assignment $T_i \mapsto \Pi_i$.

Proof. One can check that the operators Π_i satisfy the defining relations of NH_W by direct computations. Since the assignment $T_i \mapsto \pi_i$ defines a faithful representation of NH_W on $\mathbf{Z}[P]/I$, the subalgebra generated by Π_i, $i = 1, \ldots, r$ is isomorphic to NH_W.

4 Model of the equivariant K-ring

The results in the previous section are generalized to the case of the T-equivariant K-ring $K_T(G/B)$. Our construction of the model of $K_T(G/B)$ is also parallel to Lenart and Postnikov's approach [15].

Since the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ is a braided Hopf algebra in the category of the Yetter-Drinfeld modules over W, it is W-graded. Denote by w_x the W-degree of a W-homogeneous element $x \in \mathcal{B}(V_W)$. Let h be the Coxeter number and $P' := h^{-1} \cdot P \subset \mathfrak{h}^*$. The Weyl group W acts on the group algebra $\mathbf{Z}[P'] = \bigoplus_{\lambda \in P'} X^\lambda$ by $w(X^\lambda) = X^{w(\lambda)}$, $w \in W$. The twist map

$$\begin{array}{rccc} c: & \mathcal{B}(V_W) \otimes \mathbf{Z}[P'] & \rightarrow & \mathbf{Z}[P'] \otimes \mathcal{B}(V_W) \\ & x \otimes X & \mapsto & w_x(X) \otimes x \end{array}$$

gives an associative multiplication map m on $\mathcal{B}(V_W)\langle P' \rangle := \mathbf{Z}[P'] \otimes \mathcal{B}(V_W)$ as follows:

$$m : \mathbf{Z}[P'] \otimes \mathcal{B}(V_W) \otimes \mathbf{Z}[P'] \otimes \mathcal{B}(V_W) \stackrel{1 \otimes c \otimes 1}{\longrightarrow} \mathbf{Z}[P'] \otimes \mathbf{Z}[P'] \otimes \mathcal{B}(V_W) \otimes \mathcal{B}(V_W)$$
$$\stackrel{m_{\mathbf{Z}[P']} \otimes m_{\mathcal{B}}}{\longrightarrow} \mathbf{Z}[P'] \otimes \mathcal{B}(V_W),$$

where $m_{\mathbf{Z}[P']}$ and $m_{\mathcal{B}}$ are the multiplication maps on the algebras $\mathbf{Z}[P']$ and $\mathcal{B}(V_W)$ respectively. The algebras $\mathbf{Z}[P']$ and $\mathcal{B}(V_W)$ are considered as subalgebras of $\mathcal{B}(V_W)\langle P' \rangle$. We have the commutation relation

$$[\alpha] \cdot X^\lambda = X^{s_\alpha(\lambda)} \cdot [\alpha], \ \ \alpha \in \Delta, \ \lambda \in P',$$

in $\mathcal{B}(V_W)\langle P' \rangle$.

The subalgebra $\mathbf{Z}[P]^W \subset \mathbf{Z}[P']$ is regarded as a subalgebra of the character ring $R(T)$ via the isomorphism

$$\begin{array}{rccc} \iota: & \mathbf{Z}[P] & \rightarrow & R(T) \\ & e^\lambda & \mapsto & \chi^\lambda. \end{array}$$

Let us consider an $R(T)$-algebra $\mathcal{B}_T(V_W) := R(T) \otimes_{\mathbf{Z}[P]^W} \mathcal{B}(V_W)\langle P' \rangle$. We introduce the elements

$$H_{\alpha/h} := X^{\rho/h} \cdot (X^{\alpha/h} + [\alpha]) \cdot X^{-\rho/h}, \ \ \alpha \in \Delta, \ \rho := \frac{1}{2}(\sum_{\beta \in \Delta_+} \beta),$$

in the algebra $\mathcal{B}_T(V_W)$. Since the argument in the proof of [15, Theorem 10.1] is applicable to our case, Proposition 1.2 implies the following.

Lemma 4.1 The elements H_α, $\alpha \in \Delta$, satisfy the Yang-Baxter equations in the algebra $\mathcal{B}_T(V_W)$.

Let $(\beta_1, \dots, \beta_l)$ be a λ-chain for a weight $\lambda \in P$. Define the element

$$\Xi_{eq}^{[\lambda]} := H_{\beta_l} \cdots H_{\beta_1}$$

in $\mathcal{B}_T(V_W)$. The element $\Xi_{eq}^{[\lambda]}$ is independent of the choice of the λ-chain from Lemma 4.1. We also have $\Xi_{eq}^{[\lambda]} \cdot \Xi_{eq}^{[\lambda']} = \Xi_{eq}^{[\lambda+\lambda']}$ from [15, Proposition 12.2].

The braided differential operators D_α are naturally extended as $R(T)$-linear operators on $\mathcal{B}_T(V_W)$ with the conditions:
(0) $D_\alpha(X) = 0$, for $X \in \mathbf{Z}[P']$,
(1) $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(2) $D_\alpha(FF') = D_\alpha(F)F' + s_\alpha(F)D_\alpha(F')$ for $F, F' \in \mathcal{B}(V_W)\langle P' \rangle$.

Lemma 4.2 *In the algebra $\mathcal{B}_T(V_W)$, we have*

$$\cap_{\alpha \in \Delta_+} \mathrm{Ker}(D_\alpha) = R(T) \otimes_{\mathbf{Z}[P]^W} \mathbf{Q}[P'].$$

Proof. This follows immediately from Lemma 1.1.

The operator π_i defined in the previous section is extended $R(T)$-linearly to the group algebra $R(T)[P]$ with the conditions:
(1) $\pi_i(e^{\omega_j}) = \delta_{i,j} e^{\omega_i - \alpha_i}$,
(2) $\pi_i(fg) = \pi_i(f)g + s_{\alpha_i}(f)\pi_i(g)$.
Here, the action of W on $R(T)$ is assumed to be trivial.

Note that $K_T(G/B)$ is isomorphic to the quotient algebra $R(T)[P]/J$, where J is the ideal generated by the elements of form $f - \iota(f)$, $f \in \mathbf{Z}[P]^W$.

Theorem 4.1 *The subalgebra $R(T)[\Xi_{eq}^{[\lambda]} \mid \lambda \in P]$ of $\mathcal{B}_T(V_W)$ is isomorphic to the T-equivariant K-ring $K_T(G/B)$.*

Proof. Let us consider the homomorphism between $R(T)$-algebras

$$\begin{array}{cccc} \psi : & R(T)[P] & \to & R(T)[\Xi_{eq}^{[\lambda]} \mid \lambda \in P] \\ & e^\lambda & \mapsto & \Xi_{eq}^{[\lambda]}. \end{array}$$

We can see that

$$X^{\rho/h}(X^{-\alpha_i/h} - [\alpha_i])X^{-s_{\alpha_i}(\rho)/h} D_{\alpha_i}(\psi(f)) = \psi(\pi_i(f)), \ f \in R(T)[P],$$

in the same manner as the proof of Proposition 3.1. Therefore, if $f \in \mathbf{Z}[P]^W$, then $D_\alpha(\psi(f)) = 0, \forall \alpha \in \Delta_+$. From Lemma 4.2, we have $\psi(f) \in R(T) \otimes_{\mathbf{Z}[P]^W} \mathbf{Z}[P']$. Here, the constant term of $\psi(f)$ for $f \in \mathbf{Z}[P]^W$ is in $\mathbf{Z}[P]^W$ and hence equals $\iota(f)$ (see also [15, Proposition 14.5]). So we obtain $\psi(f) = 0$ for $f \in J$. On the other hand, if $f \notin J$, there exists an operator ϖ on $R(T)[P]$ obtained as a linear combination of the composites of the multiplication operators and the operators π_i such that the constant term of $\varpi(f)$ is nonzero. Now we can conclude that $\mathrm{Ker}(\psi) = J$.

5 Quantization

In this section, we briefly discuss the quantization of our model, which is expected to give a model of the quantum K-ring of the flag variety G/B. The quantum K-theory, which is introduced by Lee [13], is the K-theoretic analogue of the theory of the quantum cohomology ring. Givental and Lee [7] have shown the relationship between the quantum K-theory of the flag variety G/B and the corresponding difference Toda system by proving the K-theoretic J-function gives an eigenvector of the Hamiltonian of the difference Toda system.

Let $Q^\vee$ be the coroot lattice and $R = \mathbf{Q}[Q^\vee] = \bigoplus_{\beta \in Q^\vee} \mathbf{Q} \cdot q^\beta$ its group algebra. We consider R as the algebra generated by the quantum deformation parameters. In [10], Kirillov and the author introduced the operators $\widetilde{[\alpha]}$ acting on the algebra $\mathcal{B}(V_W) \otimes R$ in order to construct the model of the small quantum cohomology ring $QH^*(G/B)$ in terms of the Nichols-Woronowicz algebra. Let $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_l}}$ be a reduced decomposition of an element $w \in W$. Then the operator $D_w := D_{[\alpha_{i_1}]} \cdots D_{[\alpha_{i_l}]}$ does not depend on the choice of the reduced decompositon of w because the braided differential operators D_α corresponding to the simple roots α satisfy the Coxeter relations. The operator $\widetilde{[\alpha]}$, $\alpha \in \Delta_+$, is defined by

$$\widetilde{[\alpha]} := \begin{cases} [\alpha] + q^{\alpha^\vee} D_{s_\alpha}, & \text{if } l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1, \\ [\alpha], & \text{otherwise}, \end{cases}$$

where $\mathrm{ht}(\alpha^\vee) = m_1 + \cdots + m_r$ for $\alpha^\vee = m_1\alpha_1^\vee + \cdots + m_r\alpha_r^\vee$. It is easy to see $\widetilde{[\alpha]}^2 = q^{\alpha^\vee}$ for the simple roots α and $\widetilde{[\alpha]}^2 = 0$ for nonsimple positive roots. We put $\widetilde{[\alpha]} = -\widetilde{[-\alpha]}$ for a negative root α.

For the root system $\Delta = \{e_i - e_j \mid 1 \leq i,j \leq n,\ i \neq j\}$ of type A, the operators $\widetilde{[i,j]} := [e_i - e_j]$ satisfy the defining relations of the algebra $\tilde{\mathcal{E}}_n$, i.e.
(1) $\widetilde{[i,j]}^2 = 0$, for $j > i+1$, and $\widetilde{[i,i+1]}^2 = q^{e_i - e_{i+1}}$,
(2) $\widetilde{[i,j]}\widetilde{[k,l]} = \widetilde{[k,l]}\widetilde{[i,j]}$, if $\{i,j\} \cap \{k,l\} = \emptyset$,
(3) $\widetilde{[i,j]}\widetilde{[j,k]} + \widetilde{[j,k]}\widetilde{[k,i]} + \widetilde{[k,i]}\widetilde{[i,j]} = 0$.

Let us define the operator $\tilde{\Xi}^{[\lambda]}$ acting on the algebra $\mathcal{B}(V_W) \otimes R$ by replacing $[\beta_i]$ appearing the definition of the element $\Xi^{[\lambda]}$ by $\widetilde{[\beta_i]}$, i.e.

$$\tilde{\Xi}^{[\lambda]} := \tilde{h}_{\beta_l} \cdots \tilde{h}_{\beta_1}$$

for a reduced alcove path $A_0 \stackrel{-\beta_1}{\rightarrow} \cdots \stackrel{-\beta_l}{\rightarrow} A_l$ from $A_0 = A^\circ$ to $A_l = A^{-\lambda}$, where $\tilde{h}_\beta := 1 + \widetilde{[\beta]}$. The operator $\tilde{\Xi}^{[\lambda]}$ is also independent of the choice of the reduced alcove path $A_0 \stackrel{-\beta_1}{\rightarrow} \cdots \stackrel{-\beta_l}{\rightarrow} A_l$ because the operators $\tilde{h}_\alpha$ satisfy the Yang-Baxter equations. Note that we have to use the *reduced* paths in order to define the operator $\tilde{\Xi}^{[\lambda]}$, because $\tilde{h}_\alpha \tilde{h}_{-\alpha} = 1 - q^{\alpha^\vee}$ for the simple roots α.

Conjecture 5.1 The subalgebra generated by $\tilde{\Xi}^{[\lambda]}$, $\lambda \in P$, in the algebra $R\langle\widetilde{[\alpha]} \mid \alpha \in \Delta\rangle$ is isomorphic to the small quantum K-ring $QK(G/B)$.

Remark 5.1 The corresponding conjecture for the quantum quadratic algebra $\tilde{\mathcal{E}}_n$ is stated in [14]. The conjecture on the Chevalley-type formula in the quantum K-ring $QK(G/B)$ described by means of the quantum Bruhat operators is given in [15]. See also [11, Section 3].

Conjecture 5.1 for the flag variety $\mathcal{F}l_n$ of type A is proved in the forthcoming paper [12]. In fact, it can be shown that the statement holds at the level of the quantized quadratic algebra $\tilde{\mathcal{E}}_n$. The proof consists of the following two steps:
(1) To determine the relations in the quantum K-ring $QK(\mathcal{F}l_n)$ by showing that the difference operators

$$H_k := \sum_{I \subset \{1,\ldots,n\}, \sharp I = k} q^{\sum_{i \in I} \partial/\partial t_i} \prod_{i-1 \notin I, i \in I} (1 - e^{t_{i-1} - t_i}), \quad k = 1, \ldots, n,$$

form a commuting family,
(2) To show the multiplicative Dunkl elements $\tilde{\Theta}_i := \tilde{\Xi}^{[e_i]}$ satisfy the relations obtained in the step (1).

The detailed argument will appear in [12].

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