On measure contraction property of metric measure spaces*†‡

Shin-ichi OHTA
Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606-8502, JAPAN
e-mail: sohta@math.kyoto-u.ac.jp

Abstract

We introduce a measure contraction property of metric measure spaces which can be regarded as a generalized notion of the lower Ricci curvature bound on Riemannian manifolds. This property is preserved under the measured Gromov-Hausdorff convergence. Moreover, we will prove a generalization of the Bonnet-Myers theorem.

1 Introduction

The notions of lower and upper 'sectional' curvature bounds on not necessarily Riemannian metric spaces are introduced by Alexandrov by using the triangle comparison theorems, and they are called Alexandrov spaces and CAT(K)-spaces, respectively (see [ABN], [BGP], [G], [BBI], and the references therein). These spaces are quite interesting objects themselves and, furthermore, they are turned out to be useful tools to study limit spaces under the Gromov-Hausdorff convergence of sequences of Riemannian manifolds with uniform lower or upper sectional curvature bounds. Now the Alexandrov spaces and CAT(K)-spaces are ones of the most important objects in the metric geometry.

Once the importances of Alexandrov spaces and CAT(K)-spaces are understood, a natural question arises: What about the lower bound of the 'Ricci' curvature? One reason why this is a natural question is that the family of Riemannian manifolds with unifrom lower Ricci curvature and upper diameter and dimension bounds is precompact in the Gromov-Hausdorff topology ([G]). In their serial papers [CC], Cheeger and Colding investigate the limit spaces under the measured Gromov-Hausdorff convergence of sequences of Riemannian manifolds with uniform lower Ricci curvature bounds, and consider the convergence of the Laplacian (Fukaya's conjecture, [F]).

*Partially supported by the Grant-in-Aid for Scientific Research for Young Scientists (B) 16740034 from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

†Mathematics Subject Classification (2000): 28C15, 53C21, 53C23.

‡Keywords: measure contraction property, Ricci curvature, Bonnet-Myers theorem, Gromov-Hausdorff convergence, Wasserstein space.

Recently, a breakthrough on this topic, so called 'synthetic treatment of the lower Ricci curvature bound on metric measure spaces', is given by Sturm [S2] and Lott and Villani [LV] (see also [RS]). They independently introduce mutually slightly different conditions. More precisely, they consider the Wasserstein space on a metric measure space and adopt the convexity of a (family of) functional(s) on that space as a generalized notion of the lower Ricci curvature bound. However, there remains a problem on the treatment of the dimension. Sturm's condition does not contain a term of the dimension and it can be regarded as the case where the upper bound of the dimension is the infinity. In addition to it, Lott and Villani consider spaces having finite upper bounds on their dimensions (N-Ricci curvature), but it is only for the nonnegative Ricci curvature case. So that it is still unclear how to define spaces with a finite upper bound on their dimensions and with a nonzero lower Ricci curvature bound. Furthermore, some basic questions to justify their conditions are open, for instance, whether Alexandrov spaces satisfy these or not.

In this article, we introduce another kind of a generalization of the lower Ricci curvature bound, the (K,N)-*measure contraction property* (Definition 2.1, the (K,N)-*MCP* for short). Here $K \in \mathbb{R}$ is the lower bound of the Ricci curvature and $N \geq 1$ is the upper bound of the dimension, so that we can consider a situation which is not covered in [S2] and [LV] ($K \neq 0$ and $N < \infty$). This condition is defined in terms of the contraction of a measure on a set to a point, and seems simpler and more geometrically understandable than those considered in [S2] and [LV]. Indeed, we do not use the Wasserstein space to define the (K,N)-MCP, and it is not difficult to see that Alexandrov spaces satisfy the (K,N)-MCP (Proposition 2.7).

One of our main results is a generalization of the Bonnet-Myers theorem. Namely, we shall show that, if a metric measure space (X,μ) satisfies the (K,N)-MCP for some $K > 0$ and $N > 1$, then its diameter is less than or equal to $\pi/\sqrt{K}$ (Theorem 4.2). Moreover, for every point $x \in X$, the set of points at a distance of $\pi/\sqrt{K}$ from x consists of at most one point (Theorem 4.4). We also prove a generalization of the Bishop-Gromov volume comparison theorem (Theorem 5.1). In addition to these, we show that, for an n-dimensional Riemannian manifold, the (K,n)-MCP is equivalent to that its Ricci curvature is bounded from below by $(n-1)K$ (Theorem 3.2), and that the (K,N)-MCP is preserved under the measured Gromov-Hausdorff convergence (Theorem 6.8). These results as well as the (K,N)-MCP of Alexandrov spaces justify us to say that the (K,N)-MCP is a generalized notion of the lower Ricci curvature bound. Techniques developed in [RS], [S2], and [LV] play crucial roles in our discussions.

In the present article, we do not pursue the analytic property of the (K,N)-MCP, such as Poincaré inequalities, Dirichlet forms, and harmonic functions. They will be treated in other article(s).

The article is organized as follows. We give the definition of the (K,N)-MCP and consider some basic properties, such as the doubling condition, in Section 2. In Section 3, we treat the Riemannian case. Section 4 is devoted to a generalization of the Bonnet-Myers theorem. We prove a generalization of the Bishop-Gromov volume comparison theorem in Section 5. In the last section, we consider the stability of the (K,N)-MCP under the measured Gromov-Hausdorff convergence.

After this work is completed, I learn of a related work by Sturm [S3].

2 Measure contraction property

A metric space (X, d_X) is called a *length space* if it satisfies $d_X(x,y) = \inf_\gamma \mathrm{length}(\gamma)$ for all $x, y \in X$, where the infimum is taken over all rectifiable curves γ from x to y. If, for every $x, y \in X$, there exists a curve γ which satisfies $d_X(x,y) = \mathrm{length}(\gamma)$, then we say that (X, d_X) is *geodesic*. Note that, if a length space is complete and locally compact, then it is geodesic. A rectifiable curve γ in a metric space (X, d_X) is called a *geodesic* if it is locally minimizing and has a constant speed. A geodesic $\gamma : [0, l] \longrightarrow X$ is said to be *minimal* if it satisfies $\mathrm{length}(\gamma) = d_X(\gamma(0), \gamma(l))$. By taking a reparametrization of a curve which attains the distance, every two points in a geodesic metric space are joined by a (not necessarily unique) minimal geodesic.

Throughout this article, without otherwise indicated, let (X, d_X) be a length space, and let μ be a Borel measure on X such that $0 < \mu(B(x,r)) < \infty$ holds for every $x \in X$ and $r > 0$, where $B(x,r)$ (or $B^X(x,r)$) denotes the open ball with center $x \in X$ and radius $r > 0$. The closed ball with center $x \in X$ and radius $r > 0$ is denoted by $\overline{B}(x,r)$ or $\overline{B}^X(x,r)$. Henceforce, we denote $d_X(x,y)$ by $|x-y|_X$ for $x, y \in X$, and write simply X instead of (X, d_X).

As in [LV], let Γ be the set of minimal geodesics, say $\gamma : [0,1] \longrightarrow X$, in X and define the evaluationg map $e_t : \Gamma \longrightarrow X$ by $e_t(\gamma) := \gamma(t)$ for each $t \in [0,1]$. We regard Γ as a subset of the set of Lipschitz maps $\mathrm{Lip}([0,1], X)$ with the Lipschitz topology. A *dynamical transference plan* Π is a Borel probability measure on Γ, and a path $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}^2(X)$ given by $\mu_t = (e_t)_*\Pi$ is called a *displacement interpolation* associated to Π, where we define $\mathcal{P}^2(X)$ as the set of all Borel probability measures, say μ, satisfying $\int_X |x-y|_X^2 \, d\mu(y) < \infty$ for some (and hence all) $x \in X$.

For $K \in \mathbb{R}$, we define the function $\mathbf{s}_K$ on $[0, \infty)$ (on $[0, \pi/\sqrt{K})$ if $K > 0$) by

$$\mathbf{s}_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$

Definition 2.1 For $K \in \mathbb{R}$ and $N \geq 1$, a metric measure space (X, μ) is said to satisfy the (K,N)-*measure contraction property* (the (K,N)-*MCP* for short) if, for every point $x \in X$ and measurable set $A \subset X$ (provided that $A \subset B(x, \pi/\sqrt{K})$ if $K > 0$), there exists a displacement interpolation $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}^2(X)$ associated to a dynamical transference plan $\Pi = \Pi_{x,A}$ satisfying the following:

(1) We have $\mu_0 = \delta_x$ and $\mu_1 = (\mu|_A)^-$ as measures, where we denote by $(\mu|_A)^-$ the normalization of $\mu|_A$, i.e., $(\mu|_A)^- := \mu(A)^{-1} \cdot \mu|_A$;

(2) For every $t \in [0,1]$,

$$d\mu \geq (e_t)_* \left(t \left\{ \frac{\mathbf{s}_K(t|x-\gamma(1)|_X)}{\mathbf{s}_K(|x-\gamma(1)|_X)} \right\}^{N-1} \mu(A) \, d\Pi(\gamma) \right) \tag{2.1}$$

holds as measures on X, where we set $0/0 = 1$.

If there exists a measurable map $\Phi : A \longrightarrow \Gamma$ satisfying $e_0 \circ \Phi \equiv x$, $e_1 \circ \Phi = \mathrm{id}_A$, and $\Pi = \Phi_*[(\mu|_A)^-]$, then the inequality (2.1) yields that

$$d\mu \geq (e_t \circ \Phi)_* \left(t \left\{ \frac{\mathbf{s}_K(t|x-z|_X)}{\mathbf{s}_K(|x-z|_X)} \right\}^{N-1} \chi_A(z)\, d\mu(z) \right) \tag{2.2}$$

holds as measures on X. Here χ_A stands for the characteristic function on A. This is the case where, for each $y \in A$, there exists an exactly one geodesic $\gamma \in \operatorname{supp} \Pi$ from x to y.

Lemma 2.2 *The inequlity* (2.2) *is equivalent to that, for all* $t \in [0,1]$ *and measurable set* $A' \subset A$*, we have*

$$\mu\big(e_t(\Phi(A'))\big) \geq \int_{A'} t \left\{ \frac{\mathbf{s}_K(t|x-z|_X)}{\mathbf{s}_K(|x-z|_X)} \right\}^{N-1} d\mu(z). \tag{2.3}$$

Proof. Put $\Psi := e_t \circ \Phi$ and

$$d\nu := t \left\{ \frac{\mathbf{s}_K(t|x-z|_X)}{\mathbf{s}_K(|x-z|_X)} \right\}^{N-1} \chi_A(z)\, d\mu(z)$$

in this proof for simplicity. We first assume (2.2). For a measurable set $A' \subset A$, we have

$$\mu\big(\Psi(A')\big) \geq (\Psi_*\nu)\big(\Psi(A')\big) = \nu\big(\Psi^{-1}(\Psi(A'))\big) \geq \nu(A').$$

This implies (2.3). We next suppose (2.3). For a measurable set $W \subset X \setminus \Psi(A)$, we immedately obtain $\mu(W) \geq 0 = (\Psi_*\nu)(W)$. If $W \subset \Psi(A)$, then (2.3) yields that

$$\mu(W) = \mu\big(\Psi(\Psi^{-1}(W))\big) \geq \nu\big(\Psi^{-1}(W)\big) = (\Psi_*\nu)(W).$$

This completes the proof. □

The inequality (2.3) can be regarded as a generalization of the Bishop inequality under a lower Ricci curvature bound $\mathrm{Ric}_g \geq (N-1)K$ (see Theorem 3.1 below), and is a reason why we say that (2.1) is a kind of measure contraction property. We refer [S1], [KS1], [R1], and [R2] (see also [O]) for other kinds of measure contraction property of metric measure spaces. Note that a metric measure space consisting of only one point satisfies the (K,N)-MCP for all $K \in \mathbb{R}$ and $N \geq 1$. The following are clear by definition.

Lemma 2.3 (i) *Suppose that* $K \leq 0$ *or* $K > 0$ *and* $\operatorname{diam} X \leq \pi/2\sqrt{K}$*. Then the* (K,N)*-MCP of* (X,μ) *implies the* (K,N')*-MCP for all* $N' \geq N$*.*

(ii) *If* (X, d_X, μ) *satisfies the* (K,N)*-MCP and if* $a, b > 0$*, then the scaled metric measure space* $(X, a \cdot d_X, b \cdot \mu)$ *satisfies* $(K/a^2, N)$*-MCP.*

We remark that the (K,N)-MCP does not necessarily imply the (K,N')-MCP for $N' > N$ if $K > 0$. Indeed, the N-dimensional sphere $\mathbb{S}^N$ with the standard Riemannian metric satisfies the $(1,N)$-MCP, but it does not satisfy the $(1,N')$-MCP because we have

$$\left\{ \frac{\mathbf{s}_K(tr)}{\mathbf{s}_K(r)} \right\}^{N-1} < \left\{ \frac{\mathbf{s}_K(tr)}{\mathbf{s}_K(r)} \right\}^{N'-1}$$

if $\pi/\sqrt{K} - r < tr < r < \pi/\sqrt{K}$.

The following lemma is almost straightforward from the definition of the (K,N)-MCP, and will be sharpened in Section 5.

Lemma 2.4 *Let (X,μ) satisfy the (K,N)-MCP. Then, for every $x \in X$ and $0 < r \le R$ $(\le \pi/\sqrt{K}$ if $K > 0)$, we have*

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \le \frac{R}{r} \sup_{0\le\lambda\le 1} \left\{ \frac{\mathbf{s}_K(\lambda R)}{\mathbf{s}_K(\lambda r)} \right\}^{N-1}.$$

In particular, the set $S(x,r) := \{y \in X \mid |x-y|_X = r\}$ has a null measure for any $x \in X$ and $r > 0$ (provided that $r < \pi/\sqrt{K}$ if $K > 0$).

Proof. The (K,N)-MCP with $x = x$, $A = B(x,R)$, and $t = r/R$ yields that

$$\begin{aligned}
&\mu\big(B(x,r)\big) \\
&\ge \mu\big(B(x,R)\big) \frac{r}{R} \inf_{0\le\lambda\le 1} \left\{ \frac{\mathbf{s}_K(\lambda r)}{\mathbf{s}_K(\lambda R)} \right\}^{N-1} \big((e_{r/R})_* \Pi_{x,B(x,R)}\big)\big(B(x,r)\big) \\
&= \mu\big(B(x,R)\big) \frac{r}{R} \inf_{0\le\lambda\le 1} \left\{ \frac{\mathbf{s}_K(\lambda r)}{\mathbf{s}_K(\lambda R)} \right\}^{N-1} \Pi\big((e_{r/R})^{-1}[B(x,r)]\big) \\
&\ge \mu\big(B(x,R)\big) \frac{r}{R} \inf_{0\le\lambda\le 1} \left\{ \frac{\mathbf{s}_K(\lambda r)}{\mathbf{s}_K(\lambda R)} \right\}^{N-1} \Pi\big((e_1)^{-1}[B(x,R)]\big) \\
&= \frac{r}{R} \inf_{0\le\lambda\le 1} \left\{ \frac{\mathbf{s}_K(\lambda r)}{\mathbf{s}_K(\lambda R)} \right\}^{N-1} \mu\big(B(x,R)\big).
\end{aligned}$$

Here the inequality in the third line follows from $(e_0)_*\Pi = \delta_x$. Indeed, it implies

$$(e_1)^{-1}[B(x,R)] \cap \operatorname{supp}\Pi \subset (e_{r/R})^{-1}[B(x,r)].$$

This completes the proof. $\square$

In particular, the (K,N)-MCP implies the (local) doubling condition. Namely, for any $R > 0$ ($R \le \pi/\sqrt{K}$ if $K > 0$), $r \in (0,R]$, and $x \in X$, we have

$$\frac{\mu(B(x,r))}{\mu(B(x,r/2))} \le C_{K,N,R},$$

where $C_{K,N,R} < \infty$ is a constant depending only on K, N, and R. The doubling condition implies that every bounded closed ball in X is totally bounded. Therefore, if X is complete, then it is proper (i.e., all bounded closed sets are compact) and hence geodesic.

Corollary 2.5 *If (X,μ) satisfies the (K,N)-MCP and if it contains more than two points, then the measure μ is non-atomic.*

Corollary 2.6 *If (X,μ) satisfies the (K,N)-MCP, then the Hausdorff dimension of X is less than or equal to N.*

Proof. Lemma 2.4 yields that the function $f(x) := \limsup_{r\to 0} r^N \mu(B(x,r))^{-1}$ on X is locally bounded. By [AT, Theorem 2.4.3], this implies that the N-dimensional Hausdorff measure $\mathcal{H}^N$ on X is also locally bounded. Therefore the Hausdorff dimension of X is not greater than N. $\square$

We end this section with a proposition which asserts that Alexandrov spaces satisfy the MCP. As the Alexandrov space is considered as a metric space with a lower 'sectional' curvature bound, this proposition supports us for saying that the (K,N)-MCP is a generalized notion of a lower 'Ricci' curvature bound. See [BBI], [BGP], and [KS1] for the definition of and terminologies on Alexandrov spaces.

Proposition 2.7 *Let X be an n-dimensional, complete, and locally compact Alexandrov space with curvature $\geq K$, and $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on X. Then $(X,\mathcal{H}^n)$ satisfies the (K,n)-MCP.*

Proof. This easily follows from [KS1, Lemma 6.1], we give an outline of the proof for completeness. For a point $x \in X$ and a measurable set $A \subset X$, we define a map $\Phi^X = \Phi^X_{x,A} : A \longrightarrow \Gamma$ by $\Phi^X(y) := \gamma$, where $\gamma : [0,1] \longrightarrow X$ is an arbitrarily chosen minimal geodesic from x to y. Then we see that Φ^X is measurable as in the proof of [KS1, Proposition 6.1], and we put $\Pi := (\Phi^X) * [(\mu|_A)^-]$. The condition (1) in Definition 2.1 is clearly satisfied and the condition (2) follows from the curvature condition just as in [KS1, Lemma 6.1]. $\square$

3 Riemannian case

In this section, we consider the Riemannian case. See, for example, [Cl] for fundamentals on Riemannian geometry. Let (M,g) be an n-dimensional, complete Riemannian manifold without boundary and denote by d_g (or $|\cdot - \cdot|_g$) and ν_g the Riemannian distance and the Riemannian measure, respectively, on M induced from g. In addition, Ric_g stands for the Ricci tensor with respect to g and the inequality $\mathrm{Ric}_g \geq (n-1)K$ means that $\mathrm{Ric}_g(\xi,\xi) \geq (n-1)K$ holds for every $p \in M$ and $\xi \in S_pM$, where $S_pM \subset T_pM$ is the unit tangent sphere at $p \in M$. For a point $p \in M$ and a unit tangent vector $\xi \in S_pM$, we set

$$c(\xi) := \sup\{r > 0 \,|\, |p - \gamma_\xi(r)|_g = r\},$$

where we define $\gamma_\xi(r) := \exp_p r\xi$. Define, for $p \in M$,

$$\begin{aligned} C(p) &:= \{\gamma_\xi\big(c(\xi)\big) \,|\, \xi \in S_pM\}, \\ \mathsf{D}(p) &:= \{t\xi \,|\, \xi \in S_pM,\ 0 \leq r < c(\xi)\} \subset T_pM, \\ D(p) &:= \exp_p \mathsf{D}(p). \end{aligned}$$

The set $C(p)$ is called the *cut locus* of p. Recall that $\exp_p : \mathsf{D}(p) \longrightarrow D(p)$ gives a diffeomorphism and that we can represent $d\nu_g(q) = (\exp_p)_*[\mathsf{A}_p(r;\xi)\, dr\, d\xi]$ on $D(p)$, where $q = \gamma_\xi(r)$ and $\mathsf{A}_p(r;\xi)$ denotes the density of the Riemannian measure on $S(p,r)$ induced from g. Recall that we set $S(p,r) := \{q \in M \,|\, |p-q|_g = r\}$. The classical Bishop comparison theorem asserts the following ([BC], cf. [Cl, Theorem 3.8]).

Theorem 3.1 *If (M, g) satisfies $\mathrm{Ric}_g \geq (n-1)K$, then we have*

$$\frac{1}{\mathsf{A}_p(r;\xi)}\frac{d\mathsf{A}_p(r;\xi)}{dr} \leq (n-1)\frac{\mathbf{s}'_K(r)}{\mathbf{s}_K(r)}$$

for all $\xi \in S_pM$ and $r \in (0, c(\xi))$. In particular, the function $\mathsf{A}_p(r;\xi)/\mathbf{s}_K(r)^{n-1}$ is monotone non-increasing in $r \in (0, c(\xi))$.

Given a point $p \in M$ and a measurable set $A \subset M$, as in the proof of Proposition 2.7, we define a map $\Phi^M = \Phi^M_{p,A} : A \longrightarrow \Gamma$ by $\Phi^M_{p,A}(q) := \gamma$, where $\gamma : [0,1] \longrightarrow M$ is an arbitrarily chosen minimal geodesic from p to q. As $C(p)$ has a null measure, the map $\Phi^M_{p,A}$ is measurable and is uniquely determined upto a modification on a null measure set.

Theorem 3.2 *Let (M, g) be an n-dimensional, complete Riemannian manifold without boundary. Then a metric measure space (M, d_g, ν_g) satisfies the (K, n)-MCP if and only if $\mathrm{Ric}_g \geq (n-1)K$ holds.*

Proof. We first assume $\mathrm{Ric}_g \geq (n-1)K$ and fix a point $p \in M$ and a mesurable set $A \subset M$. We shall show that the map $\Phi^M = \Phi^M_{p,A}$ defined as above satisfies (2.3) with $N = n$ which implies the (K, n)-MCP. It follows from Theorem 3.1 that, for any $t \in [0,1]$ and measurable subset $A' \subset A$,

$$\begin{aligned}
\nu_g\big(e_t(\Phi^M_{p,A}(A'))\big) &= \int_{\exp_p^{-1}(A')\cap \mathsf{D}(p)} t\mathsf{A}_p(tr;\xi)\, dr\, d\xi \\
&\geq \int_{\exp_p^{-1}(A')\cap \mathsf{D}(p)} t\left\{\frac{\mathbf{s}_K(tr)}{\mathbf{s}_K(r)}\right\}^{n-1} \mathsf{A}_p(r;\xi)\, dr\, d\xi \\
&= \int_{A'} t\left\{\frac{\mathbf{s}_K(t|p-q|_g)}{\mathbf{s}_K(|p-q|_g)}\right\}^{n-1} d\nu_g(q).
\end{aligned}$$

Therefore Φ^M satisfies the inequlity (2.3).

Next we consider the converse, so that we suppose that (M, d_g, ν_g) satisfies the (K, n)-MCP. Fix $p \in M$, $\xi \in S_pM$, and an orthonormal basis $\{e_1, \ldots, e_n\}$ in T_pM with $e_1 = \xi$. We denote by k_i the sectional curvature of the plane spanned by e_1 and e_i for each $i = 2, \ldots, n$. For a small $r > 0$, it follows from

$$\frac{\mathbf{s}_K(r)}{\mathbf{s}_K(2r)} = \frac{1}{2}\left(1 + \frac{K}{2}r^2 + O(r^4)\right)$$

that

$$\begin{aligned}
\frac{\mathsf{A}_p(r;\xi)}{\mathsf{A}_p(2r;\xi)} &= \frac{1}{2^{n-1}}\prod_{i=2}^{n}\left(1 + \frac{k_i + O(r)}{2}r^2 + O(r^4)\right) \\
&= \frac{1}{2^{n-1}}\prod_{i=2}^{n}\left(1 + \frac{k_i}{2}r^2\right) + O(r^3).
\end{aligned}$$

On the other hand, it is not difficult to observe that the (K, n)-MCP implies

$$\frac{\mathsf{A}_p(r;\xi)}{\mathsf{A}_p(2r;\xi)} \geq \left\{ \frac{\mathbf{s}_K(r)}{\mathbf{s}_K(2r)} \right\}^{n-1},$$

and hence we have

$$\prod_{i=2}^{n} \left(1 + \frac{k_i}{2} r^2 \right) \geq \left\{ \frac{2\mathbf{s}_K(r)}{\mathbf{s}_K(2r)} \right\}^{n-1} + O(r^3).$$

By taking the logarithm of both sides, we find

$$\begin{aligned}\sum_{i=2}^{n} \log \left(1 + \frac{k_i}{2} r^2 \right) &\geq (n-1) \log \left(\frac{2\mathbf{s}_K(r)}{\mathbf{s}_K(2r)} \right) + O(r^3) \\ &= (n-1) \log \left(1 + \frac{K}{2} r^2 \right) + O(r^3).\end{aligned}$$

As $\log(1+s) = s + O(s^2)$, the inequality above yields that

$$\sum_{i=2}^{n} \frac{k_i}{2} r^2 \geq (n-1) \frac{K}{2} r^2 + O(r^3).$$

Dividing both sides by r^2 and letting r tend to zero, we consequently obtain

$$\mathrm{Ric}_g(\xi, \xi) = \sum_{i=2}^{n} k_i \geq (n-1)K.$$

This completes the proof. $\square$

The following are easily derived from Lemma 2.3(i) and Corollary 2.6 together with the theorem above.

Corollary 3.3 *Let (M, g) be an n-dimensional, complete Riemannian manifold without boundary.*

(i) *Suppose that $K \leq 0$ or $K > 0$ and $\operatorname{diam} M \leq \pi/2\sqrt{K}$. If (M, g) satisfies $\mathrm{Ric}_g \geq (n-1)K$ and $n \leq N$, then (M, d_g, ν_g) satisfies the (K, N)-MCP.*

(ii) *If a metric measure space (M, d_g, ν_g) satisfies the (K, N)-MCP, then we have $n \leq N$.*

4 Bonnet-Myers theorem

In this section, we shall show a generalization of the Bonnet-Myers theorem ([M]), that is, the (K, N)-MCP with $K > 0$ and $N > 1$ implies that the diameter is less than or equal to $\pi/\sqrt{K}$. By rescaling the distance, we may assume $K = 1$ (Lemma 2.3(ii)). For $x \in X$ and $s, t \geq 0$ with $s < t$, we define $A(x; s, t) := B(x, t) \setminus B(x, s)$, where we set $B(x, 0) := \emptyset$. The symbol $\theta_{\alpha,\beta}(\delta)$ denotes a function depending only on α and β with $\lim_{\delta \to 0} \theta_{\alpha,\beta}(\delta) = 0$. The following lemma will be a useful tool.

Lemma 4.1 *Let (X, μ) satisfy the $(1, N)$-MCP. Then, for any $x \in X$ and $s, t \in [0, \pi/2]$ with $s < t$, we have*

$$\mu\big(A(x; s, t)\big) \geq \mu\big(A(x; \pi - t, \pi - s)\big).$$

Proof. We may assume that $\pi/(l+2) \leq s < t \leq \pi/(l+1)$ holds for some $l \in \mathbb{N}$. Take a large $M \in \mathbb{N}$ and set $\delta = (t-s)/M$. For $1 \leq m \leq M$, we put

$$\begin{aligned}
\lambda_m &:= \frac{\pi - s - m\delta}{\pi - s - (m-1)\delta}, \\
A_m &:= A\big(x; \pi - s - m\delta, \pi - s - (m-1)\delta\big).
\end{aligned}$$

As in the proof of Lemma 2.4, the $(1, N)$-MCP implies that, for each $1 \leq m \leq M$ and $1 \leq k \leq l$,

$$\begin{aligned}
&\mu\big(A(x; \lambda_m^k(s+m\delta), \lambda_m^{k-1}(s+m\delta))\big) \\
&\geq \frac{(\pi - s - m\delta)^{k-1}}{(\pi - s - (m-1)\delta)^k}(s+m\delta)\left\{\frac{\sin(s+(m-1)\delta)}{\sin(\pi - s - m\delta)}\right\}^{N-1}\mu(A_m) \\
&\geq \frac{(\pi - s - m\delta)^{l-1}}{(\pi - s - (m-1)\delta)^l}(s+m\delta)\left\{\frac{\sin(s+(m-1)\delta)}{\sin(s+m\delta)}\right\}^{N-1}\mu(A_m). \qquad (4.1)
\end{aligned}$$

Here the second inequality follows from

$$\begin{aligned}
&\lambda_m^k(s+m\delta) \geq \lambda_m^l(s+m\delta) \\
&= \lambda_m^{l-1}\left(1 - \frac{\delta}{\pi - s - (m-1)\delta}\right)(s+m\delta) \\
&= \lambda_m^{l-2}\left(1 - \frac{\delta}{\pi - s - (m-1)\delta} - \delta\frac{\pi - s - m\delta}{(\pi - s - (m-1)\delta)^2}\right)(s+m\delta) \\
&= \cdots = (s+m\delta) - \delta\sum_{i=1}^{l}\frac{(\pi - s - m\delta)^{i-1}}{(\pi - s - (m-1)\delta)^i}(s+m\delta) \\
&\geq s + m\delta - l\delta\frac{s+m\delta}{\pi - s - (m-1)\delta} \geq s + m\delta - l\delta\frac{s+M\delta}{\pi - s - (M-1)\delta} \\
&> s + m\delta - l\delta\frac{t}{\pi - t} \geq s + m\delta - l\delta\frac{\pi/(l+1)}{\pi - \pi/(l+1)} \\
&= s + (m-1)\delta.
\end{aligned}$$

Summing up (4.1) in $k = 1, \dots, l$, we see that

$$
\begin{aligned}
&\mu\big(A(x; \lambda_m^l(s+m\delta), s+m\delta)\big) \\
&\geq l \frac{(\pi - s - m\delta)^{l-1}}{(\pi - s - (m-1)\delta)^l}(s+m\delta)\left\{\frac{\sin(s+(m-1)\delta)}{\sin(s+m\delta)}\right\}^{N-1} \mu(A_m) \\
&\geq \frac{l(s+m\delta)}{\pi - s - m\delta}\left(\frac{\pi - t}{\pi - t + \delta}\right)^l \left\{\frac{\sin s}{\sin(s+\delta)}\right\}^{N-1} \mu(A_m) \\
&= \frac{l(s+m\delta)}{\pi - s - m\delta}\big(1+\theta_l(\delta)\big)\big(1+\theta_{l,N}(\delta)\big)\mu(A_m) \\
&\geq \frac{l\pi/(l+2)}{\pi - \pi/(l+2)}\big(1+\theta_{l,N}(\delta)\big)\mu(A_m) \\
&= \frac{l}{l+1}\big(1+\theta_{l,N}(\delta)\big)\mu(A_m).
\end{aligned}
\tag{4.2}
$$

Now we estimate $\mu(A(x; s+(m-1)\delta, \lambda_m^l(s+m\delta)))$. To do this, take

$$
\tau \in [\pi - s - m\delta, \pi - s - (m-1)\delta - \rho(m,\delta)]
$$

such that

$$
\mu\big(A(x; \tau, \tau + \rho(m,\delta))\big) \geq \frac{\rho(m,\delta)}{\delta}\mu(A_m)
$$

holds, where we put

$$
\rho(m,\delta) := \frac{\pi - s - m\delta}{s+(m-1)\delta}\big\{\lambda_m^l(s+m\delta) - \big(s+(m-1)\delta\big)\big\}.
$$

We remark that $\rho(m,\delta) > 0$ and

$$
\begin{aligned}
&\frac{s+(m-1)\delta}{\tau}\big(\tau + \rho(m,\delta)\big) \\
&= s+(m-1)\delta + \frac{\pi - s - m\delta}{\tau}\big\{\lambda_m^l(s+m\delta) - \big(s+(m-1)\delta\big)\big\} \\
&\leq s+(m-1)\delta + \lambda_m^l(s+m\delta) - \big(s+(m-1)\delta\big) \\
&= \lambda_m^l(s+m\delta).
\end{aligned}
$$

Again by the $(1,N)$-MCP, we have

$$
\begin{aligned}
&\mu\big(A(x; s+(m-1)\delta, \lambda_m^l(s+m\delta))\big) \\
&\geq \frac{s+(m-1)\delta}{\tau}\left\{\frac{\sin(s+(m-1)\delta)}{\sin\tau}\right\}^{N-1}\mu\big(A(x;\tau,\tau+\rho(m,\delta))\big) \\
&\geq \frac{s+(m-1)\delta}{\pi - s - (m-1)\delta}\left\{\frac{\sin(s+(m-1)\delta)}{\sin(\pi - s - m\delta)}\right\}^{N-1}\mu\big(A(x;\tau,\tau+\rho(m,\delta))\big) \\
&= \frac{s+(m-1)\delta}{\pi - s - m\delta}\,\frac{\pi - s - m\delta}{\pi - s - (m-1)\delta}\big(1+\theta_{l,N}(\delta)\big)\mu\big(A(x;\tau,\tau+\rho(m,\delta))\big) \\
&= \big(1+\theta_{l,N}(\delta)\big)\frac{s+(m-1)\delta}{\pi - s - m\delta}\mu\big(A(x;\tau,\tau+\rho(m,\delta))\big).
\end{aligned}
$$

By the choice of τ, it implies

$$\begin{aligned}
&\mu\big(A(x; s+(m-1)\delta, \lambda^l_m(s+m\delta))\big) \\
&\geq \big(1+\theta_{l,N}(\delta)\big)\delta^{-1}\big\{\lambda^l_m(s+m\delta) - \big(s+(m-1)\delta\big)\big\}\mu(A_m) \\
&= \big(1+\theta_{l,N}(\delta)\big)\delta^{-1}\big\{\delta + (\lambda^l_m - 1)(s+m\delta)\big\}\mu(A_m).
\end{aligned}$$

Note that

$$\frac{\partial}{\partial\delta}\bigg|_{\delta=0} \lambda^l_m = l\frac{-m(\pi-s)+(m-1)(\pi-s)}{(\pi-s)^2} = -\frac{l}{\pi-s}.$$

Thus we observe

$$\lambda^l_m = 1 - \frac{l\delta}{\pi-s} + \delta\theta_l(\delta).$$

Therefore we find

$$\begin{aligned}
&\mu\big(A(x; s+(m-1)\delta, \lambda^l_m(s+m\delta))\big) \\
&\geq \big(1+\theta_{l,N}(\delta)\big)\left\{1 - \left(\frac{l}{\pi-s} + \theta_l(\delta)\right)(s+m\delta)\right\}\mu(A_m) \\
&\geq \big(1+\theta_{l,N}(\delta)\big)\left(1 - \frac{lt}{\pi-t} + \theta_l(\delta)\right)\mu(A_m) \\
&\geq \big(1+\theta_{l,N}(\delta)\big)\left(1 - \frac{l}{\pi-\pi/(l+1)}\frac{\pi}{l+1} + \theta_l(\delta)\right)\mu(A_m) \\
&\geq \big(1+\theta_{l,N}(\delta)\big)\left(\frac{1}{l+1} + \theta_l(\delta)\right)\mu(A_m) \\
&= \frac{1+\theta_{l,N}(\delta)}{l+1}\mu(A_m).
\end{aligned} \tag{4.3}$$

Combining inequalities (4.2) and (4.3), we obtain

$$\mu\big(A(x; s+(m-1)\delta, s+m\delta)\big) \geq \big(1+\theta_{l,N}(\delta)\big)\mu(A_m).$$

Summing up this inequality in $m = 1, \ldots, M$ yields

$$\mu\big(A(x; s,t)\big) \geq \big(1+\theta_{l,N}(\delta)\big)\mu\big(A(x; \pi-t, \pi-s)\big).$$

By letting δ tend to zero, we complete the proof. □

Theorem 4.2 (Bonnet-Myers theorem, I) *If a metric measure space (X,μ) satisfies the (K,N)-MCP for some $K>0$ and $N>1$, then we have* $\operatorname{diam} X \leq \pi/\sqrt{K}$.

Proof. It suffices to consider the case of $K=1$. Suppose that there exist two points $x, y \in X$ with $|x-y|_X = \pi+\varepsilon$ for some $\varepsilon > 0$. Since X is a length space, for any small $\delta \in (0,\varepsilon)$, we can take a unit speed curve $\gamma : [0, \pi+\varepsilon+\delta'] \longrightarrow X$ such that $\gamma(0) = x$, $\gamma(\pi+\varepsilon+\delta') = y$, and that $\delta' \in [0,\delta]$. If we put $z_\delta := \gamma(\varepsilon+2\delta+\delta')$, then we find

$$\varepsilon+2\delta \leq |x-z_\delta|_X \leq \varepsilon+2\delta+\delta', \qquad \pi-2\delta-\delta' \leq |z_\delta-y|_X \leq \pi-2\delta.$$

Put

$$t := \frac{\pi - \varepsilon - 2\delta - \delta'}{\pi - \delta}, \qquad A := e_t(\operatorname{supp} \Pi_{z_\delta, B(y,\delta)}).$$

Then it follows from the $(1, N)$-MCP that

$$\begin{aligned}
\mu(A) &\geq t \left\{ \frac{\sin(t(|y - z_\delta|_X + \delta))}{\sin(|y - z_\delta|_X - \delta)} \right\}^{N-1} \mu\big(B(y,\delta)\big) \big((e_t)_* \Pi_{z_\delta, B(y,\delta)}\big)(A) \\
&\geq \left(1 - \frac{\varepsilon + \delta + \delta'}{\pi - \delta}\right) \left\{ \frac{\sin(\pi - \varepsilon - 2\delta - \delta')}{\sin(\pi - 3\delta - \delta')} \right\}^{N-1} \mu\big(B(y,\delta)\big) \\
&\geq \left(1 - \frac{\varepsilon + 2\delta}{\pi - \delta}\right) \left\{ \frac{\sin(\pi - \varepsilon - 3\delta)}{\sin(\pi - 4\delta)} \right\}^{N-1} \mu\big(B(y,\delta)\big) \\
&= \left(1 - \frac{\varepsilon + 2\delta}{\pi - \delta}\right) \left\{ \frac{\sin(\varepsilon + 3\delta)}{\sin(4\delta)} \right\}^{N-1} \mu\big(B(y,\delta)\big).
\end{aligned}$$

On one hand, we observe

$$\begin{aligned}
A &\subset B\big(z_\delta, t(|z_\delta - y|_X + \delta)\big) \subset B\big(x, \varepsilon + 2\delta + \delta' + t(\pi - \delta)\big) \\
&= B(x, \pi).
\end{aligned}$$

On the other hand, we see

$$\begin{aligned}
A &\subset X \setminus \overline{B}\big(x, |x - z_\delta|_X + t(|z_\delta - y|_X - \delta)\big) \\
&\subset X \setminus \overline{B}\big(x, \varepsilon + 2\delta + t(\pi - 3\delta - \delta')\big) \\
&\subset X \setminus \overline{B}\big(x, \pi - 4\delta\big),
\end{aligned}$$

where the last implication follows from

$$\begin{aligned}
&\varepsilon + 2\delta + t(\pi - 3\delta - \delta') \\
&= \frac{1}{\pi - \delta} \big[(\varepsilon + 2\delta)\{(\pi - \delta) - (\pi - 3\delta - \delta')\} + (\pi - \delta')(\pi - 3\delta - \delta')\big] \\
&\geq \pi - 3\delta - \delta' \geq \pi - 4\delta.
\end{aligned}$$

Thus we have, by Lemma 4.1,

$$\mu(A) \leq \mu\big(A(x; \pi - 4\delta, \pi)\big) \leq \mu\big(B(x, 4\delta)\big) \leq 4^N \mu\big(B(x,\delta)\big).$$

Therefore we obtain, since $N > 1$,

$$\frac{\mu(B(x,\delta))}{\mu(B(y,\delta))} \geq 4^{-N} \left(1 - \frac{\varepsilon + 2\delta}{\pi - \delta}\right) \left\{ \frac{\sin(\varepsilon + 3\delta)}{\sin 4\delta} \right\}^{N-1} \to \infty$$

as δ tends to zero. However, this is a contradiction because we can exchange the roles of x and y. $\square$

Recall that we set $S(x,r) = \{y \in X \mid |x - y|_X = r\}$ for $x \in X$ and $r > 0$.

Lemma 4.3 *Let (X, μ) satisfy the $(1, N)$-MCP for some $N > 1$.*

(i) *For every $x \in X$, the set $S(x, \pi)$ has a null measure.*

(ii) *If $x, y \in X$ satisfies $|x - y|_X = \pi$, then we have, for any $\varepsilon \in (0, \pi/2)$,*

$$\mu\big(B(x, \varepsilon)\big) = \mu\big(B(y, \varepsilon)\big).$$

Proof. (i) We can suppose that $S(x, \pi) \neq \emptyset$, in particular, X contains more than two points. Fix an arbitrary $\varepsilon > 0$ and let $\{x_i\}_{i=1}^M$ be a maximal 2ε-discrete set in $S(x, 3\varepsilon)$, i.e., $\{x_i\}_{i=1}^M \subset S(x, 3\varepsilon)$, $|x_i - x_j|_X \geq 2\varepsilon$ holds if $i \neq j$, and $\{B(x_i, 2\varepsilon)\}_{i=1}^M$ covers $S(x, 3\varepsilon)$. Note that $B(x_i, \varepsilon)$'s are mutually disjoint. For any $y \in S(x, \pi)$, there exists a point $z \in S(x, 3\varepsilon)$ such that $|y - z|_X < \pi - 2\varepsilon$, and $|z - x_i|_X < 2\varepsilon$ holds for some i. For such i, we observe

$$\begin{aligned} |y - x_i|_X &\leq |y - z|_X + |z - x_i|_X < \pi, \\ |y - x_i|_X &\geq |y - x|_X - |x - x_i|_X = \pi - 3\varepsilon. \end{aligned}$$

Namely, we see $y \in A(x_i; \pi - 3\varepsilon, \pi)$. Combining this with Lemma 4.1(i), we obtain

$$\begin{aligned} \mu\big(S(x, \pi)\big) &\leq \mu\bigg(\bigcup_{i=1}^M A(x_i; \pi - 3\varepsilon, \pi)\bigg) \\ &\leq \sum_{i=1}^M \mu\big(A(x_i; \pi - 3\varepsilon, \pi)\big) \leq \sum_{i=1}^M \mu\big(B(x_i, 3\varepsilon)\big) \\ &\leq 3^N \sum_{i=1}^M \mu\big(B(x_i, \varepsilon)\big) = 3^N \mu\bigg(\bigcup_{i=1}^M B(x_i, \varepsilon)\bigg) \\ &\leq 3^N \mu\big(B(x, 4\varepsilon)\big) \to 0 \end{aligned}$$

as ε tends to zero by Corollary 2.5. This completes the proof.

(ii) It is a straightforward corollary to Lemma 4.1 through Theorem 4.2 and (i) of this lemma. Indeed, we have

$$\mu\big(B(x, \varepsilon)\big) \geq \mu\big(A(x; \pi - \varepsilon, \pi)\big) = \mu\big(X \setminus B(x, \pi - \varepsilon)\big) \geq \mu\big(B(y, \varepsilon)\big).$$

The converse inequality is obtained similarly. □

We remark that Lemma 4.3(i) is not covered by Lemma 2.4. Now we obtain the latter half of our generalized Bonnet-Myers theorem.

Theorem 4.4 (Bonnet-Myers theorem, II) *If a metric measure space (X, μ) satisfies the (K, N)-MCP for some $K > 0$ and $N > 1$, then, for any $x \in X$, the set $S(x, \pi/\sqrt{K})$ consists of at most one point.*

Proof. Suppose that $K = 1$ and that there exist two points $y, z \in S(x, \pi)$ satisfying $\varepsilon := |y - z|_X/2 > 0$. Then, by Lemma 4.3, Theorem 4.2, and by Lemma 4.1, we obtain

$$\begin{aligned} 2\mu\big(B(x, \varepsilon)\big) &= \mu\big(B(y, \varepsilon)\big) + \mu\big(B(z, \varepsilon)\big) = \mu\big(B(y, \varepsilon) \cup B(z, \varepsilon)\big) \\ &\leq \mu\big(A(x; \pi - \varepsilon, \pi)\big) \leq \mu\big(B(x, \varepsilon)\big). \end{aligned}$$

This contradicts to $\mu(B(x, \varepsilon)) > 0$, and hence we complete the proof. □

Corollary 4.5 *If (X,μ) satisfies the (K,N)-MCP, then it also satisfies the (K',N)-MCP for all $K' \le K$.*

5 Bishop-Gromov volume comparison theorem

This section is devoted to proving an analogue of the Bishop-Gromov volume comparison theorem. See [Cl, Theorem 3.10] for the Riemannian case. For $n \ge 2$, $K \in \mathbb{R}$, and $r > 0$ ($r \in (0, \pi/\sqrt{K})$ if $K > 0$), $\mathsf{V}_K^n(r)$ denotes the volume of a ball of radius r in an n-dimensional, simply-connected, and complete Riemannian manifold with a constant sectional curvature K.

Theorem 5.1 (Bishop-Gromov volume comparison theorem) *Let (X,μ) be a metric space satisfying the (K,n)-MCP for some $n \in \mathbb{N}$ with $n \ge 2$. Then, for any $x \in X$, the function*

$$\frac{\mu(B(x,r))}{\mathsf{V}_K^n(r)}$$

is non-increasing in $r > 0$.

Proof. The proof is based on the discretization of that in the Riemannian case. Take $r > 0$. By Theorems 4.2 and 4.4, we can suppose $r \le \pi/\sqrt{K}$ if $K > 0$. For a small $t \in (0,1)$ and any $l, m \in \mathbb{N}$ with $l < m$, it follows from the (K,n)-MCP with $x = x$, $A = A(x; t^l r, t^{l-1} r)$, and $t = t^{m-l}$ that

$$\begin{aligned}
&\mu\big(A(x; t^m r, t^{m-1} r)\big) \\
&\ge t^{m-l} \sup_{t \le s \le 1} \left\{ \frac{\mathbf{s}_K(st^{m-1}r)}{\mathbf{s}_K(st^{l-1}r)} \right\}^{n-1} \mu\big(A(x; t^l r, t^{l-1} r)\big) \\
&\ge t^{m-l} \left[\Big\{ \inf_{t \le s \le 1} \mathbf{s}_K(st^{m-1}r) \Big\} \Big/ \Big\{ \sup_{t \le s \le 1} \mathbf{s}_K(st^{l-1}r) \Big\} \right]^{n-1} \\
&\qquad \times \mu\big(A(x; t^l r, t^{l-1} r)\big).
\end{aligned}$$

Thus we have, for all $l \le j \le m-1$,

$$\begin{aligned}
&\mu\big(A(x; t^j r, t^{j-1} r)\big) \sum_{i=m}^{\infty} t^i \inf_{t \le s \le 1} \mathbf{s}_K(st^{i-1}r)^{n-1} \\
&\le \left\{ \sum_{i=m}^{\infty} \mu\big(A(x; t^i r, t^{i-1} r)\big) \right\} t^j \sup_{t \le s \le 1} \mathbf{s}_K(st^{j-1}r)^{n-1} \\
&= \mu\big(B(x, t^{m-1} r)\big) t^j \sup_{t \le s \le 1} \mathbf{s}_K(st^{j-1}r)^{n-1}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\mu\big(B(x,t^{l-1}r)\big) \\
&= \mu\big(B(x,t^{m-1}r)\big) + \sum_{j=l}^{m-1} \mu\big(A(x;t^j r, t^{j-1}r)\big) \\
&\le \left[1 + \left\{\sum_{j=l}^{m-1} t^j \sup_{t\le s\le 1} \mathbf{s}_K(st^{j-1}r)^{n-1}\right\} \Big/ \left\{\sum_{i=m}^{\infty} t^i \inf_{t\le s\le 1} \mathbf{s}_K(st^{i-1}r)^{n-1}\right\}\right] \\
&\quad \times \mu\big(B(x,t^{m-1}r)\big) \\
&\le \left[\left\{\sum_{j=l}^{\infty} t^j \sup_{t\le s\le 1} \mathbf{s}_K(st^{j-1}r)^{n-1}\right\} \Big/ \left\{\sum_{i=m}^{\infty} t^i \inf_{t\le s\le 1} \mathbf{s}_K(st^{i-1}r)^{n-1}\right\}\right] \\
&\quad \times \mu\big(B(x,t^{m-1}r)\big),
\end{aligned}$$

and hence

$$\begin{aligned}
&\mu\big(B(x,t^{l-1}r)\big) \Big/ \left\{\sum_{j=l}^{\infty} (t^{j-1}r - t^j r) \sup_{t\le s\le 1} \mathbf{s}_K(st^{j-1}r)^{n-1}\right\} \\
&\le \mu\big(B(x,t^{m-1}r)\big) \Big/ \left\{\sum_{i=m}^{\infty} (t^{i-1}r - t^i r) \inf_{t\le s\le 1} \mathbf{s}_K(st^{i-1}r)^{n-1}\right\}.
\end{aligned}$$

This completes the proof by letting t tend to 1. $\square$

6 Stability and compactness

In this section, we consider the behavior of the (K,N)-MCP under the measured Gromov-Hausdorff convergence. The Wasserstein space will play a crucial role. See [F] and [KS2] for the measured Gromov-Hausdorff convergence, and see [LV], [S2], and [V] for the Wasserstein space.

6.1 Measured Gromov-Hausdorff topology

We first recall the Gromov-Hausdorff distance between compact metric spaces. See [G] for more details. For two closed subsets A and A' in a metric space Z, the *Hausdorff distance* d_H^Z between them is defined by

$$d_H^Z(A,A') := \inf\{\varepsilon > 0 \,|\, A \subset B(A',\varepsilon),\ A' \subset B(A,\varepsilon)\}.$$

More generally, for two compact metric spaces X and Y, we define the *Gromov-Hausdorff distance* d_{GH} between them by

$$d_{GH}(X,Y) := \inf_{Z,\varphi,\psi} d_H^Z\big(\varphi(X),\psi(Y)\big),$$

where the infimum is taken over all metric spaces Z and isometric embeddings $\varphi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$. If we denote by $\mathcal{C}$ the isometric classes of compact metric spaces, then

$(\mathcal{C}, d_{GH})$ is a complete metric space. The topology of $\mathcal{C}$ induced from d_{GH} is called the *Gromov-Hausdorff topology.* It is convenient to estimate the Gromov-Hausdorff distance in terms of the ε-approximating map. For metric spaces X and X', a (not necessarily continuous) map $\varphi : X \longrightarrow X'$ is called an *ε-approximating map* for $\varepsilon \geq 0$ if it satisfies $\overline{B}^{X'}(\varphi(X), \varepsilon) \supset X'$ and if

$$\big| |\varphi(x) - \varphi(y)|_{X'} - |x - y|_X \big| \leq \varepsilon$$

holds for all $x, y \in X$. Note that a 0-approximating map is nothing but an isometry.

Lemma 6.1 *Let $X, Y \in \mathcal{C}$ and $\varepsilon > 0$.*

(i) *If $d_{GH}(X, Y) < \varepsilon$, then there exists a 2ε-approximating map from X to Y.*

(ii) *If there exists an ε-approximating map from X to Y, then we have $d_{GH}(X, Y) \leq 2\varepsilon$.*

In particular, a sequence $\{X_i\}_{i=1}^{\infty} \subset \mathcal{C}$ converges to $X \in \mathcal{C}$ if and only if there exists a sequence of ε_i-approximating maps $\varphi_i : X_i \longrightarrow X$ with $\lim_{i\to\infty} \varepsilon_i = 0$. For the later use, we recall an easily proved lemma.

Lemma 6.2 *Let $\{X_i\}_{i=1}^{\infty} \subset \mathcal{C}$ be a sequence of compact, geodesic metric spaces converging to a compact metric space $X \in \mathcal{C}$ in the Gromov-Hausdorff topology with a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ tending to zero and ε_i-approximating maps $\{\varphi_i\}_{i=1}^{\infty}$. For a sequence of minimal geodesics $\gamma_i : [0, 1] \longrightarrow X_i$, $i \in \mathbb{N}$, if the sequences of end points $\{\varphi_i(\gamma_i(0))\}_{i=1}^{\infty}$ and $\{\varphi_i(\gamma_i(1))\}_{i=1}^{\infty}$ converge to some points $x, y \in X$, respectively, then a subsequence of $\{\varphi_i \circ \gamma_i\}_{i=1}^{\infty}$ converges to a minimal geodesic from x to y uniformly.*

We next recall the measured Gromov-Hausdorff convergence introduced in [F].

Definition 6.3 (Measured Gromov-Hausdorff convergence, [F]) A directed system of metric measure spaces $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in\mathcal{A}}$ is said to converge to a metric measure space (X, μ) in the sense of the *measured Gromov-Hausdorff convergence* if there exists a directed system of positive numbers $\{\varepsilon_\alpha\}_{\alpha\in\mathcal{A}}$ satisfying the following conditions:

(1) $\{\varepsilon_\alpha\}_{\alpha\in\mathcal{A}}$ converges to zero;

(2) For each $\alpha \in \mathcal{A}$, we have a Borel, measurable, and ε_α-approximating map $\varphi_\alpha : X_\alpha \longrightarrow X$;

(3) A directed system of push-forward measures $\{(\varphi_\alpha)_*(\mu_\alpha)\}_\alpha$ converges to μ weakly, i.e., for any $f \in C(X)$, we have

$$\lim_{\alpha\in\mathcal{A}} \int_X f \, d\big((\varphi_\alpha)_*(\mu_\alpha)\big) = \int_X f \, d\mu.$$

Here $C(X)$ denotes the set of all continuous functions on X.

If we define $\mathcal{CM}$ as the isomorphic classes of all compact metric spaces equipped with Radon measures, then the measured Gromov-Hausdorff convergence gives a topology on $\mathcal{CM}$, and we call it the *measured Gromov-Hausdorff topology*. We know that this topology is Hausdorff ([F, Proposition 2.7]) and that the projection $\mathcal{CM}(V) \longrightarrow \mathcal{C}$ is proper, where we set

$$\mathcal{CM}(V) := \{(X, \mu) \in \mathcal{CM} \mid \mu(X) \le V\}$$

for $V > 0$ ([F, Proposition 2.10]). For $K \in \mathbb{R}$, $N \ge 1$, $V > 0$, and $D > 0$, we define $\mathcal{CM}(K, N, V, D) \subset \mathcal{CM}(V)$ as the isomorphic classes of compact metric measure spaces (X, μ) in $\mathcal{CM}(V)$ satisfying the (K, N)-MCP and $\operatorname{diam} X \le D$. The following is an easy corollary of Gromov's precompactness theorem ([G, Proposition 5.2]) by virtue of Theorem 5.1.

Theorem 6.4 *Let $\{(X_i, \mu_i)\}_{i=1}^{\infty} \subset \mathcal{CM}(K, N, V, D)$. Then it has a subsequence which is convergent in the measured Gromov-Hausdorff topology.*

If we denote by $(X, \mu) \in \mathcal{CM}$ that limit space, then we immediately observe $\mu(X) \le V$ and $\operatorname{diam} X \le D$. To show that (X, μ) also satisfies the (K, N)-MCP, we need to recall the Wasserstein space and some results in [LV].

6.2 Wasserstein spaces

Let X be a complete, separable metric measure space, and recall that $\mathcal{P}^2(X)$ denotes the set of all Borel probability measures, say μ, satisfying $\int_X |x-y|_X^2 \, d\mu(y) < \infty$ for some (and hence all) $x \in X$. Given two probability measures $\mu, \nu \in \mathcal{P}^2(X)$, a Borel measure q on $X \times X$ is called a *coupling* of μ and ν if, for any measurable set $A \subset X$, we have $q(A \times X) = \mu(A)$ and $q(X \times A) = \nu(A)$. For example, the product measure $\mu \times \nu$ is a coupling of μ and ν. We define the *L^2-Wasserstein distance* d_W on $\mathcal{P}^2(X)$ by

$$d_W(\mu, \nu) := \inf \left\{ \left(\int_{X \times X} |x-y|_X^2 \, dq(x,y) \right)^{1/2} \,\middle|\, q : \text{coupling of } \mu \text{ and } \nu \right\}$$

for $\mu, \nu \in \mathcal{P}^2(X)$, and we call $(\mathcal{P}^2(X), d_W)$ the *L^2-Wasserstein space* over X. Then $(\mathcal{P}^2(X), d_W)$ is a complete and separable metric space (see [S2, Proposition 2.10]). Furthermore, $(\mathcal{P}^2(X), d_W)$ is compact or a length space if and only if so is X, respectively. In particular, if X is compact and geodesic, then so is $(\mathcal{P}^2(X), d_W)$.

Proposition 6.5 (cf. [V, Theorem 7.2]) *A sequence $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}^2(X)$ converges to $\mu \in \mathcal{P}^2(X)$ with respect to d_W if and only if μ_i converges to μ weakly and*

$$\lim_{R \to \infty} \sup_{i \in \mathbb{N}} \int_{X \setminus B(x,R)} |x-y|_X^2 \, d\mu_i(y) = 0 \tag{6.1}$$

holds for some (and hence all) point $x \in X$.

We observe that (6.1) automatically holds true if X is bounded. The following two results obtained in [LV] will play key roles in our discussions.

Proposition 6.6 ([LV, Proposition 4.1, Corollary 4.3]) *If $\varphi : X \longrightarrow X'$ is a Borel, ε-approximating map, then $\varphi_* : (\mathcal{P}^2(X), d_W) \longrightarrow (\mathcal{P}^2(X'), d_W)$ is $\tilde{\varepsilon}$-approximating with*

$$\tilde{\varepsilon} = 4\varepsilon + \{\varepsilon(2 \operatorname{diam} X' + \varepsilon)\}^{1/2}.$$

In particular, if a sequence of compact metric spaces $\{X_i\}_{i=1}^\infty$ converges to a compact metric space X in the Gromov-Hausdorff topology with Borel, ε_i-approximating maps φ_i, $i \in \mathbb{N}$, then the sequence $\{(\mathcal{P}^2(X_i), d_W)\}_{i=1}^\infty$ converges to $(\mathcal{P}^2(X), d_W)$ in the Gromov-Hausdorff topology with $\tilde{\varepsilon}_i$-approximating maps $(\varphi_i)_*$.

Proposition 6.7 ([LV, Proposition 2.9]) *Let X be a compact geodesic metric space. Then any minimal geodesic in $(\mathcal{P}^2(X), d_W)$ is given by the displacement interpolation associated to some dynamical transference plan.*

6.3 Stability and compactness

All spaces in this subsection are assumed to be compact.

Theorem 6.8 (Stability) *A measured Gromov-Hausdorff limit of a sequence of metric measure spaces satisfying the (K, N)-MCP also satisfies the (K, N)-MCP.*

Proof. We first assume $K \leq 0$. Let $\{(X_i, \mu_i)\}_{i=1}^\infty \subset \mathcal{CM}$ be a sequence of metric measure spaces satisfying the (K, N)-MCP. We suppose that it converges to some metric measure space (X, μ) in the measured Gromov-Hausdorff topology, so that we have a sequence $\{\varepsilon_i\}_{i=1}^\infty$ tending to zero and a Borel, measurable, and ε_i-approximating map $\varphi_i : X_i \longrightarrow X$, $i \in \mathbb{N}$, as in Definition 6.3.

Fix a point $x \in X$ and a measurable set $A \subset X$ with $\mu(A) > 0$. For each (large) $i \in \mathbb{N}$, we choose a point $x_i \in \varphi_i^{-1}(\overline{B}^X(x, \varepsilon_i))$ and put $A_i := \varphi_i^{-1}(A)$. We remark that, by the definition of the ε_i-approximating map, $\varphi_i^{-1}(\overline{B}^X(x, \varepsilon_i))$ is not an empty set. Moreover, as $\mu(A) > 0$, we know $\mu_i(A_i) = ((\varphi_i)_*\mu_i)(A) > 0$ and hence A_i in not empty for large i. By the (K, N)-MCP, for each $i \in \mathbb{N}$, we have a dymanical transference plan $\Pi_i = \Pi_{x_i, A_i}$ such that the displacement interpolation associated to it satisfies the conditions (1) and (2) in Definition 2.1. Note that

$$(\varphi_i)_*\big((e_0)_*\Pi_i\big) = (\varphi_i)_*\delta_{x_i} = \delta_{\varphi_i(x_i)} \to \delta_x$$

and, by Proposition 6.5,

$$(\varphi_i)_*\big((e_1)_*\Pi_i\big) = (\varphi_i)_*(\mu_i|_{A_i})^- = \big([(\varphi_i)_*(\mu_i)]|_A\big)^- \to (\mu|_A)^-$$

in $(\mathcal{P}^2(X), d_W)$ as i diverges to the infinity, respectively. Thus it follows from Lemma 6.2 together with Proposition 6.6 that a subsequence of $\{(\varphi_i)_*[(e_t)_*\Pi_i]\}_{t\in[0,1]}$, $i \in \mathbb{N}$, converges to a minimal geodesic $\{\nu_t\}_{t\in[0,1]}$ between δ_x and $(\mu|_A)^-$. Again we denote this convergent subsequence by $\{(\varphi_i)_*[(e_t)_*\Pi_i]\}_{t\in[0,1]}$, $i \in \mathbb{N}$. Moreover, Proposition 6.7 implies that $\{\nu_t\}_{t\in[0,1]}$ is the displacement interpolation associated to some dynamical transference plan $\Pi = \Pi_{x,A}$ which clearly satisfies $(e_0)_*\Pi = \delta_x$ and $(e_1)_*\Pi = (\mu|_A)^-$.

Now we consider the the condition (2) in Definition 2.1. We fix $t \in (0,1)$ and put

$$d\nu_i := (e_t)_*\left(t\left\{\frac{\mathbf{s}_K(t|x-\gamma(1)|_{X_i})}{\mathbf{s}_K(|x-\gamma(1)|_{X_i})}\right\}^{N-1}\mu_i(A_i)\,d\Pi_i(\gamma)\right),$$
$$d\nu := (e_t)_*\left(t\left\{\frac{\mathbf{s}_K(t|x-\gamma(1)|_{X})}{\mathbf{s}_K(|x-\gamma(1)|_{X})}\right\}^{N-1}\mu(A)\,d\Pi(\gamma)\right)$$

on X_i and X, respectively. Since $(\varphi_i)_*[(e_t)_*\Pi_i]$ converges to $(e_t)_*\Pi$ weakly and X_i converges to X in the Gromov-Hausdorff topology, we find that $(\varphi_i)_*(\nu_i)$ converges to ν weakly as i diverges to the infinity. The (K,N)-MCP of (X_i,μ_i) yields that $\mu_i \geq \nu_i$ holds as measures for every i. Therefore we have $\mu \geq \nu$ and hence (X,μ) satisfies the (K,N)-MCP. This completes the proof in the case of $K \leq 0$.

If $K > 0$, then we take $A \subset B^X(x,\pi/\sqrt{K})$ and set, for each $i \in \mathbb{N}$,

$$A_i := \varphi_i^{-1}(A) \cap B^{X_i}(x_i,\pi/\sqrt{K}).$$

Then a completely similar discussion proves the theorem. □

Combining this with Theorem 6.4, we obtain the compactness of $\mathcal{CM}(K,N,V,D)$.

Theorem 6.9 (Compactness) *For any $K \in \mathbb{R}$, $N \geq 1$, $V > 0$, and any $D > 0$, the set $\mathcal{CM}(K,N,V,D)$ is compact in the measured Gromov-Hausdorff topology.*

6.4 Non-compact case

The discussion in the previous subsection is also applicable to the non-compact case by weakening the measured Gromov-Hausdorff convergence to the pointed one. We suppose that all metric spaces appearing in this subsection are complete.

Definition 6.10 A directed system of pointed metric measure spaces $\{(X_\alpha,\mu_\alpha,z_\alpha)\}_{\alpha\in\mathcal{A}}$ is said to converge to a pointed metric measure space (X,μ,z) in the sense of the *pointed measured Gromov-Hausdorff convergence* if there exist two directed systems $\{\varepsilon_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{r_\alpha\}_{\alpha\in\mathcal{A}}$ satisfying the following:

(1) $\{\varepsilon_\alpha\}_{\alpha\in\mathcal{A}}$ tends to zero and $\{r_\alpha\}_{\alpha\in\mathcal{A}}$ diverges to the infinity;

(2) For each $\alpha \in \mathcal{A}$, we have a Borel, measurable, and ε_α-approximating map $\varphi_\alpha : B^{X_\alpha}(z_\alpha,r_\alpha) \longrightarrow B^X(z,r_\alpha)$;

(3) A directed system of push-forward measures $\{(\varphi_\alpha)_*(\mu_\alpha)\}_{\alpha\in\mathcal{A}}$ converges to μ vaguely, i.e., for any $f \in C_0(X)$, we have

$$\lim_{\alpha\in\mathcal{A}}\int_X f\,d\big((\varphi_\alpha)_*(\mu_\alpha)\big) = \int_X f\,d\mu.$$

Here $C_0(X)$ denotes the set of all continuous functions on X whose supports are compact.

Theorem 6.11 *A pointed measured Gromov-Hausdorff limit of a sequence of pointed metric measure spaces satisfying the* (K, N)*-MCP also satisfies the* (K, N)*-MCP.*

Proof. Take a point $x \in X$ and a measurable set $A \subset X$. As X is proper, we can apply the discussion in the proof of Theorem 6.8 to each $A \cap B(x, m)$, $m \in \mathbb{N}$. This completes the proof. $\square$

References

[ABN] A. D. Alexandrov, V. N. Berestovskii, and I. G. Nikolaev, *Generalized Riemannian spaces*, Russian Math. Surveys **41** (1986), 1–54.

[AT] L. Ambrosio and P. Tilli, Topics on analysis in metric spaces, Oxford University Press, Oxford, 2004.

[BC] R. L. Bishop and R. J. Crittenden, Geometry of manifolds, Academic Press, New York-London, 1964.

[BBI] D. Burago, Yu. Burago, and S. Ivanov, A course in metric geometry, American Mathematical Society, Providence, RI, 2001.

[BGP] Yu. Burago, M. Gromov, and G. Perel'man, *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk **47** (1992), 3–52; translation in Russian Math. Surveys **47** (1992), 1–58.

[Cl] I. Chavel, Riemannian geometry—a modern introduction, Cambridge University Press, Cambridge, 1993.

[CC] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I, II, III*, J. Differential Geom. **46** (1997), 37–74; ibid. **54** (2000), 13–35; ibid. **54** (2000), 37–74.

[F] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), 517–547.

[G] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser Boston, Inc., Boston, MA, 1999.

[KS1] K. Kuwae and T. Shioya, *On generalized measure contraction property and energy functionals over Lipschitz maps*, Potential Anal. **15** (2001), 105–121.

[KS2] K. Kuwae and T. Shioya, *Convergence of spectral structures*: *a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. **11** (2003), 599–673.

[LV] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, preprint, 2004: arXiv.math.DG/0412127.

[M] S. B. Myers, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941), 401–404.

[O] S. Ohta, *Reconstructions of distances by energy forms*, preprint, 2004.

[R1] A. Ranjbar-Motlagh, *A note on the Poincaré inequality*, Studia Math. **154** (2003), 1–11.

[R2] A. Ranjbar-Motlagh, *On the Poincaré inequality for abstract spaces*, preprint, 2002.

[RS] M.-K. von Renesse and K.-T. Sturm, *Transport inequalities, gradient estimates, entropy and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), 1–18.

[S1] K.-T. Sturm, *Diffusion processes and heat kernels on metric spaces*, Ann. Probab. **26** (1998), 1–55.

[S2] K.-T. Sturm, *On the geometry of metric measure spaces*, preprint, 2004.

[S3] K.-T. Sturm, *On the geometry of metric measure spaces, II*, preprint, 2005.

[V] C. Villani, Topics in optimal transportation, American Mathematical Society, Providence, RI, 2003.