On Hyperbolic Plateaus of the Hénon Maps

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Abstract

We propose a rigorous computational method to prove the uniform hyperbolicity of discrete dynamical systems. Applying the method to the real Hénon family, we prove the existence of many regions of hyperbolic parameters in the parameter space of the family.

1 Introduction and the Statement of the Results

Consider the problem of determining the set of parameter values for which the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x) \qquad (a, b \in \mathbb{R})$$

is uniformly hyperbolic. If a dynamical system is uniformly hyperbolic, generally speaking, we can apply the so-called hyperbolic theory of the dynamical systems and obtain many results on the behavior of the system. Despite its importance, however, proving hyperbolicity is a difficult problem even for such simple dynamical systems as the Hénon maps.

The first mathematical result about the hyperbolicity of the Hénon map was obtained by Devaney and Nitecki [11]. They showed that for any fixed b, if a is sufficiently large then the non-wandering set of $H_{a,b}$ is uniformly hyperbolic and conjugate to the full shift of two symbols.

Later, Davis, MacKay and Sannami [8] conjectured that the area preserving Hénon map $H_{a,-1}$ is uniformly hyperbolic if a is taken from some intervals, and hence the dynamics is conjugate to a subshift of finite type. Describing the configuration of stable and unstable manifolds, they identified the Markov partitions for these parameter values (see also Sterling-Dullin-Meiss [22] and Hagiwara-Shudo [12]). Although the mechanism of hyperbolicity at these parameter values is clear by their observations, no mathematical proof has been obtained so far.

The purpose of this paper is to propose a general method for proving uniform hyperbolicity of discrete dynamical systems. Applying the method to the Hénon map, we obtain a computer assisted proof of the hyperbolicity of Hénon map on many parameter regions including the intervals conjectured by Davis *et al*.

Our results are summarized in the following theorems. We denote by $\mathcal{R}(H_{a,b})$ the chain recurrent set of $H_{a,b}$.

Theorem 1 (2-parameter family). *If (a, b) is taken from the closure of the colored regions of Figure 1, or if $a \notin [-1, 6.25]$ and $b \in [-1, 1]$, then $\mathcal{R}(H_{a,b})$ is uniformly hyperbolic.*

The hyperbolicity of $\mathcal{R}(H_{a,b})$ implies that $H_{a,b}$ is $\mathcal{R}$-stable on these parameter regions. We call such a region a "plateau", because no bifurcation occurs in $\mathcal{R}(H_{a,b})$ and hence numerical invariants such as the topological entropy, the number of the periodic points, etc., are constant on it.

Theorem 1 does not claim that a parameter value in the white region is a non-hyperbolic parameter. It only guarantees that the colored regions are contained in the set of hyperbolic parameter values. We can refine Theorem 1 by performing more computations, that is, the more we perform computations, the larger the colored regions will be.

It is interesting to compare Figure 1 with the bifurcation diagrams of the Hénon map given by El Hamouly and Mira [13], and by Sannami [20, 21]. There must be a bifurcation on the boundary of a hyperbolic parameter region, and in fact, the boundaries of plateaus in Figure 1 resemble the bifurcation curves given in these papers.

Since the area-preserving Hénon family $H_{a,-1}$ is of particular importance, we run a further computation restricted to it and obtain the following.

Theorem 2 (area preserving family). *If a is in one of the following intervals*:

$$
\begin{array}{ll}
[4.5383300781250, 4.5385742187500], & [4.5388183593750, 4.5429687500000], \\
[4.5623779296875, 4.5931396484375], & [4.6188964843750, 4.6457519531250], \\
[4.6694335937500, 4.6881103515625], & [4.7681884765625, 4.7993164062500], \\
[4.8530273437500, 4.8603515625000], & [4.9665527343750, 4.9692382812500], \\
[5.1469726562500, 5.1496582031250], & [5.1904296875000, 5.5366210937500], \\
[5.5659179687500, 5.6077880859375], & [5.6342773437500, 5.6768798828125], \\
[5.6821289062500, 5.6857910156250], & [5.6859130859375, 5.6860351562500], \\
[5.6916503906250, 5.6951904296875], & [5.6999511718750, \infty),
\end{array}
$$

then $\mathcal{R}(H_{a,-1})$ is uniformly hyperbolic.

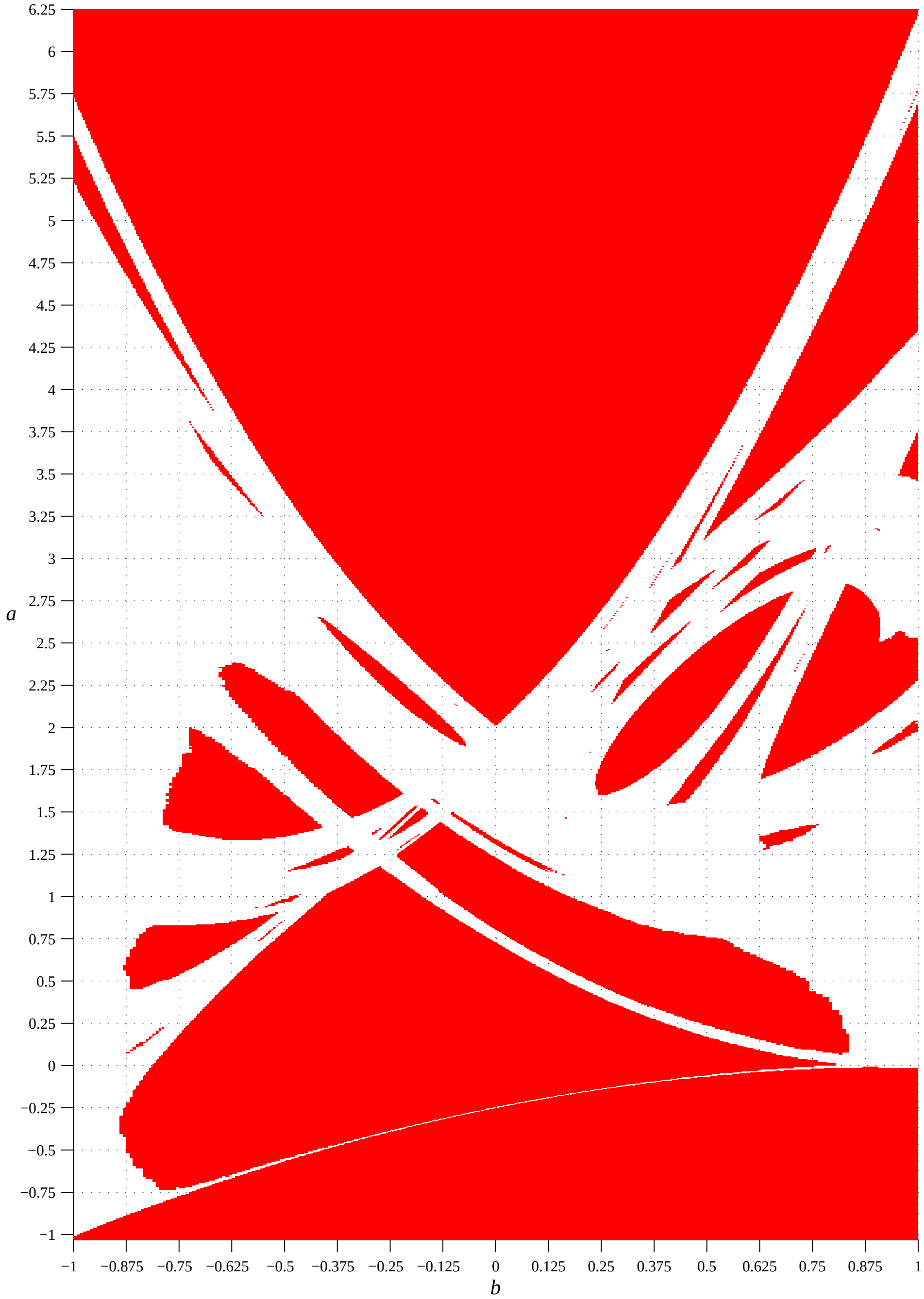

Figure 1: uniformly hyperbolic plateaus

The three intervals conjectured as hyperbolic parameter values by Davis *et al.* appear in Theorem 2, thus we can say that Theorem 2 justifies their observations.

Recently Cao, Luzzatto and Rios [6] showed that the Hénon map has a tangency and hence is non-hyperbolic if the parameter is on the boundary of the full horseshoe plateau (see also [4, 5]). This fact and Theorem 2 suggests that $H_{a,-1}$ should have a tangency when a is close to 5.699951171875. In fact, we can prove the following theorem using the rigorous computational method developed by the author and Mischaikow [2].

Proposition 3. *There exists* $a \in [5.6993102, 5.6993113]$ *such that* $H_{a,-1}$ *has a homoclinic tangency with respect to the saddle fixed point on the third quadrant.*

Consequently, Theorem 2 and Proposition 3 yields the following.

Corollary 4. *When we decrease* $a \in \mathbb{R}$ *of the area-preserving Hénon family* $H_{a,-1}$, *the first tangency occurs in the interval* $[5.6993102, 5.699951171875)$.

We remark that Hruska [14, 15] also constructed a rigorous numerical method for proving hyperbolicity of complex Hénon maps. The main difference between our method and Hruska's method is that our method does not prove hyperbolicity directly. Instead, it proves quasi-hyperbolicity, which is is equivalent to hyperbolicity under the assumption of chain recurrence. This enables us to avoid constructing a metric adapted to the hyperbolic splitting. Another peculiar feature of our algorithm is that it is based on the subdivision algorithm [9] and hence effective for inductive search of hyperbolic parameters.

The structure of the rest of this paper is as follows. The notion of quasi-hyperbolicity will be introduced in §2 and then an algorithm for proving quasi-hyperbolicity will be proposed in §3. In the last section, §4, we apply the method to the Hénon family and obtain Theorem 1 and 2.

2 Hyperbolicity and Quasi-Hyperbolicity

First we recall the definition of hyperbolicity. Let f be a diffeomorphism on a manifold M and Λ a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

Definition 5. We say f is *uniformly hyperbolic* on Λ, or Λ is a *uniformly hyperbolic invariant set* if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two

Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n|_{E^s}\| < c\lambda^n \quad \text{and} \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n$$

hold for all $n \geq 0$. Here $\|\cdot\|$ denotes a metric on M.

We note that this definition involves two constants, c and λ. If we try to prove hyperbolicity according to the definition, we must control two parameters at the same time, and the algorithm would be rather complicated. We can omit the constant c by choosing a suitable metric on M, but constructing such a metric is also a difficult problem in general. The situation is the same if we use the standard "cone fields" argument.

To avoid this computational difficulty, we introduce the notion of quasi-hyperbolicity.

Definition 6. We say f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

It is easy to see that hyperbolicity implies quasi-hyperbolicity. The converse is not true in general. However, when $f|_\Lambda$ is chain recurrent, these two notions coincide.

Theorem 7 (Churchill-Franke-Selgrade [7], Sacker-Sell [19]). *Assume that $f|_\Lambda$ is chain recurrent, that is, $\mathcal{R}(f|_\Lambda) = \Lambda$. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

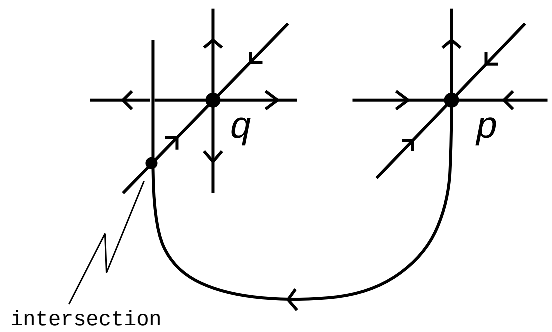

Figure 2: $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$ is quasi-hyperbolic, but is not uniformly hyperbolic.

Remark 8. The assumption of chain recurrence is essential for uniform hyperbolicity. For example, consider two hyperbolic saddle fixed points p and q in $\mathbb{R}^3$, with 1 and 2 dimensional unstable direction, respectively. Assume that the unstable manifold $W^u(p)$ of p intersects the stable manifold $W^s(q)$ of q, so that the sum of tangent spaces of these two manifolds span a 2-dimensional subspace (see Figure 2). Let $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$. Then Λ is quasi-hyperbolic, but clearly not uniformly hyperbolic because it contains fixed points with different unstable dimensions and a connecting orbit between them.

Recall that a compact set N is an isolating neighborhood [16] with respect to f if the maximal invariant set

$$\mathrm{Inv}_f N := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}$$

is contained in int N, the interior of N. An invariant set S of f is said to be isolated if there is an isolating neighborhood N such that $\mathrm{Inv}_f N = S$.

Here we note that the definition of quasi-hyperbolicity is equivalent to saying that the zero section of the tangent bundle $T\Lambda$ is an isolated invariant set with respect to $Tf : T\Lambda \to T\Lambda$. In fact, it suffice to find an isolating neighborhood that contains the zero section.

Proposition 9. *Let $K \subset M$ be a compact set containing Λ and $N \subset TK$ be a compact neighborhood of the zero section of TK. If N is an isolating neighborhood with respect to $Tf : TM \to TM$, then Λ is quasi-hyperbolic.*

Proof. For a subset S of TM and $\delta \geq 0$, we define $\delta S := \{\delta \cdot v \mid v \in S\}$. By linearity, if S is Tf-invariant so is δS. Assume $\mathrm{Inv}_{Tf} N \subset \mathrm{int}\, N$. If we show that $\mathrm{Inv}_{Tf} N$ does not contain a non-trivial orbit, then again by linearity, it follows that there is no-nontrivial bounded orbit. A standard compactness argument shows that there is $\delta > 1$ such that $\delta\, \mathrm{Inv}_{Tf} N \subset N$. Since $\delta\, \mathrm{Inv}_{Tf} N$ is Tf-invariant and contained in N, we have $\delta\, \mathrm{Inv}_{Tf} N \subset \mathrm{Inv}_{Tf} N$, by definition. It follows that if $v \in \mathrm{Inv}_{Tf} N$, we have $\delta^n v \in \mathrm{Inv}_{Tf} N$ for all $n \geq 0$. Since $\mathrm{Inv}_{Tf} N$ is bounded, v must be the zero vector. □

3 Algorithm

In this section, we assume that $M = \mathbb{R}^n$ and consider a diffeomorphism $f_a : M \to M$ that depends on k real parameters $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. Define $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ and $TF : T\mathbb{R}^n \times \mathbb{R}^k \to T\mathbb{R}^n$ by

$$F(x, a) := f_a(x) \quad \text{and} \quad TF(x, v; a) := Tf_a(x, v)$$

where $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$.

We denote by $\mathbb{F}$ the set of floating point numbers, or, the set of numbers our computer can handle. Let $\mathbb{IF}$ be the set of intervals whose end-points are in $\mathbb{F}$. Namely,

$$\mathbb{IF} := \{I = [a, b] \subset \mathbb{R} \mid a, b \in \mathbb{F}\}.$$

Similarly, we define the set of n-dimensional cubes

$$\mathbb{IF}^n := \{I_1 \times \cdots \times I_n \subset \mathbb{R}^n \mid I_i \in \mathbb{IF}\}.$$

Let $X, V \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^k$. We want to know the image of these cubes under the map F and TF, that is, $F(X \times A)$ and $TF(X \times V \times A)$. These are not an object of $\mathbb{IF}^n$ nor $\mathbb{IF}^{2n}$ in general, but we require that we can enclose these images using elements of $\mathbb{IF}^n$ and $\mathbb{IF}^{2n}$.

Assumption 10. There is a computational method that inputs cubes $X, V \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^k$, and outputs $Y \in \mathbb{IF}^n$ and $W \in \mathbb{IF}^{2n}$ such that

$$F(X \times A) \subset \operatorname{int} Y \quad \text{and} \quad TF(X \times V \times A) \subset \operatorname{int} W$$

hold rigorously.

As we will mention in the last section, for many classes of dynamical systems, this assumption can be satisfied by using rigorous interval arithmetic.

Here we recall the setting of Proposition 9. Let $K \subset \mathbb{R}^n$ be a compact set that contains Λ and $N \subset T\mathbb{R}^n$ the product of N and $[-1, 1]^n$. We assume that we can decompose K into a finite union of the elements of $\mathbb{IF}^n$, namely,

$$K = \bigcup_{i=1}^{\ell} K(i) \quad \text{where} \quad K(i) \in \mathbb{IF}^n.$$

We also decompose the fiber $[-1, 1]^n$ into a finite union of the elements of $\mathbb{IF}^n$. By constructing products of cubes contained in the decompositions of K and $[-1, 1]$, we also decompose N as

$$N = \bigcup_{j=1}^{m} N(j) \quad \text{where} \quad N(j) \in \mathbb{IF}^{2n}.$$

By Assumption 10, we can compute $Y(i) \in \mathbb{IF}^n$ and $W(j) \in \mathbb{IF}^{2n}$ such that

$$F(K(i) \times A) \subset \operatorname{int} Y(i) \quad \text{and} \quad TF(N(j) \times A) \subset \operatorname{int} W(j).$$

Next, we construct directed graphs $\mathcal{G}(F)$ and $\mathcal{G}(TF)$ from the information of $Y(i)$ and $W(j)$ as follows:

- $\mathcal{G}(F)$ has ℓ-vertices: $\{v_1, v_2, \ldots, v_\ell\}$.
- There exists an edge from v_p to v_q if and only if $Y(p) \cap K(q) \neq \emptyset$.

And similarly,

- $\mathcal{G}(TF)$ has m-vertices: $\{w_1, w_2, \ldots, w_m\}$.
- There exists an edge from w_p to w_q if and only if $W(p) \cap N(q) \neq \emptyset$.

The most important property of $\mathcal{G}(F)$ is that if there exists $x \in K(p)$ that is mapped into $K(q)$ by f_a for some $a \in A$, then there must be an edge of $\mathcal{G}(F)$ from v_p to v_q. This property also holds for $\mathcal{G}(TF)$.

We then use these directed graphs to enclose the chain recurrent set of f_a and the maximal invariant set of N. For this purpose, we define the following notions.

Definition 11. The *invariant set of a directed graph* G is a subgraph of G defined as

$$\operatorname{Inv} G := \{v \in G \mid \exists \text{ edge that ends at } v, \text{ and } \exists \text{ edge that starts at } v\}.$$

The set of *strongly connected components* of G is a subgraph of G defined as

$$\operatorname{Scc} G := \{v \in G \mid \exists \text{ path from } v \text{ to } v\}.$$

For a subgraph G of $\mathcal{G}(F)$, or $\mathcal{G}(TF)$, we define its *geometric representation* by

$$|G| := \bigcup_{v_i \in G} K(i), \quad \text{or} \quad |G| := \bigcup_{w_j \in G} N(j),$$

respectively. Obviously, $|\mathcal{G}(F)| = K$ and $|\mathcal{G}(TF)| = N$.

Proposition 12. *For any $a \in A$,*

$$\operatorname{Inv}_{f_a} N \subset |\operatorname{Inv} \mathcal{G}(F)| \quad \textit{and} \quad \operatorname{Inv}_{Tf_a} N \subset |\operatorname{Inv} \mathcal{G}(TF)|.$$

Furthermore, for any $a \in A$, if $\mathcal{R}(f_a)$ is contained in int K *then*

$$\mathcal{R}(f_a) \subset |\operatorname{Scc} \mathcal{G}(F)|$$

holds.

Proof. The first claim follows immediately from the construction of $\mathcal{G}(TF)$. We prove the latter claim. Since $F(K(i) \times \{a\}) \subset \operatorname{int} Y(i)$ holds for all i and the number of cubes in K is finite, we can choose $\varepsilon > 0$ such that if $x \in K(i)$ and y is a point with $d(f_a(x), y) < \varepsilon$ then y must be contained in $Y(i)$. Here d denotes a fixed metric of $\mathbb{R}^n$. This implies that if such y is contained in $K(j)$, there must be an edge from v_i to v_j. Let $x \in \mathcal{R}(f_a)$. From the assumption, there exists p such that $x \in K(p)$. Since $\mathcal{R}(f_a) \subset \operatorname{int} K$, we can assume that these is an ε-chain form x to itself that is contained in K, by choosing smaller ε, if necessary. It follows that there must be a path of $\mathcal{G}(F)$ from v_p to itself and therefor $x \in |\operatorname{Scc} \mathcal{G}(F)|$. This proves the claim. □

For the computation of $\operatorname{Inv} G$, the algorithm of Szymczak [24] can be used. There is also an algorithm for computing $\operatorname{Scc} G$ that is standard in the algorithmic graph theory (see [23], for example).

Combining Proposition 9 and Proposition 12, we obtain the following theorem.

Theorem 13. *If $|\operatorname{Inv} \mathcal{G}(TF)| \subset \operatorname{int} N$, then f_a is quasi-hyperbolic on Λ_a for all $a \in A$.*

Now we describe an algorithm to prove the quasi-hyperbolicity of invariant sets. The algorithm involves the subdivision algorithm [9], that is, if it fails to prove quasi-hyperbolicity, then it subdivide all of the cubes in K and N to have a better approximation of the invariant set.

Let Λ_a be the invariant set of our interest, that depends on a. We assume that $\Lambda_a \subset \operatorname{int} K$ holds for all $a \in A$.

Algorithm 14. (For proving quasi-hyperbolicity for all $a \in A$)

1. If we know that $\Lambda_a \subset \mathcal{R}(f_a)$ holds for all $a \in A$, compute $\operatorname{Scc} \mathcal{G}(F)$ and replace K with $|\operatorname{Scc} \mathcal{G}(F)|$. Otherwise compute $\operatorname{Inv} \mathcal{G}(F)$ and replace K with $|\operatorname{Inv} \mathcal{G}(F)|$.

2. Replace N with $K \times [-1, 1]^n$ and decompose N into cubes using the product of the decomposition of K and $[-1, 1]^n$.

3. Compute $\operatorname{Inv} \mathcal{G}(TF)$.

4. If $|\operatorname{Inv} \mathcal{G}(TF)| \subset \operatorname{int} N$ then stop.

5. Otherwise, refine the decomposition of K and $[-1, 1]$ by bisecting all cubes and then goto the step 1.

The difference of $\mathrm{Scc}\,\mathcal{G}(F)$ and $\mathrm{Inv}\,\mathcal{G}(F)$ in the step 1 of Algorithm 14 is sometimes crucial. This is because $|\mathrm{Inv}\,\mathcal{G}(F)|$ may contain a connecting orbit between the hyperbolic basic set of different indices, for example a connecting orbit from a saddle periodic point to a sink, that violates quasi-hyperbolicity.

The following theorem immediately follows from Theorem 13.

Theorem 15. *If Algorithm* 14 *stops, then f_a is quasi-hyperbolic on Λ_a for all $a \in A$.*

Note that if there is $a \in A$ for which Λ_a is not quasi-hyperbolic, then Algorithm 14 never stops. Practically, therefore, Algorithm 14 is useful only when A is very small or we are confident that all $a \in A$ are quasi-hyperbolic parameters.

Thus, when we want to apply our method for larger A, the algorithm should involve an automatic selection of parameter values. This selection is also realized with the subdivision algorithm. Assume A is decomposed into a finite union of the elements of $\mathbb{IF}^k$,

$$A := \bigcup_i A(i) \quad \text{where} \quad A(i) \in \mathbb{IF}^k.$$

Denote the set of cubes in the decomposition of A by $\mathcal{A}$.

Algorithm 16. (Adaptive selection of quasi-hyperbolic parameters)

1. Start with $\mathcal{A} = \{A\}$.

2. Choose a cube $A(i)$ from $\mathcal{A}$ according to the selection rule.

3. Load the data of N, K and their decompositions for $A(i)$.

4. Apply the step 1, 2, and 3 of Algorithm 14.

5. If $|\mathrm{Inv}\,\mathcal{G}(TF)| \subset \mathrm{int}\,N$ then remove $A(i)$ from $\mathcal{A}$ and goto the step 2.

6. Otherwise, bisect $A(i)$ into two cubes and add these new cubes to $\mathcal{A}$. Save the data of N, K and their decompositions as the data for these two cubes and remove $A(i)$ from $\mathcal{A}$. Then goto the step 2.

In the step 2 of Algorithm 16, we do not specify the rule for selecting $A(i)$. This is because the effectiveness of a rule depends on the case. One example of such a rule is selecting $A(i)$ with smaller number of cubes in N and K. Since the computational cost of step 1, 2 and 3 of Algorithm 14 depends on the number of cubes, this rules makes our computation concentrated on parameter values on which the computation is fast. This rule works sufficiently well for general purpose. But when we want to find a hyperbolic region with very weak hyperbolicity, for example, we want to distribute our computational effort to whole of the parameter space. In this case, we should select all cubes in $\mathcal{A}$ sequentially.

4 Application to the Hénon Maps

In this section, we apply the method developed above to the chain recurrent set $\mathcal{R}(H_{a,b})$ of the Hénon family $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x)$ where $a, b \in \mathbb{R}$.

For this purpose, $\mathcal{R}(H_{a,b})$ must be compact and to perform actual computations, the size of it must be explicitly known. Further, in order to use Theorem 7, we need to check that the dynamics restricted to $\mathcal{R}(H_{a,b})$ is chain recurrent. This is not trivial in this case because the phase space is noncompact.

First we recall the numbers defined by Devaney and Nitecki [11]. Let

$$R(a, b) := \frac{1}{2}(1 + |b| + \sqrt{(1 + |b|)^2 + 4a}),$$
$$S(a, b) := \{(x, y) \in \mathbb{R}^2 : |x| \leq R(a, b), |y| \leq R(a, b)\}.$$

Then we can prove the following.

Lemma 17. *$\mathcal{R}(H_{a,b}) \subset S(a, b)$, and $H_{a,b}$ restricted to $\mathcal{R}(H_{a,b})$ is chain recurrent.*

Proof. If $x \notin S(a, b)$, we can choose $\varepsilon_0 > 0$ so small that if $\varepsilon < \varepsilon_0$ then all ε-chain thorough x must diverge to infinity and hence, x can not be chain recurrent (this is a special case of Corollary 2.7 of [3]). The proof for the second claim is the same as that for the compact case (see [18] for example), because we can choose a compact neighborhood S' of $S(a, b)$ and $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then all ε-chain from $x \in \mathcal{R}$ to x must be contained in S'. □

In the case of Hénon map, Assumption 10 can be satisfied using the interval arithmetics on a CPU that satisfies IEEE754 standard for binary floating-point arithmetic. This is also the case for an arbitrary polynomial map of $\mathbb{R}^n$. In our computation, we use the PROFIL/BIAS interval arithmetic package [17].

We can restrict our computation to the case $(a, b) \in [-1, 12] \times [-1, 1]$, because otherwise it follows from the proof of Devaney and Nitecki [11] that $\mathcal{R}(H_{a,b})$ is hyperbolic or empty.

Therefore, we start with $A := [-1, 12] \times [-1, 1]$ and $K := [-8, 8] \times [-8, 8]$, so $N = K \times [-1, 1]^2$. Then Lemma 17 implies $\mathcal{R}(H_{a,b}) \subset \operatorname{int} K$ holds for all $(a, b) \in A$. Then we prove Theorem 1 by applying Algorithm 16.

To obtain Theorem 2, we fix $b = -1$ and start the computation with $A := [4, 12]$. The set K and N are the same as the computation for Theorem 1.

All of the source codes used in these computations are available at the home page of the author [1]. It uses the data structure and the subdivision algorithm implemented in the GAIO [10] package.

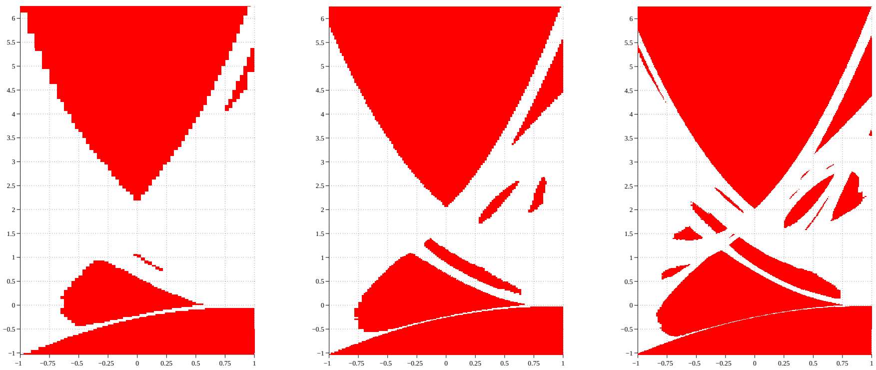

Figure 3: results after 1, 10 and 100 hour computation (from left to right)

Finally, we mention about the computational cost of the method. To achieve Theorem 1, we need 1000 hour computation with PowerPC 970 (2GHz) CPU. With the same CPU, 260 hours were used for Theorem 2. If we restrict our computations to the regions with stronger hyperbolicity, then the computations are much faster. For example, we can obtain the hyperbolic plateaus illustrated in Figure 3 by 1, 10 and 100 hour computation.

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