Hom stacks

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Abstract

We study Hom 2-functors parameterizing 1-morphisms of algebraic stacks, and prove that it is representable by an algebraic stack under certain conditions, using Artin's criterion. As an application we study Picard 2-functors which parameterizes line bundles on algebraic stacks.

1 Introduction

Let S be an affine noetherian scheme over an excellent Dedekind domain. Let $\mathscr{X}$ and $\mathscr{Y}$ be separated algebraic stacks of finite type over S. The Hom 2-functor $\mathscr{HOM}(\mathscr{X}, \mathscr{Y})$ is a contravariant 2-functor from the category of affine noetherian schemes over S to the 2-category of groupoids given by

$$\mathscr{HOM}(\mathscr{X}, \mathscr{Y})(T) = \mathrm{HOM}_T(\mathscr{X} \times_S T, \mathscr{Y} \times_S T).$$

The right hand side is the groupoid of 1-morphisms.

The purpose of this paper is to show the following theorem:

Theorem 1.1. *If $\mathscr{X}$ is proper and flat over S, the 2-functor $\mathscr{H} = \mathscr{HOM}(\mathscr{X}, \mathscr{Y})$ is an algebraic stack in Artin's sense [Ar2].*

Here "in Artin's sense" means that the diagonal $\mathscr{H} \to \mathscr{H} \times_S \mathscr{H}$ is representable and locally of finite type.

It is already known (see [Ol1, 2.1]) that if X is a proper flat algebraic space and Y is a separated algebraic space of finite type, the functor $\mathscr{HOM}(X, Y)$ is representable by an algebraic space. Moreover if X and Y are quasi-projective schemes, $\mathscr{HOM}(X, Y)$ is also a quasi-projective scheme. This is proved by the fact that the map

$$\begin{array}{rcl} \mathscr{HOM}(X, Y) & \to & \mathrm{Hilb}(X \times Y) \\ f & \mapsto & \text{graph of } f \end{array}$$

is representable by an open immersion.

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Unfortunately, we can not use this technique in the case of algebraic stacks, because we do not have "Hilbert stacks" for algebraic stacks yet. The Quot functors of Olsson and Starr ([OS],[Ol3]) does not work for our purpose. The functor $\mathrm{Quot}_{O_{\mathscr{X}\times\mathscr{Y}}}$ parameterizes closed substacks of $\mathscr{X}\times\mathscr{Y}$, but graphs of 1-morphisms are not closed substacks in general, even if stacks $\mathscr{X}$ and $\mathscr{Y}$ are separated. For instance, the graph of $\mathrm{id} : \mathscr{X}\to\mathscr{X}$ is the diagonal $\mathscr{X}\to\mathscr{X}\times\mathscr{X}$, which is not a closed immersion unless $\mathscr{X}$ is representable by an algebraic space.

Olsson [Ol1] studied this problem when $\mathscr{X}$ and $\mathscr{Y}$ are Deligne-Mumford stacks. He investigated the map

$$\mathscr{HOM}(\mathscr{X},\mathscr{Y})\to\mathscr{HOM}(\underline{\mathscr{X}},\underline{\mathscr{Y}})$$

mapping a morphism to that of coarse moduli spaces. Even this technique does not work for Artin stacks, because they do not have coarse moduli spaces in general.

We prove Theorem 1.1 by verifying Artin's condition [Ar2] directly. The most essential part of the proof is the deformation theory of morphisms of algebraic stacks, based on the author's previous work [Ao].

As an application, we prove that the Picard 2-functor [LM, 14.4.7] that parameterizes line bundles on an algebraic stack is representable by an algebraic stack in Artin's sense. This is a generalization of Artin's results on algebraic spaces ([Ar1, 7.3], [Ar2, Appendix 2]).

1.2 Conventions and notations

In this paper we refer to [LM] for definitions and basic properties of algebraic stacks. Especially we assume all algebraic stacks are quasi-separated [LM, 4.1] unless mentioned. Algebraic stacks as in Artin's definition [Ar2, 5.1] is called "algebraic stack in Artin's sense".

We denote schemes and algebraic spaces by Italic letters like X, Y and T, and algebraic stacks by script letters like $\mathscr{X},\mathscr{Y}$ and $\mathscr{T}$. Subscriptions like $\mathscr{X}_T$ mean base change $\mathscr{X}\times_S T$. Superscripts like $X^\bullet$ are used to denote simplicial algebraic spaces.

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2 Deformation of morphisms of algebraic stacks

In this section we study the deformation theory of 1-morphisms of algebraic stacks. This is a generalization of Illusie's work [Il, III 2.2].

2.1 Definitions and Statements

Deformations of 1-morphisms are defined as follows. Let $\mathscr{X}$ and $\mathscr{Y}$ be separated algebraic stacks over a scheme T and $f : \mathscr{X} \to \mathscr{Y}$ a 1-morphism over T. Consider the 2-commutative diagram of solid arrows:

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "i"'] \arrow[dr, "f"] \arrow[ddr] & & \widetilde{\mathscr{X}} \arrow[dr, dotted, "\widetilde{f}"] \arrow[ddr] & \\
& \mathscr{Y} \arrow[rr, "j"'] \arrow[d] & & \widetilde{\mathscr{Y}} \arrow[d] \\
& T \arrow[rr, "k"'] & & \widetilde{T}.
\end{tikzcd}$$

Here i, j and k are closed immersions defined by square-zero ideals I, J and K. Then a deformation of f is a pair $(\widetilde{f}, \lambda)$ where $\widetilde{f}$ is a 1-morphism from $\widetilde{\mathscr{X}}$ to $\widetilde{\mathscr{Y}}$ over $\widetilde{T}$ and $\lambda : \widetilde{f} \circ i \Rightarrow j \circ f$ is a 2-isomorphism. A morphism from $(\widetilde{f}, \lambda)$ to $(\widetilde{g}, \mu)$ is a 2-morphism $\alpha : \widetilde{f} \Rightarrow \widetilde{g}$ such that 2-morphisms

$$i^*\alpha \circ \mu, \lambda : \widetilde{f} \circ i \Rightarrow j \circ f$$

are equal.

We denote the category of deformations of f by $\mathrm{Defm}_T(f)$ and the set of its isomorphic classes by $\overline{\mathrm{Defm}}_T(f)$.

In this section we prove the following generalization of [Il, III 2.2.4].

Theorem 2.1.1.

(1) *There exists an obstruction* $o \in \mathrm{Ext}^1(Lf^*L_{\mathscr{Y}/T}, I)$ *whose vanishing is equivalent to the existence of a deformation.*

(2) *If* $o = 0$*, the set* $\overline{\mathrm{Defm}}_T(f)$ *is a torsor under* $\mathrm{Ext}^0(Lf^*L_{\mathscr{Y}/T}, I)$.

(3) *The automorphism group of any deformation of* f *is isomorphic to* $\mathrm{Ext}^{-1}(Lf^*L_{\mathscr{Y}/T}, I)$.

In the proof of Theorem 2.1.1, we need the deformation theory of morphisms of schemes over algebraic stacks.

Let $\mathscr{T}$ be an algebraic stack, $x : X \to \mathscr{T}$ and $y : Y \to \mathscr{T}$ schemes over $\mathscr{T}$, and $f : X \to Y$ a morphism of schemes with $y \circ f = x$. Consider the diagram of solid arrows:

$$\begin{tikzcd}
X \arrow[rr, "i"'] \arrow[dr, "f"] \arrow[ddr, "x"'] & & \widetilde{X} \arrow[dr, dotted, "\widetilde{f}"] \arrow[ddr, "\widetilde{x}"'] & \\
& Y \arrow[rr, "j"'] \arrow[d, "y"] & & \widetilde{Y} \arrow[d, "\widetilde{y}"] \\
& \mathscr{T} \arrow[rr, "k"'] & & \widetilde{\mathscr{T}}.
\end{tikzcd}$$

Here i, j and k are closed immersions defined by square-zero ideals I, J and K. Then we define a deformation of f to be a pair $(\widetilde{f}, \gamma)$ where $\widetilde{f}$ is a morphism $\widetilde{X} \to \widetilde{Y}$ which satisfies $\widetilde{f} \circ i = j \circ f$ and γ is a 2-isomorphism $\widetilde{y} \circ \widetilde{f} \Rightarrow \widetilde{x}$ whose restriction $y \circ f \Rightarrow x$ is equal to the identity.

We denote the set of deformations of f by $\mathrm{Defm}_{\mathscr{T}}(f)$.

Proposition 2.1.2.

(1) *There exists an obstruction $o \in \mathrm{Ext}^1(Lf^* L_{Y/\mathscr{T}}, I)$ whose vanishing is equivalent to the existence of a deformation.*

(2) *If $o = 0$, $\mathrm{Defm}_T(f)$ is a torsor under $\mathrm{Ext}^0(Lf^* L_{Y/\mathscr{T}}, I)$.*

Remark 2.1.3. The torsor actions and isomorphisms in Theorem 2.1.1 and Proposition 2.1.2 are functorial on $\mathscr{X}, \mathscr{Y}$ and T etc. For example, if $T \to U$ is a morphism of schemes, we have natural "forgetting" map

$$C : \overline{\mathrm{Defm}_T(f)} \to \overline{\mathrm{Defm}_U(f)}$$

and the group homomorphism

$$D : \mathrm{Ext}^0(Lf^* L_{\mathscr{Y}/T}, I) \to \mathrm{Ext}^0(Lf^* L_{\mathscr{Y}/U}, I)$$

induced by the morphism $L_{\mathscr{Y}/U} \to L_{\mathscr{Y}/T}$ [LM, 17.3(3)]. Then for any $[\widetilde{f}] \in \overline{\mathrm{Defm}_T(f)}$ and $\sigma \in \mathrm{Ext}^0(Lf^* L_{\mathscr{Y}/T}, I)$, we have

$$C(\sigma \cdot [\widetilde{f}]) = D(\sigma) \cdot C([\widetilde{f}]).$$

Note that this is true for schemes and simplicial algebraic spaces (See the proof of [Il, III 2.2.4]). We prove a special case of this for Proposition 2.1.2 which is necessary for the proof of Theorem 2.1.1. A proof for general case is straightforward.

2.2 Proof of Proposition 2.1.2

The strategies of proofs of Theorem 2.1.1 and Proposition 2.1.2 are the same as those of [Ao] and [Ol2].

Step 1: Choose good presentations of algebraic stacks and make associated simplicial algebraic spaces.

Step 2: Compare deformations in the 2-category of algebraic stacks and those in the category of simplicial algebraic spaces.

Step 3: Compare the Ext groups.

Proof of Proposition 2.1.2. Let $P^0 : T^0 \to \mathscr{T}$ be a presentation with T^0 affine. By [Ol2, 1.4], the obstruction for existence of a deformation of T^0 to $\widetilde{\mathscr{T}}$ is in $\mathrm{Ext}^2(L_{T^0/\mathscr{T}}, {P^0}^* I)$ and the set of isomorphism classes of such deformations is a torsor under $\mathrm{Ext}^1(L_{T^0/\mathscr{T}}, P^{0*}I)$. Both of these groups are zero because $T_0 \to \mathscr{T}$ is smooth and T^0 is affine. Therefore there exists a unique deformation $\widetilde{T}^0 \to \widetilde{\mathscr{T}}$.

Let $T^\bullet = \mathrm{cosq}_0(T^0 \to \mathscr{T})$ and $\widetilde{T}^\bullet = \mathrm{cosq}_0(\widetilde{T}^0 \to \widetilde{T})$. Consider the diagram obtained by base changes $T^\bullet \to \mathscr{T}$ and $\widetilde{T}^\bullet \to \widetilde{\mathscr{T}}$:

$$\begin{tikzcd}
X^\bullet \arrow[rr, "i^\bullet"] \arrow[dr, "f^\bullet"] \arrow[ddr, "x^\bullet"'] & & \widetilde{X}^\bullet \arrow[dr, dotted, "\widetilde{f}^\bullet"] \arrow[ddr, "\widetilde{x}^\bullet"'] & \\
& Y^\bullet \arrow[rr, "j^\bullet"] \arrow[d, "y^\bullet"] & & \widetilde{Y}^\bullet \arrow[d, "\widetilde{y}^\bullet"] \\
& T^\bullet \arrow[rr, "k^\bullet"] & & \widetilde{T}^\bullet.
\end{tikzcd}$$

Then by construction $\widetilde{X^\bullet} \cong \mathrm{cosq}_0(\widetilde{X^0} \to \widetilde{X})$ and $\widetilde{Y^\bullet} \cong \mathrm{cosq}_0(\widetilde{Y^0} \to \widetilde{Y})$. Therefore $\widetilde{f^\bullet} : \widetilde{X^\bullet} \to \widetilde{Y^\bullet}$ descends to a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Thus we can define a map $A' : \mathrm{Defm}_{T^\bullet}(f^\bullet) \to \mathrm{Defm}_{\mathscr{T}}(f)$.

The map A' is bijective: the inverse is obtained by the base change.

Let $I^\bullet = \ker(O_{\widetilde{X}^\bullet} \to O_{X^\bullet})$. By the construction of cotangent complex [LM, 17.5], the homomorphisms

$$P_X^{\bullet\,*} : \mathrm{Ext}^i(Lf^* L_{X/\mathscr{T}}, I) \to \mathrm{Ext}^i(f^{\bullet *} L_{X^\bullet/T^\bullet}, I^\bullet)$$

are isomorphisms for all i.

By [Il, III 2.2.4], the obstruction for the existence of deformation of $f^\bullet$ is in $\mathrm{Ext}^1(f^{\bullet *} L_{X^\bullet/T^\bullet}, I^\bullet)$ and the set $\mathrm{Defm}(f^\bullet)$ is a torsor under $\mathrm{Ext}^0(f^{\bullet *} L_{X^\bullet/T^\bullet}, I^\bullet)$. These proves the proposition. $\square$

Next we prove that the action of Ext groups are functorial on $\mathscr{T}$.

Let $f : X \to Y$ be a morphism over $\mathscr{T}$ as in Proposition 2.1.2 and $\mathscr{T} \to U$ a morphism to a scheme. Here we consider a deformation diagram:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"] \arrow[ddr, "x"'] \arrow[dddr, bend right] & & \widetilde{X} \arrow[dr, dotted, "\widetilde{f}"] \arrow[ddr, "\widetilde{x}"'] \arrow[dddr, bend right] & \\
& Y \arrow[rr] \arrow[d, "y"] & & \widetilde{Y} \arrow[d, "\widetilde{y}"] \\
& \mathscr{T} \arrow[rr] \arrow[d] & & \widetilde{\mathscr{T}} \arrow[d] \\
& U \arrow[rr] & & \widetilde{U}
\end{tikzcd}$$

Proposition 2.2.1. *The natural map*

$$C : \mathrm{Defm}_{\mathscr{T}}(f) \to \mathrm{Defm}_U(f)$$

is compatible with the homomorphism of groups

$$D : \mathrm{Ext}^0(Lf^*L_{Y/\mathscr{T}}, I) \to \mathrm{Ext}^0(f^*L_{Y/U}, I).$$

Proof. Let $T^0 \to \mathscr{T}$ be a presentation and $T^\bullet = \mathrm{cosq}_0(T^0 \to \mathscr{T})$. Consider the diagram obtained by base change:

$$\begin{tikzcd}
& X^\bullet \arrow[dl, "P_X^\bullet"'] \arrow[drrr, "f^\bullet"] \arrow[ddrr] & & \\
X \arrow[drr, "f"] \arrow[ddrr] & & & Y^\bullet \arrow[dl, "P_Y^\bullet"] \arrow[d] \\
& & Y \arrow[d] & T^\bullet \arrow[dl] \\
& & \mathscr{T} \arrow[d] & \\
& & U. &
\end{tikzcd}$$

The map C factors as

$$\begin{aligned}
&\mathrm{Defm}_{\mathscr{T}}(f) \xrightarrow{C_1} \mathrm{Defm}_{T^\bullet}(f^\bullet) \xrightarrow{C_2} \mathrm{Defm}_U(f^\bullet) \\
&\xrightarrow{C_3} \mathrm{Defm}_U(P_Y^\bullet \circ f^\bullet) = \mathrm{Defm}_U(f \circ P_X^\bullet) \xrightarrow{C_4} \mathrm{Defm}_U(f)
\end{aligned}$$

and D factors as

$$\begin{aligned}
&\mathrm{Ext}^0(Lf^*L_{Y/\mathscr{T}}, I) \xrightarrow{D_1} \mathrm{Ext}^0(f^{\bullet *}L_{Y^\bullet/T^\bullet}, I^\bullet) \xrightarrow{D_2} \mathrm{Ext}^0(f^{\bullet *}L_{Y^\bullet/U}, I^\bullet) \\
&\xrightarrow{D_3} \mathrm{Ext}^0((P_Y^\bullet \circ f^\bullet)^* L_{Y/U}, I^\bullet) = \mathrm{Ext}^0((f \circ P_X^\bullet)^* L_{Y/U}, I^\bullet) \\
&\xrightarrow{D_4} \mathrm{Ext}^0(f^* L_{Y/U}, I).
\end{aligned}$$

The compatibility of isomorphisms C_1 and D_1 is obvious by the definition of the action of $\mathrm{Ext}^0(Lf^*L_{Y/\mathscr{T}}, I)$ in the proof of Proposition 2.1.2. That of C_2 and D_2 follows from the case of simplicial algebraic spaces. For C_3 and D_3, it follows from the definition of the morphism $P_Y^{\bullet *} L_{Y/U} \to L_{Y^\bullet/U}$ [Il, II 1.2.7]. For C_4 and D_4, it is trivial. □

2.3 Proof of Theorem 2.1.1: Step 1

Let $P_Y : Y^0 \to \mathscr{Y}$ be a presentation of $\mathscr{Y}$, $\mathscr{X}' = \mathscr{X} \times_{\mathscr{Y}} Y^0$ and $X^0 \to \mathscr{X}'$ a presentation of $\mathscr{X}'$. Then the composition $P_X : X^0 \to \mathscr{X}' \to \mathscr{X}$ is a presentation of $\mathscr{X}$. We may assume X^0 and Y^0 are affine. Since $X^0 \to \mathscr{X}$ and $Y^0 \to \mathscr{Y}$ are smooth, we have the unique deformations $\widetilde{X^0} \to \widetilde{\mathscr{X}}$ and $\widetilde{Y^0} \to \widetilde{\mathscr{Y}}$.

Let $X^\bullet = \mathrm{cosq}_0(X^0 \to \mathscr{X})$ etc. We obtain the following diagram:

$$\begin{tikzcd}
X^\bullet \arrow[rr] \arrow[dr, "f^\bullet"] \arrow[dd, "P_X^\bullet"'] & & \widetilde{X}^\bullet \arrow[dr, dotted, "\widetilde{f}^\bullet"] \arrow[dd, "\widetilde{P_X^\bullet}"'] & \\
& Y^\bullet \arrow[rr] \arrow[dd, "P_Y^\bullet"] & & \widetilde{Y}^\bullet \arrow[dd, "\widetilde{P_Y^\bullet}"] \\
\mathscr{X} \arrow[rr] \arrow[dr, "f"] \arrow[ddr, "x"'] & & \widetilde{\mathscr{X}} \arrow[dr, dotted, "\widetilde{f}"] \arrow[ddr, "\widetilde{x}"'] & \\
& \mathscr{Y} \arrow[rr] \arrow[d, "y"] & & \widetilde{\mathscr{Y}} \arrow[d, "\widetilde{y}"] \\
& T \arrow[rr] & & \widetilde{T}.
\end{tikzcd}$$

Let $I^\bullet = \ker(O_{\widetilde{X}^\bullet} \to O_{X^\bullet}) \cong P_X^{\bullet\,*} I$.

2.4 Proof of Theorem 2.1.1: Step 2

The map

$$A : \mathrm{Defm}_T(f^\bullet) \to \overline{\mathrm{Defm}_T(f)}$$

is defined by sending $\widetilde{f}^\bullet : \widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ to the morphism of associated stacks $\widetilde{f} : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{Y}}$.

Proposition 2.4.1. *The map A is surjective.*

Proof. Fix $[\widetilde{f}] \in \overline{\mathrm{Defm}_T(f)}$. First we claim that $[\widetilde{f}]$ is in the image of A if $\mathrm{Defm}_{\mathscr{Y}}(f^0)$ is not empty. To see this, let $(\widetilde{f^0}, \gamma) \in \mathrm{Defm}_{\mathscr{Y}}(f^0)$. We define $\widetilde{f}^\bullet = \mathrm{cosq}_0(\widetilde{f}^0, \gamma) : \widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ as follows. Since $\widetilde{X}^\bullet$ and $\widetilde{\mathscr{Y}}^\bullet$ are the images of cosq, by the similar discussion as in [Ao, 3.1.3], to give $\widetilde{f^\bullet}$ it suffices to give $\widetilde{f}^1 : \widetilde{X^1} \to \widetilde{Y^1}$. This is equivalent to give a triple $(\widetilde{f}^0 \circ p_1, \widetilde{f}^0 \circ p_2, \epsilon)$, where

$$\epsilon : P_Y \circ \widetilde{f}^0 \circ p_1 \Rightarrow P_Y \circ \widetilde{f}^0 \circ p_2$$

is a 2-morphism. Now we put $\epsilon = p_2^* \gamma \circ p_1^* \gamma^{-1}$. Then $A(\widetilde{f}^\bullet) = [\widetilde{f}]$.

By Proposition 2.1.2 the obstruction for the existence of $(\widetilde{f^0}, \gamma)$ is in $\mathrm{Ext}^1(Lf^{0*} L_{Y^0/\mathscr{Y}}, I^0)$. This group is zero because X^0 is affine and $L_{Y^0/\mathscr{Y}}$ is quasi-isomorphic to a locally free sheaf $\Omega_{Y^0/\mathscr{Y}}$. ☐

Corollary 2.4.2. *The obstruction for existence of deformation of f is in* $\mathrm{Ext}^1(f^{\bullet *} L_{Y^\bullet/T}, I^\bullet)$.

For each $[\widetilde{f}] \in \overline{\mathrm{Defm}_T(f)}$, let C be the composition of maps

$$\mathrm{Defm}_{\mathscr{Y}}(f^0) \overset{\mathrm{cosq}_0}{\xrightarrow{\sim}} \mathrm{Defm}_{\mathscr{Y}}(f^\bullet) \overset{\text{“forget”}}{\longrightarrow} \mathrm{Defm}_T(f^\bullet).$$

By Proposition 2.2.1, this is compatible with the group homomorphism

$$D : \mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0) \to \mathrm{Ext}^0(f^{\bullet *}L_{Y_\bullet/T}, I^\bullet).$$

Proposition 2.4.3. *The set* $\overline{\mathrm{Defm}_T(f)}$ *is the set of* $\mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0)$*-orbits in* $\mathrm{Defm}_T(f^\bullet)$ *by the action induced by* D*.*

Proof. Suppose that $\widetilde{f^\bullet}, \widetilde{g^\bullet} \in \mathrm{Defm}_T(f^\bullet)$ satisfy $A(\widetilde{f^\bullet}) = A(\widetilde{g^\bullet}) = [\widetilde{f}]$. Then there exists $(\widetilde{f^0}, \gamma), (\widetilde{g^0}, \delta) \in \mathrm{Defm}_{\mathscr{Y}}(f^0)$ such that $C(\widetilde{f^0}, \gamma) = \widetilde{f^\bullet}$ and $C(\widetilde{g^0}, \delta) = \widetilde{g^\bullet}$. Since $\mathrm{Defm}_{\mathscr{Y}}(f^0)$ is a $\mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0)$-torsor, there exists $\sigma \in \mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0)$ such that $\sigma \cdot (\widetilde{f^0}, \gamma) = (\widetilde{g^0}, \delta)$. Hence $D(\sigma) \cdot \widetilde{f^\bullet} = \widetilde{g^\bullet}$.

Conversely, suppose that $\widetilde{f^\bullet}, \widetilde{g^\bullet} \in \mathrm{Defm}_T(f^\bullet)$ satisfy $D(\sigma) \cdot \widetilde{f^\bullet} = \widetilde{g^\bullet}$ for some $\sigma \in \mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0)$. Let $[\widetilde{f}] = A(\widetilde{f^\bullet})$ and choose $(\widetilde{f^0}, \gamma) \in \mathrm{Defm}_{\mathscr{Y}}(f^0)$ such that $C(\widetilde{f^0}, \gamma) = \widetilde{f^\bullet}$. Then $C(\sigma \cdot (\widetilde{f^0}, \gamma)) = D(\sigma) \cdot \widetilde{f^\bullet} = \widetilde{g^\bullet}$. Therefore $A(\widetilde{g^\bullet}) = [\widetilde{f}]$. □

Proposition 2.4.4. *Fix an object* $\widetilde{f}$ *of* $\mathrm{Defm}_T(f)$*. Then* $\mathrm{Aut}(\widetilde{f})$*, the group of automorphisms of deformations, is isomorphic to* $\ker(D)$*.*

Proof. Fix $\widetilde{f^\bullet} \in \mathrm{Defm}_T(f^\bullet)$ such that $A(\widetilde{f^\bullet}) = [\widetilde{f}]$ and $(\widetilde{f}^0, \gamma) \in C^{-1}(\widetilde{f^\bullet})$.

First we identify $\mathrm{Aut}(\widetilde{f})$ with a subset of $\mathrm{Defm}_{\mathscr{Y}}(f^0)$ and construct set-theoretical bijection from $\mathrm{Aut}(\widetilde{f})$ to $C^{-1}(\widetilde{f^\bullet})$. Let $\alpha \in \mathrm{Aut}(\widetilde{f})$ and let β be the composition of 2-morphisms

$$\widetilde{P_Y} \circ \widetilde{f^0} \overset{\gamma^{-1}}{\Longrightarrow} \widetilde{f} \circ \widetilde{P_X} \overset{\widetilde{P_X}^*\alpha}{\Longrightarrow} \widetilde{f} \circ \widetilde{P_X} \overset{\gamma^{-1}}{\Longrightarrow} \widetilde{P_Y} \circ \widetilde{f^0}.$$

Then the triple $(\widetilde{f}^0, \widetilde{f}^0, \beta)$ defines a morphism

$$d_\alpha : \widetilde{X}^0 \to \widetilde{Y^0} \times_{\widetilde{\mathscr{Y}}} \widetilde{Y^0} = \widetilde{Y}^1.$$

This is an element of $\mathrm{Defm}_{Y^0}(\Delta \circ f^0)$. Here Y^1 is a scheme over Y^0 by $p_1 : Y^1 \to Y^0$.

$$\begin{tikzcd}
X^0 \arrow[rr] \arrow[dr, "\Delta\circ f^0"] \arrow[ddr, "f^0"'] & & \widetilde{X^0} \arrow[dr, "d_\alpha"] \arrow[ddr, "\widetilde{f^0}"'] & \\
& Y^1 \arrow[rr] \arrow[d, "p_1"'] & & \widetilde{Y^1} \arrow[d, "p_1"] \\
& Y^0 \arrow[rr] \arrow[u, bend right, "\Delta"'] & & \widetilde{Y^0}
\end{tikzcd}$$

The map

$$\begin{array}{rcl} p_1^* : \mathrm{Defm}_{\mathscr{Y}}(f^0) = \mathrm{Defm}_{\mathscr{Y}}(p_1 \circ \Delta \circ f^0) & \to & \mathrm{Defm}_{Y^0}(\Delta \circ f^0) \\ (\widetilde{f^0}', \gamma') & \mapsto & (\widetilde{f}^0, \widetilde{f^0}', {\gamma'}^{-1} \circ \gamma) \end{array}$$

is a bijection and compatible with isomorphism

$$\begin{aligned} p_1^* : \mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0) &= \mathrm{Ext}^0(L(p_1 \circ \Delta \circ f^0)^* L_{Y^0/\mathscr{Y}}, I^0) \\ &\xrightarrow{\sim} \mathrm{Ext}^0(L(\Delta \circ f^0)^* L_{Y^1/Y^0}, I^0) \end{aligned}$$

induced by $p_1^* L_{Y^0/\mathscr{Y}} \cong L_{Y^1/Y^0}$.

Now $(\widetilde{f^0}', \sigma')$ is in $C^{-1}(\widetilde{f}^\bullet)$ if and only if $\widetilde{f^0}' = \widetilde{f}^0$ and $p_2^*\gamma' \circ p_1^*{\gamma'}^{-1} = p_2^*\gamma \circ p_1^*\gamma^{-1}$. The latter is equivalent to

$$p_1^*({\gamma'}^{-1}\gamma) = p_2^*({\gamma'}^{-1}\gamma),$$

which implies the existence of $\alpha \in \mathrm{Aut}(\widetilde{f})$ such that $\gamma' \circ \gamma^{-1} = \gamma \circ P_X^* \alpha \circ \gamma^{-1}$.

Thus we can identify $\mathrm{Aut}(\widetilde{f})$ with $C^{-1}(\widetilde{f}^\bullet)$ as subsets of $\mathrm{Defm}_{\mathscr{Y}}(f^0)$.

Next we see that the group structure of $\mathrm{Aut}(f)$ is compatible with that of $\ker(D)$ acting on $C^{-1}(\widetilde{f}^\bullet)$. The composition $\alpha \circ \alpha'$ corresponds to the morphism

$$d_{\alpha \circ \alpha'} = (\widetilde{f^0}, \widetilde{f^0}, \gamma \circ \widetilde{P_X}^* \alpha \circ \widetilde{P_X}^* \alpha' \circ \gamma^{-1}) : \widetilde{X^0} \to \widetilde{Y^1}.$$

This is equal to the composition

$$\widetilde{X}^0 \xrightarrow{(d_{\alpha'}, d_\alpha)} \widetilde{Y}^1 \times_{p_1 \widetilde{Y}^0 p_2} \widetilde{Y}^1 = \widetilde{Y}^2 \xrightarrow{p_{13}} \widetilde{Y}^1.$$

$$\begin{tikzcd}
 & \widetilde{Y^2} = \widetilde{Y^0} \times_{\widetilde{\mathscr{Y}}} \widetilde{Y^0} \times_{\widetilde{\mathscr{Y}}} \widetilde{Y^0} \arrow[d, shift right=6, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[d, shift left=6, "p_{23}"] \\
\widetilde{X^0} \arrow[ur, "{(d_{\alpha'}, d_\alpha)}"] \arrow[r, "{d_\alpha, d_{\alpha'}}", "d_{\alpha \circ \alpha'}"'] \arrow[dr, "\widetilde{f^0}"'] & \widetilde{Y^1} = \widetilde{Y^0} \times_{\widetilde{\mathscr{Y}}} \widetilde{Y^0} \arrow[u, shift right=3] \arrow[u, shift left=3] \arrow[d, shift right=3, "p_1"'] \arrow[d, shift left=3, "p_2"] \\
 & \widetilde{Y^0} \arrow[u, "\Delta"]
\end{tikzcd}$$

On the other hand, the group structure of

$$\mathrm{Ext}^0((\Delta \circ f^0)^* L_{Y_1/Y_0}, I^0) \cong \mathrm{Der}_{B^0}(B^1, I^0)$$

is given by taking sum of derivations $D_\alpha, D_{\alpha'} : B^1 \to I^0$. Here B^i denotes the coordinate ring of Y^i. Pulling back by $p_{12} : \widetilde{Y^2} \to \widetilde{Y^1}$, we identify D_α with a derivation

$$\begin{array}{rcl} B^2 = B_1 \otimes_{p_1^* B^0 p_2^*} B^1 & \xrightarrow{D_\alpha} & I^0 \\ x \otimes y & \mapsto & D_\alpha(x \otimes y) = x D_\alpha(1 \otimes y). \end{array}$$

Pulling back by $p_{23}:\widetilde{Y^2}\to\widetilde{Y^1}$, $D_{\alpha'}$ is identified with

$$\begin{array}{ccc} B^2 = B_1\otimes_{p_1^*B^0p_2^*}B^1 & \xrightarrow{D_{\alpha'}} & I^0 \\ x\otimes y & \mapsto & D_{\alpha'}(x\otimes y) = yD_{\alpha'}(x\otimes 1). \end{array}$$

The morphism $(d_{\alpha'}, d_\alpha)$ as above corresponds to a derivation

$$\begin{array}{ccc} B^3 = B_1\otimes_{p_1^*B^0p_2^*}B^1\otimes_{p_1^*B^0p_2^*}B^1 & \xrightarrow{D} & I^0 \\ x\otimes y\otimes 1 & \mapsto & yD_{\alpha'}(x\otimes 1) \\ 1\otimes y\otimes z & \mapsto & yD_\alpha(1\otimes z). \end{array}$$

Then the morphism $d_{\alpha\circ\alpha'}$ corresponds to the composition:

$$\begin{array}{ccccc} B^2 = B_1\otimes_{p_1^*B^0p_2^*}B^1 & \to & B^3 & \xrightarrow{D} & I^0 \\ x\otimes y & \mapsto & x\otimes 1\otimes y & \mapsto & D((x\otimes 1\otimes 1)(1\otimes 1\otimes y)) \\ & & & & = yD(x\otimes 1\otimes 1) + xD(1\otimes 1\otimes y) \\ & & & & = D_\alpha(x\otimes y) + D_{\alpha'}(x\otimes y) \end{array}$$

Thus group structures of $\mathrm{Aut}(\widetilde{f})$ and $\mathrm{Der}_{B^0}(B^1, I^0)$ are compatible. □

2.5 Proof of Theorem 2.1.1: Step 3

The following lemma completes the proof of Theorem 2.1.1.

Lemma 2.5.1.

(1) *There is an isomorphism*

$$\mathrm{Ext}^1(f^{\bullet *}L_{Y^\bullet/T}, I^\bullet) \xrightarrow{\sim} \mathrm{Ext}^1(Lf^*L_{\mathscr{Y}/T}, I).$$

(2) *The cokernel of* $D:\mathrm{Ext}^0(Lf^{0*}L_{Y^0/\mathscr{Y}}, I^0)\to \mathrm{Ext}^0(f^{\bullet *}L_{Y^\bullet/T}, I^\bullet)$ *is isomorphic to* $\mathrm{Ext}^0(Lf^*L_{\mathscr{Y}/T}, I)$.

(3) *The kernel of* D *is isomorphic to* $\mathrm{Ext}^{-1}(Lf^*L_{\mathscr{Y}/T}, I)$.

Proof. The morphisms

$$Y^\bullet \to \mathscr{Y}\to T$$

induces a triangle in $D(O_{Y^\bullet})$

$$LP_Y^{\bullet *}L_{\mathscr{Y}/T}\to L_{Y^\bullet/T}\to L_{Y^\bullet/\mathscr{Y}}\to LP_Y^{\bullet *}L_{\mathscr{Y}/T}[1],$$

and this in turn induces a long exact sequence

$$\begin{array}{ccccccc} & & & & 0 & \to & \mathrm{Ext}^{-1}(Lf^{\bullet *}LP_Y^{\bullet *}L_{\mathscr{Y}/T}, I^\bullet) \\ \to & \mathrm{Ext}^0(Lf^{\bullet *}L_{Y^\bullet/\mathscr{Y}}, I^\bullet) & \to & \mathrm{Ext}^0(f^{\bullet *}L_{Y^\bullet/T}, I^\bullet) & \to & \mathrm{Ext}^0(Lf^{\bullet *}LP_Y^{\bullet *}L_{\mathscr{Y}/T}, I^\bullet) \\ \to & \mathrm{Ext}^1(Lf^{\bullet *}L_{Y^\bullet/\mathscr{Y}}, I^\bullet) & \to & \mathrm{Ext}^1(f^{\bullet *}L_{Y^\bullet/T}, I^\bullet) & \to & \mathrm{Ext}^1(Lf^{\bullet *}LP_Y^{\bullet *}L_{\mathscr{Y}/T}, I^\bullet) \\ \to & \mathrm{Ext}^2(Lf^{\bullet *}L_{Y^\bullet/\mathscr{Y}}, I^\bullet) & \to & \cdots & & \end{array}$$

By the similar discussion as in [Ol2, 4.7],

$$\mathrm{Ext}^i(Lf^{\bullet *}L_{Y^\bullet/\mathscr{Y}}, I^\bullet) \cong \mathrm{Ext}^i(Lf^{0^*}L_{Y^0/\mathscr{Y}}, I^0)$$

and the right hand side is zero for $i > 0$. The isomorphism $P_X^{\bullet *} : D^+(O_{\mathscr{X}}) \to D^+(O_{X^\bullet})$ induces isomorphisms

$$\mathrm{Ext}^i(Lf^{\bullet *}LP_Y^{\bullet *}L_{\mathscr{Y}/T}, I^\bullet) \cong \mathrm{Ext}^i(LP_X^{\bullet *}Lf^*L_{\mathscr{Y}/T}, I^\bullet) \cong \mathrm{Ext}^i(Lf^*L_{\mathscr{Y}/T}, I).$$

□

3 Artin's criterion

In this section we prove Theorem 1.1 by verifying the following Artin's criterion [Ar2, 5.3].

(1) $\mathscr{H}$ is a limit-preserving stack.

(2) $\mathscr{H}$ satisfies Schlessinger's conditions.

(S1) If $A' \to A$ and $B \to A$ are homomorphisms of noetherian rings over S and $A' \to A$ is a small extension, then for any $f \in \mathscr{H}(A)$ the natural functor

$$\mathscr{H}_f(A' \times_A B) \to \mathscr{H}_f(A') \times \mathscr{H}_f(B)$$

is an equivalence of categories. Here $\mathscr{H}_f(R)$ denotes the subcategory of $\mathscr{H}(R)$ consisting of objects g such that $g|_A \simeq f$ and morphisms α such that $\alpha|_A = \mathrm{id}_f$.

(S2) If M is a finite A-module and $f \in \mathscr{H}(A)$, then

$$D_f(M) = \mathrm{Ob}\,\mathscr{H}_f(A+M)/\sim$$

is a finite A-module.

(3) Compatibility with completion.
If A is a complete local noetherian ring with maximal ideal m, the functor

$$\mathscr{H}(A) \to \varprojlim_n \mathscr{H}(A/m^{n+1})$$

is an equivalence.

(4) Conditions on modules of obstruction, deformations and infinitesimal automorphisms.
For any $f \in \mathscr{H}(A)$ and a finite A-module M, there exists a module of obstructions $O_f(M)$, a modules of deformations $D_f(M)$ and a modules of infinitesimal automorphisms $\mathrm{Aut}_f(M)$ which satisfy the following conditions:

(a) compatibility with étale localization:
If $A \to B$ is étale and g is a image of f in $\mathscr{H}(B)$,

$$D_g(M \otimes B) \cong D_f(M) \otimes_A B$$

etc.

(b) compatibility with completion:
If m is a maximal ideal of A and $\hat{A}$ is a completion with respect to m,

$$D_f(M) \otimes \hat{A} \cong \varprojlim D_f(M/m^n M)$$

etc.

(c) constructibility:
There is a open dense set of points of finite type $A \to k(p)$ such that

$$D_f(M) \otimes k(p) \cong D_f(M \otimes k(p)).$$

etc.

(5) For any $f \in \mathscr{H}(A)$ and $\alpha \in \mathrm{Aut}(f)$, if $\alpha|_k = \mathrm{id}$ for dense set of points of finite type $A \to k$, then $\alpha = \mathrm{id}$.

3.1 Preliminaries

We can reduce many properties of $\mathscr{H}$ to that of $\mathscr{Y}$ by the following observations.

Lemma 3.1.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks over S and $X \to \mathscr{X}$ a presentation of $\mathscr{X}$. Let $X^1 = X^0 \times_{\mathscr{X}} X^0$. Then the category $\mathrm{HOM}_S(\mathscr{X}, \mathscr{Y})$ is equivalent to the following category:*

- *An object is a pair (f^0, α) where f^0 is an object of $\mathscr{Y}(X^0)$ and $\alpha : p_1^* f^0 \Rightarrow p_2^* f^0$ is a morphism in $\mathscr{Y}(X^1)$.*
- *A morphism from (f^0, α) to (g^0, β) is a morphism $\gamma : f^0 \Rightarrow g^0$ in $\mathscr{Y}(X^0)$ such that $p_2^* \gamma \circ \alpha = \beta \circ p_1^* \gamma$ in $\mathscr{Y}(X^1)$.*

Proof. This follows immediately from the fact that $\mathscr{X}$ is a stack associated to the groupoid $X^1 \rightrightarrows X^0$. □

Lemma 3.1.2. *Let $y : \mathscr{Y} \to S$ be an algebraic stack over a scheme S, $\varphi : T \to S$ a morphism of schemes and $x : \mathscr{X}_T \to T$ an algebraic stack over T. Then the natural functor*

$$\mathrm{HOM}_T(\mathscr{X}_T, \mathscr{Y}_T) \to \mathrm{HOM}_S(\mathscr{X}_T, \mathscr{Y})$$

is an equivalence of categories.

Proof. If $\mathscr{X}_T$ is a scheme, this is clear by the construction of fiber products [LM, 2.2.2]. In the general case, let $X^0 \to \mathscr{X}_T$ be a presentation and $X^1 = X^0 \times_{\mathscr{X}} X^0$. Then by the case of schemes we have

$$\mathscr{Y}_T(X^0) \simeq \mathscr{Y}(X^0)$$
$$\mathscr{Y}_T(X^1) \simeq \mathscr{Y}(X^1).$$

The result follows from Lemma 3.1.1. □

3.2 Limit preserving stack

Fix a presentation $X^0 \to \mathscr{X}$ and let $X^1 = X^0 \times_{\mathscr{X}} X^0$. Then if $\{U_i \to U\}$ is an étale covering, so is $\{X^k_{U_i} \to X^k_U\}$ for $k = 0, 1$. The conditions of stacks for $\mathscr{H}$ follows from those of $\mathscr{Y}$:

(1) Let f and g be objects of $\mathscr{H}(U)$ and $\varphi, \psi : f \Rightarrow g$ be morphisms in $\mathscr{H}(U)$. Suppose that $\varphi|_i = \psi|_i$ in $\mathscr{H}(U_i)$ for all i. By Lemma 3.1.2, φ and ψ are identified with morphisms in $\mathrm{HOM}(\mathscr{X}_U, \mathscr{Y})$. Let φ' and ψ', morphisms in $\mathscr{Y}(X^0_U)$ corresponding to φ and ψ by Lemma 3.1.1. Then $\varphi'|_{X^0_{U_i}} = \psi'|_{X^0_{U_i}}$ for all i imply $\varphi' = \psi'$. Hence $\varphi = \psi$.

(2) Let f and g be objects of $\mathscr{H}(U)$ and $\varphi_i : f|_i \Rightarrow g|_i$ morphisms in $\mathscr{H}(U_i)$. Suppose that $\varphi_i|_{ij} = \varphi_j|_{ij}$ for all i and j. Let (f^0, α) and (g^0, β) be pairs corresponding to f and g, and φ'_i morphisms in $\mathscr{Y}(X^0_{U_i})$ corresponding to φ_i. Then $\varphi'_i|_{X^0_{U_{ij}}} = \varphi'_j|_{X^0_{U_{ij}}}$ imply existence of $\psi' : f^0 \Rightarrow g^0$ in $\mathscr{Y}(X^0_U)$ such that $\psi'|_{X_{U_i}} = \varphi'_i$. Since

$$p_2^*\psi'|_{X_{U_i}} \circ \alpha|_{X_{U_i}} = \beta|_{X_{U_i}} \circ p_1^*\psi'|_{X_{U_i}}$$

hold for all i,

$$p_2^*\psi' \circ \alpha = \beta \circ p_1^*\psi'$$

and ψ' corresponds to a morphism $\psi : f \Rightarrow g$ in $\mathscr{H}(U)$ such that $\psi|_i = \varphi_i$.

(3) Let f_i be objects of $\mathscr{H}(U_i)$ and $\varphi_{ij} : f_i|_{ij} \Rightarrow f_j|_{ij}$ morphisms in $\mathscr{H}(U_{ij})$ which satisfy cocycle conditions:

$$\varphi_{jk}|_{ijk} \circ \varphi_{ij}|_{ijk} = \varphi_{ik}|_{ijk}.$$

Let (f^0_i, α_i) be pairs corresponding to f_i and φ'_{ij} morphisms in $\mathscr{Y}(X^0_{U_{ij}})$ corresponding to φ_{ij}. Then by the cocycle conditions

$$\varphi'_{jk}|_{X^0_{U_{ijk}}} \circ \varphi'_{ij}|_{X^0_{U_{ijk}}} = \varphi'_{ik}|_{X^0_{U_{ijk}}},$$

there exists an object f^0 of $\mathscr{Y}(X^0_U)$ and morphisms $\psi'_i : f^0|_{X^0_{U_i}} \Rightarrow f^0_i$ such that $\varphi'_{ij} \circ \psi'_i|_{X^0_{U_{ij}}} = \psi'_j|_{X^0_{U_{ij}}}$. Let

$$\beta_i = p_2^*{\psi'_i}^{-1} \circ \alpha_i \circ p_1^*\psi'_i : p_1^*f^0|_{X^1_{U_i}} \Rightarrow p_2^*f^0|_{X^1_{U_i}}.$$

Then

$$\begin{aligned}
\beta_i|_{X_{U_{ij}}} &= p_2^*{\psi'_i}^{-1}|_{X_{U_{ij}}} \circ \alpha_i|_{X_{U_{ij}}} \circ p_1^*\psi'_i|_{X_{U_{ij}}} \\
&= p_2^*{\psi'_i}^{-1}|_{X_{U_{ij}}} \circ p_2^*{\varphi'_{ij}}^{-1} \circ \alpha_j|_{X_{U_{ij}}} \circ p_1^*\varphi'_{ij} \circ p_1^*\psi'_i|_{X_{U_{ij}}} \\
&= p_2^*{\psi'_j}^{-1}|_{X_{U_{ij}}} \circ \alpha_j|_{X_{U_{ij}}} \circ p_1^*\psi'_j|_{X_{U_{ij}}} \\
&= \beta_j|_{X_{U_{ij}}}.
\end{aligned}$$

Therefore there exists $\beta : p_1^* f^0 \Rightarrow p_2^* f^0$ in $\mathscr{Y}(X_U^1)$ such that $\beta|_{X_{U_i}} = \beta_i$. The pair (f^0, β) defines an object f of $\mathscr{H}(U)$. The morphism ψ_i' satisfies

$$p_2^* \psi_i' \circ \beta|_{X_{U_i}} = \alpha_i \circ p_1^* \psi_i'.$$

Therefore ψ_i' corresponds to $\psi_i : f|_i \Rightarrow f_i$ such that $\varphi_{ij} \circ \psi_i|_{ij} = \psi_j|_{ij}$.

$\mathscr{H}$ is limit-preserving by [LM, 4.18].

3.3 Schlesinger's conditions

First, let $\varphi : A' \to A$ and $\psi : B \to A$ be homomorphisms of noetherian rings over S and suppose φ is a small extension. Let $f \in \mathscr{H}(A)$. By Lemma 3.1.2, the condition (S1') on $\mathscr{H}$ is equivalent to the equivalence

$$\mathrm{HOM}_f(\mathscr{X}_{A' \times_A B}, \mathscr{Y}) \xrightarrow{\sim} \mathrm{HOM}_f(\mathscr{X}_{A'}, \mathscr{Y}) \times \mathrm{HOM}_f(\mathscr{X}_B, \mathscr{Y}).$$

Let $X^0 \to \mathscr{X}$ be a presentation. Since $\mathscr{X}$ is of finite type over noetherian base, we may assume X^0 is a noetherian affine scheme $\operatorname{Spec} R$.

Lemma 3.3.1. *The homomorphism*

$$\begin{array}{rcl} \pi : R \otimes (A' \times_A B) & \to & (R \otimes A') \times_{R \otimes A} (R \otimes B) \\ r \otimes (a', b) & \mapsto & (r \otimes a', r \otimes b) \end{array}$$

is an isomorphism.

Proof. The kernel of the projection $A' \times_A B \to B$ is isomorphic to $\ker \varphi$ and the kernel of $(R \otimes A') \times_{R \otimes A} (R \otimes B) \to R \otimes B$ is isomorphic to $\ker(\mathrm{id}_R \otimes \varphi)$. Since R is flat, the horizontal sequences of the following diagram are exact:

$$\begin{tikzcd} 0 \arrow[r] & R \otimes \ker \varphi \arrow[r] \arrow[d, equal] & R \otimes (A' \times_A B) \arrow[r] \arrow[d, "\pi"'] & R \otimes B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R \otimes \ker \varphi \arrow[r] & (R \otimes A') \times_{R \otimes A} (R \otimes B) \arrow[r] & R \otimes B \arrow[r] & 0. \end{tikzcd}$$

It is easy to check that this diagram commutes. Therefore π is an isomorphism. $\square$

Let $X^1 = X^0 \times_{\mathscr{X}} X^0$ and (f^0, α) a pair correspond to $f : \mathscr{X} \to \mathscr{Y}$ as in Lemma 3.1.1. By the condition (S1') for $\mathscr{Y}$ and Lemma 3.3.1, we have an equivalence

$$\mathscr{Y}_{f^0}(X^0_{A' \times_A B}) \xrightarrow{\sim} \mathscr{Y}_{f^0}(X^0_{A'}) \times \mathscr{Y}_{f^0}(X^0_B)$$

Since the functor $\mathrm{Isom}(p_1^* f^0, p_2^* f^0)$ is represented by an algebraic space, we also have

$$\mathrm{Isom}_\alpha(p_1^* f^0_{X^0_{A' \times_A B}}, p_2^* f^0_{X^0_{A' \times_A B}}) \xrightarrow{\sim} \mathrm{Isom}_\alpha(p_1^* f^0_{X^0_{A'}}, p_2^* f^0_{X^0_{A'}}) \times \mathrm{Isom}_\alpha(p_1^* f^0_{X^0_B}, p_2^* f^0_{X^0_B})$$

These equivalences proves (S1').

By Theorem 1.1, we have

$$D_{f_{X_0}}(M) \cong \mathrm{Ext}^0(Lf^*_{A_0} L_{\mathscr{X}_{A_0}/A_0}, x^*_{A_0} M).$$

This is a finite A_0 module because $Lf^*_{A_0} L_{\mathscr{X}_{A_0}/A_0}$ is coherent and $\mathscr{X}_{A_0}$ is proper over A_0. This proves (S2).

3.4 Compatibility with completion

Let $A_n = A/m^{n+1}$. The functor

$$\mathscr{H}(A) \to \varprojlim \mathscr{H}(A_n).$$

is equal to the functor

$$\pi : \mathrm{HOM}_A(\mathscr{X}_A, \mathscr{Y}_A) \to \varprojlim \mathrm{HOM}_{A_n}(\mathscr{X}_{A_n}, \mathscr{Y}_{A_n}).$$

First note that π is a bijection if $\mathscr{X}$ and $\mathscr{Y}$ are schemes [EGA, I 10.6.1].

To see π is fully faithful, let f, g be objects of the left hand side. Fix a presentation $Y^0_A \to \mathscr{Y}_A$ and let $\mathscr{X}_f = \mathscr{X}_A \times_{f\mathscr{Y}_A} Y^0_A$, $\mathscr{X}_g = \mathscr{X}_A \times_{g\mathscr{Y}_A} Y^0_A$ and $\mathscr{X}' = \mathscr{X}_f \times_{\mathscr{X}_A} \mathscr{X}_g$. Fix a presentation $X^0_A \to \mathscr{X}'$ of $\mathscr{X}'$. Then the composition $X^0_A \to \mathscr{X}_A$ is a presentation of $\mathscr{X}_A$. Let $f^0, g^0 : X^0_A \to Y^0_A$ be morphisms induced by f and g.

$$\begin{tikzcd}
& X^0_A \arrow[d] \arrow[rrd, bend left, "f^0"] \arrow[rrd, bend right, "g^0"'] & & \\
& \mathscr{X}' \arrow[dl] \arrow[dr] & & Y^0_A \arrow[dd] \\
\mathscr{X}_f \arrow[rrru] \arrow[dr] & & \mathscr{X}_g \arrow[ru] \arrow[dl] & \\
& \mathscr{X}_A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & \mathscr{Y}_A
\end{tikzcd}$$

Let $X^1_A = X^0_A \times_{\mathscr{X}_A} X^0_A$ and $Y^1_A = Y^0_A \times_{\mathscr{Y}_A} Y^0_A$. Then the set of 2-morphisms $\mathrm{Hom}(f, g)$ is equal to the set of morphisms $\alpha : X^0_A \to Y^1_A$ such that $p_1 \circ \alpha =$

$f^0, p_2 \circ \alpha = g^0$ and $\alpha \circ p_1 = \alpha \circ p_2$.

$$\begin{tikzcd}
X_A^1 \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & Y_A^1 \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
X_A^0 \arrow[ur, dotted, "\alpha"] \arrow[r, bend left, "f^0"] \arrow[r, bend right, "g^0"'] \arrow[d] & Y_A^0 \arrow[d] \\
\mathscr{X}_A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathscr{Y}_A
\end{tikzcd}$$

The sets $\mathrm{Hom}(X_A^0, Y_A^1)$, $\mathrm{Hom}(X_A^0, Y_A^0)$ and $\mathrm{Hom}(X_A^1, Y_A^1)$ are equal to limits of its reductions, hence the set $\mathrm{Hom}(f, g)$ is also equal to the limit of its reductions.

To see π is essentially surjective, let $\{f_n\}$ be an object of the right hand side. Let $X_{A_0}^0 \to \mathscr{X}_{A_0}$ and $Y_{A_0}^0 \to \mathscr{Y}_{A_0}$ be presentations such that $X_{A_0}^0$ and $Y_{A_0}^0$ are affine and f_0 lifts to $f_0^0 : X_{A_0}^0 \to Y_{A_0}^0$. Let $X_{A_0}^1 = X_{A_0}^0 \times_{\mathscr{X}_{A_0}} X_{A_0}^0$, $Y_{A_0}^1 = Y_{A_0}^0 \times_{\mathscr{Y}_{A_0}} Y_{A_0}^0$ and $f_0^1 : X_{A_0}^1 \to Y_{A_0}^1$ a morphism induced by f_0 and f_0^0.

For each n, by [Ol2, 1.4], there exists a unique deformation X_n^0 (resp. Y_n^0) of $X_{A_0}^0$ (resp. $Y_{A_0}^0$) to $\mathscr{X}_{A_n}$ (resp. $\mathscr{Y}_{A_n}$). By Theorem 2.1.1 there exists a unique deformation $f_n^0 : X_{A_n}^0 \to Y_{A_n}^0$ of f_0^0. Let $X^0 = \varinjlim X_n^0$ and $Y^0 = \varinjlim Y_n^0$. These are schemes over A and $X^0 \otimes A_n \cong X_n^0$, $Y^0 \otimes A_n \cong Y_n^0$ for each n. Since the map

$$\mathrm{Hom}(X^0, Y^0) \to \varprojlim \mathrm{Hom}(X_n^0, Y_n^0)$$

is a bijection, we have a morphism $f^0 : X^0 \to Y^0$.

Let $X^1 = X^0 \times_{\mathscr{X}_A} X^0$ and $Y^1 = Y^0 \times_{\mathscr{Y}_A} Y^0$. By the similar discussion there is a morphism $f^1 : X^1 \to Y^1$. Now the pair (f^0, f^1) induces a morphism $f : \mathscr{X}_A \to \mathscr{Y}_A$ whose restriction is isomorphic to $\{f_n\}$.

Remark 3.4.1. This discussion will be clearer if we use the theory of "formal algebraic stacks" [Iw].

3.5 Conditions on modules

By Theorem 2.1.1, the modules $O_f(M)$, $D_f(M)$ and $\mathrm{Aut}_f(M)$ are represented as follows:

$$O_f(M) = \mathrm{Ext}^1(Lf^* L_{Y_A/A}, x_A^* M)$$
$$D_f(M) = \mathrm{Ext}^0(Lf^* L_{Y_A/A}, x_A^* M)$$
$$\mathrm{Aut}_f(M) = \mathrm{Ext}^{-1}(Lf^* L_{Y_A/A}, x_A^* M)$$

Here x_A denotes the structural morphism $\mathscr{X}_A \to \mathrm{Spec}\, A$.

The compatibility with étale localization is equivalent to that the maps

$$\mathrm{Ext}^i(Lf^* L_{\mathscr{Y}_B/B}, I \otimes B) \to \mathrm{Ext}^i(Lf^* L_{\mathscr{Y}_A/A}, I) \otimes B \quad (i = -1, 0, 1)$$

are isomorphisms for any étale localization $A \to B$. Since $L_{B/A} = 0$, we have $L_{\mathscr{Y}/B} \cong L_{\mathscr{Y}/A}$, which induces the desired isomorphisms.

The compatibility with completion follows from 3.4.

The constructibility of these modules follows from the semicontinuity theorem for proper algebraic stacks (Theorem A.1).

3.6 Quasi-separation of the diagonal

Let $f \in \mathscr{H}(A)$, $\alpha \in \mathrm{Aut}(f)$ and suppose that $\alpha|_k = \mathrm{id}$ for dense set of points $A \to k$. Fix a presentation $P : X^0 = \operatorname{Spec} R \to \mathscr{X}$. Then $P^*\alpha$ is an automorphism of ${P_A}^* f \in \mathscr{Y}(X_A^0)$. The set of points $R \otimes A \to k'$ which factors through $R \otimes k$ with $\alpha|_k = \mathrm{id}$ is dense in X_A^0, and $P^*\alpha|_{k'} = \mathrm{id}$ on such points. Hence $P^*\alpha = \mathrm{id}$ because $\mathscr{Y}$ is a quasi-separated stack. This implies $\alpha = \mathrm{id}$.

4 A remark on quasi-separation

It is hard to show that the stack $\mathscr{H}$ is quasi-separated, in other words, it is an algebraic stack in the sence of [LM, 4.1]. In the case of Deligne-Mumford stacks, Olsson [Ol1] needed some extra hyposeses on corse moduli spaces to prove this. In our case we have the following partial result.

Proposition 4.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ as in Theorem 1.1. Suppose that $\mathscr{X} = X$ is representable by an algebraic space and $\mathscr{Y}$ has a proper presentation $Y^0 \to \mathscr{Y}$. Then the stack $\mathscr{H} = \mathscr{HOM}(X, \mathscr{Y})$ is quasi-separated.*

Proof. What we have to show is that if f and g are objects of $\mathscr{HOM}(X, \mathscr{Y})(T)$, then the algebraic space $\mathrm{Isom}_T(f, g)$ is separated and quasicompact over T.

Let $X_f = X_T \times_{f\mathscr{Y}_T} Y_T^0$, $X_g = X_T \times_{g\mathscr{Y}_T} Y_T^0$, $X_T^0 = X_f \times_{X_T} X_g$ and $f^0, g^0 :$ $X_T^0 \to Y_T^0$ morphisms induced by f and g.

$$\begin{tikzcd}
 & X_T^0 \arrow[rr, bend left, "f^0"] \arrow[rr, bend right, "g^0"'] \arrow[dl] \arrow[dr] & & Y_T^0 \arrow[dd] \\
X_f \arrow[urrr] \arrow[dr] & & X_g \arrow[ur] \arrow[dl] & \\
 & X_T \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & \mathscr{Y}_T
\end{tikzcd}$$

Let $X_T^1 = X_T^0 \times_{\mathscr{X}_T} X_T^0$ and $Y_T^1 = Y_T^0 \times_{\mathscr{Y}_T} Y_T^0$. Then X_T^0 and X_T^1 are proper and flat algebraic spaces over T. Therefore the functors $\mathscr{HOM}(X_T^0, Y_T^1)$, $\mathscr{HOM}(X_T^0, X_T^1)$ and $\mathscr{HOM}(X_T^1, Y_T^1)$ are representable by separated algebraic spaces over T. The algebraic space $\mathrm{Isom}_T(f, g)$ can be identified with a closed subspace of $\mathscr{HOM}(X_T^0, Y_T^1)$ whose point α satisfies $p_1 \circ \alpha = f^0, p_2 \circ \alpha = g^0$ and $\alpha \circ p_1 = \alpha \circ p_2$. Hence $\mathrm{Isom}_T(f, g)$ is separated and quasicompact. $\square$

5 Application: the Picard stack

Let $\mathscr{X}$ be an algebraic stack over S. The Picard 2-functor $\mathscr{P}ic_{\mathscr{X}}$ from the category of affine noetherian schemes over S to the 2-category of groupoids is defined by

$$\mathscr{P}ic_{\mathscr{X}}(T) = \text{the category of line bundles on } \mathscr{X}_T.$$

as in [LM, 14.4.7]. Then we have

Theorem 5.1. *If $\mathscr{X}$ is proper and flat over S, then $\mathscr{P}ic_{\mathscr{X}}$ is an algebraic stack in Artin's sense.*

Proof. To give a line bundle on $\mathscr{X}$ is equivalent to give a morphism $\mathscr{X} \to B\mathbb{G}_m$. Here $B\mathbb{G}_m$ denotes the classifying stack of the multiplicative group $\mathbb{G}_m$. Therefore

$$\mathscr{P}ic_{\mathscr{X}} = \mathscr{HOM}(\mathscr{X}, B\mathbb{G}_m).$$

This is an algebraic stack in Artin's sense by Theorem 1.1. $\square$

A The semicontinuity theorem for proper algebraic stacks

Let $x : \mathscr{X} \to T$ be a proper algebraic stack over an affine scheme $T = \operatorname{Spec} A$ and $\mathscr{F}$ a coherent sheaf on $\mathscr{X}$. Suppose that T is reduced and $\mathscr{F}$ is flat over T. For each point t of T, let $\mathscr{X}_t$ be the fiber over t and $\mathscr{F}_t = \mathscr{F} \otimes_{O_T} k(t)$.

Theorem A.1.

(1) *The function on T defined by*

$$t \mapsto \dim_{k(t)} H^i(\mathscr{X}_t, \mathscr{F}_t)$$

is upper semi-continuous on Y.

(2) *There is an open subscheme $U \subset X$ in which*

$$R^i x_* \mathscr{F} \otimes_{O_T} k(t) \to H^i(\mathscr{X}_t, \mathscr{F}_t)$$

is an isomorphism.

The proof is almost the same as one in [Mu, 5]. The key is the following lemma:

Lemma A.2. *Let $\mathscr{X}$, T and $\mathscr{F}$ be as above. For each positive integer N, there is a complex*

$$K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^N \to 0$$

of finitely generated projective A-modules and isomorphisms

$$H^i(\mathscr{X} \times_T \operatorname{Spec} B, \mathscr{F} \times_A B) \xrightarrow{\sim} H^i(K^\bullet \otimes_A B) \quad (0 < i < N)$$

functorial on A-algebra B.

Remark A.3. This is a generalization of the second theorem in [Mu, 5]. The first theorem in [Mu, 5] which claims direct images of proper schemes are coherent also holds in the case of proper algebraic stacks [Fa, Theorem 1]. We have to limit $i < N$ because cohomological dimension of an algebraic stack may be infinite. Note that Lemma 1 and Lemma 2 in the proof of [Mu, 5] concern only modules on A, and the same discussion applies to our case.

Proof of Lemma A.2. Let $P^0 : X^0 \to \mathscr{X}$ be a presentation with X^0 affine and $X^\bullet = \operatorname{cosq}_0(X^0 \to \mathscr{X})$. Then by cohomological descent, we have an isomorphism

$$H^i(\mathscr{X}, \mathscr{F}) \simeq H^i(X^\bullet, P^{\bullet *}\mathscr{F}).$$

Since X^0 is affine and $\mathscr{X}$ is separated, X^n is affine for all n and $H^i(X^n, P^{n*}\mathscr{F}) = 0$ for $i > 0$. Let

$$C^n = H^0(X^n, P^{n*}\mathscr{F})$$

and $C^\bullet$ be the alternating cochain. Then we have

$$H^i(\mathscr{X}, \mathscr{F}) \simeq H^i(C^\bullet).$$

Note that $H^i(C^\bullet)$ is a finite A-module because $\mathscr{F}$ is coherent. Moreover, for any A-algebra B,

$$P^0_B : X^0_B := X^0 \times_T \operatorname{Spec} B \to \mathscr{X} \times_T \operatorname{Spec} B =: \mathscr{X}_B$$

is a presentation from affine scheme and

$$H^0(X^n_B, P^{n*}_B \mathscr{F} \otimes_A B) \simeq H^0(X^n, P^{n*}\mathscr{F}) \otimes_A B$$

because $\mathscr{F}$ is flat. Therefore we have functorial isomorphisms

$$H^i(\mathscr{X}_B, \mathscr{F} \otimes_A B) \simeq H^i(C^\bullet \otimes_A B) \quad (i > 0).$$

Now replace $C^\bullet$ by its truncation $\tau_{\leq N} C^\bullet$ and construct $K^\bullet$ by descending induction as in [Mu, 5 Lemma 1]. This is the desired complex. □

Fix N sufficiently large. Then by Lemma A.2, We can reduce Theorem A.1 to statements in homological algebra as in corollaries of [Mu, 5]. Proofs of these corollaries also works for our case.

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