Regular projectively Anosov flows on three manifolds

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March 8, 2005

Abstract: We give the complete classification of C^2-regular projectively Anosov flows on closed three dimensional manifolds. More precisely, we show that if the manifold is connected then such a flow must be either an Anosov flow or represented as a finite union of $T^2 \times I$-models.
Keywords: projectively Anosov system, conformally Anosov systems, bi-contact structures
MSC2000: Primary 37D30, Secondary 57R30

1 Introduction

Codimension one foliations and contact structures play important roles in the study of topology and geometry of three dimensional manifolds. In [9], Eliashberg and Thurston combined the theories of these objects together as that of confoliations. One of the fundamental results is that any codimension one foliation on a three dimensional manifold except $\mathcal{F} = \{S^2 \times \{*\}\}$ on $S^2 \times S^1$ can be perturbed into a positive (or negative) contact structure as a plane field.

They also introduced a special class of perturbations of foliations, so called linear perturbations. Suppose a foliation generated by a plane field ξ. *A linear perturbation* of ξ is a one parameter family $\{\text{Ker } \alpha_t\}_{t \in (-\epsilon, \epsilon)}$ of plane fields defined by a family of 1-forms $\{\alpha_t\}$ with $\xi = \text{Ker } \alpha_0$ and $(d/dt)(\alpha_t \wedge d\alpha_t) > 0$. Eliashberg and Thurston observed that if the kernel of $\beta = d\alpha_t/dt|_{t=0}$ also generates a foliation, then $(\text{Ker } (\alpha + t\beta), \text{Ker } (\alpha - t\beta))$ is a pair of mutually transverse positive and negative contact structures for any $t \neq 0$. Independently, Mitsumatsu [13] also studied the same deformation for invariant foliations of Anosov flows and he called such a pair of contact structures *a bi-contact structure*. Mitsumatsu, and Eliashberg and Thurston observed that bi-contact structures correspond to *projectively Anosov flows* (or *conformally Anosov flows* in [9]), which are the main objects of this paper.

*Supported by Grant-in-Aid for Encouragement of Young Scientists (B)

A flow $\Phi = \{\Phi^t\}_{t\in\mathbb{R}}$ on a three dimensional manifold M is called a *projectively Anosov* flow (or $\mathbb{P}$A flow) if it has no stationary points and admits a decomposition $TM = E^u + E^s$ by continuous plane fields such that

- $E^u(z) \cap E^s(z) = T\Phi(z)$ for any $z \in M$, where $T\Phi$ is the line field tangent to the orbits of Φ,
- $D\Phi^t(E^\sigma(z)) = E^\sigma(\Phi^t(z))$ for any $\sigma \in \{u, s\}$, $z \in M$, and $t \in \mathbb{R}$, and
- there exist two constants $C > 0$ and $\lambda \in (0,1)$ such that

  $$\|D\hat{\Phi}^t|_{(E^s/T\Phi)(z)}\| \cdot \|(D\hat{\Phi}^t|_{(E^u/T\Phi)(z)})^{-1}\| \leq C\lambda^t$$

  for any $z \in M$ and $t \geq 0$, where $D\hat{\Phi} = \{D\hat{\Phi}^t\}_{t\in\mathbb{R}}$ is the induced flow on $TM/T\Phi$.

We call the decomposition $TM = E^u + E^s$ *a* $\mathbb{P}$A *splitting*. If it satisfies stronger inequalities

$$\|D\hat{\Phi}^t|_{(E^s/T\Phi)(z)}\| \leq C\lambda^t, \ \|(D\hat{\Phi}^t|_{(E^u/T\Phi)(z)})^{-1}\| \leq C\lambda^t$$

for any $z \in M$ and $t \geq 0$, then the flow is called an *Anosov flow* and the splitting is called a *weak-Anosov splitting* [1]. We remark that a variant of a $\mathbb{P}$A splitting localized at a flow invariant set is called a *dominated splitting*, which plays important roles in the modern theory of dynamical systems. See [4] for example.

Any $\mathbb{P}$A splitting is integrable, however, is not smooth in general. A $\mathbb{P}$A flow (or an Anosov flow) with a C^r-smooth $\mathbb{P}$A splitting is called *C^r-regular*. When a $\mathbb{P}$A flow is C^∞-regular, we simply say it is regular. From the viewpoint of deformations of foliations, regular $\mathbb{P}$A flows correspond to linear deformations $\{\text{Ker } \alpha_t\}$ of a foliation such that the derivative $d\alpha_t/dt|_{t=0}$ generates another foliation.

Regular Anosov flows on three dimensional manifolds are completely classified by Ghys.

Theorem 1.1 ([10]). *Up to finite covering, any regular Anosov flow on a three dimensional closed manifold is smoothly equivalent to either the suspension flow of a two dimensional hyperbolic toral automorphism or a quasi-Fuchsian flow on the unit tangent bundle of closed surface of genus greater than one.*

It is natural to ask whether the similar classification exists for regular $\mathbb{P}$A flows or not. In [16], Noda gave a classification of regular $\mathbb{P}$A flows with an invariant torus on a $\mathbb{T}^2$-bundle over S^1. After that, he and Tuboi gave a classification for certain manifolds, which can be summarized as follows.

Theorem 1.2 ([16],[17],[18], and [21]). *Any regular* $\mathbb{P}$A *flow on a Seifert manifold or a $\mathbb{T}^2$-bundle over S^1 must be either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$-models.*

[1] It is different from but equivalent to the common definition of an Anosov flow as pointed out by Doering [8, Proposition 1.1].

Roughly speaking, a $\mathbb{T}^2 \times I$-model is a flow on $\mathbb{T}^2 \times [0,1]$ which is transverse to $\mathbb{T}^2 \times \{z\}$ for any $z \in (0,1)$ and is equivalent to a linear flow on each boundary. See [16] for the precise definition. The author also approached the classification from another direction. In [2], he showed that any regular $\mathbb{P}$A flow on *any* closed three dimensional manifold without non-hyperbolic periodic orbits is equivalent to one of the flows that they classified.

In [17], Noda conjectured that there are no other regular $\mathbb{P}$A flows. The main theorem of this paper gives a solution of this conjecture and classify three dimensional regular $\mathbb{P}$A flows completely.

Main Theorem. *Any C^2-regular $\mathbb{P}$A flow on a closed and connected there dimensional manifold must be either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$ models.*

The theorem gives an answer to a conjecture posed by Mitsumatsu (Conjecture 4.3.3 in [14]) immediately.

Corollary 1.3. *Any bi-contact structure associated with a regular $\mathbb{P}$A flow consists of tight contact structures.*

The proof is divided into two parts. In Section 2, we show a dichotomy on dynamics of regular $\mathbb{P}$A flows. Namely, either the set of periodic orbits is dense in the manifold, or any positive or negative orbit converges to an invariant torus with rotational dynamics. We can see that the latter implies that the flow is represented by $\mathbb{T}^2 \times I$-models. In Sections 3 and 4, we show the former implies that the flow is Anosov. It is done by proving the hyperbolicity of all periodic orbits.

Acknowledgment The author would like to thank Prof. Takashi Inaba, who pointed out that we can apply the stability theory and the level theory of Cantwell and Conlon in our proof.

2 A dichotomy on dynamics

We fix a C^2-regular $\mathbb{P}$A flow Φ on a closed and connected three dimensional manifold M. Let $TM = E^u + E^s$ be a $\mathbb{P}$A splitting associated to Φ and $\mathcal{F}^\rho$ the foliations generated by E^ρ for $\rho \in \{u, s\}$. Without loss of generality, we can assume that both $\mathcal{F}^u$ and $\mathcal{F}^s$ are transversely orientable. For a compact Φ-invariant set Λ, we define *the stable set* $W^s(\Lambda)$ and *the unstable set* $W^u(\Lambda)$ by

$$\begin{aligned} W^s(\Lambda) &= \left\{ z \in M \mid \lim_{t \to +\infty} d(\Phi^t(z), \Lambda) \to 0 \right\} \\ W^u(\Lambda) &= \left\{ z \in M \mid \lim_{t \to -\infty} d(\Phi^t(z), \Lambda) \to 0 \right\}. \end{aligned}$$

For $\rho \in \{u, s\}$, let $\Omega_*^\rho(\Phi)$ be the union of all closed leaves of $\mathcal{F}^\rho$ on which the flow Φ is topologically conjugate to a linear flow. Remark that $\Omega_*^u(\Phi)$ is a finite union of Φ-invariant tori and $W^s(\Omega_*^u(\Phi))$ is an open neighborhood of $\Omega_*^u(\Phi)$. Similarly, $\Omega_*^s(\Phi)$ is a finite union of Φ-invariant tori and $W^u(\Omega_*^s(\Phi))$ is an open neighborhood of $\Omega_*^s(\Phi)$.

For a foliation $\mathcal{G}$ on M, we denote the leaf of a foliation through a point $z \in M$ by $\mathcal{G}(z)$. We also denote the orbit $\{\Phi^t(z) \mid z \in \mathbb{R}\}$ of a point $z \in M$ by $\mathcal{O}(z)$ and the set of periodic points of Φ by $\mathrm{Per}(\Phi)$. We say a periodic point z_0 is *s-regular* when there exists an embedded compact annulus $A \subset \mathcal{F}^s(z_0)$ such that $\Phi^t(A) \subset \mathrm{Int}\, A$ for any $t > 0$ and $\bigcap_{t>0} \Phi^t(A) = \mathcal{O}(z_0)$. Similarly, we say a periodic point z_0 is *u-regular* when there exists an embedded compact annulus $A \subset \mathcal{F}^u(z_0)$ such that $\Phi^{-t}(A) \subset \mathrm{Int}\, A$ for any $t > 0$ and $\bigcap_{t>0} \Phi^{-t}(A) = \mathcal{O}(z_0)$. We also say z_0 is *ρ-irregular* if z_0 is not ρ-regular for $\rho = u, s$.

The aim of this section is to prove the following proposition.

Proposition 2.1. *Either one of the followings hold:*

1. $M = W^s(\Omega_*^u(\Phi)) \cup \Omega_*^s(\Phi) = W^u(\Omega_*^s(\Phi)) \cup \Omega_*^u(\Phi)$.
2. $M = \overline{\mathrm{Per}(\Phi)}$ *and any periodic point of* Φ *is s- and u-regular.*

It is not hard to show that the former implies that Φ is equivalent to one of known models. Namely,

Proposition 2.2. *If* $M = W^s(\Omega_*^u(\Phi)) \cup \Omega_*^s(\Phi) = W^u(\Omega_*^s(\Phi)) \cup \Omega_*^u(\Phi)$*, then* Φ *is represented by a finite union of* $\mathbb{T}^2 \times I$*-models.*

Proof. Fix a connected component T_0 of $\Omega_*^s(\Phi)$ and a connected component U of $W^u(T_0) \setminus T_0$. Take a subset B of $W^u(T_0)$ which is diffeomorphic to $\mathbb{T}^2 \times [0,1]$ so that $T_0 \subset \partial B$ and $B \setminus T_0 \subset U$. Let T_* be the component of ∂B different from T_0. Notice that $W^s(\Omega_*^u(\Phi))$ is the disjoint union of the stable sets of the connected components of $\Omega_*^u(\Phi)$. Since T_* is connected and contained in $W^s(\Omega_*^u(\Phi))$, we have $T_* \subset W^s(T_1)$ for some connected component T_1 of $\Omega_*^u(\Phi)$.

Take a neighborhood B_* of T_1 which is diffeomorphic to $\mathbb{T}^2 \times [0,1]$ so that $B_* \subset W^s(T_1)$. Then, we have $\Phi^{t_0}(T_*) \subset \mathrm{Int}\, B_*$ for some $t_0 > 0$. Since $\Phi^{t_0}(T_*)$ separates two boundary components of B_* in B_*, it must be incompressible in B_*. In particular, it is isotopic to T_1 in B_* It implies that there exists a subset B_1 of M which is diffeomorphic to $\mathbb{T}^2 \times [0,1]$, and satisfies $\partial B_1 = T_0 \cup T_1$ and $\mathrm{Int}\, B_1 = U$.

Inductively, we can take sequences $(T_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ so that T_n is a connected component of $\Omega_*^u(\Phi) \cup \Omega_*^s(\Phi)$, B_n is a subset of M which is diffeomorphic to $\mathbb{T}^2 \times [0,1]$, $\partial B_n = T_n \cup T_{n+1}$, and $B_n \cap B_{n+1} = T_{n+1}$ for any n. Since $\Omega_*^u(\Phi)$ and $\Omega_*^s(\Phi)$ contain only finitely many tori, we have $T_n = T_m$ for some $n \neq m$. It implies that M is a $\mathbb{T}^2$-bundle over S^1. By Noda's classification [16], Φ is represented by a finite union of $\mathbb{T}^2 \times I$-models. $\square$

2.1 Return maps

We introduce the concept of return maps. For a finite set Σ, let π_x and π_y be the projections from $[-2,2]^2 \times \Sigma$ to the first and the second components respectively. We say a subset R of $[-2,2]^2 \times \Sigma$ is a *rectangle* if it has the form $[x_-, x_+] \times [y_-, y_+] \times \sigma_0$.

We call a C^2-embedding $\psi : [-2,2]^2 \times \Sigma \to M$ with a finite set Σ a *canonical cross section* if

- Im ψ is transverse to $T\Phi$,
- $\psi([-2,2] \times y \times \sigma) \subset \mathcal{F}^s(\psi(x,y,\sigma))$ and $\psi(x \times [-2,2] \times \sigma) \subset \mathcal{F}^u(\psi(x,y,\sigma))$ for any $(x,y,\sigma) \in [-2,2]^2 \times \Sigma$, and
- both $\{\Phi^t(z) \mid t > 0\}$ and $\{\Phi^t(z) \mid t < 0\}$ intersect with $\psi((-1,1)^2 \times \Sigma)$ for any $z \in M$.

It is easy to see that the flow Φ admits such an embedding.

Fix a canonical cross section $\psi : [-2,2]^2 \times \Sigma \to M$. We call a C^2-diffeomorphism $r : R \to R'$ between two rectangles R and R' a *return* associated to (Φ, ψ) if there exists a positive valued continuous function τ on R such that $\Phi^{\tau(w)}(\psi(w)) = \psi \circ r(w)$ for any $w \in R$. The function τ is called *the return time* associated to R. Note that τ is uniquely determined since any return of a $\mathbb{P}$A flow cannot be the identity map. For a return $r : R \to R'$, we can take C^2-diffeomorphisms $r_x : \pi_x(R) \to \pi_x(R')$ and $r_y : \pi_y(R) \to \pi_y(R')$ so that $r(x,y,\sigma) = (r_x(x), r_y(y), \sigma')$ for any $(x,y,\sigma) \in R$. We call the pair (r_x, r_y) *the xy-decomposition* of r. For a return r associated to (Φ, ψ), the map r^{-1} is a return associated to (Φ^{-1}, ψ). For a family $\mathcal{R} = \{r_k\}_{k=1}^{k_*}$ of returns, we write $\mathcal{R}^{-1}$ for a family $\{r_k^{-1}\}_{k=1}^{k_*}$ of returns associated to (Φ^{-1}, ψ).

We say a family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ of returns is *full* when

- $[-1,1]^2 \times \Sigma$ is contained in both $\bigcup_{k=1}^{k_*} R_k$ and $\bigcup_{k=1}^{k_*} R'_k$, and
- there exists a constant $\Delta > 0$ such that if $w \in R_k \cap ([-1,1]^2 \times \Sigma)$ satisfies $r_k(w) \in [-1,1]^2 \times \Sigma$, then $Q_\Delta(w) \subset R_k$ and $Q_\Delta(r_k(w)) \subset R'_k$, where $Q_\Delta(x,y,\sigma) = [x-\Delta, x+\Delta] \times [y-\Delta, y+\Delta] \times \sigma$.

It is easy to see that any canonical cross section admits a full family of returns.

Fix a full family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ of returns associated to (Φ, ψ). For a subset Λ of $[-2,2]^2 \times \Sigma$, we call a sequence $(k(n))_{n=1}^{n_*}$ an *$\mathcal{R}$-admissible* sequence for Λ if $r_{k(n)} \circ \cdots \circ r_{k(1)}|_\Lambda$ is well-defined for any $n = 1, 2, \ldots, n_*$. We say an $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^{n_*}$ for a point w of $[-1,1]^2 \times \Sigma$ is *fine* when $r_{k(n)} \circ \cdots \circ r_{k(1)}(w) \subset [-1,1]^2 \times \Sigma$ for any $n = 1, 2, \ldots, n_*$.

For $\Delta_1 > 0$, we say an $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^{n_*}$ for an interval I is *$(\mathcal{R}, \Delta_1)$-admissible* if $|r_{k(n)} \circ \cdots \circ r_{k(1)}(I)| \leq \Delta_1$, where $|J|$ is the length of an interval J. We call a sequence $(I_i = [x_i, x'_i] \times y_i \times \sigma_i)_{i \geq 1}$ of intervals in $[-2,2]^2 \times \Sigma$ *a Δ_1-family* if there exists a family $\{(k_i(n))_{n=1}^{n_i}\}_{i \geq 1}$ of sequences such that $(k_i(n))_{n=1}^{n_i}$ is an $(\mathcal{R}, \Delta_1)$-admissible sequence for I_i for any $i \geq 1$, n_i tends to infinity as $i \to \infty$, and $\limsup |r_{k_i(n_i)} \circ \cdots r_{k_i(1)}(I_i)| > 0$.

The following is the keystone to control the topology of the stable and unstable foliations.

Lemma 2.3. *There exists a constant $\Delta_1 > 0$ such that any Δ_1-family $\{I_i\}_{i=1}^{\infty}$ of intervals admits a sequence $\{z_i \in \psi(I_i)\}_{i=1}^{\infty}$ accumulating to a point of $\Omega_*^u(\Phi)$ or an s-irregular periodic point.*

Proof. Notice that almost all arguments in the proof of Proposition 3.1 of [19] (or Proposition 4.2 of [1]) work even if non-hyperbolic periodic orbits exist. They allow us to take a constant $\Delta_1 > 0$ such that if an interval $I = [x, x'] \times y \times \sigma$ admits an $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence $(k(n))_{n=1}^{\infty}$ then $\psi(I) \subset W^u(\Omega_*^s(\Phi))$ or $\mathrm{Int}\ \psi(I) \cap W^u(\mathcal{O}(z_*)) \neq \emptyset$ for some s-irregular periodic point z_*.

Let $(I_i)_{i\geq 1}$ be a Δ_1-family of intervals and $\{(k_i(n))_{n=1}^{n_i}\}_{i\geq 1}$ the corresponding family of sequences. Put $J_i = r_{k_i(n_i)} \circ \cdots \circ r_{k_i(1)}(I_i)$ and $k_i'(n) = k(n_i - n + 1)$ for any $n = 1, \ldots, n_i$. Then, $(k_i'(n))_{n=1}^{n_i}$ is an $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence for J_i. By taking subsequences if it is necessary, we can assume that J_i converges to an interval $J_* = [\bar{x}, \bar{x}'] \times \bar{y} \times \bar{\sigma}$ and there exist sequences $(k'(n))_{n=1}^{\infty}$ and $(i_n)_{n\geq 1}$ such that i_n tends to infinity as $n \to \infty$ and $k_i'(n) = k'(n)$ for any $n \geq 1$ and $i = 1, \ldots, i_n$. It is easy to check that $(k'(n))_{n=1}^{\infty}$ is an $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence for J_*

By the choice of the constant Δ_1, there exists $x_* \in (\bar{x}, \bar{x}')$ such that $\psi(x_*, \bar{y}, \bar{\sigma}) \in W^u(\Omega_*^s(\Phi))$ or $\psi(x_*, \bar{y}, \bar{\sigma}) \in W^u(\mathcal{O}(z_*))$ for some s-irregular periodic point z_*. Hence, we can take a neighborhood U of $\bar{y}$ such that $\overline{\bigcup_{t\geq T} \Phi^{-t}(x_* \times U \times \bar{\sigma})}$ converges to a connected component of $\Omega_*^s(\Phi)$ or $\mathcal{O}(z_*)$ as $T \to \infty$. It follows the lemma immediately. □

2.2 Local dynamics at periodic points

Put $\mathrm{Per}_h(\Phi) = \mathrm{Per}(\Phi) \backslash \{\Omega_*^u(\Phi) \cup \Omega_*^s(\Phi)\}$. The main aim of this subsection is to show that any point of $\mathrm{Per}_h(\Phi)$ is u- and s-regular. It is a main step of the proof of Proposition 2.1, and is done by a variant of the argument in [2].

Fix a canonical cross section $\psi : [-2, 2]^2 \times \Sigma \to M$. For a periodic point $z_0 = \psi(x_0, y_0, \sigma_0)$, we call a return $r : R \to R'$ *the first return* of z_0 if $(x_0, y_0, \sigma_0) \in \mathrm{Int}\ R \cap \mathrm{Int}\ R'$, $r(x_0, y_0, \sigma_0) = (x_0, y_0, \sigma_0)$, and the return time τ_R satisfies $\tau_R(w) = \inf\{t > 0 \mid \Phi^t \circ \psi(w) \in R'\}$ for any $w \in R$.

We say a point z of a topological space X is accessible from a subset A of X when there exists a continuous map $l : [0, 1] \to X$ such that $l(1) = z$ and $l(t) \in A$ for any $t \in [0, 1)$.

Lemma 2.4. *Let z_0 and z_1 be periodic points of Φ and suppose that z_1 is accessible from $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^\rho(z_1)$ for $\rho \in \{u, s\}$. Then, there exists an embedded closed annulus $A \subset \mathcal{F}^\rho(z_0)$ satisfying $\partial A = \mathcal{O}(z_0) \cup \mathcal{O}(z_1)$ and* $\mathrm{Int}\ A \subset W^s(\mathcal{O}(z_0)) \cap W^u(\mathcal{O}(z_1))$. *In particular, we have $\mathcal{F}^\rho(z_0) = \mathcal{F}^\rho(z_1)$.*

Proof. Without loss of generality, we can assume that $z_0 = \psi(x_0, y_0, \sigma_0)$ for some $(x_0, y_0, \sigma_0) \in [-1, 1]^2 \times \Sigma$. We prove the lemma for the case $\rho = u$ since the proof for the other case is similar. Let $r : R \to R'$ be the first return map of z_0 and τ the return time associated to r. Put $V = \{\Phi^t \circ \psi(w) \mid w \in R, t \in [0, \tau(w)]\}$ and let $\mathcal{G}$ be the restriction of $\mathcal{F}^\rho$ on V. It is easy to see that $\mathcal{G}$ is diffeomorphic to the foliation $\{x \times \pi_y(R) \times [0, 1]/\sim\}_{x \in \pi_x(R)}$ on $(\pi_x(R) \times \pi_y(R) \times [0, 1])/(x, y, 1) \sim$

$(r_x(x), r_y(y), 0)$. Hence, a leaf of $\mathcal{G}$ is non-contractible if and only if it contains a periodic point of Φ. See Figure 1.

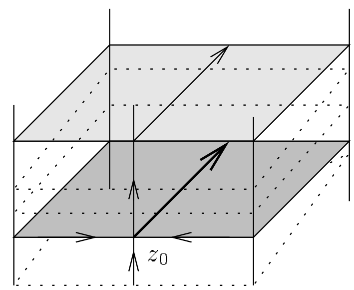

Figure 1: A neighborhood V

Since z_1 is accessible from $W^s(\mathcal{O}(z_0))$, there exist a simple closed curve γ transverse to $T\Phi$ and an embedded closed annulus $A_1 \subset \mathcal{F}^\rho(z_1)$ such that $\partial A_1 = \gamma \cup \mathcal{O}(z_1)$ and $\gamma \cup \mathrm{Int}\, A \subset W^s(\mathcal{O}(z_0))$. The curve $\Phi^t(\gamma)$ is contained in a leaf L of $\mathcal{G}$ for any sufficiently large $t > 0$. By Poincaré-Bendixon's theorem, L must be non-contractible, and hence, it contains a periodic point. Since $\gamma \in W^s(\mathcal{O}(z_0))$, we obtain $L = \mathcal{G}(z_0)$. It implies that the existence of the required embedded annulus. $\square$

For $w = (x, y, \sigma) \in [-1, 1]^2 \times \Sigma$ and $\delta > 0$, we define intervals $I^s_\delta(w)$ and $I^u_\delta(w)$ by

$$I^s_\delta(w) = [x - \delta, x + \delta] \times y \times \sigma, \; I^u_\delta(w) = x \times [y - \delta, y + \delta] \times \sigma.$$

Lemma 2.5. *Let z_0 be an s-regular periodic point. Then, there exist a constant $\delta_{z_0} > 0$ such that any fine $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^\infty$ for $w \in ([-1, 1]^2 \times \Sigma) \cap \psi^{-1}(W^s(\mathcal{O}(z_0)))$ satisfies $\psi(I^s_\delta(r_{k(n)} \circ \cdots \circ r_{k(1)}(w))) \subset W^s(\mathcal{O}(z_0))$ for some $n \geq 1$. In particular, $\mathcal{F}^u(z) \cap W^s(\mathcal{O}(z_0))$ is an open subset of $\mathcal{F}^u(z)$ for any $z \in W^s(\mathcal{O}(z_0))$.*

Proof. Fix $w_0 = (x_0, y_0, \sigma_0) \in [-1, 1]^2 \times \Sigma$ so that $\psi(w_0) \in \mathcal{O}(z_0)$. Let $r : R \to R'$ be the first return of w_0 and (r_x, r_y) the xy-decomposition of r. Since z_0 is s-regular, there exists $I_x \subset \pi_x(R)$ such that $r_x(I_x) \subset \mathrm{Int}\ I_x$ and $\bigcap_{n \geq 0} r_x^n(I_x) = \{x_0\}$. Put $\Lambda_y = \{y \in \bigcap_{n \geq 0} r_y^{-n}([-2, 2]) \mid \lim_{n\to\infty} r_y^n(y) = y_0\}$ and take $\delta > 0$ so that $[x - \delta, x + \delta] \subset I_x$ for any $x \in r_x(I_x)$. It is easy to see that $\psi(I_x \times \Lambda_y \times \sigma_0) \subset W^s(\mathcal{O}(z_0))$. In particular, $\psi(I^s_\delta(w)) \subset W^s(\mathcal{O}(z_0))$ for any $w \in r_x(I_x) \times \Lambda_y \times \sigma_0$.

There exist a neighborhood U_0 of $\mathcal{O}(z_0)$ and a constant $T_0 > 0$ such that $\psi(r_x(I_x) \times \Lambda_y \times \sigma_0) \neq \emptyset$ for any $z \in W^s(\mathcal{O}(z_0)) \cap \bigcap_{t \geq 0} \Phi^{-t}(U_0)$. Hence, we can take $\delta_z > 0$ so that $I^s_{\delta_z}(w) \subset W^s(\mathcal{O}(z_0))$ for any $w \in [-1, 1]^2 \times \Sigma$ with $\psi(w) \in W^s(\mathcal{O}(z_0)) \cap \bigcap_{t \geq 0} \Phi^{-t}(U_0)$. It is easy to see that the constant δ_{z_0} satisfies the required condition. $\square$

Lemma 2.6. *The followings hold for any s-regular periodic point z_0:*

1. *$\mathcal{F}^s(z) \subset W^s(\mathcal{O}(z_0))$ for any $z \in W^s(\mathcal{O}(z_0)) \backslash \mathcal{F}^s(z_0)$.*

2. *$\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ is diffeomorphic to $S^1 \times \mathbb{R}$.*

3. *If $\mathcal{F}^s(z_0) \not\subset W^s(\mathcal{O}(z_0))$, then there exist an s-irregular periodic point $z_1 \in \mathcal{F}^s(z_0)$ and an embedded closed annulus $A \subset \mathcal{F}^s(z_0)$ such that $\partial A = \mathcal{O}(z_0) \cup \mathcal{O}(z_1)$ and $\mathrm{Int}\, A \subset W^s(\mathcal{O}(z_0)) \cap W^u(\mathcal{O}(z_1)))$.*

Proof. Since z_0 is s-regular, we can take an embedded closed annulus $A_0 \subset \mathcal{F}^s(z_0)$ such that $\Phi^t(A_0) \subset \mathrm{Int}\ A_0$ for any $t > 0$ and $\bigcap_{t>0} \Phi^t(A_0) = \mathcal{O}(z_0)$. Then, $W_0 = \bigcup_{t>0} \Phi^{-t}(A_0)$ is a connected component of $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^s(z_0)$ which is diffeomorphic to $S^1 \times \mathbb{R}$.

Fix a leaf L of $\mathcal{F}^s$ with $L \cap W^s(\mathcal{O}(z_0)) \neq \emptyset$ and take a connected component W of $L \cap W^s(\mathcal{O}(z_0))$. It is sufficient to show that if $W \neq L$ then there exists a periodic point $z_1 \in L \backslash U$ which is accessible from W. In fact, if such z_1 exists, then Lemma 2.4 implies that there exists an embedded closed annulus $A \subset \mathcal{F}^s(z_0)$ with $\partial A = \mathcal{O}(z_0) \cup \mathcal{O}(z_1)$ and $\mathrm{Int}\ A \subset W^s(\mathcal{O}(z_0)) \cap W^u(\mathcal{O}(z_1)))$ In particular, z_1 is s-irregular and $W = W_0 \subset \mathcal{F}^s(z_0)$.

Suppose that $W \neq L$. Then, there exists $z_0 = \psi(x_0, y_0, \sigma_0) \in \psi([-1,1]^2 \times \Sigma) \cap (L \backslash W)$ which is accessible from W. Put $I_x = [x_0, x] \times y_0 \times \sigma_0$ for $x > x_0$. Without loss of generality, we can assume that $\psi(\mathrm{Int}\ I_{x_0'}) \subset W$ for some $x_0' > x_0$.

Fix a full family $\mathcal{R} = \{r_k : R_k \to R_k'\}_{k=1}^{k_*}$ of returns and let $\Delta_1 > 0$ and $\delta_{z_0} > 0$ be the constants obtained in Lemmas 2.3 and 2.5. Put $\Delta = \min\{\Delta_1, \delta_{z_0}, x_0' - x_0\}$ and $C_1 = \sup\{\|Dr_k\| \mid k = 1, \dots, k_*\}$. We claim that for any $x \in (x_0, x_0 + \Delta)$, $I_x = [x_0, x] \times y_0 \times \sigma_0$ admits an $(\mathcal{R}, \Delta)$-admissible sequence $(k_x(n))_{n=1}^{n_x}$ such that $|r_{k_x(n_x)} \circ r_{k_x(1)}(I_x)| \geq C_1^{-1}\Delta$ and $C_1^{n_x}(x - x_0) > \Delta$. In fact, take a fine $\mathcal{R}$-admissible sequence $(k_x(n))_{n=1}^{\infty}$ for (x, y_0, σ_0). Then, $|r_{k_x(n')} \circ r_{k_x(1)}(I_x)| \leq C_1^n(x - x_0)$ if $(k(n))_{n=1}^{n'}$ is an $\mathcal{R}$-admissible sequence for I_x. Since $\psi(I_\Delta^s(r_{k_x(n)} \circ r_{k_x(1)}(x))) \subset W^s(\mathcal{O}(z_0))$ and $\psi(x) \notin W^s(\mathcal{O}(z_0))$, there exists $n_x \geq 1$ such that $(k(n))_{n=1}^{n_x}$ is an $(\mathcal{R}, \Delta)$-admissible sequence for I_x with $|r_{k_x(n_x+1)} \circ r_{k_x(1)}(I_x)| \geq C_1^{-1}\Delta$. It is easy to see the sequence $(k_x(n))_{n=1}^{n_x}$ satisfies the required conditions.

By the above claim, $\{I_{C_1^{-n}\Delta}\}_{i \geq 0}$ is a Δ_1-family. Lemma 2.3 implies that z_0 is a point of $\Omega_*^s(\Phi)$ or an s-irregular periodic point. If the former holds, then $\mathcal{F}^s(\psi(w_0))$ is contained in $\Omega_*^s(\Phi)$. However, $\Omega_*^s(\Phi)$ does not intersects with $W^s(\mathcal{O}(z_0))$. Therefore, $\psi(w_0)$ is an s-irregular periodic point. □

Recall that we say a leaf of a codimension one foliation is *semi-proper* when it does not accumulate to itself from at least one side. We also say a leaf is *proper* when it does not accumulate to itself from both sides.

Lemma 2.7. *Let $\mathcal{G}$ be a C^2 codimension one foliation of a closed three dimensional manifold. Then, any semi-proper leaf of $\mathcal{G}$ diffeomorphic to $S^1 \times \mathbb{R}$ has trivial holonomy.*

Proof. Let L be a leaf of $\mathcal{G}$ which is diffeomorphic to $S^1 \times \mathbb{R}$. Note that the end set of L consists of two points. By the level theory of Cantwell and Conlon [5] L is either proper or contained in an exceptional local minimal set. However, Duminy's theorem (See [7] for the proof) implies that the end of a semi-proper leaf in an exceptional local minimal set must be a Cantor set. Hence, the leaf L is proper. By a theorem of Cantwell and Conlon [6, Theorem 1], L has trivial holonomy. $\square$

Now, we show the main result of this subsection.

Proposition 2.8. *Any point of* $\mathrm{Per}_h(\Phi)$ *is s- and u-regular.*

Proof. We show that any $z_0 \in \mathrm{Per}_h(\Phi)$ is u-regular. Once it is done, then we apply it to the flow $\Phi^{-1} = \{\Phi^{-t}\}$ and obtain that any $z_0 \in \mathrm{Per}_h(\Phi)$ is s-regular.

Suppose that z_0 is u-irregular. If $\mathcal{F}^s$ has trivial holonomy along $\mathcal{O}(z_0)$, then $\mathrm{Per}(\Phi) \cap \mathcal{F}^u(z_0)$ contains a closed annulus, whose boundary consists of u-irregular periodic points. Hence, we can assume that $\mathcal{F}^s$ has non-trivial holonomy along $\mathcal{O}(z_0)$ by replacing z_0 if it is necessary. Without loss of generality, we also assume $z_0 = \psi(w_0)$ for some $w_0 = (x_0, y_0, \sigma_0) \in [-1,1]^2 \times \Sigma$.

Take the first return $r : R \rightarrow R'$ of z_0 and let (r_x, r_y) be the xy-decomposition of r. Since z_0 is u-irregular, it is s-regular. Hence, we can assume that $I_x = \pi_x(R)$ satisfies $r_x(I_x) \subset \mathrm{Int}\ I_x$ and $\bigcap_{n \geq 0} r^n(I_x) = \{x_0\}$. Since z_0 is u-irregular, we can take $y_1 \in \pi_y(R) \backslash \{y_0\}$ such that $|r_y(y_1) - y_0| \leq |y_1 - y_0|$. It implies that there exists a compact interval $I_y \subset \pi_y(R)$ such that $r(I_y) \subset I_y$ and $y_0 \in \partial I_y$. Put $J_x = \psi(I_x \times y_0 \times \sigma_0)$ and $J_y = \psi(x_0 \times I_y \times \sigma_0)$.

Put $W = \bigcup_{t \geq 0} \Phi^{-t}(J_x)$. Lemma 2.6 implies that $W = W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^s(z_0)$ and it is diffeomorphic to $S^1 \times \mathbb{R}$. Since $\bigcup_{t>0} \Phi^t(\mathrm{Int}\ J_y) \cap \mathrm{Im}\ \psi \subset \mathrm{Int}\ J_y$, we have $W \cap \mathrm{Int}\ J_y = \emptyset$. If $\mathcal{F}^s(z_0)$ coincide with W, then it must be a semi-proper leaf of $\mathcal{F}^s$. In particular, it has trivial holonomy by Lemma 2.7. It contradicts the choice of z_0. Therefore, we obtain $W \neq \mathcal{F}^s(z_0)$.

By Lemma 2.6, there exist an s-irregular periodic point $z_1 \in \mathcal{F}^s(z_0)$ and an embedded closed annulus $A^s \subset \mathcal{F}^s(z_0)$ such that $\partial A^s = \mathcal{O}(z_0) \cup \mathcal{O}(z_1)$ and $\mathrm{Int}\ A^s \subset W^s(z_0)$. Let t_i be the period of z_i and put $\lambda_i^\rho = \|D\Phi^{t_i}|_{(E^\rho/T\Phi)(z_i)}\|$ for $i = 0, 1$ and $\rho \in \{u, s\}$. Notice that the orientation of the orbits of z_0 and z_1 must be opposite since z_0 is u-irregular, z_1 is u-regular, and $\mathcal{F}^s(z_0) = \mathcal{F}^s(z_1)$. In particular, we have $\lambda_0^u \cdot \lambda_1^u = 1$. See Figure 2.

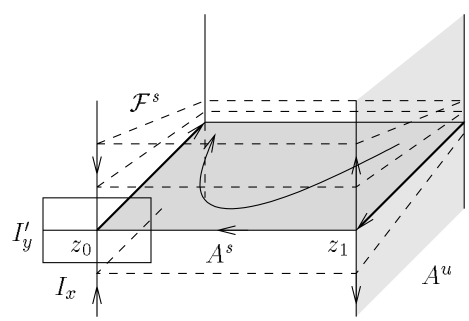

Figure 2: Annuli A^s and A^u

Since z_1 is s-irregular and Φ is a $\mathbb{P}$A flow, we have $1 \leq \lambda_1^s < \lambda_1^u$, and hence, $\lambda_0^s < \lambda_0^u < 1$. The latter implies that there exists a compact interval $I'_y \subset \pi_y(R)$ such that $r_y(I'_y) \subset \mathrm{Int}\ I'_y$ and $\bigcap_{n \geq 0} r_y^n(I'_y) = \{y_0\}$.

Put $J'_y := \psi(x_0 \times (I'_y \backslash \{y_0\}) \times \sigma_0)$. Then, we have $\bigcup_{t>0} \Phi^t(J'_y) \cap \mathrm{Im}\ \psi \subset J'_y$ and $J'_y \subset W^s(\mathcal{O}(z_0))$. The former implies $J'_y \cap W = \emptyset$, and hence, $J'_y \subset W^s(\mathcal{O}(z_0)) \backslash \mathcal{F}^s(z_0)$.

Take an embedded annulus $A^u \subset \mathcal{F}^u(z_1)$ such that $\mathcal{F}^s(z) \cap J'_y \neq \emptyset$ for any $z \in A^u \backslash \mathcal{O}(z_1)$. Then, Lemma 2.6 implies $A^u \backslash \mathcal{O}(z_1) \subset W^s(\mathcal{O}(z_0))$. In particular, z_1 is accessible from $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^u(z_1)$. Applying Lemma 2.4 to z_0, z_1, and $\rho = u$, we obtain an embedded annulus A in $\mathcal{F}^u(z_0) = \mathcal{F}^u(z_1)$. such that $\partial A = \mathcal{O}(z_0) \cap \mathcal{O}(z_1)$. It implies $\lambda_0^s \cdot \lambda_1^s = 1$ since the orientation of $\mathcal{O}(z_0)$ and $\mathcal{O}(z_1)$) in A must be opposite. However, it contradicts the inequalities $\lambda_0^u \cdot \lambda_1^u = 1$, $\lambda_0^s < \lambda_0^u$, and $\lambda_1^s < \lambda_1^u$ □

Corollary 2.9. *The leaf $\mathcal{F}^\rho(z_0)$ coincides with $W^\rho(\mathcal{O}(z_0))$ and it is diffeomorphic to $S^1 \times \mathbb{R}$ for any $z_0 \in \mathrm{Per}_h(\Phi)$ and $\rho \in \{u, s\}$.*

Proof. Since z_0 is u- and s-regular, it is clear that $W^s(\mathcal{O}(z_0)) \subset \mathcal{F}^s(z_0)$. By Lemma 2.6, non-existence of s-irregular periodic point in $\mathrm{Per}_h(\Phi)$ implies that $\mathcal{F}^s(z_0)$ is a subset of $W^s(\mathcal{O}(z_0))$ and is diffeomorphic to $S^1 \times \mathbb{R}$. The proof for $\mathcal{F}^u(z_0)$ is the same. □

2.3 Proof of Proposition 2.1

First, we show the Birkoff-Smale theorem in our setting.

Lemma 2.10. $\mathcal{F}^s(z_0) \cap \mathcal{F}^u(z_0) \subset \overline{\mathrm{Per}_h(\Phi)}$ *for any* $z_0 \in \mathrm{Per}_h(\Phi)$.

Proof. Take a canonical cross section $\psi : [-2, 2]^2 \times \Sigma \to M$. Without loss of generality, we can assume that $z_0 = \psi(w_0)$ for some $w_0 = (x_0, y_0, \sigma_0) \in [-1, 1]^2 \times \Sigma$. Let $r : R \to R'$ be the first return of z_0 and (r_x, r_y) the xy-decomposition of r. Put $I_x = \pi_x(R)$ and $I_y = \pi_y(R)$. Since z_0 is u- and s-regular, we can assume that $r(I_x) \subset \mathrm{Int}\ I_x$, $I_y \subset \mathrm{Int}\ r_y(I_y)$, $\bigcap_{n \geq 0} r^n(I_x) = \{x_0\}$, and $\bigcap_{n \geq 0} r^{-n}(I_y) = \{y_0\}$.

Fix $z_1 \in \mathcal{F}^s(z_0) \cap \mathcal{F}^u(z_0)$. By Corollary 2.9, we have $\mathcal{F}^\rho(z_0) = W^\rho(\mathcal{O}(z_0))$. Hence, there exist $t_- < t_+$, $x_1 \in I_x$, and $y_1 \in I_y$ such that $\Phi^{t_-}(z_1) = \psi(x_0, y_1, \sigma_0)$ and $\Phi^{t_+}(z_1) = \psi(x_1, y_0, \sigma_0)$. For any neighborhood $U \subset R$ of $\psi(x_0, y_1, \sigma_0)$, we can take a return $r_1 : R_1 \to R'_1$ so that $(x_0, y_1, \sigma_0) \in \mathrm{Int}\ R_1 \subset U$, $(x_1, y_0, \sigma_0) \in \mathrm{Int}\ R_2 \subset U$, and $r_1(x_0, y_1, \sigma_0) = (x_1, y_0, \sigma_0)$. Let $(r_{1,x}, r_{1,y})$ be the xy-decomposition of r_1. We can see that $r^n(\pi_x(R'_1)) \subset \pi_x(R_1)$ and $r^{-n}(\pi_y(R_1)) \subset \pi_y(R'_1)$ for some $n \geq 1$. See Figure 3. Then, there exists $(x_*, y_*) \in R_1$ such that $r_x^n \circ r_{1,x}(x_*) = (x_*)$ and $r_y^n \circ r_{1,y}(y_*) = (y_*)$. Since $\psi(x_*, y_*, \sigma_0)$ is a periodic point of Φ and the neighborhood U can be arbitrary small, we obtain that $z_1 \in \overline{\mathrm{Per}_h(\Phi)}$. □

Now, we show Proposition 2.1. If $\mathrm{Per}_h(\Phi) = \emptyset$, then Theorem B of [1] implies that the non-wandering set of Φ coincides with $\Omega_*^u(\Phi) \cup \Omega_*^s(\Phi)$. It is clear that $M = W^s(\Omega_*^u(\Phi)) \cup \Omega_*^s(\Phi) = W^u(\Omega_*^s(\Phi)) \cup \Omega_*^u(\Phi)$ in this case.

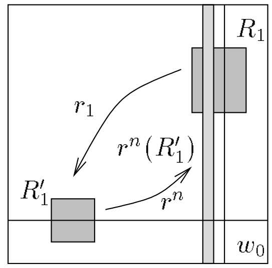

Figure 3: Proof of Lemma 2.10

Suppose $\mathrm{Per}_h(\Phi) \neq \emptyset$. By Lemma 2.10, we have $\mathcal{F}^u(z_0) \cap \mathcal{F}^u(z_0) \subset \overline{\mathrm{Per}_h(\Phi)}$ for any $z_0 \in \mathrm{Per}_h(\Phi)$. Since $\mathcal{F}^u$ and $\mathcal{F}^s$ are mutually transverse, it implies that $\mathcal{F}^s(z) \cap \mathcal{F}^u(z) \subset \overline{\mathrm{Per}_h(\Phi)}$ for any $z \in \overline{\mathrm{Per}_h(\Phi)}$.

We claim that $\mathcal{F}^u(z) \subset \overline{\mathrm{Per}_h(\Phi)}$ for any $z \in \overline{\mathrm{Per}_h(\Phi)}$. If it does not hold, then there exists $z_1 \in \overline{\mathrm{Per}_h(\Phi)}$ which is accessible from $\mathcal{F}^u(z_1) \backslash \overline{\mathrm{Per}_h(\Phi)}$. It implies that $\mathcal{F}^s(z_1)$ is a semi-proper leaf. However, it contradicts Lemma 2.7 since z_1 is u-regular.

Applying the claim for the flow Φ^{-1}, we also have $\mathcal{F}^s(z) \subset \overline{\mathrm{Per}_h(\Phi)}$ for any $z \in \overline{\mathrm{Per}_h(\Phi)}$. It implies that $\overline{\mathrm{Per}_h(\Phi)}$ is a non-empty open subset of M, and hence, $M = \overline{\mathrm{Per}_h(\Phi)}$.

3 Markov families and the redcution to one dimensional dynamics

For $\rho \in \{u, s\}$, let $\mathrm{Per}^{\rho}_*(\Phi)$ be the set of periodic point z_* with $\|D\hat{\Phi}^{t_*}|_{(E^\rho/T\Phi)(z_*)}\| = 1$, where t_* is the period of z_*. Put $\mathrm{Per}_*(\Phi) = \mathrm{Per}^u_*(\Phi) \cup \mathrm{Per}^s_*(\Phi)$.

In this section, we fix a C^2-regular $\mathbb{P}$A flow Φ and assume that $M = \overline{\mathrm{Per}(\Phi)}$ and any periodic point of Φ is s- and u-regular. In Subsection 3.1, we show Φ admits a kind of Markov partitions. Such a partition allows us to reduce the family of return maps to a one-dimensional dynamical system. In Subsection 3.2, we apply a theorem of Mañé to the reduced system and estimate $\|D\hat{\Phi}^t|_{E^u/T\Phi}(z)\|$. One of the consequences is that $\Phi_*(\Phi)$ contains only finitely many periodic orbits. The other is the flow is Anosov if $\mathrm{Per}_*(\Phi)$ is empty [2].

3.1 Markov families of returns

Fix a canonical cross-section $\psi : [-2,2]^2 \times \Sigma \rightarrow M$. Recall that $I^s_\delta(w)$ and $I^u_\delta(w)$ be intervals $[x-\delta, x+\delta] \times y \times \sigma$ and $x \times [y-\delta, y+\delta] \times \sigma$ respectively. for $\delta > 0$ and $w = (x, y, \sigma) \in [-1,1]^2 \times \Sigma$. The following lemma asserts that we can regard $I^s_\delta(w)$ and $I^u_\delta(w)$ as the local stable and the unstable manifolds for returns if δ is sufficiently small.

[2]It also follows from Theorem B of [1].

Lemma 3.1. *Let $\{r_k\}_{k=1}^{k_*}$ be a full family of returns and $\epsilon_* > 0$ a given constant. There exist a constant $\Delta_{\mathcal{R}} > 0$ and a sequence $(\epsilon_n)_{n=1}^{\infty}$ which satisfy the followings:*

1. *$\epsilon_n \in (0, \epsilon_*)$ for any $n \geq 1$ and tends to 0 as $n \to \infty$.*

2. *Any fine $\mathcal{R}$-admissible sequence $(k_n)_{n=1}^{\infty}$ for $w \in [-1,1]^2 \times \Sigma$ is also $\mathcal{R}$-admissible for the interval $I^s_{\Delta_{\mathcal{R}}}(w)$ and it satisfies $r_{k_n} \circ \cdots \circ r_{k_1}(I^s_{\Delta_{\mathcal{R}}}(w)) \subset I^s_{\epsilon_n}(r_{k_n} \circ \cdots r_{k_1}(w))$ for any $n \geq 1$.*

3. *Any fine $\mathcal{R}^{-1}$-admissible sequence $(k_n)_{n=1}^{\infty}$ for $w \in [-1,1]^2 \times \Sigma$ is also $\mathcal{R}^{-1}$-admissible for the interval $I^u_{\Delta_{\mathcal{R}}}(w)$ and it satisfies $r_{k_n}^{-1} \circ \cdots \circ r_{k_1}^{-1}(I^u_{\Delta_{\mathcal{R}}}(w)) \subset I^u_{\epsilon_n}(r_{k_n}^{-1} \circ \cdots r_{k_1}^{-1}(w))$ for any $n \geq 1$.*

Proof. It is enough to show the existence of $\Delta_{\mathcal{R}}$ and $(\epsilon_n)_{n \geq 1}$ which satisfies the first and the second conditions.

Let Δ_0 be the constant in the definition of a full family $\mathcal{R}$ of returns. Take $\Delta_1 > 0$ in Lemma 2.3. Remark that there is no Δ_1-family $\{I_i\}$ of intervals since any point of $\mathrm{Per}_h(\Phi)$ is u- and s-regular by Proposition 2.8.

Put $\Delta = \min\{\epsilon_*, \Delta_1\}$. We claim that there exists $\Delta_{\mathcal{R}} > 0$ such that any fine $\mathcal{R}$-admissible sequence $(k_n)_{n \geq 1}$ for $w \in [-1,1]^2 \times \Sigma$ is also an $(\mathcal{R}, \Delta)$-admissible sequence for $I^s_{\Delta_{\mathcal{R}}}(w)$. In fact, if it does not hold, then for any $\delta \in (0, \Delta)$ there exist $\delta' \in (0, \delta)$ and $w \in [-1,1]^2 \times \Sigma$ and an $(\mathcal{R}, \Delta)$ admissible sequence $(k(n))_{n=1}^{n_*}$ such that $|r_{k(n_*)} \circ \cdots r_k(1)(I^s_{\delta'}(w))| = \Delta$. Hence, we can take sequences $(w_i)_{i=1}^{\infty}$ in $[-1,1]^2 \times \Sigma$ and $(\delta_i)_{i=1}^{\infty}$ in $(0, \Delta)$ so that δ_i tends to zero as $i \to \infty$, and $\{I^s_{\delta_i}(w_i)\}_{i=1}^{\infty}$ is a Δ_1-family. However, it contradicts the choice of Δ_1.

It is easy to see that if the constant $\delta_{\mathcal{R}}$ that is obtained in the above claim does not satisfies the second assertion of the lemma, then we can take a Δ_1-family of intervals. However, it contradicts the choice of Δ_1. $\square$

For a rectangle $R = I \times J \times \sigma$ and a point $w = (x, y, \sigma)$ of R, we define two intervals $I^s(R, w)$ and $I^u(R, w)$ by $I^s(R, w) = I \times y \times \sigma$ and $I^u(R, w) = x \times J \times \sigma$. We call a family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ of returns *a Markov family* if there exists a $\{0,1\}$-valued $(k_* \times k_*)$-matrix $A_{\mathcal{R}} = (a_{ij})$ such that

1. $R'_i \subset \bigcup_{a_{ij}=1} R_j$,

2. $I^s(R'_i, w) \subset I^s(R_j, w)$ and $I^u(R_j, w) \subset I^u(R'_i, w)$ for $w \in R'_i \cap R_j$ with $a_{ij} = 1$,

3. $\mathrm{Int}\, R_j \cap \mathrm{Int}\, R_{j'} = \emptyset$ if $a_{ij} = a_{ij'} = 1$ and $j \neq j'$, and

4. there exists a sequence $(\epsilon_n)_{n=1}^{\infty}$ such that ϵ_n tends to zero as $n \to \infty$ and if a sequence $(k(m))_{m=0}^{n}$ and $w \in R_{k(0)}$ satisfies $a_{k(n)k(n+1)} = 1$ and $r_{k(n)} \circ \cdots \circ r_{k(0)}(w) \in R_{k(n+1)}$ for any $m = 0, \ldots, n-1$, then

$$\left| I^u(R_0, w) \cap \left(\bigcap_{m=0}^{n-1} (r_{k(m)} \circ \cdots \circ r_{k(1)})^{-1} (R_{k(m+1)}) \right) \right| \leq \epsilon_n.$$

We call $A_{\mathcal{R}}$ *the transition matrix* of $\mathcal{P}$.

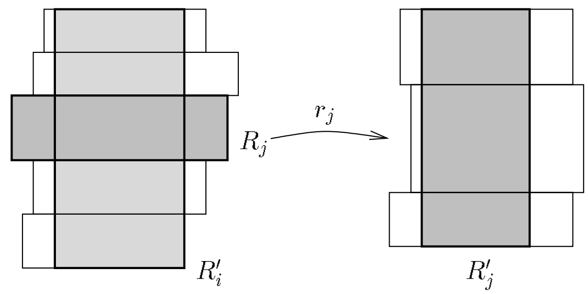

Figure 4: Markov family

Lemma 3.2. *Any canonical cross-section admits a Markov family of returns.*

Proof. Fix a full family $\mathcal{P} = \{p_l : P_l \to P'_l\}_{l=1}^{l_*}$ of returns associated to (Φ, ψ). Let $\Delta_{\mathcal{R}}$ be the constant obtained in Lemma 3.1 for some $\epsilon_* > 0$. Since $M = \overline{\mathrm{Per}(\Phi)}$, we can take a family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ of returns such that $\bigcup_{k=1}^{k_*} R_k \supset [-1,1]^2 \times \Sigma$ and

- $R'_k \subset [-1,1]^2 \times \Sigma$, $R_k \subset P_{l_k}$, and $r_k = p_{l_k}|_{R_k}$ for some l_k,
- the diameters of R_k and R'_k is less than $\Delta_{\mathcal{R}}$, and
- every boundary segment of ∂R_k intersects with $\psi^{-1}(\mathrm{Per}(\Phi))$ in its interior

for any k. Lemma 3.1 allows we can apply the proof of Theorem 2 in [20, Appendix 2]. In fact, we obtain a Markov family as a subdivision of $\mathcal{R}$. $\square$

3.2 The reduced one-dimensional map

Fix a canonical cross-section ψ and a Markov family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ of returns with the transition matrix $A_{\mathcal{R}} = (a_{ij})$. Let π_y be the projection defined by $\pi_y(x, y, \sigma) = y$. Put $I'_k = \pi_y(R'_k) \times k$ for $k \in \{1, \ldots, k_*\}$ and $I_* = \bigcup_{k=1}^{k_*}$. Take a partition $\{I_{ij} \mid a_{ij} = 1\}$ of I_* so that $\overline{I_{ij}} = \pi_y(R_j) \times i$ for any i and j with $a_{ij} = 1$.

We define a map $f : I_* \to I_*$ so that $f(y, i) = ((\pi_y \circ r_j)(x, y, \sigma)) \times j$ for $(x, y, \sigma) \in R_j \cap R'_i$ with $a_{ij} = 1$ and $y \in I_{ij}$. For $y \in I_*$ and $n \geq 0$, let $I(y, n)$ be the set of $y' \in I_*$ satisfying $y' \in I_{i_m j_m}$ for any $0 \leq m \leq n$ if $y \in I_{i_m j_m}$. It is easy to see that $f^n(I(y, m+n)) \subset I(y, m)$.

Lemma 3.3. *If $I(y, m) \subset f^n(I(y, m))$ for $y \in I_*$ $m \geq 0$, and $n \geq 1$, then $I(y, m)$ contains a fixed point of f^n.*

Proof. It is easy to see that the restriction of f^n on $I(y,m)$ extend to a continus map $\bar{f}^n_{y,m}$ on $\overline{I(y,m)}$ uniquely for any y, m and n. In particular, if $I(y,m) \subset f^n(I(y,m))$ then the map $\bar{f}^n_{y,m}$ has a fixed point.

It is easy to see that $\bar{f}^{m+1}_{y,m}(\partial(I(y,m))) \subset \partial I_*$ for any $y \in I_*$ and $m \geq 0$. Hence, we have $\partial I(y,m) \cap \mathrm{Per}(\bar{f}^n_{y,m}) \subset \partial I_*$. On the other hand, the construction of the partition $\{I_{ij}\}$ implies $\partial I_* \cap \overline{I_{ik}} \subset I_{ik}$. □

Put $\delta_n = \sup\{|I(y,n)| \mid y \in I_*\}$. The condition 4 in the definition of a Markov family implies that δ_n tends to zero as $n \to \infty$. In particular, we have $\bigcup_{n\geq 0} I(y,n) = \{y\}$ for any $y \in I_*$.

In the rest of the subsection, we estimate $\|D\Phi^t|_{E^u/T\Phi(z)}\|$ by a variation of a theorem of Mañé [12] to f. For a C^2 map $g : I \to I'$ between intervals, we define the **distortion** $\mathrm{dist}(g,I)$ of g by

$$\mathrm{dist}(g,I) = \sup\{\log|Dg(y)| - \log|Dg(y')| \mid y, y' \in I\}.$$

We define *the intersection multiplicity* of a family $\{S_i\}$ of subsets of a set S by $\sup_{x\in X} \#\{i \mid x \in X_i\}$, where $\#S$ is the cardinality of a set S. Notice that $C_1 = (\sum_{k=1}^{k_*} |I'_k|) \sup\{|D(\log|Df|)|\}$ is finite. Hence, if the intersection multiplicity of a family $\{\mathrm{Int}\ f^m(I)\}_{m=0}^{n-1}$ of subintervals of I_* is at most l, then $\mathrm{dist}(f^n, I) \leq C_1 l$.

Proposition 3.4. *There exists a sequence $(K_n)_{n\geq 1}$ such that K_n tends to infinity as $n \to \infty$ and $|(f^n)'(y_*)| \geq K_n$ for any periodic point y_* of period n.*

Proof. We follow the proof of Theorem 1 of [11]. Since $M = \overline{\mathrm{Per}(\Phi)}$, there exists a periodic point $y_{ij} \in \mathrm{Int}\ I_{ij}$ of period $m_{ij} \geq 1$ for any i,j with $a_{ij} = 1$. Put $J^n_{ij} = I(y_{ij}, n) \backslash I(y_{ij}, n+1)$. Let C_2 be the constant satisfying $|J| \geq C_2$ for any connected component J of J^2_{ij} with $a_{ij} = 1$.

Fix a periodic point $y_* \in I_*$ of period n_0. Suppose $y_* \in J^{m_0}_{i_0 j_0}$. Without loss of generality, we can assume that y_* is the point closest to $y_{i_0 j_0}$ in $\{f^n(y_*) \mid 0 \leq n \leq n_0 - 1\}$. Let J_* be the minimal compact interval that contains y_* and a connected component of $J^{m_0+1}_{i_0 j_0}$. For $0 \leq n \leq n_0$, let J^n_* be the connected component of $f^{-n}(J_*)$ that contains $f^{n_0-n}(y_*)$. It is easy to see that $J^n_* \cap \{f^m(y_*) \mid 0 \leq m \leq n_0 - 1\} = \{f^{n_0-n}(y_*)\}$ for any $0 \leq n \leq n_0$. It implies that the intersection multiplicity of $\{J^n_*\}_{n=0}^{n_0-1}$ is at most two. Since $f^n(J_*) \subset J^{m_0-n}_{i_0 j_0} \cup J^{m_0-n+1}_{i_0 j_0}$ for any $1 \leq n \leq m_0 - 1$, the intersection multiplicity of $\{f^n(J_*)\}_{n=0}^{m_0-1}$ is also at most two. Therefore, we have

$$\begin{aligned} \mathrm{dist}(f^{n_0}, J^{n_0-m_0}_*) &\leq \mathrm{dist}(f^{n_0-m_0}, J^{n_0-m_0}_*) + dist(f^{m_0}, J_*) \\ &\leq 2C_1 + 2C_1 = 4C_1. \end{aligned}$$

It implies that

$$\begin{aligned} |(f^{n_0})'(y_*)| = |(f^{n_0})'(f^{m_0}(y_*))| &\geq \exp(-4C_1)\frac{|f^{m_0}(J_*)|}{|J^{n_0-m_0}_*|} \\ &\geq \exp(-4C_1)C_2 \cdot \delta^{-1}_{n_0}, \end{aligned}$$

and the last term tends to infinity as $n_0 \to \infty$. □

Corollary 3.5. $\mathrm{Per}_*^u(\Phi)$ *contains only finitely many orbits.*

Proof. Since any periodic point of Φ is s-regular, the set $\psi(I^s(w, R_k'))$ contains at most one periodic point for any $w \in R_k'$. It implies that there exists a one-to-one correspondence H between $\mathrm{Per}(f)$ and $\psi(\bigcup_{k=1}^{k_*} R_k') \cap \mathrm{Per}(\Phi)$. It is easy to see that $|(f^n)'(y)| = 1$ if and only if $\|D\hat{\Phi}^t|_{(E^u/T\Phi)(H(y))}\| = 1$ for any $y \in \mathrm{Per}(f)$, where n is the period of y and t is that of $H(y)$. Hence, the corollary follows from Proposition 3.4. $\square$

The next is the main results of this subsection.

Proposition 3.6. *For any given constant $\alpha > 0$ and any neighborhood U_* of $\mathrm{Per}_*^u(\Phi)$, there exists $T \geq 0$ such that any $z \in M \setminus \bigcap_{t \geq T}(U_*)$ satisfies*

$$\sup\{\|D\Phi^t|_{(E^u/T\Phi)(z)}\| \mid t \geq 0\} \geq \alpha.$$

Corollary 3.7. *If a C^2-regular $\mathbb{P}$A flow Φ on a 3-dimensional manifold satisfies $\overline{\mathrm{Per}(\Phi)} = M$ and $\mathrm{Per}_*^u(\Phi) = \mathrm{Per}_*^s(\Phi) = \emptyset$, the it is an Anosov flow.*

Proof. Proposition 3.6 implies that any $z \in M$ satisfies $\|D\hat{\Phi}^{t_z}|_{(E^u/T\Phi)(z)}\| > 2$ for some $t_z > 0$. By the compactness of M, there exists $T > 0$ such that $\|D\hat{\Phi}^T|_{(E^u/T\Phi)(z)}\| > 2$ for any $z \in M$. Similarly, we can also take $T' > 0$ such that $\|D\hat{\Phi}^{-T'}|_{(E^s/T\Phi)(z)}\| > 2$ for any $z \in M$. $\square$

In the proof of the proposition, we follow the argument in Theorem 5.1 of [15, Chapter III]. Let $P_* = \bigcup_{k=1}^{k_*}(\pi_y(\psi^{-1}(\mathrm{Per}_*^u(\Phi)) \cap R_k') \times k)$. Remark that P_* coincides the set of periodic points p_* of f with $|(f^n)(p_*)| = 1$, where n is the period of p_*.

Lemma 3.8. *There exists $N_1 \geq 1$ such that any $y \in I_* \setminus P_*$ satisfies $I(f^{n_1}(y), N_1) \cap P_* = \emptyset$ for some $n_1 \geq 0$.*

Proof. Since P_* is finite, there exists $N_1 \geq 1$ such that $I(y_*, N_1 - 1)$ contains at most one point of P_* for any $y_* \in I_*$.

Fix $y \in I_*$ such that $I(f^n(y), N_1) \cap P_* \neq \emptyset$ for any $n \geq 0$. Let y_n be the unique point of P_* in $I(f^n(y), N_1)$ for $n \geq 0$. Since both $f(y_n)$ and y_{n+1} is contained in $I(f^{n+1}(y), N_1 - 1)$, we have $y_{n+1} = f(y_n)$. In particular, $f^n(y_0) \in I(f^n(y), N_1)$ for any $n \geq 1$. Since $\bigcap_{n \geq 0} f^{-n}(I(y, N_1)) = \{y\}$, we have $y = y_0 \in P_*$. $\square$

Lemma 3.9. *There exist $N_2 \geq 1$ and $C_2 >$ such that*

$$|(f^n)'(y)| \geq \exp(-C_2)\frac{|I(f^n(y), N_2)|}{|I(y, N_2 + n)|}$$

for any $y \in I_*$ and $n \geq 1$ with $I(f^n(y), N_2) \cap P_* = \emptyset$.

Proof. By Proposition 3.4, there exists a sequence $(K_n)_{n\geq 1}$ such that K_n tends to infinity as $n\to\infty$, and $|(f^n)'(y)| \geq K_n$ for any periodic point y of period n. Take $n_1 \geq 1$ so that $K_n \geq 2\exp(C_1)$ for any $n \geq n_1$. We also take $\lambda_0 < 1$ and $N_2 \geq 1$ so that $|(f^n)'(y)| \geq \lambda_0$ for any $n \leq n_1$, $y_0 \in \mathrm{Fix}(f^n)\backslash P_*$, and $y \in I(y, N_2)$.

We claim that there exists $\lambda_0 > 1$ such that if $y_0 \in I_*$ satisfies $f^n(y_0) \in I(y_0, N_2)$ and $I(y_0, N_2) \cap P_* = \emptyset$, then $|(f^n)'(y)| \geq \lambda_0$ for any $y \in I(y_0, N_2 + n)$. We can assume that $f^m(y_0) \notin I(y_0, N_2)$ for $1 \leq m \leq n-1$ without loss of generality. Then, the intersection multiplicity of $\{f^m(I(y_0, N_2 + n - m))\}_{m=0}^{n-1}$ is one and there exists a periodic point $y_* \in I(y_0, N_2 + n)\backslash P_*$ of period n. If $n \geq n_1$, then we have $|(f^n)'(y)| \geq \exp(-C_1)|(f^n)'(y_*)| \geq 2$. If $n \leq n_1$, then it is clear that $|(f^n)'(y)| \geq \lambda_0$.

We say an interval $I \subset I_*$ is (λ, n)-*compatible* when

- $\mathrm{Int}\, f^i(I) \cap \mathrm{Int}\, f^j(I) = \emptyset$ or $f^i(I) \subset f^j(I)$ for any $0 \leq i < j \leq n$, and
- if $f^i(I) \cup f^j(I) \subset f^k(I)$ for $0 \leq i < j \leq k \leq n$, then $|f^j(I)| \geq \lambda |f^i(I)|$.

By Lemma 5.7 of [15, Chapter III], we have $\sum_{m=0}^{n} |f^m(I)| \leq \lambda(\lambda-1)^{-1}(\sum_{k=1}^{k_*} |I'_k|)$ for any (λ, n)-compatible interval I. In particular, there exists $C_2 > 1$ such that $\mathrm{dist}(f^n, I) \leq C_2$ for any (λ_0, n)-compatible interval I.

We show that $I(y, N_2 + n)$ is (λ_0, n)-compatible if $y \in I_*$ and $n \geq 1$ satisfy $I(f^n(y), N_2) \cap P_* = \emptyset$. Once it is done, the proof is completed. First, it is clear that the first condition holds. Suppose integers $i < j \leq k \leq n$ satisfy $f^i(I(y, N_2 + n)) \cup f^j(I(y, N_2 + n)) \subset f^k(I(y, N_2 + n))$. Since $f^j(y) \in I(f^k(y), N_2) = I(f^i(y), N_2)$, we can apply the claim above to $y_0 = f^i(y)$, $n = j - i$ and $I(y_0, N_2) = I(f^i(y), N_2)$. It implies that $|f^j(I(y, N_2 + n))| \geq \lambda_0 |f^i(I(y, N_2 + n))|$. Therefore, $I(y, N_2 + n)$ is (λ_0, n) compatible. □

Proof of Proposition 3.6. It is enough to show that for any given $\alpha > 0$ there exists $N \geq 1$ such that $\sup\{|(f^n)'(y)| \mid n \geq 0\} \geq \alpha$ for any $y \in I_*^\infty \backslash f^{-N}(P_*)$.

Let N_1, N_2 and C_2 be the numbers obtained in Lemmas 3.8 and 3.9. Fix $N \geq N_2$ so that $|I(y, N_1)| \geq \alpha \exp(C_2)|I(y', N_1 + N)|$ for any $y, y' \in I_*$.

Take $y \in I_* \backslash f^N(P_*)$. Then, Lemma 3.8 implies $I(f^{n+N}(y), N_1) \cap P_* = \emptyset$ for some $n \geq 1$. By Lemma 3.9, we obtain

$$|(f^{n+N})'(y)| \geq \exp(-C_2)\frac{|I(f^{n+N}(y), N_1)|}{|I(y, N_1 + n + N)|} \geq \exp(-C_2)\frac{|I(f^{n+N}(y), N_1)|}{|I(y, N_1 + N)|} \geq \alpha.$$

□

4 Regular $\mathbb{P}$A flows without invariant tori

In this section, we fix a C^2-regular $\mathbb{P}$A flow and assume that $M = \overline{\mathrm{Per}(\Phi)}$ and any periodic point of Φ is s- and u-regular. The goal is to show that $\mathrm{Per}_*(\Phi)$ is empty. The author recommend that the readers should refer to [3], which provides a sketch of the proof for the case that Φ admits a global cross-section.

First of all, we fix a good parameter change of Φ and a family of coordinates associated to the flow. Remark that $\mathcal{F}^u$, $\mathcal{F}^s$ and $\mathrm{Per}_*(\Phi)$ does not depend on the parameter change of Φ.

Fix a neighborhood U_* of $\mathrm{Per}_*(\Phi)$ and a C^2-foliation $\mathcal{G}$ on U_* so that $T_zM = T_z\mathcal{G} \oplus T\Phi(z)$ for any $z \in U_*$. Recall that $\mathrm{Per}_*(\Phi)$ contains only finitely many periodic orbits by Corollary 3.5. We replace Φ by its parameter change if it is necessary, and assume that

1. all $z_* \in \mathrm{Per}_*(\Phi)$ have the same period T_*, and
2. $D\Phi^t(T_z\mathcal{G}) = T_{\Phi^t(z)}\mathcal{G}$ for any $t \geq 0$ and $z \in \bigcap_{t'=0}^{t} \Phi^{-t'}(U_*)$.

Let X be the vector field that generates Φ. For each $\rho \in \{u, s\}$, we fix a C^2 unit vector field Y^ρ so that $\{X, Y^\rho\}$ is a framing of E^ρ and $Y^\rho(z) \in T_z\mathcal{G}$ if $z \in U_*$. We replace the norm $\|\cdot\|$ on TM so that $\{X, Y^s, Y^u\}$ forms an orthonormal framing of TM. Remark that $D\Phi^t(X(z)) = X(\Phi^t(z))$, and hence, $\|D\Phi^t(X(z))\| = 1$ for any $z \in M$ and $t \in \mathbb{R}$.

Let $\{e_x(w), e_y(w), e_s(w)\}$ be the natural basis of $T_w\mathbb{R}^3$ at $w = (x, y, s) \in \mathbb{R}^3$. For $w \in [-2, 2]^3$ and $\delta > 0$, we define cones $C_x(w, \delta)$ and $C_y(w, \delta)$ in $T_w\mathbb{R}^3$ by

$$
\begin{aligned}
C_x(w,\delta) &= \{ae_s(w) + be_x(w) \mid |a| \leq \delta|b|\},\\
C_y(w,\delta) &= \{ae_s(w) + be_y(w) \mid |a| \leq \delta|b|\}.
\end{aligned}
$$

We call an embedding $\varphi_\sigma : [-2, 2]^3 \to M$ *a canonical coordinate* if

$$
\begin{aligned}
D\varphi_\sigma^{-1}(X(\varphi(w))) &\in \{ae_s(w) \mid a > 0\},\\
D\varphi_\sigma^{-1}(Y^s(\varphi(w))) &\in C_x(w, 1/4),\\
D\varphi_\sigma^{-1}(Y^u(\varphi(w))) &\in C_y(w.1/4)
\end{aligned}
$$

for any $w \in [-2, 2]^2$, We can take a finite family $\{\varphi_\sigma\}_{\sigma\in\Sigma}$ of canonical coordinates so that

1. $\bigcup_{\sigma\in\Sigma} \varphi_\sigma((-1, 1)^3) = M$,
2. the map $\psi(x, y, \sigma) = \varphi_\sigma(x, y, 0)$ is a canonical cross-section associated to Φ, and
3. if $\mathrm{Im}\ \varphi_\sigma \cap \mathrm{Per}_*(\Phi) \neq \emptyset$, then $\mathrm{Im}\ \varphi_\sigma \cap \mathrm{Per}_*(\Phi) = \varphi_\sigma(0 \times 0 \times [-2, 2])$, $\mathrm{Im}\ \varphi_\sigma \subset \bigcap_{t=-T_*}^{T_*} \Phi^{-t}(U_*)$, and $\varphi_\sigma([-2, 2]^2 \times s) \subset \mathcal{G}(\varphi_\sigma(0, 0, s))$ for any $s \in [-2, 2]$.

We put $\Sigma_* = \{\sigma \in \Sigma \mid \mathrm{Im}\ \varphi_\sigma \cap \mathrm{Per}_*(\Phi) \neq \emptyset\}$. Remark that $D\varphi_\sigma^{-1}(Y^s)$ is parallel to e_x and $D\varphi_\sigma^{-1}(Y^u)$ is parallel to e_y on $\mathrm{Im}\ \varphi_\sigma$ for $\sigma \in \Sigma_*$.

For $\rho \in \{u, s\}$ and $t \in \mathbb{R}$, we define a vector field Y_t^ρ by $Y_t^\rho(\Phi^t(z)) = D\Phi^t(Y^\rho(z))$. For $\sigma \in \Sigma$, we define functions $\pi_{\sigma,x}$, $\pi_{\sigma,y}$, and $\pi_{\sigma,s}$ on $\mathrm{Im}\ \varphi_\sigma$ by $(\pi_{\sigma,x}(z), \pi_{\sigma,y}(z), \pi_{\sigma,s}(z)) = (x, y, s)$ for $z = \varphi_\sigma(x, y, s)$.

In Subsection 4.1, we show that the curves tangent to Y_t^u satisfy a kind of uniform continuity as graphs of functions in canonical coordinates. It allows the argument in [3] to work well. In fact, in Subsection 4.2, we estimate the distortion of a holonomy map of $\mathcal{F}^u$ in two ways and the comparition of them implies $\mathrm{Per}_*(\Phi) = \emptyset$. Combined with Propositions 2.1, 2.2, and Corollary 3.7, it completes the proof of the main theorem.

4.1 Quasi-invariant vector fields

For $\rho \in \{u, s\}$, $z \in M$, and $\delta > 0$, we define a cone $C^\rho(z, \delta)$ in $T_z M$ by

$$C^\rho(z,\delta) = \{aX(z) + bY^\rho(z) \mid |a| \leq \delta|b|\}.$$

We also define functions $\hat{a}$ and $\hat{b}$ on $M \times \mathbb{R}$ by

$$Y_t^u(z) = \hat{a}(z,t)(\Phi^t(z)) + \hat{b}(z,t)Y^u(\Phi^t(z)).$$

Lemma 4.1. *$D\Phi^t(C^u(z,\delta)) \subset C^u(\Phi^t(z), \alpha\delta)$ for any $z \in M$, $t \geq 0$, $\alpha > 2|\hat{b}(z,t)|^{-1}$, and $\delta > |\hat{a}(z,t)|$.*

Proof. Proof is by elementary calculation. Since $D\Phi^t(X(z))) = X(\Phi^t(z))$, we have

$$D\Phi^t(aX(z) + bY^u(z)) = (a + b \cdot \hat{a}(z,t))X(z) + b \cdot \hat{b}(z,t)Y^u(z).$$

for $a, b \in \mathbb{R}$. If $\alpha > 2|\hat{b}(z,t)|^{-1}$, $\delta > |\hat{a}(z,t)|$, and $|a| \leq \delta|b|$, then

$$\begin{aligned} |a + b \cdot \hat{a}(z,t)| &\leq |b|(\delta + |\hat{a}(z,t)|) \\ &\leq (2^{-1}\alpha \cdot |\hat{b}(z,t)|) \cdot |b| \cdot (2\delta) = (\alpha\delta)|b \cdot \hat{b}(z,t)|. \end{aligned}$$

It implies the required inclusion. ☐

The aim of this subsection is to show the following.

Proposition 4.2. *There exists $\Delta_1 \in (0, 1/4)$ such that*

1. *If a curve $J \subset M$ is tangent to Y_t^u for $t \geq 0$, and satisfies $J \subset \mathrm{Im}\ \varphi_\sigma$ and $|\pi_{\sigma,y}(J)| \leq \Delta_1$ for $\sigma \in \Sigma$, then $|\pi_{\sigma,s}(J)| \leq 1/4$, and*
2. *if a curve $J \subset M$ is tangent to Y_{-t}^s for $t \geq 0$, and satisfies $J \subset \mathrm{Im}\ \varphi_\sigma$ and $|\pi_{\sigma,x}(J)| \leq \Delta_1$ for $\sigma \in \Sigma$, then $|\pi_{\sigma,s}(J)| \leq 1/4$.*

We prepare two lemmas to prove the proposition. The first allows us to control the expansion of cones in a small neighborhood of $\mathrm{Per}_*(\Phi)$. The second asserts the existence of the uniform lower bound of the angle between Y_t^s and X outside any given neighborhood of $\mathrm{Per}_*(\Phi)$.

For any subset V of M, we define *the escape-time function* $\tau_V^E : V \to \{0 \leq t \leq \infty\}$ by

$$\tau_V^E(z) = \inf\{t > 0 \mid \Phi^t(z) \notin V\}.$$

Lemma 4.3. *Suppose that $z_* \in \mathrm{Per}_*(\Phi)$ and a neighborhood U of $\mathcal{O}(z_*)$ are given. There exist a neighborhood $V \subset U$ of $\mathcal{O}(z_*)$ and a function T_V on $\{\alpha > 0\}$ such that if $z \in V$ satisfies $\tau_V^E(z) \geq T_V(\alpha)$ then*

$$D\Phi^{\tau_V^E(z)}(C^u(z,\delta)) \subset C^u(\Phi^{\tau_V^E(z)}(z), \alpha\delta)$$

for any $\delta > 0$.

Proof. Without loss of generality, we can assume that $z_* = \varphi_\sigma(0,0,0)$ for some $\sigma \in \Sigma_*$. Let $r : R \to R'$ be the first return of z_* and (r_x, r_y) the xy-decomposition of r. We remark that $\Phi^{T_*}(\psi(x,y,\sigma)) = \psi \circ r(x,y,\sigma)$ for any $(x,y,\sigma) \in R$, Since z_* is u- and s-regular, we have $|r_x(x)| < |x|$ and $|r_y^{-1}(y)| < |y|$ if $x, y \neq 0$. Take a subrectangle $R_0 = I_x \times I_y \times \sigma$ of R so that $(0,0,0) \in \mathrm{Int}\, R_0$ and $V = \bigcup_{0 \leq t < T_*} \Phi^{-t}(\psi(R_0))$ satisfies $V \cap \psi([-2,2]^2 \times \sigma) = \psi(R_0)$. Put $I_n = r_y^{-n}(I_y)$ for $n \geq 0$.

We claim that there exists a sequence $(K_n)_{n \geq 1}$ such that $|(r_y^n)'(y)| \geq K_n$ for any $n \geq 1$ and $y \in I_n \backslash I_{n+1}$. Put $I_n^+ = I_n \cap [0,2]$ and $I_n^- = I_n \cap [-2,0]$. Since the intersection multiplicity of $\{I_n^\rho \backslash I_{n+1}^\rho\}_{n \geq 1}$ is one for $\rho \in \{+,-\}$, there exists $C_1 > 0$ such that $\mathrm{dist}(r_y^n, I_n^\rho \backslash I_{n+1}^\rho) \geq C_1$ for any $n \geq 1$ and $\rho \in \{+,-\}$. Hence, we have

$$|(r_n^y)'(y)| \geq \exp(-C_1) \frac{|I_0^\rho \backslash I_1^\rho|}{|I_n^\rho \backslash I_{n+1}^\rho|}$$

for any $y \in I_n^\rho \backslash I_{n+1}^\rho$. Since I_n converges to $\{0\}$, the right term tends to infinity as $n \to \infty$.

Put $B_n = \psi(I_x \times (I_n \backslash I_{n+1}) \times \sigma)$. If $z \in V$ satisfies $\tau_V^E(z) < \infty$, then there exist $t_*(z) \in [0, T_*]$ and $n_*(z) \geq 0$ such that $z \in \Phi^{-t_*(z)}(B_{n_*(z)})$. Notice that $\Phi^{t_*(z)+nT_*-t}(z)$ is contained in a subset $\Phi^{-t}(B_{n_*(z)-n})$ of V for any $z \in V$, $n \leq n_*(z)+1$, and $t \in [0, T_*)$. Since $\tau_V^E(z) = 0$ for any $z \in B_0$, it implies $n_*(z)T_* + t_*(z) = \tau_V^E(z)$.

Since both Y^u and $Y_{nT_*}^u$ are parallel to $D\varphi_\sigma \circ e_y$ on $\psi([-2,2]^2 \times \sigma)$, there exists $C_2 > 1$ such that

$$\|D\Phi^{nT_*}(Y^u(z))\| \geq C_2^{-1} |(r_y^n)'(y)| \geq K_n$$

for any $z = \psi(x,y,\sigma) \in B_n$. Since $D\Phi_{nT_*} X(z) = X(\Phi^{nT_*}(z))$, it implies

$$D\Phi^{nT_*}(C^u(z,\delta)) \subset C^u(\Phi^{nT_*}(z), C_2 K_n^{-1} \delta)$$

for any $z \in B_n$ and $\delta > 0$. By Lemma 4.1, we can take $C_3 > 0$ so that $D\Phi^t(C^u(z,\delta)) \subset C^u(\Phi^t(z), C_3\delta)$ for any $z \in M$, $\delta > 0$, and $t \in [0, T_*]$. Hence, if $z \in V$ satisfies $\tau_V^E(z) < \infty$, then

$$D\Phi^{\tau_V^E(z)}(C^u(z,\delta)) \subset C^u(\Phi^{\tau_V^E(z)}(z), C_2 C_3 K_{n_*(z)}^{-1} \delta)$$

for any $\delta > 0$. Since K_n tends to infinity as $n \to \infty$ and $n_*(z) = T_*^{-1}(\tau_V^E(z) - t_*(z)) \geq T_*^{-1} \cdot \tau_V^E(z) - 1$, this completes the proof. $\square$

Lemma 4.4. *For any given neighborhood U of $\mathrm{Per}_*(\Phi)$, there exist $\delta_* > 0$ such that if $z \in M$ satisfies $\Phi^t(z) \notin U$ for $t \geq 0$, then $Y_t^u(z) \subset C^u(z, \delta_*)$.*

Proof. To prove the lemma, it is sufficient to show that there exist $\delta_*, \delta'_* > 0$ such that if $z \in M$ satisfies $\Phi^t(z) \notin U$ for $t \geq 0$, then $D\Phi^t(C^u(z, \delta'_*)) \subset C^u(\Phi^t(z, \delta_*))$.

By Lemma 4.3, there exists an open neighborhood V_0 of $\mathrm{Per}_*(\Phi)$ and a function T_{V_0} such that $\overline{V_0} \subset U$ and

$$D\Phi^{\tau^E_{V_0}}(z)(C^u(z,\delta)) \subset C^u(\Phi^{\tau^E_{V_0}(z)}(z), \alpha\delta) \tag{1}$$

for any $\alpha > 0$, $\delta > 0$, and $z \in V_0$ with $\tau^E_{V_0} \geq T_{V_0}(\alpha)$. Put $V_* = \bigcap_{t \geq 0} \Phi^{-t}(V_0)$. By Proposition 3.6, there exists $T_1 > 0$ such that $\sup\{\hat{b}(z,t) \mid t \geq 0\} > 2$ for any $z \in M \backslash \Phi^{-T_1}(V_*)$. By Lemma 4.1, we can take $\delta_1 > 0$ and $\alpha_1 > 0$ so that $D\Phi^{T_1}(C^u(z,\delta)) \subset C^u(\Phi^{T_1}(z), \alpha_1\delta_1)$ for any $z \in M$ and $\delta \geq \delta_1$.

We define a function τ_1^* on $\Phi^{-T_1}(V_0)$ by $\tau_1^*(z) = \tau^E_{V_0}(\Phi_1^T(z)) + T_1$. Put $T_2 = T_{V_0}(\alpha_1^{-1}) + T_1$ and take an open set $V_1 = \{z \in \Phi^{-T_1}(V_0) \mid \tau_1^*(z) > T_2\}$. By the inclusion (1), we have

$$D\Phi^{\tau_1^*(z)}(C^u(z,\delta)) \subset C^u(\Phi^{\tau_1^*(z)}(z), \delta) \tag{2}$$

for any $\delta \geq \delta_1$ and $z \in V_1$.

Notice that $\sup\{\hat{b}(z,t) \mid t \geq 0\} > 2$ for any $z \in M \backslash V_1$ since $\Phi^{-T}(V_*) \subset V_1$. By Lemma 4.4 and the compactness of $M \backslash V_1$, there exist $0 < \tau_1 < \tau_2$, $\delta'_* > \delta_1$, and a function $\tau_2^* : M\backslash V_1 \rightarrow [\tau_1, \tau_2]$ such that

$$D\Phi^{\tau_2^*(z)}(C^u(z,\delta)) \subset C^u(\Phi^{\tau_2^*(z)}(z), \delta) \tag{3}$$

for any $\delta \geq \delta'_*$ and $z \in M \backslash V_1$. By Lemma 4.1, we can take δ_* so that $D\Phi^t(C^u(z, \delta'_*)) \subset C^u(\Phi^t(z), \delta_*)$ for any $z \in M$ and $t \in [0, \max\{T_1, \tau_2\}]$.

Fix $t > 0$ and $z_0 \in M \backslash \Phi^{-t}(U)$. By the inclusions (2) and (3), there exists a sequence $(t_i)_{i=0}^{i_*+1}$ such that $t_0 = 0$, $t_{i_*} < t \leq t_{i_*+1}$ and each t_i satisfies

1. $\Phi^{t_i}(z_0) \in V_1$ and $t_{i+1} = t_i + \tau_1^*(\Phi^{t_i}(z_0)) \geq t_i + T_2$, or
2. $\Phi^{t_i}(z_0) \in M \backslash V_1$ and $t_{i+1} = t_i + \tau_2^*(\Phi^{t_i}(z_0)) \geq t_i + \tau_1$.

Remark that $D\Phi^{t_{i*}}(C^u(z, \delta'_*)) \subset C^u(\Phi^{t_{i*}}(z_0), \delta'_*)$ by the inclusions (2) and (3).

We see $t - t_{i_*} \leq \max\{T_1, \tau_2\}$. In fact, if $\Phi^{t_{i*}}(z_0) \in V_1$, then $\Phi^{t'}(z_0) \in V_0$ for $t' \in [t_{i_*} + T_1, t_{i_*+1}]$. Since $\Phi^t(z_0) \notin U$, it implies $t - t_{i_*} \leq T_1$. If $\Phi^{t_{i*-1}}(z_0) \in V_1$, then $t - \tau_{i_*} \leq \tau_2^*(\Phi^{t_{i*}}(z_0)) \leq \tau_2$. Therefore, we obtain $D\Phi^t(C^u(z, \delta'_*)) \subset C^u(\Phi^t(z_0), \delta_*)$ by the choice of δ_*. □

Proof of Proposition 4.2. We only prove the former assertion. The latter is given by applying the former to the flow Φ^{-1}.

Take $\delta_0 > 0$ so that

$$D\varphi_\sigma^{-1}(Y_t^u(\varphi_\sigma(w))) \subset C_y(w, \delta_0/4) \tag{4}$$

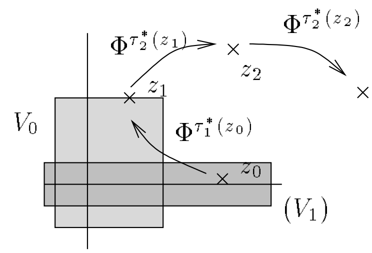

Figure 5: Proof of Lemma 4.4

for any $w \in [-2,2]^3$, $\sigma \in \Sigma$, and $t \in [0, T_*]$.

For $\sigma \in \Sigma_*$, let $r_\sigma : R_\sigma \to R'_\sigma$ be the first return of $(0,0,\sigma)$ and $(r_{\sigma,x}, r_{\sigma,y})$ the xy-decomposition of r_σ. Remark that

$$\Phi^{T_*}(\varphi_\sigma(x,y,s)) = \varphi_\sigma(r_{\sigma,x}(x), r_{\sigma,y}(y), s)$$

for any $(x,y,\sigma) \in R$ and $s \in [-2,2]$, and $|r_{\sigma,x}(x)| < |x|$ and $|r_{\sigma,y}^{-1}(y)| < |y|$ if $x, y \neq 0$. Fix intervals I_x and I_y so that $(0,0) \in \mathrm{Int}\ I_x \times I_y$ and $I_x \times I_y \times \sigma \subset R_\sigma$ for any $\sigma \in \Sigma_*$.

Take a neighborhood U_1 of $\mathrm{Per}_*(\Phi)$ so that $U_1 \cap \mathrm{Im}\ \varphi_\sigma = \emptyset$ for $\sigma \in \Sigma \backslash \Sigma_*$ and $U_1 \cap \mathrm{Im}\ \varphi_{\sigma_*} \subset \varphi_{\sigma_*}(r_{\sigma,x}(I_x) \times I_y \times [-2,2])$ for $\sigma_* \in \Sigma_*$. By Lemma 4.4, there exists $\delta_1 > 0$ such that if $w \in [-2,2]^3$ and $\sigma \in \Sigma$ satisfy $\varphi_\sigma(w) \notin U_1$ then

$$D\varphi_\sigma^{-1}(Y^u(\varphi_\sigma(w))) \subset C_y(w, \delta_1/4) \tag{5}$$

for any $t \geq 0$.

Fix $\epsilon_* \in (0, \min\{\delta_0^{-1}, \delta_1^{-1}\}/4)$ so that $[-2\epsilon_*, 2\epsilon_*] \subset I_y$ and put $B_\sigma^n = \varphi_\sigma(r_{\sigma,x}^n(I_x) \times [-2\epsilon_*, 2\epsilon_*] \times [-2,2])$ for $\sigma \in \Sigma_*$ and $n \geq 0$. Take a neighborhood $U_2 \subset U_1$ of $\mathrm{Per}_*(\Phi)$ so that $U_2 \cap \mathrm{Im}\ \varphi_\sigma \subset B_\sigma^1$ for any $\sigma \in \Sigma_*$. By Lemma 4.4 again, there exists $\delta_2 > 0$ such that if $w \in [-2,2]^3$ and $\sigma \in \Sigma$ satisfy $\varphi_\sigma(w) \notin U_2$, then

$$D\varphi_\sigma(Y_t^u(\varphi_\sigma(w))) \subset C_y(w, \delta_2/4) \tag{6}$$

for any $t \geq 0$. Put $\Delta_1 = \min\{\delta_2^{-1}, \epsilon_*\}$.

Suppose a curve $J \subset M$ is tangent to Y_t^u for $t \geq 0$, and satisfies $J \subset \mathrm{Im}\ \varphi_\sigma$ and $|\pi_{\sigma,y}(J)| \leq \Delta_1$ for $\sigma \in \Sigma$. Then, there exists a function h_J such that $\varphi_\sigma^{-1}(J) = \{(x_0, y, h_J(y)) \mid y \in \pi_{\sigma,y}(J)\}$, where $\pi_{\sigma,x}(J) = \{x_0\}$.

If $J \cap U_2 = \emptyset$, then the inclusion (6) implies that $|\pi_{\sigma,s}(J)| \leq (\delta_2/4)|\pi_{\sigma,y}(J)| \leq 1/4$.

Suppose $J \cap U_2 \neq \emptyset$. Then, σ is an element of Σ_* and $J \subset B_\sigma^n \backslash B_\sigma^{n+1}$ for some $n \geq 1$. It implies

$$\Phi^{-mT_*}(J) = \varphi_\sigma(\{(r_{\sigma,x}^{-m}(x_0), r_{\sigma,y}^{-m}(y), h_J(y)) \mid y \in \pi_{\sigma,y}(J)\})$$

for any $0 \leq m \leq n$. In particular, we have $J \subset B_\sigma^{n-m} \backslash B_\sigma^{n-m+1}$, and

$$|\pi_{\sigma,s}(\Phi^{-mT_*}(J))| = |\pi_{\sigma,s}(J)|,\ |\pi_{\sigma,y}(\Phi^{-mT_*}(J))| \leq |\pi_{\sigma,s}(J)| \leq 4\epsilon_*.$$

Take $N \geq 1$ so that $NT_* \leq t \leq (N+1)T_*$. If $n \geq N$, then $\Phi^{-NT_*}(J)$ is tangent to Y^u_{t-NT*} and is contained in $B^0_\sigma \subset \mathrm{Im}\ \varphi_\sigma$. Hence, we have

$$\begin{aligned}
|\pi_{\sigma,s}(J)| &= |\pi_{\sigma,s}(\Phi^{-NT_*}(J))| \\
&\leq (\delta_0/4)|\pi_{\sigma,y}(\Phi^{-NT_*}(J))| \leq \delta_0\epsilon_* \leq 1/4
\end{aligned}$$

by the inclusion (4) and $\epsilon_* \leq \delta_0^{-1}/4$.

If $n < N$, then $\Phi^{-nT_*}(J) \subset B^0_\sigma \backslash B^1_\sigma$ and hence, $\Phi^{-nT_*}(J) \cap U_1 = \emptyset$. Since $\Phi^{-nT_*}(J)$ is tangent to $Y^u_{t-nT_*}$, we have

$$\begin{aligned}
|\pi_{\sigma,s}(J)| &= |\pi_{\sigma,s}(\Phi^{-nT_*}(J))| \\
&\leq (\delta_1/4)|\pi_{\sigma,y}(\Phi^{-nT_*}(J))| \leq \delta_1\epsilon_* \leq 1/4
\end{aligned}$$

by the inclusion (5) and the inequality $\epsilon_* \leq \delta_1^{-1}/4$. □

4.2 Hyperbolicity of periodic orbits

The goal is the following proposition, which completes the proof of the main theorem combining with Propositions 2.1,2.2 and Corollary 3.7.

Proposition 4.5. $\overline{\mathrm{Per}_*(\Phi)} = \emptyset$.

We need some preparation to prove the proposition. Suppose $\mathrm{Per}^s_*(\Phi) \neq \emptyset$ and fix $z_* = \varphi_{\sigma_*}(0,0,0) \in \mathrm{Per}^s_*(\Phi)$. Let $\Delta_1 \in (0, 1/4)$ be the constant obtained in Proposition 4.2. For $t \geq 0$, we define a map $H_t : [0,\Delta_1]^2 \to M$ so that $H_t(0,y) = \varphi_{\sigma_*}(0,y,0)$, $\pi_{\sigma_*,x}(H_t(x,y)) = x$, and $H_t([0,\Delta_1] \times y)$ is a curve tangent to Y^s_{-t} for any $(x,y) \in [0,\Delta_1]^2$.

Lemma 4.6. *H_t is well-defined and satisfies the followings:*

1. $H_t([0,\Delta_1] \times y) \subset \varphi_{\sigma_*}([0,\Delta_1] \times [-1/4, 1/4] \times 0)$ *for any* y.
2. $H_t(0 \times [0,\Delta_1]) = \varphi_{\sigma_*}(0 \times [0,\Delta_1] \times 0)$ *is tangent to* Y^u_0.
3. $H_{nT_*}(x,0) = \varphi_{\sigma_*}(x,0,0)$ *for any* $n \geq 0$.

Proof. The first assertion is a consequence of the choice of Δ_1 The second follows from the fact that Y^u_0 is parallel to $D\varphi_{\sigma_*} \circ e_s$ on $\mathrm{Im}\ \varphi_{\sigma_*}$.

Put $J = \varphi_{\sigma_*}([0,\Delta_1] \times 0 \times 0)$. Then, we have $\Phi^{nT_*}(J) \subset J$. Since Y^s_0 is parallel to $D\varphi_{\sigma_*} \circ e_x$ on $\mathrm{Im}\ \varphi_{\sigma_*}$, the interval $\Phi^{nT_*}(J)$ is tangent to Y^s_0. It implies the last assertion of the lemma. □

Put

$$C_\Sigma = \sup\{\|D\varphi_\sigma\|, \|D\varphi_\sigma^{-1}\| \mid \sigma \in \Sigma\}.$$

It is easy to see that

- $C_\Sigma^{-1}|J| \leq |\varphi_\sigma(J)| \leq C_\Sigma |J|$ for any interval $J \subset \mathrm{Im}\ \varphi_\sigma$, and
- if an interval $J \subset M$ satisfies $J \cap \varphi_\sigma([-3/2, 3/2]^2) \neq \emptyset$ and $|J| \leq (4C_\Sigma)^{-1}$, then $J \subset \varphi_\sigma((-2,2)^3)$.

For $t \geq 0$, $y \in [0, \Delta_1]$, and $\Delta \in (0, \Delta_1)$, we define an interval $J_t(y; \Delta)$ tangent to Y_0^s by $J_t(y; \Delta) = \Phi^t \circ H_t([0, \Delta] \times y)$.

Lemma 4.7. *There exists $\Delta_2 \in (0, \Delta_1)$ such that $|J_t(y; \Delta_2)| \leq (4C_\Sigma)^{-1}$ for any $t \geq 0$ and $y \in [0, \Delta_1]$.*

Proof. Fix a full family $\{r_k : R_k \to R_k'\}_{k=1}^{k_*}$ of returns associated to (Φ, ψ). Let τ_k be the return time of r_k. Put $T_+ = \sup\{\tau_k(w) \mid k = 1, \ldots, k_*, w \in R_k\}$ and $C_1 = \sup\{\|D\Phi^t\| \mid t \in [0, T_+]\}$. Since $D\varphi_\sigma^{-1}(Y_{-t}^s(\varphi_\sigma(w)))$ is transverse to the xs-plane, there exists $\Delta > 0$ such that if an interval J is tangent to Y_{-t}^s for $t \in [0, T_+]$ and satisfies $J \cap \varphi_\sigma([-1,1]^3) \neq \emptyset$ and $|\pi_{\sigma,x}(J)| \leq \Delta$ for $\sigma \in \Sigma$, then $|J| \leq (4C_1C_\Sigma)^{-1}$.

Suppose $(0, 0, \sigma_*) \in \mathrm{Int}\ R_{k_0}$. By Lemma 3.1, we can take $\Delta_2 > 0$ so that any fine $\mathcal{R}$-admissible sequence for $(0, y, \sigma_*)$ with $y \in [0, \Delta_1]$ is also an $(\mathcal{R}, \Delta)$-admissible sequence for $[0, \Delta_2] \times y \times k_0$.

Fix $t \geq 0$ and $y \in [0, \Delta_1]$. Take a fine $\mathcal{R}$-admissible sequence $(k(n))_{n \geq 1}$ for $(0, y, \sigma_*)$. Put $w_0 = (0, y, \sigma_*)$, $t_0 = 0$ $w_n = r_{k(n)} \circ \cdots \circ r_{k(1)}(w_0)$ and $t_n = \sum_{m=1}^n \tau_{k(m)}(w_{m-1})$ for any $n \geq 1$. Take $n_* \geq 0$ so that $t_{n_*} \leq t < t_{n_*+1}$. It is easy to see $\Phi^{t_{n_*}}(\varphi_{\sigma_*}(0, y, 0)) = \psi(w_{n_*})$ and $0 \leq t - t_{n_*} \leq T_+$. It implies that

$$|\pi_{\sigma',x} \circ \Phi^{t_{n_*}} \circ H_T([0, \Delta_2] \times y)| \leq \Delta,$$

where $R'_{k(n_*)} \subset [-2, 2]^2 \times \sigma'$. Since $\Phi^{t_{n_*}} \circ H_T([0, \Delta_2] \times y)$ is tangent to $Y_{-t+t_{n_*}}^s$, we have

$$|\Phi^{t_{n_*}} \circ H_t([0, \Delta_2] \times y)| \leq (4C_1C_\Sigma)^{-1}.$$

It is easy to see that the lemma follows from the choice of C_1. □

Since $M = \overline{\mathrm{Per}(\Phi)}$ and $\mathrm{Per}_*(\Phi)$ contains only finitely many periodic orbits, there exists $(x_h, y_h) \in [0, \Delta_2] \times [0, \Delta_1]$ such that $z_h = H_t(x_h, y_h)$ is a point of $\mathrm{Per}(\Phi) \backslash \mathrm{Per}_*(\Phi)$. Put $J_0 = \varphi_{\sigma_*}([0, x_h] \times 0 \times 0)$ and $\gamma_t = \Phi^t \circ \varphi_{\sigma_*}(0 \times [0, y_h] \times 0)$. We define a map $h_t : J_0 \to J_t(y_h; x_h)$ by

$$h_t(\varphi_\sigma(x, 0, 0)) = \Phi^t(H_t(x, y_h)).$$

Remark that $\Phi^{-t} \circ h_t$ and $h_t \circ \Phi^{-t}$ are the holonomy maps of $\mathcal{F}^u$ between J_0 and $\Phi^{-t}(J_t(y_h; x_h))$ along γ_0, and between $\Phi^t(J_0)$ and $J_t(y_h; x_h)$ along γ_t. See figure 6.

Lemma 4.8. *There exists $T_h \geq 0$ such that $(\mathrm{dist}(h_{nT_h}, J_0))_{n \geq 0}$ is a bounded sequence.*

Proof. Suppose $H_t(x_h, y_h) = \varphi_{\sigma_*(x_h, y_h, s_h)}$. Since $z_h = \varphi_{\sigma_*}(x_h, y_h, s_h)$ is a hyperbolic periodic point, there exist maps $g : [0, x_h] \to [0, x_h]$, $\tau : [0, x_h] \to \{t > 0\}$, and a constant $\lambda \in (0, 1)$ such that $g(x_h) = x_h$, $\Phi^{\tau(x)}(\varphi_{\sigma_*}(x, y_h, s_h)) = \varphi_{\sigma_*}(g(x), y_h, s_h)$, and $0 < g'(x) < \lambda$ for any $x \in [0, x_h]$. Put $T_h = \tau(x_h)$. Then, we see that $\Phi^{nT_h}(z_h) = z_h$ and $J_{nT_h}(y_h; x_h) = \varphi_{\sigma_*}(g^n([0, x_h]) \times y_h \times s_h)$. In fact, the former is clear, and the latter follows from the fact that Y_0^s is parallel to $D\varphi_{\sigma_*}(e_x)$ on $\mathrm{Im}\ \varphi_{\sigma_*}$, and hence, $J_{nT_h}(y_h; x_h) \subset \varphi_{\sigma_*}([-2, 2] \times y_h \times s_h)$.

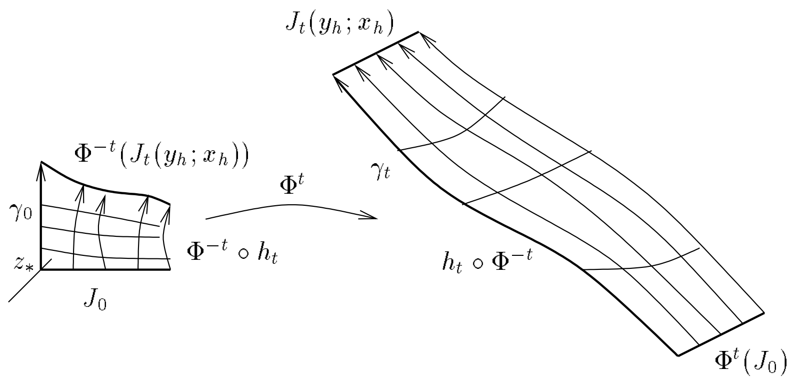

Figure 6: The holonomy maps $\Phi^{-t} \circ h_t$ and $h_t \circ \Phi^{-t}$

Put $C_h = \sup\{|D(\log|Dg(x)|)| \mid x \in [0, x_h]\}$. Then, we have

$$\begin{aligned} \operatorname{dist}(g^n, [0, x_h]) &\leq \sum_{m=0}^{n-1} \operatorname{dist}(g, g^m([0, x_h])) \\ &\leq \sum_{m=0}^{n-1} C_h |g^m([0, x_h])| \leq \sum_{m=0}^{n-1} C_h \lambda^m x_h < C_h(1-\lambda)^{-1} x_h. \end{aligned}$$

We define maps l_0 and l_1 from $[0, x_h]$ to M by $l_0(x) = \varphi_{\sigma_*}(x, 0, 0)$ and $l_1(x) = \varphi_{\sigma_*}(x, y_h, s_h)$. It is easy to see

$$h_{nT_h}(z) = \Phi^{nT_h} \circ H_{nT_1}(l_0^{-1}(x), y_h) = l_1 \circ g^n \circ l_0^{-1}(z)$$

for any $z \in J_0 = \varphi_{\sigma_*}([0, x_h] \times 0 \times 0)$. Since $(\operatorname{dist}(g^n, [0, x_h]))_{n \geq 1}$ is a bounded sequence, the sequence $(\operatorname{dist}(h_{nT_h}, J_0))_{n \geq 0}$ also is. $\square$

Lemma 4.9. *The family $\{\operatorname{dist}(h_t \circ \Phi^{-t}, \Phi^t(J_0))\}_{t \geq 0}$ is bounded.*

Proof. For $t > 0$, there exist sequences $(y_n(t))_{n=0}^{n(t)}$ in $[0, \Delta_1]$ and $(\sigma_n(t))_{n=0}^{n(t)}$ in Σ such that

- $y_0(t) = 0$, $\sigma_0(t) = \sigma_*$, $y_{n(t)}(t) = \Delta_1$,
- $y_n(t) < y_{n+1}(t)$ and $\sigma_n(t) \neq \sigma_{n+1}(t)$ for any n, and
- $\Phi^t \circ H_t(0 \times [y_n(t), y_{n+1}(t)] \times 0)$ is contained in $\varphi_{\sigma_n(t)}([-1, 1]^3)$ for any n.

Let $L_n(t)$ be the connected component of $\{y \in [0, \Delta_1] \mid \Phi^t \circ H_t(0, y) \in \varphi_{\sigma_n(t)}([-3/2, 3/2]^3)\}$ that contains $y_n(t)$. Put $I_n(t) = \Phi^t \circ H_t(0 \times L_n(t))$ and $B_n(t) = \bigcup_{y \in L_n(t)} J_t(y; x_h)$. Note that $L_n(t)$ contains $[y_n(t), y_{n+1}(t)]$ and $I_n(t)$ is the connected component of $\gamma_t \cap \varphi_{\sigma_n(t)}([-3/2, 3/2]^3)$ that contains $\Phi^t \circ$

$H_t(0, y_n(t))$. Since $|J_t(y; x_h)| \leq (4C_\Sigma)^{-1}$ and $J_t(y; x_h) \cap \varphi_{\sigma_n(t)}([-3/2, 3/2]^3) \neq \emptyset$ for any $y \in L_n(t)$, we have $B_n(t) \subset \mathrm{Im}\ \varphi_{\sigma_n(t)}$. It implies

$$\pi_{\sigma_n(t)}(B_n(t)) = \pi_{\sigma_n(t),x}(J_t(y_n(t); x_h)) \times \pi_{\sigma_n(t),y}(I_n(t))$$

where $\pi_\sigma(z) = (\pi_{\sigma,x}(z), \pi_{\sigma,y}(z))$.

Since Y_0^s is a C^2 vector field transverse to $\mathcal{F}^u$, there exists $C_1 > 0$ such that $\mathrm{dist}(\pi_{\sigma,x}, J) \leq C_1|\pi_{\sigma,x}(J)|$ and $\mathrm{dist}((\pi_{\sigma,x}|_J)^{-1}, \pi_{\sigma,x}(J)) \leq C_1|\pi_{\sigma,x}(J)|$ for any interval J which is tangent to Y_0^s and is contained in $\mathrm{Im}\ \varphi_\sigma$. It is easy to see that

$$\mathrm{dist}(h_t \circ \Phi^{-t}, \Phi^t(J_0)) \leq 2C_1 \sum_{n=0}^{n(t)} |\pi_{\sigma_n(t),x}(J_t(y_n(t); x_h))| \tag{7}$$

for any $t \geq 0$. Hence, it is sufficient to show that the latter sum is bounded by a constant.

Fix $T_1 > 0$ so that $\gamma_t \not\subset \mathrm{Im}\ \varphi_\sigma$ for any $\sigma \in \Sigma$ and $t \geq T_1$. We claim that $|\varphi_{\sigma_n(t),x}(I_n(t))| \geq \Delta_1$ for any $t \geq T_1$ and $n = 0, \ldots, n(t)$. Suppose $|\pi_{\sigma_n(t),x}(I_n(t))| < \Delta_1$. Since $I_n(t)$ is tangent to Y_t^u and $I_n(t) \cap \varphi_{\sigma_n(t)}([-1,1]^3)$ contains $\Phi^t \circ H_t(0, y_n(t))$, Proposition 4.2 implies that $I_n(t) \subset \varphi_{\sigma_n(t)}([-5/4, 5/4]^3)$. It contradicts that $I_n(t)$ is a connected component of $\gamma_t \cap \varphi_{\sigma_n(t)}([-3/2, 3/2]^3)$ and $\gamma_t \not\subset \mathrm{Im}\ \varphi_{\sigma_n(t)}$.

The claim implies

$$|\pi_{\sigma_n(t),x}(J_t(y_n(t); x_h))| \leq \Delta_1 \cdot \mathrm{Area}(\pi_{\sigma_n(t)}(B_n(t))), \tag{8}$$

where Area is the Lebesgue measure on $\mathbb{R}^2$. Hence, it is sufficient to show that the intersection multiplicity of $\{B_n(t) \mid \sigma_n(t) = \sigma\}$ is less than $8C_\Sigma^2$ for any $t \geq T_1$ and $\sigma \in \Sigma$. In fact, the inequality (8) implies that the sum $\sum_{n=0}^{n(t)} |\pi_{\sigma_n(t),x}(J_t(y_n(t); x_h))|$ is bounded by $(8C_\Sigma^2) \cdot (4\#\Sigma)$. The proof is completed by the inequality (7).

Suppose that intersection multiplicity of $\{B_n(t) \mid \sigma_n(t) = \sigma\}$ is at least $8C_\Sigma^2$ for $t_0 \geq T_1$ and $\sigma_0 \in \Sigma$. Then, there exist $(x_0, y_0) \in [-2,2]^2$ and $s_1, s_2 \in [-2,2]$ such that $0 < s_2 - s_1 < (4C_\Sigma^2)^{-1}$ and $\varphi_{\sigma_0}(x_0, y_0, s_i) \in \Phi^t(\mathrm{Im}\ H_{t_0})$ for $i = 1, 2$. Put $L = \varphi_{\sigma_0}(x_0 \times y_0 \times [s_1, s_2])$. Since L is tangent to X and $\|D\Phi^t(X(z))\| = \|X(\Phi^t(z))\| = 1$ for any z, we have $|\Phi^{-t_0}(L)| = |L| \leq (4C_\Sigma)^{-1}$. Since $\partial\Phi^{-t_0}(L)$ is contained in a subset $\mathrm{Im}\ H_{t_0}$ of $\varphi_{\sigma_*}([-1,1]^3)$, we have $\Phi^{-t_0}(L) \subset \mathrm{Im}\ \varphi_{\sigma_*}$. In particular, $\varphi_{\sigma_*}^{-1} \circ \Phi^{-t_0}(L)$ is an interval parallel to the s-axis. Such an interval intersects with $\mathrm{Im}\ H_{t_0}$ at most once. It contradicts $\partial\Phi^{-t_0}(L) \subset \mathrm{Im}\ H_{t_0}$. □

Proof of Proposition 4.5. Suppose $\mathrm{Per}_*^s(\Phi)$ is non-empty. Take periodic points $z_* = \varphi_{\sigma_*}(0,0,0) \in \mathrm{Per}_*(\Phi)$ and $z_h = \varphi_{\sigma_*}(x_h, y_h, s_h) \notin \mathrm{Per}_*(\Phi)$, and an interval $J_0 = \varphi_{\sigma_*}([0, x_h] \times 0 \times 0)$ tangent to Y_0^s as above. Notice that $\Phi^{T_*}(J_0) \subset J_0$ and $\bigcap_{n \geq 0} \Phi^{nT_*}(J_0) = \{z_*\}$. Since $\|D(\Phi^{T_*}|_{J_0})(z_*)\| = \|D\hat{\Phi}^{T_*}|_{(E^s/T\Phi)(z_*)}\| = 1$, we obtain $\mathrm{dist}(\Phi^t|_{J_0}, J_0)$ tends to infinity as $t \to \infty$. However, it contradicts Lemmas 4.8 and 4.9. Therefore, $\mathrm{Per}_*^s(\Phi)$ is empty. Applying it to the flow Φ^{-1}, we obtain that $\mathrm{Per}_*^u(\Phi)$ also is. □

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