The secondary Novikov-Shubin invariants of groups and quasi-isometry

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Abstract

We define new L^2-invariants which we call the secondary Novikov-Shubin invariants. We calculate the first secondary Novikov-Shubin invariants of finitely generated groups by using random walk on Cayley graphs and see in particular that these are invariant under quasi-isometry.

1 Introduction

In this paper we study the secondary Novikov-Shubin invariants. These new L^2-invariants are naturally defined by a modifying the original definition of the Novikov-Shubin invariants (Section 2). By using the secondary Novikov-Shubin invariants, we can study density functions whose Novikov-Shubin invariants are infinite. It is known that the first Novikov-Shubin invariants of finitely generated groups classify infinite virtually nilpotent groups ([5, Lemma 2.46.]). By using the first secondary Novikov-Shubin invariants, we would like to study finitely generated groups which are not virtually nilpotent. We prove the following in Section 4.

Theorem 1.1.

Let G be an infinite amenable finitely generated group and $0 < a < 1$. Then,
(i) $\beta_1(G) = 0 \Leftrightarrow p(n) \not\preceq \exp(-n^b)$ $(0 < \forall b < 1)$,
(ii) $\beta_1(G) = \frac{2a}{1-a} \Leftrightarrow p(n) \preceq \exp(-n^b)$ $(0 < \forall b < a)$ *and* $p(n) \not\preceq \exp(-n^b)$ $(a < \forall b < 1)$,
(iii) $\beta_1(G) = \infty \Leftrightarrow p(n) \preceq \exp(-n^b)$ $(0 < \forall b < 1)$.

In particular the first secondary Novikov-Shubin invariants of finitely generated groups are invariant under quasi-isometry.

Here $\beta_1(G)$ is the first secondary Novikov-Shubin invariant of G (Section 3) and $p(n)$ is the asymptotic equivalence class of the probability of return after n steps for random walk on Cayley graph of G (Section 4).

Example 1.2.

If U is a non-trivial finite group and $d = 1, 2, \ldots$, then the asymptotic equivalence class of $U \wr \mathbb{Z}^d$ is $\exp(-n^{d/d+2})$ ([8, Theorem 3.5.]). *Thus*

$$\beta_1(U \wr \mathbb{Z}^d) = d.$$

In particular any positive integer can occur as the first secondary Novikov-Shubin invariants of finitely generated groups. In the case where $d = 1$, we know the spectral density function of $U \wr \mathbb{Z}$ ([3, Corollary 3.], [2, Theorem 5.], [1, Theorem 1.1.]). *Hence we can also get*

$$\beta_1(U \wr \mathbb{Z}) = 1$$

by a direct calculation.

Remark 1.3.

The asymptotic equivalence class of $\mathbb{Z} \wr \mathbb{Z}$ is $\exp(-n^{1/3}(\ln(n))^{2/3})$ ([8, Theorem 3.11.]). *Thus*

$$\beta_1(\mathbb{Z} \wr \mathbb{Z}) = 1.$$

Though $\exp(-n^{1/3}(\ln(n))^{2/3})$ *and* $\exp(-n^{1/3})$ *are not asymptotically equivalent, their first secondary Novikov-Shubin invariants are equal.*

Gromov indicates that the Novikov-Shubin invariants of a certain class of groups may be invariant under quasi-isometry ([4, p.241]). Naturally we can formulate the following conjecture.

Conjecture 1.4.

The secondary Novikov-Shubin invariants of groups of finite type are invariant under quasi-isometry.

The author does not know whether these conjectures are true. The Novikov-Shubin invariants of amenable groups are studied by Roman Sauer ([9]).

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2 The secondary Novikov-Shubin invariants of density functions

Definition 2.1.

We say that a function $F : [0, \infty) \to [0, \infty]$ is a density function if F is monotone non-decreasing and right-continuous. If F and F' are two density functions, we write $F \preceq F'$ if there exist $C > 0$ and $\epsilon > 0$ such that $F(\lambda) \le F'(C\lambda)$ holds for all $\lambda \in [0, \epsilon]$. We say that F and F' are dilatationally equivalent (in signs $F \simeq F'$) if $F \preceq F'$ and $F' \preceq F$. We say that F is Fredholm if there exists $\lambda > 0$ such that $F(\lambda) < \infty$, in which case we write $F^{\perp}(\lambda) := F(\lambda) - F(0)$. If F and F' are two Fredholm density functions and $F^{\perp} \simeq F'^{\perp}$, we say that F and F' are dilatationally equivalent up to L^2-Betti numbers.

If F and F' are two Fredholm density functions which are dilatationally equivalent, surely F and F' are dilatationally equivalent up to L^2-Betti numbers.

We will recall the definition of the L^2-Betti numbers and the Novikov-Shubin invariants of density functions. For the details, we refer to [5, Chapter 1, 2.].

Definition 2.2.

Let F be a Fredholm density function. The L^2-Betti number of F is

$$b^{(2)}(F) := F(0).$$

Its Novikov-Shubin invariant is

$$\alpha(F) := \liminf_{\lambda \to 0+} \frac{\ln(F(\lambda) - F(0))}{\ln(\lambda)},$$

provided that $F(\lambda) > b^{(2)}(F)$ holds for all $\lambda > 0$. Otherwise, we put $\alpha(F) := \infty^+$.

The L^2-Betti numbers of density functions are invariant under dilatational equivalence and the Novikov-Shubin invariants of density functions are invariant under dilatational equivalence up to L^2-Betti numbers ([5, Chapter 2.]).

Definition 2.3.

Let F be a Fredholm density function. The secondary Novikov-Shubin invariant of F is

$$\beta(F) := \liminf_{\lambda \to 0+} \frac{-\ln(-\ln(F(\lambda) - F(0)))}{\ln(\lambda)},$$

provided that $F(\lambda) > b^{(2)}(F)$ holds for all $\lambda > 0$. Otherwise, we put $\beta(F) := \infty^+$.

Lemma 2.4.

Let F and F' be two Fredholm density functions. Then,

$$F^\perp \preceq F'^\perp \Rightarrow \beta(F) \geq \beta(F').$$

In particular the secondary Novikov-Shubin invariants of density functions are invariant under dilatational equivalence up to L^2-Betti numbers.

Proof. Since $F^\perp \preceq F'^\perp$, there exist $C > 0$ and $\epsilon > 0$ such that $F^\perp(\lambda) \leq F'^\perp(C\lambda)$ holds for all $\lambda \in [0, \epsilon]$. Hence,

$$\frac{\ln(-\ln(F^\perp(\lambda)))}{-\ln(\lambda)} \geq \frac{\ln(-\ln(F'^\perp(C\lambda)))}{-\ln(\lambda)} = \frac{\ln(-\ln(F'^\perp(C\lambda)))}{-\ln(C\lambda)} \cdot \frac{-\ln(C\lambda)}{-\ln(C\lambda) + \ln(C)}.$$

Since

$$\frac{-\ln(C\lambda)}{-\ln(C\lambda) + \ln(C)} \to 1 \ (\lambda \to 0+),$$

we have

$$\beta(F) \geq \beta(F').$$

□

We find the following relationship between the Novikov-Shubin invariants and the secondary Novikov-Shubin invariants.

Lemma 2.5.

Let F be a Fredholm density function. Then,
(i) $\alpha(F) = \infty^+ \Leftrightarrow \beta(F) = \infty^+$
(ii) $\alpha(F) < \infty \Rightarrow \beta(F) = 0$

Proof. (i) is clear by definition. We will prove (ii), that is,

$$\beta(F) > 0 \text{ and } \beta(F) \neq \infty^+ \Rightarrow \alpha(F) = \infty.$$

Since

$$\beta(F) := \liminf_{\lambda \to 0+} \frac{\ln(-\ln(F^\perp(\lambda)))}{-\ln(\lambda)},$$

for $\forall \epsilon > 0$, $1 > \exists \lambda_0 > 0$ such that

$$\beta(F) - \epsilon \leq \inf_{\lambda \in (0, \lambda_0]} \frac{\ln(-\ln(F^\perp(\lambda)))}{-\ln(\lambda)}.$$

Hence for $\forall \lambda \in (0, \lambda_0]$,

$$\beta(F) - \epsilon \leq \frac{\ln(-\ln(F^\perp(\lambda)))}{-\ln(\lambda)}.$$

When we take

$$\epsilon = \frac{1}{2}\beta(F),$$

then for $\forall \lambda \in (0, \lambda_0]$,

$$\frac{1}{2}\beta(F)(-\ln(\lambda)) \leq \ln(-\ln(F^\perp(\lambda))).$$

Thus

$$\frac{\exp(\frac{1}{2}\beta(F)(-\ln(\lambda)))}{-\ln(\lambda)} \leq \frac{-\ln(F^\perp(\lambda))}{-\ln(\lambda)}.$$

Since

$$\frac{\exp(\frac{1}{2}\beta(F)(-\ln(\lambda)))}{-\ln(\lambda)} \to \infty \ (\lambda \to 0+),$$

we have

$$\alpha(F) = \infty.$$

□

We see that any possible value can occur as the secondary Novikov-Shubin invariant of a density function.

Example 2.6. *Let us define density functions F_s for $s \in [0, \infty] \sqcup \{\infty^+\}$ by $F_s(0) = 0$ and for $\lambda > 0$ by*

$$\begin{aligned}
F_0(\lambda) &= \lambda, \\
F_s(\lambda) &= \exp\left(-\frac{1}{\lambda^s}\right), \\
F_\infty(\lambda) &= \exp\left(-\exp\left(\frac{1}{\lambda}\right)\right), \\
F_{\infty^+}(\lambda) &= 0.
\end{aligned}$$

Then we can check for $s \in [0,\infty] \sqcup \{\infty^+\}$,

$$\beta(F_s) = s.$$

3 The secondary Novikov-Shubin invariants of groups

Definition 3.1.

Let X be a free G-CW-complex of finite type. We define its cellular p-th spectral density function and its cellular p-th secondary Novikov-Shubin invariant by the the cellular L^2-chain complex $C_*^{(2)}(X)$ of X as follows:

$$\begin{aligned} F_p(X) &:= F(c_p|_{\mathrm{im}(c_{p+1})^\perp} : \mathrm{im}(c_{p+1})^\perp \to C_{p-1}^{(2)}(X)) \\ &:= \mathrm{tr}_{\mathcal{N}(G)}\, E_{\lambda^2}^{c_p^* c_p|_{\mathrm{im}(c_{p+1})^\perp}}, \\ \beta_p(X) &:= \beta(F_p(X)), \end{aligned}$$

where $\mathcal{N}(G)$ is the group von Neumann algebra of G.

The dilatational equivalence class of $F_p(X)$ is invariant under G-homotopy equivalence ([5, Theorem 2.55. (1)]). Hence $\beta_p(X)$ is also.

Remark 3.2. *In the case when X is a cocompact free proper G-manifold without boundary and with G-Riemannian metric, by using L^2-de Rham complex, we can define its analytic spectral density function and its analytic secondary Novikov-Shubin invariant. However when we regard X as a free G-CW-complex of finite type, its cellular spectral density function and its analytic one are dilatationally equivalent* ([5, Theorem 2.68.]). *Hence its cellular secondary Novikov-Shubin invariant is the same as its analytic one.*

Definition 3.3.

Let n be a non-negative integer or $n = \infty$. Define $\mathcal{F}_n$ to be the class of groups for which BG are CW-complexes which have a finite number of p-dimensional cells for $p \leq n$, where BG are the classifying spaces of G.

Example 3.4.

$$\begin{aligned} G \in \mathcal{F}_0 &\Leftrightarrow G : a\ discrete\ group, \\ G \in \mathcal{F}_1 &\Leftrightarrow G : a\ finitely\ generated\ group, \\ G \in \mathcal{F}_2 &\Leftrightarrow G : a\ finitely\ presented\ group, \\ G \in \mathcal{F}_\infty &\Leftrightarrow G : a\ group\ of\ finite\ type. \end{aligned}$$

Definition 3.5.

Let $G \in \mathcal{F}_n$. Then for $1 \leq p \leq n$,

$$\begin{aligned} F_p(G) &:= F_p(EG), \\ b_p^{(2)}(G) &:= b_p^{(2)}(EG), \\ \alpha_p(G) &:= \alpha_p(EG), \\ \beta_p(G) &:= \beta_p(EG), \end{aligned}$$

where EG is the classifying space for free proper G-actions.

4 The first secondary Novikov-Shubin invariants of groups

Let G be a finitely generated group and S be a finite set of generaters of G. The Cayley graph $C_S(G)$ of (G,S) is the following connected one-dimensional free G-CW-complex. Its 0-skelton is G. For each element $s \in S$ we attach a free equivalent G-cell $G \times D^1$ by the attaching map $G \times S^0 \to G$ which sends $(g,-1)$ to g and $(g,1)$ to gs. We will study the first secondary Novikov-Shubin invariant of $C_S(G)$. We can identify

$$c_S : C_1^{(2)}(C_S(G)) \to C_0^{(2)}(C_S(G))$$

with

$$\bigoplus_{s\in S} r_{s^{-1}-1} : \bigoplus_{s\in S} l^2(G) \to l^2(G),$$

where $r_h(\sum_{g\in G}\lambda_g g) := \sum_{g\in G}\lambda_g gh^{-1}$ $(h,g\in G,\ \lambda_g\in\mathbb{C})$.

The following is clear since $F_1(X)$ and $F_1(C_S(G))$ are dilatationally equivalent up to L^2-Betti numbers ([5, Lemma 2.45, Theorem 2.55. (1)]).

Lemma 4.1.

Let G be a finitely generated group and let X be a connected free G-CW-complex of finite type. Then for any finite set S of generaters of G, we have

$$\beta_1(X) = \beta_1(C_S(G)).$$

In particular $\beta_1(C_S(G))$ is independent of the choice of the finite set S of generaters and we have

$$\beta_1(G) = \beta_1(C_S(G)).$$

Moreover we will prove that $\beta_1(G)$ is invariant under quasi-isometry. We can assume that S is symmetric, i.e. $s\in S$ implies $s^{-1}\in S$ and S does not contain the unit element of G. We will recall simple random walk on $C_S(G)$. The probability distribution is

$$p : G \to [0,1],\ g \mapsto \begin{cases} |S|^{-1} & \text{if } g \in S, \\ 0 & \text{if } g \notin S. \end{cases}$$

Thus the transition probability operator is

$$P = \sum_{s\in S}\frac{1}{|S|} r_{s^{-1}} : l^2(G) \to l^2(G),$$

in particular,

$$P = id - \frac{1}{2|S|} c_S c_S^*.$$

Then for $n \in \mathbb{Z}_{\geq 0}$

$$p(n) := \operatorname{tr}_{\mathcal{N}(G)} P^n$$

is the probability of return after n steps for random walk on Cayley graph. It is well known that $p(n)$ on $n \in 2\mathbb{Z}_{\geq 0}$ is a non-increasing function.

In the following, we regard $p(n)$ as the function which is defined only on the even numbers.

Definition 4.2.

Let u, v be two positive non-increasing functions defined on the positive real axis. We write $u \preceq v$ if there exists $C \geq 1$ such that

$$\forall t > 0, \ u(t) \leq C v(t/C).$$

We say that u and v are asymptotically equivalent (in signs $u \simeq v$) if $u \preceq v$ and $v \preceq u$. When a function is defined only on the even numbers, we extend it to the positive real axis by linear interpolation. We will use the same notation for the original function and its extension.

Remark 4.3.

The asymptotic equivalence class of $p(n)$ is invariant under quasi-isometry ([7, Theorem 1.2])*.*

In particular the asymptotic equivalence class of $p(n)$ is independent of the choice of the finite symmetric set S of generaters of G.

Theorem 4.4.

Let G be a finitely generated group and $0 < b < 1$. Then,
(i) *G is non-amenable or finite $\Leftrightarrow \beta_1(G) = \infty^+$,*
(ii) *G is infinite amenable and $p(n) \preceq \exp(-n^b) \Rightarrow \beta_1(G) \geq \frac{2b}{1-b}$,*
(iii) *G is infinite amenable and $p(n) \not\preceq \exp(-n^b) \Rightarrow \beta_1(G) \leq \frac{2b}{1-b}$.*

Proof. If G is finite, then obviously $\beta_1(G) = \infty^+$. Hence we can assume that G is infinite. We have

$$F(\lambda) := \operatorname{tr}_{\mathcal{N}(G)}(\chi_{[1-\lambda,1]}(P)) = F_1^{\perp}(C_S(G))(\sqrt{2|S|\lambda}).$$

Indeed, because $\operatorname{tr}_{\mathcal{N}(G)} E_0^{c_S c_S^*} = 0$ when G is infinite ([5, Theorem 1.35. (8)]),

$$\begin{aligned} F_1^{\perp}(C_S(G))(\sqrt{2|S|\lambda}) = \quad & \operatorname{tr}_{\mathcal{N}(G)} E_{2|S|\lambda}^{c_S^* c_S} - \operatorname{tr}_{\mathcal{N}(G)} E_0^{c_S^* c_S} \\ = \operatorname{tr}_{\mathcal{N}(G)} E_{2|S|\lambda}^{c_S c_S^*} - \operatorname{tr}_{\mathcal{N}(G)} E_0^{c_S c_S^*} = \quad & \operatorname{tr}_{\mathcal{N}(G)} E_{\lambda}^{\frac{1}{2|S|} c_S c_S^*} \\ = \operatorname{tr}_{\mathcal{N}(G)}(\chi_{[0,\lambda]}(\tfrac{1}{2|S|} c_S c_S^*)) = \quad & \operatorname{tr}_{\mathcal{N}(G)}((\chi_{[0,\lambda]} \circ f)(P)) \\ = \operatorname{tr}_{\mathcal{N}(G)}(\chi_{[1-\lambda,1]}(P)), \quad & \end{aligned}$$

where $f(\mu) := 1 - \mu$. Then

$$\beta_1(G) = 2\beta(F).$$

(i) is clear since it is well known that the spectrum of P contains 1 if and only if G is amenable. For $n \in 2\mathbb{Z}_{>0}$,

$$(1-\lambda)^n(\chi_{[-1,-1+\lambda]} + \chi_{[1-\lambda,1]})(P) \leq P^n.$$

Hence

$$(1-\lambda)^n F(\lambda) \leq p(n). \tag{1}$$

Also

$$P^n \leq (1-\lambda)^n \chi_{(-1+\lambda,1-\lambda)}(P) + (\chi_{[-1,-1+\lambda]} + \chi_{[1-\lambda,1]})(P).$$

Claim 4.5.

When $-1 \in \sigma(P)$, we have

$$\mathrm{tr}_{\mathcal{N}(G)}(\chi_{[-1,-1+\lambda]}(P)) = \mathrm{tr}_{\mathcal{N}(G)}(\chi_{[1-\lambda,1]}(P)).$$

We will prove this after the proof of this theorem.
By Claim 4.5, when $-1 \in \sigma(P)$, for $\lambda \in [0,1]$

$$p(n) \leq (1-\lambda)^n + 2F(\lambda)$$

When $-1 \notin \sigma(P)$, for $\lambda \in [0, 1 + \inf \sigma(P))$,

$$p(n) \leq (1-\lambda)^n + 2F(\lambda).$$

Hence if $\lambda > 0$ is sufficiently small,

$$p(n) \leq (1-\lambda)^n + 2F(\lambda). \tag{2}$$

We will prove (ii). By (1) and $p(n) \preceq \exp(-n^b)$, for $0 < \exists C \leq 1$

$$\begin{aligned} F(\lambda) &\leq \frac{p(n)}{(1-\lambda)^n} \ (\forall n \in 2\mathbb{Z}_{>0}) \\ &\leq \frac{C^{-1}\exp(-Cn^b)}{(1-\lambda)^n} \ (\forall n : \text{sufficiently large even number}). \end{aligned}$$

Hence we have

$$CF(\lambda) \leq \frac{\exp(-Cn^b)}{(1-\lambda)^n} \ (\forall n : \text{sufficiently large even number}). \tag{3}$$

For $0 < \forall \epsilon < b$, we put

$$n_\lambda := \left[\left[\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}\right]\right],$$

where $[[v]]$ is the gratest even number not greater than v. Then

$$\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)} - 2 < n_\lambda \leq \left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}.$$

When $\lambda > 0$ is sufficiently small,

$$\frac{1}{2^{1/b}}\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)} < n_\lambda \leq \left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}. \tag{4}$$

By (3) and (4), when we sufficiently reduce $\lambda > 0$ if necessary,

$$\begin{aligned}
CF(\lambda) &\leq \frac{\exp(-Cn_\lambda^b)}{(1-\lambda)^{n_\lambda}} \\
&< \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\frac{1}{(1-\lambda)^{\left(\frac{1}{\lambda}\right)^{1/(1-b+\epsilon)}}} \\
&= \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\frac{1}{\left\{(1-\lambda)^{\frac{1}{\lambda}}\right\}^{\left(\frac{1}{\lambda}\right)^{(b-\epsilon)/(1-b+\epsilon)}}} \\
&\leq \exp\left(-\frac{C}{2}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\exp\left(2\left(\frac{1}{\lambda}\right)^{(b-\epsilon)/(1-b+\epsilon)}\right) \\
&= \exp\left\{\left(-\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\left(\frac{C}{2} - 2\left(\frac{1}{\lambda}\right)^{-\epsilon/(1-b+\epsilon)}\right)\right\} \\
&\leq \exp\left\{\frac{C}{4}\left(-\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{-\ln(-\ln CF(\lambda))}{\ln(\lambda)} &\geq \frac{-\ln\left\{\frac{C}{4}\left(\frac{1}{\lambda}\right)^{b/(1-b+\epsilon)}\right\}}{\ln(\lambda)} \\
&= \frac{\frac{b}{1-b+\epsilon}\ln\left(\frac{1}{\lambda}\right) + \ln\left(\frac{C}{4}\right)}{-\ln(\lambda)} \\
&\to \frac{b}{1-b+\epsilon} \ (\lambda \to 0+).
\end{aligned}$$

Thus for $0 < \forall\epsilon < b$,

$$\beta(F) = \beta(CF) \geq \frac{b}{1-b+\epsilon}.$$

Hence

$$\beta(F) \geq \frac{b}{1-b}.$$

Next we will prove (iii). Since $p(n) \not\lesssim \exp(-n^b)$,

$$1 \geq \forall C > 0, \forall N > 0, \exists n \geq N \ (n \in 2\mathbb{Z}_{>0}) \ s.t. \ p(n) > \frac{1}{C}\exp(-Cn^b).$$

When we fix $1 \geq C > 0$, we put $\Lambda_C := \left\{n \in 2\mathbb{Z}_{>0} | p(n) > \frac{1}{C}\exp(-Cn^b)\right\}$. By (2), for $n \in \Lambda_C$ and $\lambda > 0$ which is sufficiently small,

$$2F(\lambda) \geq p(n) - (1-\lambda)^n > \frac{1}{C}\exp(-Cn^b) - (1-\lambda)^n.$$

Hence

$$2F(\lambda) \geq \exp(-Cn^b) - (1-\lambda)^n. \tag{5}$$

For $\forall r > \frac{b}{1-b}$,

$$n_\lambda := \left[\left[\frac{1}{\lambda^{r/b}}\right]\right] + 2.$$

Then for λ which is sufficiently small,

$$\frac{1}{\lambda^{r/b}} < n_\lambda \leq \frac{1}{\lambda^{r/b}} + 2 \leq \frac{2^{1/b}}{\lambda^{r/b}}. \tag{6}$$

By (5) and (6), when we sufficiently reduce $\lambda > 0$ which satisfies $n_\lambda \in \Lambda_C$ if necessary,

$$\begin{aligned}
& 2F(\lambda) \\
\geq\ & \exp(-Cn_\lambda^b) - (1-\lambda)^{n_\lambda} \\
>\ & \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - (1-\lambda)^{\frac{1}{\lambda^{r/b}}} \\
=\ & \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - \left\{(1-\lambda)^{\frac{1}{\lambda}}\right\}^{\frac{1}{\lambda^{(r-b)/b}}} \\
\geq\ & \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right) - \exp\left(-\left(\frac{1}{\lambda}\right)^{\frac{r-b}{b}}\right) \\
=\ & \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right)\left(1 - \exp\left\{2C\left(\frac{1}{\lambda}\right)^r - \left(\frac{1}{\lambda}\right)^{(r-b)/b}\right\}\right) \\
=\ & \exp\left(-2C\left(\frac{1}{\lambda}\right)^r\right)\left(1 - \exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C - \left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right) \\
>\ & 0.
\end{aligned}$$

Hence

$$-\ln(2F(\lambda)) \leq 2C\left(\frac{1}{\lambda}\right)^r - \ln\left(1 - \exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C - \left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right). \tag{7}$$

Let $1 > \delta > 0$. Then for $\lambda > 0$ which is sufficiently small,

$$1 - \exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C - \left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right] \geq 1 - \delta > 0.$$

Since

$$-\ln\left(1 - \exp\left[\left(\frac{1}{\lambda}\right)^r\left\{2C - \left(\frac{1}{\lambda}\right)^{r((1-b)/b)-1}\right\}\right]\right) \leq -\ln(1-\delta) =: D$$

and (7), we have

$$-\ln(2F(\lambda)) \leq 2C\left(\frac{1}{\lambda}\right)^r + D.$$

Thus

$$\begin{aligned}\frac{-\ln(-\ln(F(\lambda)))}{\ln(\lambda)} &\leq \frac{\ln(2C\left(\frac{1}{\lambda}\right)^r + D + \ln(2))}{-\ln(\lambda)} \\ &\to r\ (\lambda \to 0+).\end{aligned}$$

Because $\beta(F)$ is defined by using "lim inf",

$$\beta(F) \leq r.$$

Thus we have

$$\beta(F) \leq \frac{b}{1-b}.$$

□

Now Theorem 1.1. is clear.

Finally we will prove claim 4.5.

Lemma 4.6. *Let G be a finitely generated group and S be a finite symmetric set of generaters of G where $e \notin S$. Then,*

$$\sigma(P) \ni -1 \Rightarrow \exists f : G \to S^1 : \textit{group homomorphism s.t. } f(s) = -1\ (\forall s \in S),$$

where $P := \frac{1}{|S|}\sum_{s\in S} r_{s^{-1}}$.

Proof. Since $\sigma(P) \ni -1$,

$$\exists(\xi_n)_{n\in\mathbb{N}} \subset l^2(G)\ s.t.\ \|\xi_n\| = 1, \|P\xi_n + \xi_n\| \to 0\ (n \to \infty).$$

Then,

$$\begin{aligned}|\langle P\xi_n + \xi_n, \xi_n\rangle| &\leq \|P\xi_n + \xi_n\|\|\xi_n\| \\ &\to 0\ (n \to \infty).\end{aligned}$$

Since

$$\begin{aligned}2|S|\langle P\xi_n + \xi_n, \xi_n\rangle &= 2\sum_{s\in S}\langle r_s\xi_n + \xi_n, \xi_n\rangle \\ &= \sum_{s\in S}\langle (r_s + r_{s^{-1}} + 2)\xi_n, \xi_n\rangle\end{aligned}$$

and $r_s + r_{s^{-1}} + 2$ is a positive operator, we have

$$\langle (r_s + r_{s^{-1}} + 2)\xi_n, \xi_n\rangle \to 0\ (n \to \infty).$$

Because $r_{s^{-1}}$ is unitary,

$$\begin{aligned}\langle (r_s + r_{s^{-1}} + 2)\xi_n, \xi_n\rangle &= \langle (r_s + 1)\xi_n, \xi_n\rangle + \langle (r_{s^{-1}} + 1)\xi_n, \xi_n\rangle \\ &= \langle (r_s + 1)\xi_n, \xi_n\rangle + \langle \xi_n, (r_s + 1)\xi_n\rangle \\ &= 2\,\mathrm{Re}\langle (r_s + 1)\xi_n, \xi_n\rangle \\ &= 2\,\mathrm{Re}\langle (1 + r_{s^{-1}})\xi_n, r_{s^{-1}}\xi_n\rangle.\end{aligned}$$

Thus for $\forall s \in S$,

$$\begin{aligned}\|r_s\xi_n + \xi_n\|^2 &= \operatorname{Re}\|r_s\xi_n + \xi_n\|^2 \\ &= \operatorname{Re}\langle (r_s+1)\xi_n, (r_s+1)\xi_n\rangle \\ &= \operatorname{Re}\langle (r_s+1)\xi_n, r_s\xi_n\rangle + \operatorname{Re}\langle (r_s+1)\xi_n, \xi_n\rangle \\ &\to 0\ (n \to \infty).\end{aligned}$$

Here we define for $\forall s_{i_1}, s_{i_2}, \ldots, s_{i_m} \in S$

$$f(s_{i_1}s_{i_2}\cdots s_{i_m}) := \lim_{n\to\infty} \langle r_{s_{i_1}s_{i_2}\cdots s_{i_m}}\xi_n, \xi_n\rangle.$$

This is well-defined since for $\forall s \in S$

$$\lim_{n\to\infty}\langle r_s\xi_n, \xi_n\rangle = -1$$

and for $\forall s_{i_1}, s_{i_2}, \ldots, s_{i_{m+1}} \in S$

$$\begin{aligned} & |\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, \xi_n\rangle + \langle r_{s_{i_1}s_{i_2}\cdots s_{i_{m+1}}}\xi_n, \xi_n\rangle| \\ = & |\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, \xi_n\rangle + \langle r_{s_{i_1}} r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, \xi_n\rangle| \\ = & |\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, \xi_n\rangle + \langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, r_{s_{i_1}^{-1}}\xi_n\rangle| \\ = & |\langle r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n, (1 + r_{s_{i_1}^{-1}})\xi_n\rangle| \\ \leq & \|r_{s_{i_2}\cdots s_{i_{m+1}}}\xi_n\| \|(1 + r_{s_{i_1}^{-1}})\xi_n\| \\ \to & 0\ (n\to\infty).\end{aligned}$$

□

Proof of Claim 4.5. For $\forall \xi = \sum_{g\in G}\xi_g g \in l^2(G)$, we define

$$U(\xi) := \sum_{g\in G} f(g)\xi_g g.$$

This is unitary on $l^2(G)$ and $U(e) = e$ ($e \in l^2(G)$). Moreover we have $Ur_s = -r_sU$. Indeed, by Lemma 4.6,

$$\begin{aligned} Ur_s(\xi) &= \sum_{g\in G} f(gs^{-1})\xi_g gs^{-1} \\ &= \sum_{g\in G} f(g)f(s^{-1})\xi_g gs^{-1} \\ &= -\sum_{g\in G} f(g)\xi_g gs^{-1} \\ &= -r_sU(\xi).\end{aligned}$$

Hence we have

$$UPU^{-1} = -P.$$

Since U is unitary,

$$\begin{aligned} U\chi_{[-1,-1+\lambda]}(P)U^{-1} &= UE_\lambda^P U^{-1} \\ &= E_\lambda^{UPU^{-1}} \\ &= \chi_{[-1,-1+\lambda]}(UPU^{-1}) \\ &= \chi_{[-1,-1+\lambda]}(-P) \\ &= \chi_{[1-\lambda,1]}(P). \end{aligned}$$

Thus,

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)}(\chi_{[-1,-1+\lambda]}(P)) &= \langle \chi_{[-1,-1+\lambda]}(P)e, e\rangle \\ &= \langle \chi_{[-1,-1+\lambda]}(P)U^{-1}e, U^{-1}e\rangle \\ &= \langle U\chi_{[-1,-1+\lambda]}(P)U^{-1}e, e\rangle \\ &= \langle \chi_{[1-\lambda,1]}(P)e, e\rangle \\ &= \operatorname{tr}_{\mathcal{N}(G)}(\chi_{[1-\lambda,1]}(P)). \end{aligned}$$

□

References

[1] W. Dicks; T. Schick, The spectral measure of certain elements of the complex group ring of a wreath product. Geom. Dedicata 93 (2002), 121–137.

[2] R. Grigorchuk; P. Linnell; T. Schick; A. Żuk, On a question of Atiyah. C. R. Acad. Sci. Paris Ser. I Math. 331 (2000), no. 9, 663–668.

[3] R. Grigorchuk; A. Żuk, The lamplighter group as a group generated by a 2-state automaton, and its spectrum. Geom. Dedicata 87 (2001), no. 1-3, 209–244.

[4] M. Gromov, Asymptotic invariants of infinite groups. Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[5] W. Lück, L^2-invariants: theory and applications to geometry and K-theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 44. Springer-Verlag, Berlin, 2002.

[6] S. Oguni, The zero-in-the-spectrum conjecture and finitely presented groups. Preprint, 2004.

[7] C. Pittet; L. Saloff-Coste, On the stability of the behavior of random walks on groups. J. Geom. Anal. 10 (2000), no. 4, 713–737.

[8] C. Pittet; L. Saloff-Coste, On random walks on wreath products. Ann. Probab. 30 (2002), no. 2, 948–977.

[9] R. Sauer, L^2-Invariants of Groups Discrete Measured Groupoids. Preprint, 2002.

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