SYMMETRY CHARACTERIZATION OF QUASISYMMETRIC SIEGEL DOMAINS BY CONVEXITY OF CAYLEY TRANSFORM IMAGES

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Abstract. In this paper we characterize symmetric Siegel domains among quasisymmetric Siegel domains by means of the Cayley transform introduced by Dorfmeister. We show that a quasisymmetric Siegel domain is symmetric if and only if its Cayley transform image is convex.

1. Introduction

In the previous paper [7], we characterized symmetric tube domains (or symmetric Siegel domains of type I) among homogeneous ones by convexity of Cayley transform images. The present paper is a continuation of [7] to non-tube type Siegel domains. Specifically, we treat here quasisymmetric Siegel domains. This class of Siegel domains is strictly wider than the class of symmetric ones, and one can find a complete list of classification of irreducible quasisymmetric Siegel domains in [14, p.240] and [15]. There are already some works on characterization of symmetric domains among quasisymmetric domains: a characterization by the non-positivity of the sectional curvature with respect to the Bergman metric [1], an algebraic one by means of the Jordan algebra representations associated with the domain [2, Subsection 3.7], [14, Theorem V.3.5], and one by means of the infinitesimal automorphisms of the domain [3, Theorem 3.3], [14, Proposition V.4.8].

In [3] Dorfmeister defined a Cayley transform for a quasisymmetric Siegel domain by using the Jordan algebra structure and the Jordan algebra representation attached to the domain. When the domain is symmetric, this Cayley transform can be identified with (the inverse of) the Cayley transform introduced by Korányi and Wolf in [9], where their terminology is a Lie-theoretic one.

Our main theorem states that for an irreducible quasisymmetric Siegel domain D, the Cayley transform image of D is convex if and only if D is symmetric.

We now describe the organization of this paper. In Section 2, we collect a few basic facts concerning quasisymmetric Siegel domains. The definition of our Cayley transform $\mathcal{C}$ is given and the main theorem is stated. In Section 3, the “only if” part of the main theorem is proved. Our tool is a criterion due to Dorfmeister (see Proposition 2.5). In Section 4 we verify that the Cayley transform image is convex for the symmetric case. The main task is to identify our Cayley transform with the Cayley transform defined in terms of the structure of Jordan triple system introduced in the ambient vector space of D (see [10] and [14]). Then the image is, up to a linear isomorphism, the open unit ball for a certain norm, and the required convexity follows immediately.

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2. Preliminaries

2.1. **Quasisymmetric Siegel domains.** Let V be a finite-dimensional vector space over $\mathbb{R}$. An open convex cone $\Omega \subset V$ is called a *homogeneous convex cone* if Ω contains no entire straight line (not necessarily passing through the origin) and the linear automorphism group $G(\Omega)$ of Ω defined by

$$G(\Omega) := \{g \in GL(V) \mid g\Omega = \Omega\}$$

acts transitively on Ω. Let $\Omega \subset V$ be a homogeneous convex cone. Put $W := V_{\mathbb{C}}$, the complexification of V. We denote by $w \mapsto w^*$ the complex conjugation of W with respect to V. Let U be a finite-dimensional vector space over $\mathbb{C}$. We assume that a Hermitian sesquilinear map $Q : U \times U \to W$ (complex linear in the first variable and antilinear in the second variable) is Ω-positive:

$$Q(u,u) \in \overline{\Omega} \setminus \{0\} \text{ for any } u \in U \setminus \{0\}. \tag{2.1}$$

The *Siegel domain* D defined by these data is

$$D := \{(u,w) \in U \times W \mid \operatorname{Re} w - \tfrac{1}{2}Q(u,u) \in \Omega\}. \tag{2.2}$$

Let κ be the Bergman kernel of D, that is, the reproducing kernel of the Bergman space of D. By [6, Lemma 5.1] we know that κ is expressed in the following form by means of a strictly positive C^∞ function η on Ω which has a holomorphic extension to the tube domain $\Omega + iV$:

$$\kappa(z_1, z_2) = \eta\left(w_1 + w_2^* - Q(u_1, u_2)\right) \qquad (z_j = (u_j, w_j) \in D).$$

Moreover, by [6, §1] η is homogeneous: there exists an integer k such that

$$\eta(\lambda x) = \lambda^k \eta(x) \qquad (\lambda > 0,\ x \in \Omega). \tag{2.3}$$

Let us take any $E \in \Omega$ and fix it. We know by [4, §2] that the bilinear form

$$\langle x|y\rangle_\eta := D_x D_y \log \eta(E) \qquad (x, y \in V)$$

defines a positive definite inner product on V, where $D_v f(u) := \frac{d}{dt} f(u+tv)\big|_{t=0}$ $(u, v \in V)$ for smooth functions f on V. We extend $\langle\cdot|\cdot\rangle_\eta$ to W by complex bilinearity and denote it by the same symbol. We define a sesquilinear form $(\cdot|\cdot)_U$ on U by

$$(u_1|u_2)_U := \langle Q(u_1,u_2)|E\rangle_\eta \qquad (u_1, u_2 \in U). \tag{2.4}$$

Then $(\cdot|\cdot)_U$ is a positive definite Hermitian inner product on U. For every $w \in W$, we define a complex linear operator $\varphi(w)$ on U by

$$(\varphi(w)u_1|u_2)_U = \langle Q(u_1,u_2)|w\rangle_\eta \qquad (w \in W,\ u_1, u_2 \in U). \tag{2.5}$$

Evidently the assignment $w \mapsto \varphi(w)$ is also complex linear. By definition we have $\varphi(E) = \mathrm{Id}$. We introduce a product on V by

$$\langle xy|z\rangle_\eta = -\tfrac{1}{2} D_x D_y D_z \log \eta(E) \qquad (x, y, z \in V). \tag{2.6}$$

Since η is a C^∞ function, this product is commutative. Moreover we see by (2.3) that E is a unit element.

From now on we assume that D is *quasisymmetric*, that is, we assume that the vector space V with the product xy defined by (2.6) is a Jordan algebra. This means that, in addition to the commutativity of the product, we have $x(x^2 y) = x^2(xy)$ for all $x, y \in V$. Furthermore, this Jordan algebra is Euclidean in the sense of [5]. Indeed, $\langle\cdot|\cdot\rangle_\eta$ is an associative inner product: every Jordan multiplication operator is self-adjoint relative to $\langle\cdot|\cdot\rangle_\eta$. Moreover by [3, Theorem 2.1] Ω is self-dual with respect to $\langle\cdot|\cdot\rangle_\eta$. In this paper we suppose further that D is irreducible, so that Ω is also irreducible (see [8, Theorem 6.3]) and V is simple. We have the following proposition due to Dorfmeister [3, Theorem 2.1 (6)] (see also [11, Proposition 4.5]), where we note that W is a complex Jordan algebra in a natural way.

Proposition 2.1. *The linear map $\varphi : w \mapsto \varphi(w)$ is a $*$-representation of the Jordan algebra W:*

$$\varphi(w^*) = \varphi(w)^* \qquad (w \in W), \tag{2.7}$$

$$\varphi(w_1 w_2) = \tfrac{1}{2}\left(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)\right) \qquad (w_1, w_2 \in W), \tag{2.8}$$

where, if A is a complex linear operator on U, then A^* stands for the adjoint operator of A with respect to $(\cdot|\cdot)_U$.

Let $E_1, \dots, E_r$ be a Jordan frame of V, that is, a complete system of orthogonal primitive idempotents. Clearly, these elements also form a Jordan frame of W. Let us put $U_k := \varphi(E_k)U$ $(k = 1, \dots, r)$, and recall the Hermitian inner product $(\cdot|\cdot)_U$ defined by (2.4).

Lemma 2.2. *The operators $\varphi(E_k)$ $(k = 1, \dots, r)$ are orthogonal projections onto U_k, and we have an orthogonal direct sum $U = U_1 \oplus \cdots \oplus U_r$.*

Proof. It is evident from (2.7) and (2.8) that the operators $\varphi(E_k)$ are mutually orthogonal self-adjoint idempotents with $\varphi(E_1) + \cdots + \varphi(E_r) = \mathrm{Id}$. $\square$

The Euclidean Jordan algebra V has the *Peirce decomposition*: if we put

$$\begin{aligned} V(a, \tfrac{1}{2}) &:= \{v \in V \mid av = \tfrac{1}{2}v\} \qquad (a \in V),\\ V_{ii} &:= \mathbb{R}E_i \qquad (i = 1, \dots, r),\\ V_{kj} &:= V(E_k, \tfrac{1}{2}) \cap V(E_j, \tfrac{1}{2}) \qquad (1 \le j < k \le r), \end{aligned}$$

then V decomposes into the following orthogonal direct sum:

$$V = \bigoplus_{1 \le j \le k \le r} V_{kj}.$$

We put $W_{kj} := (V_{kj})_{\mathbb{C}}$ $(1 \le j \le k \le r)$.

Lemma 2.3. *For $w_{kj} \in W_{kj}$ $(k > j)$, one has*
(1) $\varphi(w_{kj})U_j \subset U_k$.
(2) $\varphi(w_{kj})U_k \subset U_j$.
(3) $\varphi(w_{kj})U_l = 0$ *if* $l \neq j, k$.

Proof. We have $E_k w_{kj} = \frac{1}{2} w_{kj}$. Hence

$$\varphi(w_{kj}) = \varphi(E_k)\varphi(w_{kj}) + \varphi(w_{kj})\varphi(E_k). \tag{2.9}$$

Applying (2.9) to $u_j \in U_j$, we get $\varphi(w_{kj})u_j = \varphi(E_k)\varphi(w_{kj})u_j$, so that we have $\varphi(w_{kj})u_j \in U_k$, and the statement (1) holds. By a similar argument, we get (2). To prove (3), let $l \neq j, k$. Application of (2.9) to the vector $u_l \in U_l$ yields $\varphi(w_{kj})u_l = \varphi(E_k)\varphi(w_{kj})u_l$. Similarly we have $\varphi(w_{kj})u_l = \varphi(E_j)\varphi(w_{kj})u_l$. Hence $\varphi(w_{kj})u_l \in U_j \cap U_k = \{0\}$. □

Let $\langle v_1|v_2\rangle_{\mathrm{tr}} := \mathrm{tr}(v_1 v_2)$ be the inner product of the Euclidean Jordan algebra V defined by the trace function of V. We know by [5, Proposition II.4.3] that $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$ is associative. Since $E_1, \ldots, E_r$ are primitive idempotents, we have $\langle E_k|E_k\rangle_{\mathrm{tr}} = 1$ $(k = 1, \ldots, r)$. Since V is assumed to be simple, it follows from [5, Proposition III.4.1] that the associative inner products $\langle\cdot|\cdot\rangle_\eta$ and $\langle\cdot|\cdot\rangle_{\mathrm{tr}}$ are proportional to each other. We put $\beta_0 := \|E_k\|_\eta^2 > 0$, independent of k. Then we have clearly

$$\langle\cdot|\cdot\rangle_\eta = \beta_0 \langle\cdot|\cdot\rangle_{\mathrm{tr}}. \tag{2.10}$$

We introduce a positive definite Hermitian inner product $(\cdot|\cdot)_W$ on W by

$$(w_1|w_2)_W := \langle w_1|w_2^*\rangle_\eta \qquad (w_1, w_2 \in W).$$

Then the subspaces $\{W_{kj}\}_{1\le j\le k\le r}$ are orthogonal to each other with respect to $(\cdot|\cdot)_W$ and we have by (2.5)

$$(\varphi(w)u_1|u_2)_U = (Q(u_1, u_2)|w^*)_W \qquad (w \in W,\ u_1, u_2 \in U). \tag{2.11}$$

Moreover we have

$$(xy|z)_W = (y|x^* z)_W. \tag{2.12}$$

Proposition 2.4. (1) *For $k > j, u_j \in U_j, u_k \in U_k$, one has $Q(u_j, u_k) \in W_{kj}$.*
(2) *If $u_j \in U_j$, then one has $Q(u_j, u_j) = \beta_0^{-1} \|u_j\|_U^2 E_j$.*

Proof. (1) Let $w \in W$ be arbitrary. Then by (2.12), (2.11) and (2.8)

$$\begin{aligned}(E_j Q(u_j, u_k)|w)_W &= (\varphi(E_j w^*)u_j|u_k)_U \\ &= \tfrac{1}{2}(\varphi(w^*)u_j|\varphi(E_j)u_k)_U + \tfrac{1}{2}(\varphi(w^*)u_j|u_k)_U \\ &= \tfrac{1}{2}(Q(u_j, u_k)|w)_W,\end{aligned}$$

where the last equality follows from Lemma 2.2. Thus $E_j Q(u_j, u_k) = \frac{1}{2} Q(u_j, u_k)$. In a similar way we obtain $E_k Q(u_j, u_k) = \frac{1}{2} Q(u_j, u_k)$. Hence $Q(u_j, u_k) \in W_{kj}$.
(2) An argument similar to (1) shows $E_j Q(u_j, u_j) = Q(u_j, u_j)$. Hence $Q(u_j, u_j) \in W_{jj} = \mathbb{C}E_j$. Let us put $Q(u_j, u_j) = \lambda E_j$ $(\lambda \in \mathbb{C})$. By (2.4) we have

$$\|u_j\|_U^2 = \langle Q(u_j, u_j)|E\rangle_\eta = \lambda \|E_j\|_W^2.$$

Therefore we get $\lambda = \|E_j\|_W^{-2} \|u_j\|_U^2$, which completes the proof. □

To prove our main theorem, we quote the following criterion.

Proposition 2.5 ([2, Corollary 1]). *The irreducible quasisymmetric Siegel domain D is symmetric if and only if there exists a Jordan frame $f_1, \ldots, f_r$ of V such that with $\widetilde{U_k} := \varphi(f_k)U$ we have $\varphi(Q(u_1, u_2))u_1 = 0$ for all $u_1 \in \widetilde{U_1}$ and $u_2 \in \widetilde{U_2}$.*

2.2. **Cayley transform and main theorem.** We put

$$\mathscr{I} := \{(u, w) \in U \times W \mid w + E \text{ is invertible in the Jordan algebra } W\}.$$

If $(u, w) \in \overline{D}$ (the closure of D), then by (2.1) and (2.2) we have $w + E \in \Omega + iV$, so that $w + E$ is invertible. Hence

$$\overline{D} \subset \mathscr{I}. \tag{2.13}$$

Moreover, it is clear that $\mathscr{I}$ is an open set. We define the *Cayley transform* $\mathcal{C}$ by

$$\mathcal{C}(u, w) := \left(2\varphi\left((w + E)^{-1}\right) u,\ (w - E)(w + E)^{-1}\right) \qquad ((u, w) \in \mathscr{I}).$$

We set

$$\widetilde{\mathscr{I}} := \{(u, w) \in U \times W \mid E - w \text{ is invertible in the Jordan algebra } W\}.$$

The inverse map of $\mathcal{C}$ is given by

$$\mathcal{C}^{-1}(u, w) = \left(\varphi\left((E - w)^{-1}\right) u,\ (E + w)(E - w)^{-1}\right) \qquad ((u, w) \in \widetilde{\mathscr{I}}).$$

Now our main theorem is stated as follows:

Theorem 2.6. *Let D be an irreducible quasisymmetric Siegel domain. Then, $\mathcal{C}(D)$ is convex if and only if D is symmetric.*

3. Proof of the "only if" part of the main theorem

Let D be an irreducible quasisymmetric Siegel domain. In this section we show that the convexity of $\mathcal{C}(D)$ implies that D is symmetric. Before proceeding, we note that the Shilov boundary of D coincides with the set Σ:

$$\Sigma = \{(u, w) \in U \times W \mid \operatorname{Re} w = \tfrac{1}{2} Q(u, u)\}.$$

Let us assume that $\mathcal{C}(D)$ is convex. Let j, k be integers with $1 \leq j < k \leq r$. Let us take any non-zero $u_j \in U_j, u_k \in U_k$. We consider the following two points $z_1, z_2 \in \Sigma$:

$$\begin{aligned} z_1 = (u_{z_1}, w_{z_1}) &:= \left(u_j + u_k,\ \tfrac{1}{2} Q(u_j + u_k, u_j + u_k) + i \operatorname{Im} Q(u_j, u_k)\right), \\ z_2 = (u_{z_2}, w_{z_2}) &:= \left(-u_j + u_k,\ \tfrac{1}{2} Q(-u_j + u_k, -u_j + u_k) - i \operatorname{Im} Q(u_j, u_k)\right). \end{aligned}$$

By Proposition 2.4 (2) we have

$$\begin{aligned} z_1 &= \left(u_j + u_k,\ (2\beta_0)^{-1} \|u_j\|_U^2 E_j + (2\beta_0)^{-1} \|u_k\|_U^2 E_k + Q(u_j, u_k)\right), \\ z_2 &= \left(-u_j + u_k,\ (2\beta_0)^{-1} \|u_j\|_U^2 E_j + (2\beta_0)^{-1} \|u_k\|_U^2 E_k - Q(u_j, u_k)\right). \end{aligned}$$

We shall compute the Cayley transforms of z_1, z_2. For simpilicity we put

$$\begin{aligned} q_{jk} &:= \langle Q(u_j,u_k)|Q(u_j,u_k)\rangle_\eta\,, \\ \delta_j &:= 1+(2\beta_0)^{-1}\left\|u_j\right\|_U^2, \qquad \delta_k := 1+(2\beta_0)^{-1}\left\|u_k\right\|_U^2, \\ \tau &:= \delta_j\delta_k-(2\beta_0)^{-1}q_{jk}. \end{aligned} \tag{3.1}$$

Since $z_1 \in \overline{D}$, the element $w_{z_1}+E$ is invertible in W by (2.13). Hence [12, Lemma 10.2] together with (2.10) gives $\tau \neq 0$ and

$$(w_{z_1}+E)^{-1} = \sum_{m\neq j,k} E_m + \tau^{-1}\left(\delta_k E_j + \delta_j E_k - Q(u_j,u_k)\right).$$

Therefore we have by Lemmas 2.2, 2.3 and Proposition 2.4 that

$$\varphi\left((w_{z_1}+E)^{-1}\right)u_{z_1} = \tau^{-1}\left(\delta_k u_j+\delta_j u_k - \varphi(Q(u_j,u_k))(u_j+u_k)\right).$$

Thus we get

$$\begin{aligned} \mathcal{C}(z_1) = \Big(&2\tau^{-1}\left(\delta_k u_j+\delta_j u_k-\varphi(Q(u_j,u_k))(u_j+u_k)\right), \\ &-\sum_{m\neq j,k} E_m + \left(1-2\tau^{-1}\delta_k\right)E_j + \left(1-2\tau^{-1}\delta_j\right)E_k + 2\tau^{-1}Q(u_j,u_k)\Big). \end{aligned}$$

A similar argument gives

$$\begin{aligned} \mathcal{C}(z_2) = \Big(&2\tau^{-1}\left(-\delta_k u_j+\delta_j u_k-\varphi(Q(u_j,u_k))(u_j-u_k)\right), \\ &-\sum_{m\neq j,k} E_m + \left(1-2\tau^{-1}\delta_k\right)E_j + \left(1-2\tau^{-1}\delta_j\right)E_k - 2\tau^{-1}Q(u_j,u_k)\Big). \end{aligned}$$

We consider the midpoint ξ of $\mathcal{C}(z_1)$ and $\mathcal{C}(z_2)$: $\xi = (u_\xi, w_\xi) := \frac{1}{2}(\mathcal{C}(z_1)+\mathcal{C}(z_2))$. We have

$$\begin{aligned} u_\xi &= 2\tau^{-1}\left(\delta_j u_k - \varphi(Q(u_j,u_k))u_j\right), \\ w_\xi &= -\sum_{m\neq j,k} E_m + (1-2\tau^{-1}\delta_k)E_j + (1-2\tau^{-1}\delta_j)E_k. \end{aligned}$$

Since $\mathcal{C}$ is continuous in the open set $\mathscr{I}$ which contains $\overline{D}$, we see that $\mathcal{C}(\overline{D})$ is convex. Hence $\xi \in \mathcal{C}(\overline{D})$, so that $\mathcal{C}^{-1}(\xi) \in \overline{D}$.

We shall compute $\xi' := \mathcal{C}^{-1}(\xi)$. First we have

$$(E-w_\xi)^{-1} = \sum_{m\neq j,k} 2^{-1}E_m + \left(\tau(2\delta_k)^{-1}E_j + \tau(2\delta_j)^{-1}E_k\right).$$

Since $u_\xi \in U_k$ by Lemma 2.3 and Proposition 2.4, it follows from Proposition 2.2 that if we put $\xi' = (u_{\xi'}, w_{\xi'})$, then

$$u_{\xi'} = u_k - \delta_j^{-1}\varphi(Q(u_j,u_k))u_j, \qquad w_{\xi'} = (\tau\delta_k^{-1}-1)E_j + (\tau\delta_j^{-1}-1)E_k.$$

Since $\xi' \in \overline{D}$, we have

$$\operatorname{Re} w_{\xi'} - \tfrac{1}{2}Q\left(u_{\xi'}, u_{\xi'}\right) \in \overline{\Omega}.$$

Hence it follows from Proposition 2.4 (2) that

$$\begin{aligned}&\left((\operatorname{Re}\tau)\delta_k^{-1}-1\right)E_j\\&\qquad+\left((\operatorname{Re}\tau)\delta_j^{-1}-1-(2\beta_0)^{-1}\left\|u_k-\delta_j^{-1}\varphi(Q(u_j,u_k))u_j\right\|_U^2\right)E_k\in\overline{\Omega}.\end{aligned}$$

Therefore the coefficient of E_k must be non-negative:

$$(\operatorname{Re}\tau)\delta_j^{-1}-1-(2\beta_0)^{-1}\left\|u_k-\delta_j^{-1}\varphi(Q(u_j,u_k))u_j\right\|_U^2\geq 0. \tag{3.2}$$

Here by (2.5), we have

$$\begin{aligned}&\left\|u_k-\delta_j^{-1}\varphi(Q(u_j,u_k))u_j\right\|_U^2\\&\qquad=\|u_k\|_U^2+\delta_j^{-2}\|\varphi(Q(u_j,u_k))u_j\|_U^2-2\delta_j^{-1}\operatorname{Re}(\varphi(Q(u_j,u_k))u_j|u_k)_U\\&\qquad=\|u_k\|_U^2+\delta_j^{-2}\|\varphi(Q(u_j,u_k))u_j\|_U^2-2\delta_j^{-1}\operatorname{Re}q_{jk}.\end{aligned}$$

By this equality and (3.1) the left-hand side of (3.2) equals

$$\begin{aligned}&\left(\delta_j\delta_k-(2\beta_0)^{-1}\operatorname{Re}q_{jk}\right)\delta_j^{-1}\\&\qquad\quad-1-(2\beta_0)^{-1}\left(\|u_k\|_U^2+\delta_j^{-2}\|\varphi(Q(u_j,u_k))u_j\|_U^2-2\delta_j^{-1}\operatorname{Re}q_{jk}\right)\\&\qquad=(2\beta_0)^{-1}\delta_j^{-2}\left(\delta_j\operatorname{Re}q_{jk}-\|\varphi(Q(u_j,u_k))u_j\|_U^2\right).\end{aligned}$$

Hence we have

$$\left(1+(2\beta_0)^{-1}\|u_j\|_U^2\right)\operatorname{Re}q_{jk}-\|\varphi(Q(u_j,u_k))u_j\|_U^2\geq 0. \tag{3.3}$$

Since $\langle\cdot|\cdot\rangle_\eta$ is complex bilinear, we can find $\theta\in\mathbb{R}$ such that

$$\operatorname{Re}\left\langle Q(e^{i\theta}u_j,u_k)|Q(e^{i\theta}u_j,u_k)\right\rangle_\eta=0.$$

Replacing u_j by $e^{i\theta}u_j$ in (3.3), we get $\|\varphi(Q(u_j,u_k))u_j\|_U=0$, i.e., $\varphi(Q(u_j,u_k))u_j=0$. Now Proposition 2.5 tells us that D is symmetric.

4. The case of symmetric domains.

In this section we verify that the Cayley transform image of a symmetric Siegel domain D is convex, keeping to the notation in Section 2. Since symmetric Siegel domains are quasisymmetric, we can define a linear map $\varphi: W\to\operatorname{End}_{\mathbb{C}}U$ and equip W with a complex Jordan algebra structure with unit element E as we did in Section 2.

4.1. **Jordan triple system associated with a symmetric Siegel domain.** Symmetric Siegel domains are described in terms of Jordan triple systems (JTS). Here we recall that a finite-dimensional complex vector space Z is called a Hermitian JTS if Z is endowed with a real trilinear map $\{\cdot,\cdot,\cdot\}: Z\times Z\times Z\to Z$ such that

(JTS1) $\{x,y,z\}$ is complex linear in x,z and antilinear in y.

(JTS2) $\{x,y,z\}=\{z,y,x\}$.

(JTS3) $\{a,b,\{x,y,z\}\}=\{\{a,b,x\},y,z\}-\{x,\{b,a,y\},z\}+\{x,y,\{a,b,z\}\}$.

For $x,y\in Z$, we define $x\square y\in\operatorname{End}_{\mathbb{C}}Z$ by $(x\square y)z:=\{x,y,z\}$. A Hermitian JTS Z is said to be positive if the trace form $(x,y)\mapsto\operatorname{Tr}(x\square y)$ is positive definite.

We now describe how a structure of JTS is introduced in the ambient vector space of D. We put $Z := U \oplus W$. Define a real trilinear map $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ by

$$\begin{aligned}\{x+a, y+b, z+c\} \\ := \tfrac{1}{2}\varphi(c)\varphi(b^*)x + \big(\tfrac{1}{2}\varphi(a)\varphi(b^*)z + \tfrac{1}{2}\varphi(Q(x,y))z + \tfrac{1}{2}\varphi(Q(z,y))x\big) \\ + \big((ab^*)c + a(b^*c) - b^*(ac) + \tfrac{1}{2}Q(x,\varphi(c^*)y) + \tfrac{1}{2}Q(z,\varphi(a^*)y)\big),\end{aligned} \tag{4.1}$$

where $x, y, z \in U$ and $a, b, c \in W$. Clearly (JTS1) and (JTS2) hold. We see by [14, Chapter V, §4, Exercise 5 (c)] that $\{\cdot,\cdot,\cdot\}$ satisfies (JTS3), so that Z is a Hermitian JTS. Proposition III.4.2 of [5] together with (2.10) gives $\operatorname{Tr} L(xy) = (r\beta_0)^{-1}(\dim V)\,\langle x|y\rangle_\eta$. Hence it follows from [14, Chapter V, §4, Exercise 5 (a)] that for $x \in U, a \in W$,

$$\operatorname{Tr}((x+a)\Box(x+a)) = (r\beta_0)^{-1}(\dim V + \tfrac{1}{2}\dim U)\big(\|a\|_W^2 + \|x\|_U^2\big).$$

This shows that Z is positive.

Now we have introduced a structure of positive Hermitian Jordan triple system in Z. The following equalities hold:

$$\begin{aligned} ww' &= \{w, E, w'\} \qquad (w, w' \in W), \\ Q(u,u') &= 2\{u, u', E\} \qquad (u, u' \in U), \\ \varphi(w)u &= 2\{w, E, u\} \qquad (w \in W,\ u \in U). \end{aligned} \tag{4.2}$$

4.2. **Convexity of the Cayley transform image.** To verify that the Cayley transform image of a symmetric Siegel domain is convex, we quote some results of [10]. Though the contents of [10] are written in terms of Jordan pairs, we can translate them easily into the language of Jordan triple systems (see [10, 2.9]). If we write $[\cdot,\cdot,\cdot]$ for $\{\cdot,\cdot,\cdot\}$ used in [10], the translation is as follows:

$$[x, y, z] = 2\{x, \overline{y}, z\}. \tag{4.3}$$

Then, the positive Hermitian JTS Z gives a Jordan pair with a positive Hermitian involution by [10, Corollary 3.16]. It should be noted that our translation is different from [10, 2.9] by the multiplication constant 2. This modification is made for E to be a tripotent in the sense of [10].

An element $e \in Z$ is called a tripotent if $\{e, e, e\} = e$. We know by [10, Corollary 3.12] that every $x \in Z$ can be written uniquely as $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ where the e_i are pairwise orthogonal non-zero tripotents which are real linear combinations of powers of x, and the λ_i satisfy $0 < \lambda_1 < \cdots < \lambda_n$. We call the λ_i the eigenvalues of x. We denote by $|x|$ the largest eigenvalue of x. Then we see by [10, Theorem 3.17] that $|\cdot|$ is a norm on Z, called the spectral norm. Let us denote by B the open unit ball for the spectral norm.

By (4.2), E is a tripotent. Definition (4.1) of the triple product shows that U (resp. W) is the $\frac{1}{2}$-eigenspace (resp. 1-eigenspace) of the operator $E\Box E$. In particular, E is a maximal tripotent, and $Z = U \oplus W$ gives the Peirce decomposition of Z with respect to E. By (4.2) and (4.3), the Jordan algebra structure on W defined at the beginning of [10, §10] coincides with ours. Moreover the product

$\circ$ defined in [10, §10] is $w \circ u = \varphi(w)u\,(w \in W,\, u \in U)$. Hence we see by [10, Proposition 10.3] that the Cayley transform γ_E defined in [10, §10] corresponding to E is

$$\gamma_E(u,w) = (\sqrt{2}\varphi((E-w)^{-1})u,\ (E+w)(E-w)^{-1}).$$

Now [10, Corollary 10.9] tells us that $\gamma_E(B) = D$. If we define a linear map T on $U \times W$ by $T(u,w) := (\sqrt{2}u, w)\,(u \in U, w \in W)$, then $\gamma_E = \mathcal{C}^{-1} \circ T$. Therefore we get $\mathcal{C}(D) = T(B)$, which clearly implies that $\mathcal{C}(D)$ is convex.

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