ON THE GEOMETRY OF WIMAN'S SEXTIC

NAOKI INOUE AND FUMIHARU KATO

ABSTRACT. We give a new version of W.L. Edge's construction of the linear system of plane sextics containing Wiman's sextic, by means of configuration space of 5 points on projective line. This construction reveals out more of the inner beauty of the hidden geometry of Wiman's sextic. Furthermore, it allows one to give a friendly proof for the fact that the linear system is actually a pencil, the fact that is important in both Edge's and our constructions.

1. INTRODUCTION

Consider the action of the symmetric group $\mathfrak{S}_5$ of five letters on the projective 5-space $\mathbb{P}^5$ defined over an algebraically closed field K of characteristic not equal to 2, 3, 5, induced from the 6-dimensional irreducible representation. Calculating the symmetric square of the representation, one sees that there exists a quadratic form in $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2))$, unique up to scalar, that is invariant by any element of $\mathfrak{S}_5$. This gives rise to the unique quadratic hypersurface in $\mathbb{P}^5$, which is stable under the action of $\mathfrak{S}_5$.

There is, on the other hand, the famous surface embedded in $\mathbb{P}^5$, the Del Pezzo quintic surface, on which the group $\mathfrak{S}_5$ acts equivariantly with the action on $\mathbb{P}^5$ as above. The intersection of the hypersurface with the Del Pezzo quintic surface defines a curve, denoted by $\widetilde{W}$. By the 4-point blow-up map, it is mapped to a certain curve on $\mathbb{P}^2$, which is stable under the $\mathfrak{S}_5$-action by Cremona transformations. This curve is actually an irreducible 4-nodal sextic, which we denote by W. The actual equation for W with respect to a suitably chosen homogeneous coordinate is given as follows:

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2 = 0.$$

The curve W is first discovered by Wiman in the end of 19th century [5]. This curve is, needless to say, interesting in its own light, for it has a lot of symmetries; the normalization of W is actually isomorphic to $\widetilde{W}$ as above, which is, therefore, a non-singular projective curve of genus 6 having the automorphism group isomorphic to $\mathfrak{S}_5$. But more attractive is the inner beauty of the rich geometry hidden behind the curve W. In his 1981 papers [2][3] W.L. Edge, unsatisfied with Wiman's original description of the curve, gave projective geometric characterization of the Wiman's sextic, which reveals out rich geometric background and several nice properties. In the first work Edge constructed in a purely projective geometric manner a pencil L of plane sextics on which the group $\mathfrak{S}_5$ acts by Cremona transformations. As the non-trivial action of $\mathfrak{S}_5$ on projective line is only possible by the signature action, there are precisely two members in L that are stable under the action. One of them is a union of 6 lines, and the other is the Wiman's sextic. Notice that, in order to find the Wiman's sextic in L, it was important to know that his linear system L is actually a pencil.

In this note we are going to recast Edge's construction in a slightly different manner. The main points of our method are that we regard the Del Pezzo quintic surface as the configuration space of 5 points on $\mathbb{P}^1$, and that we consider the so-called *pentagonal coordinates* (studied by M. Yoshida [1]) on the surface, which has 6 variables X, Y, Z, U, V, W together with 6 dummy ones $X^*, Y^*, Z^*, U^*, V^*, W^*$, and gives the anti-canonical embedding into $\mathbb{P}^5$. There are several benefits arising from these points. First, by means of the configuration space of 5 points, the natural $\mathfrak{S}_5$-action becomes visible; moreover, the action can be quite explicitly described in terms of the pentagonal coordinates. This allows one to understand the construction more transparently. Secondly, our method allows to give a friendly proof of the fact that the linear system L is a pencil, the fact for which Edge only gives a short explanation. Finally, as the reader will find soon, the pentagonal coordinates turns out to be the most optimal coordinate system in the sense that, in terms of it, the Wiman's sextic has the very beautiful and simple defining equation; indeed, it is

$$X^2 + X^{*2} + Y^2 + Y^{*2} + Z^2 + Z^{*2} + U^2 + U^{*2} + V^2 + V^{*2} + W^2 + W^{*2} = 0$$

(cf. 3.19 below). Note that our coordinate system is different from Edge's one in [3].

The composition of this note is as follows: In the next section, we will briefly recall Edge's construction of the linear system L and the Wiman's sextic W. In Section 3 we will perform our way of the construction. In the last section (Section 4) we give the proof of the fact that the linear system is a pencil.

2. Review of Edge's construction

In this section we will briefly review Edge's construction of Wiman's sextic [2].

General Convention.

2.1. Throughout this paper, K denotes an algebraically closed field with $\mathrm{char}(K) \neq 2, 3, 5$. By $V = K^{n+1}$ we denote the vector space consisting of all column vectors ${}^{\mathrm{t}}(a_0, \ldots, a_n)$ of height $n+1$. Set $\mathbb{P}^n = \mathrm{Proj}\,\mathrm{Sym}_K V^*$. The set of K-rational points $\mathbb{P}^n(K)$ thus consists of homothecy classes of column vectors; such a point will be written as ${}^{\mathrm{t}}(a_0 : \cdots : a_n)$, or more simply, $(a_0 : \cdots : a_n)$, if there is no danger of confusion. The group $\mathbf{PGL}_{n+1}(K)$ naturally acts on $\mathbb{P}^n$ from the left. On K-rational points, the action is described as follows: For $A = (a_{ij})$ and $x = (x_0 : \cdots : x_n)$, we have $Ax = (y_0 : \cdots : y_n)$, where ${}^{\mathrm{t}}(y_0, \cdots, y_n) = A \cdot {}^{\mathrm{t}}(x_0, \cdots, x_n)$.

2.2. In the sequel, simply by a point of a K-scheme we always mean a K-rational point. Accordingly, for a K-scheme X, writing $x \in X$ means that x is a K-rational point of X, that is, $x \in X(K)$.

2.3. We will be concerned with some elementary projective plane geometry. For two distinct points $p_0, p_1 \in \mathbb{P}^2$, we denote by $p_0 * p_1$, the so-called *join*, the line spanned by these points. For two different lines ℓ_0, ℓ_1 on $\mathbb{P}^2$, we likewise denote by $\ell_0 * \ell_1$ the join, that is, the unique intersection point of them.

Cremona transformation.

2.4. Recall that a *Cremona transformation* on $\mathbb{P}^n$ is a rational selfmap of $\mathbb{P}^n$ that has the rational inverse. They evidently form a group by composition, which contains $\mathrm{Aut}(\mathbb{P}^n) = \mathbf{PGL}_{n+1}(K)$ as a subgroup. The Cremona transformations of the following kind will be of particular importance: Let $p_0, p_1, p_2, q \in \mathbb{P}^2$ be four points in general position (i.e., no three of them sit on a line), and Q the blow-up of $\mathbb{P}^2$ at the three points p_0, p_1, p_2. Let E_i $(i = 0, 1, 2)$ be the resulting exceptional line over p_i, and C_i the strict transform of the line $p_j * p_k$, where $\{i, j, k\} = \{0, 1, 2\}$. Blow-down the (-1)-curves C_0, C_1, C_2, and coordinate the resulting $\mathbb{P}^2$ in such a way that the image of C_i is p_i and that q is mapped to q. Thus we get a Cremona tramsformation, which we denote by $J_{\{p_0,p_1,p_2\},q}$. Note that $J_{\{p_0,p_1,p_2\},q}$ is involutive, i.e., $(J_{\{p_0,p_1,p_2\},q})^2 = \mathrm{id}$. The following proposition is easy to verify, and the proof is left to the reader:

Proposition 2.5. *The Cremona transformation $J_{\{p_0,p_1,p_2\},q}$ maps the pencil of all lines passing through p_i linearly isomorphically onto itself for each $i = 0, 1, 2$, and maps the pencil of all lines passing through q linearly isomorphically onto the pencil of all conics passing through p_0, p_1, p_2, q and vise versa.*

Hessian duad and Hessian pair.

2.6. Consider a set of three distinct points $\{p_0, p_1, p_2\}$ of $\mathbb{P}^1$. The *Hessian duad* of the set $\{p_0, p_1, p_2\}$ is the set of two points $\{q^+, q^-\}$ of $\mathbb{P}^1$ characterized by one of the following equivalent conditions:

(1) The points q^+ and q^- are the fixed points of the linear transformation of $\mathbb{P}^1$ induced from a cyclic permutation of $\{p_0, p_1, p_2\}$.
(2) $q = q^+$ and $q = q^-$ are the solutions for the equation $\mathrm{cr}(p_0, p_1, p_2, q) = -\omega, -\omega^2$, where ω is the primitive cubic root of unity.
(3) If we choose $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\varphi(\{p_0, p_1, p_2\}) = \{1, \omega, \omega^2\}$, then $\varphi(\{q^+, q^-\}) = \{0, \infty\}$.

Here, $\mathrm{cr}(p_0, p_1, p_2, q)$ denotes the *cross ratio* defined by

$$\mathrm{cr}(p_0, p_1, p_2, q) = \frac{q - p_1}{q - p_2} \cdot \frac{p_0 - p_2}{p_0 - p_1},$$

where, now, the points are displayed in terms of inhomogeneous coordinate.

Note that the definition of Hessian duad depends only on the set $\{p_0, p_1, p_2\}$. One can similarly define Hessian duad of three distinct points on a line in $\mathbb{P}^2$. In duality, one defines likewise the notion of *Hessian pair* of the set of three distinct lines sitting in a pencil. The following proposition is easy to see, and the proof is left to the reader:

Proposition 2.7. *Let p be a point of $\mathbb{P}^2$, and $\{\ell_0, \ell_1, \ell_2\}$ a set of distinct three lines passing through p. Let ℓ be a line that does not contain p, and set $q_i = \ell_i * \ell$ for $i = 0, 1, 2$. Then the following conditions for two lines $\ell^\pm$ passing through p are equivalent:*

(1) *$\{\ell^+, \ell^-\}$ is the Hessian pair of $\{\ell_0, \ell_1, \ell_2\}$.*
(2) *$\{\ell^+ * \ell, \ell^- * \ell\}$ is the Hessian duad of $\{q_0, q_1, q_2\}$ on the line ℓ.*

The line configuration $(\Pi + \mathrm{H})$.

2.8. Now we begin the construction. The first step of the construction is to give a certain line configuration, which we denote symbolically by $(\Pi + \mathrm{H})$, on $\mathbb{P}^2$ that is determined by a set of 4 points in general position; since the construction is entirely linear, change of the set of the points only gives rise to the linear change of the configuration, and hence, $(\Pi + \mathrm{H})$ is unique up to linear transformation.

Let $\{p_1, p_2, p_3, p_4\}$ be a set of 4 points of $\mathbb{P}^2$ in general position. This set gives rise to the so-called *quadrangle* on $\mathbb{P}^2$, which is the line configuration consisting of 6 lines $L_{ij} = L_{ji} = p_k * p_l$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. It has the points p_i's as its *vertices*, and the 3 *diagonal points* $p_{ij} = p_{kl} = L_{ij} * L_{kl}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Note that, on the line L_{ij}, there are 3 distinguished points p_k, p_l, p_{kl} ($\{i, j, k, l\} = \{1, 2, 3, 4\}$), where the first two are triple points, and the last one is a double point. We denote by Π the line configuration consisting of the 6 lines L_{ij}.

2.9. Let $\{i,j,k,l\} = \{1,2,3,4\}$. The 3 lines L_{kl}, L_{jl}, L_{jk} pass through the vertex p_i, and one can consider the Hessian pair $\{H_i^+, H_i^-\}$ of them. We have in total 8 lines of the form $H_i^{\pm}$, which we call the *Hessian lines* of the quadrangle. The line configuration consisting of the 8 lines $H_i^{\pm}$ is denoted by H.

On the line L_{ij}, one can consider the Hessian duad $\{q_{ij}^+, q_{ij}^-\}$ of $\{p_k, p_l, p_{kl}\}$. By 2.7, it coincides with the pair $\{H_i^+ * L_{ij}, H_i^- * L_{ij}\}$, and furthermore, with the pair $\{H_j^+ * L_{ij}, H_j^- * L_{ij}\}$. In this way, we get 12 points of the form $q_{ij}^{\pm}$. We call them the *Hessian points* of the quadrangle Π.

Definition 2.10. The *line configuration* $(\Pi + \mathrm{H})$ is the collection of 6 lines L_{ij} (edges of the quadrangle) and the 8 lines $H_i^{\pm}$ (hessian lines), hence in total 14 lines. It has in total the 19 distinguished points, viz., 4 vertices p_i, 3 diagonal points p_{ij}, and 12 Hessian points q_{ij}^+.

Cremona action by $\mathfrak{S}_5$ on $\mathbb{P}^2$.

2.11. Let H be the group of the linear transformations induced by permutations of four points p_1, p_2, p_3, p_4. Thus, H is a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ isomorphic to $\mathfrak{S}_4$. Moreover, it is obvious that the line configuration $(\Pi + \mathrm{H})$ is stable under the action by H.

To understand the symmetry more in detail, let us introduce the following 5 pencils:

- α_0: the pencil of conics passing through p_1, p_2, p_3, p_4.
- α_i $(i = 1,2,3,4)$: the pencil of lines passing through p_i.

The group H acts on the set $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ as the permutations of the last four elements.

Set $J_i = J_{\{p_j,p_k,p_l\},p_i}$ for $\{i,j,k,l\} = \{1,2,3,4\}$. By 2.5, we readily see the following:

Proposition 2.12. *The Cremona transformation J_i gives rise to the transposition $(\alpha_0\alpha_i)$ on the set $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.*

2.13. Now set $G = \langle H, J_1 \rangle$ as a subgroup of the group of all Cremona transformations. Since G acts on the set $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, there exists a map $G \to \mathfrak{S}_5$. As the image contains all transpositions, this map is surjective; one can check, moreover, without so much pain that it is injective, and thus that the group G is isomorphic to $\mathfrak{S}_5$. Hence we get the Cremona action of $\mathfrak{S}_5$ on $\mathbb{P}^2$. Note that the group G contains all J_i $(i = 1,2,3,4)$ as conjugates of J_1 by suitable elements in H. By the construction of J_i's (as in 2.4), we see the following:

Proposition 2.14. *The line configuration* $(\Pi + \mathrm{H})$ *is stable under the action of* G.

The linear system L.

Definition 2.15. Define L to be the linear system of curves on $\mathbb{P}^2$ spanned by general 4-nodal sextics C having the nodes at the vertices p_1, p_2, p_3, p_4 such that the following conditions are satisfied:

(1) For each $i = 1,2,3,4$, the nodal tangents at p_i coincide with $H_i^{\pm}$.
(2) For each $i,j = 1,2,3,4$ $(i \neq j)$, C passes through the points $q_{ij}^{\pm}$.

Note that, due to 2.14, the linear system L is acted on by the group $G \cong \mathfrak{S}_5$.

Wiman's sextic W.

2.16. Edge claims in [2], with a short explanation, that the linear system L is a pencil. We will give an algebro-geometric proof for this fact in Section 4. Assuming this fact, as well as that the action of G on L is not the trivial one, one can characterize the Wiman's sextic curve W as the unique irreducible member of L that is stable under the action of the whole group G. In fact, since the action of G on L is non-trivial, G acts on $L \cong \mathbb{P}^1$ via the map

$$G \cong \mathfrak{S}_5 \longrightarrow \mathfrak{S}_5/\mathfrak{A}_5 \cong \{\pm 1\},$$

namely;

(1) general member of L is stable under the action of $\mathfrak{A}_5$;
(2) there exist exactly 2 members (corresponding to the fixed point in $\mathbb{P}^1$ of the action $z \mapsto -z$) which are stable under the whole group $G = \mathfrak{S}_5$.

As we will see in the next section that, in fact, one of the members as in (2) is Π, and the other one is an irreducible one, which is nothing but Wiman's sextic.

3. Construction by configuration space

Configuration space.

3.1. We denote by $X(2,n)$ the K-scheme representing the set of all configurations of n rational points on the projective line $\mathbb{P}^1$, that is,

$$X(2,n) = \mathbf{PGL}_2 \backslash [(\mathbb{P}^1)^n - \Delta],$$

where Δ is the locus of coincidence of at least two points. Here, $(\mathbb{P}^1)^n$ is acted on by $\mathbf{PGL}_2$ diagonally, and hence, the natural $\mathfrak{S}_n$-action that permutes the factors descends to that on $X(2,n)$. It is known that $X(2,n)$ is an $(n-3)$-dimensional non-singular quasi-projective scheme on K. This space comes more visible when one describes it in terms of coordinates: Let $\mathbf{Mat}_{2,n}$ be the affine scheme of all $2 \times n$ matrices. It has the n-tuples of column vectors of the form ${}^{\mathrm{t}}(x_i, y_i)$ $(i = 0, \ldots, n-1)$ as the coordinate system. Let $D(ij)$ denotes the determinant of the (i,j)-minor:

$$D(ij) = x_i y_j - x_j y_i.$$

Then we have

$$X(2,n) \cong \mathbf{PGL_2} \backslash [\mathbf{Mat}_{2,n} - \widetilde{\Delta}]/(\mathbf{G}_{\boldsymbol{m}})^n,$$

where $\widetilde{\Delta}$ is the closed subscheme defined by $\prod_{i<j} D(ij)$, and $(\mathbf{G}_{\boldsymbol{m}})^n$ acts on $\mathbf{Mat}_{2,n} - \widetilde{\Delta}$ columnwise. The $\mathfrak{S}_n$-action is simply given by the permutation of indices $\{0, 1, \ldots, n-1\}$.

3.2. The space $X(2,n)$ has the nice projective compactification $\overline{X}(2,n)$, which is the K-scheme classifying all *stable configurations*; see [4, Def. 3.7/Prop. 3.4] for the definition of stable configuration. Denote the open subscheme of $(\mathbb{P}^1)^n$ of stable configurations by $(\mathbb{P}^1)^n_{\text{stable}}$. Then the scheme $\overline{X}(2,n)$ is given by

$$\overline{X}(2,n) = \mathbf{PGL}_2 \backslash (\mathbb{P}^1)^n_{\text{stable}}.$$

It is well-known that $\overline{X}(2,n)$ is a non-singular projective K-scheme of dimension $n-3$, which contains $X(2,n)$ as a dense open subscheme.

Examples 3.3. We will be only concerned with the configuration spaces $X(2,n)$ with $n = 4$ and $n = 5$.

(1) If $n = 4$, then $X(2,4)$ is the projective line deprived of three points. Given a rational point of $X(2,4)$, or what amounts to the same, a 4-tuple (p_0, p_1, p_2, p_3) of rational points of $\mathbb{P}^1$ (displayed in terms of inhomogeneous coordinate), one considers the cross ratio $\mathrm{cr}(p_0, p_1, p_2, p_3)$, which gives rise to the open immersion $X(2,4) \hookrightarrow \mathbb{P}^1$. The compactification $\overline{X}(2,4)$ is simply a projective line that fills in the three missing points of $X(2,4)$.

Another realization is given by the immersion $X(2,4) \hookrightarrow \mathbb{P}^2$ by the map

$$p \longmapsto (D(01)(p)D(23)(p) : D(02)(p)D(13)(p) : D(03)(p)D(12)(p)).$$

It maps $X(2,4)$ isomorphically onto the locally closed subset consisting of points $(x : y : z)$ such that none of x, y, z vanishes and that the *Plücker relation*

$$x - y + z = 0$$

is satisfied.

(2) If $n = 5$, it is known that $\overline{X}(2,5)$ is the Del Pezzo quintic surface. A configuration of 5 points $(p_0, p_1, p_2, p_3, p_4)$ is stable if and only if there exists no subset $I \subset \{0,1,2,3,4\}$ with $|I| = 3$ such that $p_i = p_j = p_k$. This allows one to describe the boundary $\overline{X}(2,5) - X(2,5)$. In fact, the boundary consists of 10 lines ℓ_{ij} defined by the equation $D(ij) = 0$. The line ℓ_{ij} is, therefore, the locus of the points representing the configurations of 5 points $(p_0, p_1, p_2, p_3, p_4)$ with $p_i = p_j$; it is thus isomorphic to $\overline{X}(2,4)$, hence to $\mathbb{P}^1$ (3.3 (1)). Each ℓ_{ij} intersects exactly 3 other such lines transversally; indeed, the lines ℓ_{ij} and ℓ_{kl} intersect if and only if $\{i,j\} \cap \{k,l\} = \emptyset$.

The linear system $\widetilde{L}$.

3.4. We are going to define a linear system $\widetilde{L}$ on the non-singular projective surface $S = \overline{X}(2,5)$. Consider the natural $\mathfrak{S}_5$-action on $\overline{X}(2,5)$ introduced in 3.1. By $\sigma \in \mathfrak{S}$, the line ℓ_{ij} is mapped linearly to $\ell_{\sigma(i)\sigma(j)}$. Hence, in particular, the divisor

$$\widetilde{\Pi} = \textstyle\sum_{i<j} \ell_{ij}$$

is stable under the $\mathfrak{S}_5$-action.

3.5. Let $\{i,j,k,l,m\} = \{0,1,2,3,4\}$. Then the line ℓ_{ij} intersects ℓ_{kl}, ℓ_{lm}, and ℓ_{mk}, and these intersection points gives rise to the Hessian duad $\{\widetilde{q}^{+}_{ij}, \widetilde{q}^{-}_{ij}\}$ (2.6). We have 20 such points on $\widetilde{\Pi}$ in total. Since the divisor $\widetilde{\Pi}$ is stable under the action by $\mathfrak{S}_5$, so is the set of all those points $\widetilde{q}^{\pm}_{ij}$.

Definition 3.6. Define the linear system $\widetilde{L}$ on $\overline{X}(2,5)$ by

$$\begin{aligned} \widetilde{L} &= \left|\widetilde{\Pi} - \textstyle\sum_{i<j}(\widetilde{q}^{+}_{ij} + \widetilde{q}^{-}_{ij})\right| \\ &= \left\{\begin{matrix}\text{effective divisors linearly equivalent to } \widetilde{\Pi} \\ \text{which pass through all the points } \widetilde{q}^{\pm}_{ij}\end{matrix}\right\}. \end{aligned}$$

As the divisor $\widetilde{\Pi}$ and the set $\{\widetilde{q}^{+}_{ij}, \widetilde{q}^{-}_{ij}\}_{i<j}$ are stable under the action by $\mathfrak{S}_5$, the linear system $\widetilde{L}$ is acted on by $\mathfrak{S}_5$.

Pentagonal coordinates.

3.7. We are going to study the linear system $\widetilde{L}$ in more detail. To do this, we are to introduce a useful coordinates on the surface $\overline{X}(2,5)$ following [1].

Given an ordered set (a,b,c,d,e) of indices such that $\{a,b,c,d,e\} = \{0,1,2,3,4\}$, we put

$$\langle abcde \rangle = D(ab)D(bc)D(cd)D(de)D(ea).$$

Clearly, we have $\langle abcde \rangle = \langle bcdea \rangle$ and $\langle abcde \rangle = -\langle edcba \rangle$. One $\langle abcde \rangle$ is, therefore, fixed up to sign by the subgroup of $\mathfrak{S}_5$ isomorphic to $\mathfrak{D}_5$, and hence, there exist precisely 12 such symbols.

3.8. There are several relations among $\langle abcde\rangle$'s. We first find linear ones as follows: Consider the Kleinian subgroup $\mathfrak{K}_4 = \{1, (ab), (cd), (ab)(cd)\}$ in $\mathfrak{S}_4$ acting on the set $\{a, b, c, d\}$. Then we have

$$\text{(L)} \qquad \textstyle\sum_{\sigma\in\mathfrak{K}_4} \langle abcde\rangle^\sigma = \langle abcde\rangle + \langle bacde\rangle + \langle abdce\rangle + \langle badce\rangle = 0,$$

which follows from Plücker's identity $D(ab)D(cd) - D(ac)D(bd) + D(ad)D(bc) = 0$. There are precisely 6 linearly independent such relations.

3.9. For a given $\langle abcde\rangle$, its *dual* is defined to be $\langle acebd\rangle$, that is, the unique one such that any couple of adjacent indices in the former are not adjacent in the latter. We denote it by $\langle abcde\rangle^*$. Observe that we have $(\langle abcde\rangle^*)^* = -\langle abcde\rangle$, hence, despite the name, the formation of taking dual is involutive only up to sign. Note that the product $\langle abcde\rangle\langle abcde\rangle^*$ is the product of all 10 $D(ij)$'s up to sign, and hence, they are all equal up to sign; checking the sign, we easily see

$$\text{(Q)} \qquad \langle abcde\rangle\langle abcde\rangle^* + \langle abced\rangle\langle abced\rangle^* = 0.$$

This gives rise to 5 linearly independent quadratic relations.

3.10. There is yet another series of relations, which is actually given by cubics of $\langle abcde\rangle$'s. They are generated, by the $\mathfrak{S}_5$-action, by the one that looks like

$$\text{(C)} \qquad \langle abcde\rangle\langle abecd\rangle\langle abdec\rangle - \langle abdce\rangle\langle abedc\rangle\langle abced\rangle = 0.$$

We have, thus, 10 such cubic relations.

Definition 3.11. We set

$$\begin{array}{rclrcl} X^* &=& \langle 01234\rangle, & X &=& \langle 03142\rangle, \\ Y^* &=& \langle 01423\rangle, & Y &=& \langle 02134\rangle, \\ Z^* &=& \langle 01342\rangle, & Z &=& \langle 04123\rangle, \\ U^* &=& \langle 10234\rangle, & U &=& \langle 13042\rangle, \\ V^* &=& \langle 10423\rangle, & V &=& \langle 12034\rangle, \\ W^* &=& \langle 10342\rangle, & W &=& \langle 14023\rangle. \end{array}$$

These gives rise to a homogeneous coordinate $(X : Y : Z : U : V : W)$ on $\overline{X}(2,5)$. In terms of this, the relations (L), (Q), and (C) are now read off as follows:

$$\text{(L)} \qquad \left\{ \begin{array}{rcl} X^* &=& -U - Y - Z \\ Y^* &=& -V - Z - X \\ Z^* &=& -W - X - Y \\ U^* &=& -X - V - W \\ V^* &=& -Y - W - U \\ W^* &=& -Z - U - V, \end{array} \right.$$

$$\text{(Q)} \qquad XX^* = YY^* = ZZ^* = -UU^* = -VV^* = -WW^*,$$

$$\text{(C)} \qquad \left\{ \begin{array}{rclrcl} X^*Y^*Z^* &=& -U^*V^*W^*, & XYV^* &=& -Z^*UW, \\ XYU^* &=& Z^*VW, & Y^*VW &=& -XZU^*, \\ Y^*UV &=& -XZW^*, & X^*Y^*W &=& ZU^*V^*, \\ YZV^* &=& -X^*UW, & XV^*W^* &=& Y^*Z^*U, \\ X^*UV &=& -YZW^*, & X^*Z^*V &=& YU^*W^*. \end{array} \right.$$

Theorem 3.12. (1) *The homogeneous coordinate* $(X : Y : Z : U : V : W)$ *on* $S = \overline{X}(2,5)$ *gives rise to a closed immersion* $\Phi\colon S = \overline{X}(2,5) \hookrightarrow \mathbb{P}^5$ *onto the closed subvariety defined by the relations* (L), (Q), *and* (C), *which is nothing but the embedding by the anti-canonical class* $-K_S$.

(2) *Consider the homogeneous coordinate* $(x' : y' : z')$ *given by*

$$\begin{aligned} x' &= D(12)D(03)D(04), \\ y' &= D(13)D(04)D(02), \\ z' &= D(14)D(02)D(03). \end{aligned}$$

Then it gives rise to the proper birational morphism $\Psi\colon S = \overline{X}(2,5) \to \mathbb{P}^2$ *that blows down the* 4 *lines* ℓ_{0i} $(i = 1, 2, 3, 4)$.

Proof. We first show (2). Due to the definition of the stable configuration (as in 3.3 (2)), one sees immediately that the values x', y', z' cannot be all zero at the same time. Hence it gives the morphism Ψ as above. To understand it, we may limit ourselves to the locus of $p = (p_0, p_1, p_2, p_3, p_4)$ with $z'(p) \neq 0$.

If $D(01)(p) \neq 0$ and $D(04)(p) \neq 0$, then one can normalize p as $p = (\infty, 0, u, v, 1)$, where $(u, v) \neq (1, 1), (0, 0)$. Then the map Ψ on this locus looks like $\Psi(p) = (u : v : 1)$, hence is one-to-one. In particular, it is a birational morphism. Since S is proper, Φ is a proper mapping.

If in turn $D(04)(p) = 0$ (the case $D(01)(p) = 0$ is similar), then p can be normalized as $p = (\infty, 0, u, 1, \infty)$, by which we get $\Psi(p) = (0 : 0 : 1)$. The inverse image of $(0 : 0 : 1)$ on this locus us an affine line. Actually, this gives one affine patch in the blow-up at $(0 : 0 : 1)$; indeed, by the affine coordinate $(x'/z', y'/z')$ on $\mathbb{P}^2$, one gets $(x'/z') = u(y'/z')$. By this, one easily deduces (2).

For (1), we refer to [1] for the proof that the rational map Φ actually gives a closed immersion onto the prescribed closed subvariety. To show that the other statement of (1), one calculates (due to (2)) $K_S \sim \Psi^* K_{\mathbb{P}^2} + \sum_{i=1}^{4} \ell_{0i}$, whence deducing that $-K_S$ is the divisor of

$$\frac{D(12)D(13)D(14)D(02)^2D(03)^2D(04)^2}{D(01)D(02)D(03)D(04)}$$

(where the numerator stands for $x'y'z'$), that is, in view of Plücker relation (cf. 3.3 (1)), equal to $ZW/(Z + W)$. Hence we see that $-K_S \sim \mathscr{O}_S(1)$ as desired. □

3.13. Since the surface $S = \overline{X}(2,5)$ is acted on by the group $\mathfrak{S}_5$, it is to be checked whether the action is equivalent, through the mapping Φ, to the Cremona action on $\mathbb{P}^2$ as in 2.13. Set $p_1 = (1 : 1 : 1)$, $p_2 = (1 : 0 : 0)$, $p_3 = (0 : 1 : 0)$, $p_4 = (0 : 0 : 1)$. Then the map Ψ in 3.12 (2) maps the line ℓ_{0i} $(i = 1, 2, 3, 4)$ to the point p_i and the line ℓ_{ij} $(i, j \neq 0)$ linearly onto the line L_{ij}.

Proposition 3.14. *The natural* $\mathfrak{S}_5$*-action on* $S = \overline{X}(2,5)$ *is, through the map* Ψ*, equivalent to the Cremona* $\mathfrak{S}_5$*-action on* $\mathbb{P}^2$*; that is, for any* $\sigma \in \mathfrak{S}_5$*, we have* $\Psi \circ \sigma = \sigma \circ \Psi$.

Proof. The desired equality $\Psi \circ \sigma = \sigma \circ \Psi$ are to be checked for the transpositions (01), (12), (23), and (34), which suffices to show the proposition. Let us first check the case $\sigma =$ (01). In this case, $\Psi \circ \sigma(p) = (D(02)D(13)D(14) : D(03)D(14)D(12) : D(04)D(12)D(13))$. Dividing out the entries by $D(12)D(03)D(04)D(02)D(13)D(14)$, one gets $\Phi \circ \sigma(p) = (\frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'})$, which is nothing but $J_1(x' : y' : z')$ as in 2.11. In case $\sigma = (12)$, one calculates $\Psi \circ \sigma(p) = (-x' : y' - x' : z' - x')$; as the linear transformation $(x' : y' : z') \mapsto (-x' : y' - x' : z' - x')$ is the unique one that exchanges p_1 and p_2 and fixes the other two, we get the desired equality in this case. The other cases are similar. □

The linear system $\widetilde{L}$ as conic section.

Proposition 3.15. *The divisor $\widetilde{\Pi}$ is a conic-cut in $\mathbb{P}^5$ with respect to the embedding Φ; i.e., there exists a quadratic hypersurface C in $\mathbb{P}^5$ such that $\Phi(\widetilde{\Pi}) = C \cdot \Phi(S)$. In particular, we have $\widetilde{\Pi} \sim \mathscr{O}_S(2)$.*

Proof. It suffices to invoke the fact that $\widetilde{\Pi}$ is the zero set of $XX^* = -X(U+Y+Z)$. □

3.16. It is, therefore, natural to ask whether all members of $\widetilde{L}$ could be obtained in this way. To check this, we look at the exact sequence

$$0 \longrightarrow \mathscr{I}_S(2) \longrightarrow \mathscr{O}_{\mathbb{P}^5}(2) \longrightarrow \mathscr{O}_S(2) \longrightarrow 0,$$

where $\mathscr{I}_S$ is the defining ideal of $\Phi(S)$. The associated cohomology exact sequence begins with

$$(*) \qquad 0 \longrightarrow \mathrm{H}^0(\mathbb{P}^5, \mathscr{I}_S(2)) \longrightarrow \mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) \longrightarrow \mathrm{H}^0(S, \mathscr{O}_S(2)).$$

Since $\mathscr{O}_S(2) \cong \mathscr{O}_S(\Pi)$, what to prove is the following

Proposition 3.17. *The map $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) \to \mathrm{H}^0(S, \mathscr{O}_S(2))$ is surjective. Hence, in particular, every member of $\widetilde{L}$ is a conic-cut in $\mathbb{P}^5$.*

Proof. In the exact sequence $(*)$ in 3.16, the cohomology group in the middle is the space of all quadratic forms, whence having dimension 21, while the first one is the subspace of the quadratic forms whose zero sets contain $\Phi(S)$. Such a quadratic form should belong to $\mathscr{I}_S$, and hence is a linear combination of the known quadratic relations in (Q) in 3.11. As there are exactly 5 linearly independent such quadratic relations, we have $\dim \mathrm{H}^0(\mathbb{P}^5, \mathscr{I}_S(2)) = 5$. Therefore, it suffices to show that the dimension of $\mathrm{H}^0(S, \mathscr{O}_S(2))$ is 16. Since $\mathscr{O}_S(2) \sim -2K_S$, we deduce by Riemann-Roch Theorem

$$\begin{aligned} \chi(\mathscr{O}_S(2)) &= \tfrac{1}{2}(-2K_S \cdot (-3K_S)) + \xi(\mathscr{O}_S) \\ &= \tfrac{1}{2} \cdot 30 + 1 = 16. \end{aligned}$$

On the other hand, since $-K_S$ is ample, we have $\dim \mathrm{H}^2(S, \mathscr{O}_S(2)) = \dim \mathrm{H}^0(S, 3K_S) = 0$. Moreover, by Kodaira-Deligne-Illusie vanishing theorem, we have $\dim \mathrm{H}^1(S, \mathscr{O}_S(2)) = \dim \mathrm{H}^1(S, 3K_S) = 0$ (for, when K is of positive characteristic, S is liftable to the Witt ring of K). Hence $\dim \mathrm{H}^0(S, \mathscr{O}_S(2)) = 16$ as desired. □

3.18. Consider the subspace V of the quadratic forms in $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2))$ defined as follows:

$$V = \left\{ \begin{array}{l} \text{quadratic forms } Q \text{ such that the conic} \\ Q = 0 \text{ passes through the 20 points } \widetilde{q}_{ij}^{\pm} \end{array} \right\} \subset \mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)).$$

Clearly, V contains $\mathrm{H}^0(\mathbb{P}^5, \mathscr{I}_S(2))$ as the 5-dimensional subspace. The following theorem completely determines the structure of V:

Theorem 3.19. *Set*

$$\begin{aligned} F &= X^2 + Y^2 + Z^2 + U^2 + V^2 + W^2 \\ &\quad + X^{*2} + Y^{*2} + Z^{*2} + U^{*2} + V^{*2} + W^{*2}, \\ G &= XX^* + YY^* + ZZ^* - UU^* - VV^* - WW^*. \end{aligned}$$

Then F and G sit in V, spanning an $\mathfrak{S}_5$-stable 2-dimensional subspace V_0 such that $V = V_0 \oplus \mathrm{H}^0(\mathbb{P}^5, \mathscr{I}_S(2))$.

The proof of the theorem will be done in the next section.

Corollary 3.20. (1) *The birational morphism* Ψ (*as in* 3.12 (2)) *gives rise to the linear isomorphism* $\widetilde{L} \xrightarrow{\sim} L$ *of linear systems, where* L *is the linear system defined in* 2.15 *with* $\{p_1, p_2, p_3, p_4\} = \{(0:0:1), (0:1:0), (1:0:0), (1:1:1)\}$.

(2) *The linear system* $\widetilde{L}$ (*and hence* L *also*) *is a pencil. Moreover, it is spanned by the two members* $\widetilde{\Pi} = \{G = 0\}$ *and* $\widetilde{W} = \{F = 0\}$.

(3) *General members of* $\widetilde{L}$ *are stable under the action of* $\mathfrak{A}_5$, *while only* $\widetilde{\Pi}$ *and* $\widetilde{W}$ *are the members that are stable under the action of the whole* $\mathfrak{S}_5$.

Proof. In view of 3.17, we see that $\widetilde{L}$ is isomorphic to $\mathbb{P}(V_0)$, and hence is a pencil. Moreover, it is spanned by $\widetilde{\Pi}$ and $\widetilde{W}$. By the birational morphism Ψ, the divisor $\widetilde{\Pi}$ is obviously mapped to the union Π of the 6 lines of the quadrangle spanned by the 4 points $\{p_1, p_2, p_3, p_4\} = \{(0:0:1), (0:1:0), (1:0:0), (1:1:1)\}$. By 3.12 (2) and the $\mathfrak{S}_5$ action, we see that each line ℓ_{ij} is a (-1)-curve. Hence one calculates the self-intersection number

$$\widetilde{\Pi}^2 = 10 \cdot (-1) + 10 \cdot 3 = 20,$$

which is exactly the number of the points $\widetilde{q}^{\pm}_{ij}$. Hence, in particular, these points are the only base points for the general members of the linear system $\widetilde{L}$. As it is easily checked that $\widetilde{W}$ does not contain any of the lines ℓ_{ij}, one deduces that all intersections of general members of $\widetilde{L}$ with $\widetilde{\Pi}$ are transversal. Hence they are mapped by Ψ to curves having nodes at the 4 points p_i $(i = 1, 2, 3, 4)$. Moreover, their degree is $2 \cdot 5 - 4 = 6$, and hence, they are sextics. By the definition of $\widetilde{L}$, these curves satisfy the conditions (1) (2) in 2.15, and hence belong to L. Thus we get a linear map $\widetilde{L} \to L$. One can construct (by taking the strict transforms) the inverse mapping $L \to \widetilde{L}$. This proves (1) and (2).

To show (3), it suffices to observe that F and G are invariant up to sign under the $\mathfrak{S}_5$-action and that the action on the pencil $\widetilde{L}$ is non-trivial (cf. 2.16). But these assertions are clear, for, while F is fixed by any element of $\mathfrak{S}_5$, G is fixed only by even permutations and mapped to $-G$ by transpositions. $\square$

Wiman's sextic: Conclusion.

3.21. As in 3.20 we have two particular members $\widetilde{\Pi}$ and $\widetilde{W}$ of the linear system $\widetilde{L}$, now known to be a pencil. By the linear isomorphism $\widetilde{L} \xrightarrow{\sim} L$, $\widetilde{\Pi}$ is mapped to the union of 6 lines Π. It then follows that the image W of $\widetilde{W}$ must be the Wiman's sextic.

It is an easy but tedious job to recover the defining equation of the curve W. Let $(x' : y' : z')$ be the homogeneous coordinate of $\mathbb{P}^2$ defined as in 3.12, and then consider the linear change

$$x = -x' - y' + z', \quad y = x' - y' + z', \quad z = -x' + y' + z'$$

of coordinates (so that the set $\{p_1, p_2, p_3, p_4\}$ coincides with the set of points $(\pm 1 : \pm 1 : \pm 1)$. Then we see that the polynomial F is transformed into the following one:

$$2[x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2].$$

3.22. One sees easily, by means of the coordinates $X, Y, \ldots, W$, that the 6-dimensional representaion $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(1))$ of $\mathfrak{S}_5$ is the unique irreducible one. The induced action on the space of quadratic forms $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2))$ is isomorphic to the second symmetric product of the first one, and hence the irreducible decomposition can be easily calculated:

$$\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) = (\mathrm{triv}) \oplus (\mathrm{sgn}) \oplus (4^+) \oplus (5^+)^2 \oplus (5^-),$$

where ((triv) (resp. (sgn)) is the trivial (resp. signature) representation, and (n^+) denotes the n-dimensional irreducible representation such that the signature of the trace of transpositions is $\pm$. By a slightly more calculation one sees that the component (5^-) is the subspace $\mathrm{H}^0(\mathbb{P}^5, \mathscr{I}_S(2))$ (having the basis $XX^* - YY^*$, $YY^* - ZZ^*$, $ZZ^* + UU^*$, $UU^* - VV^*$, $VV^* - WW^*$). The subspace V_0 is the direct sum of the first two components; F is a basis of the trivial part, and G of the signature part.

4. Proof of Theorem 3.19

In this section, we prove Theorem 3.19. The proof is divided into several steps.

4.1. First we write the equation in $\mathbb{P}^5$ of lines, which can be easily done by the fact $\ell_{ij} = \{D(ij) = 0\}$. For the later use, we divide these equations into 3 types:

$$\begin{cases} \ell_{04} & : \quad Y = Z = U = W = 0, \\ \ell_{13} & : \quad X = Y = U = W = 0, \\ \ell_{02} & : \quad X = Y = V = W = 0, \\ \ell_{14} & : \quad X = Z = V = W = 0, \\ \ell_{03} & : \quad X = Z = U = V = 0, \\ \ell_{12} & : \quad Y = Z = U = V = 0. \end{cases} \tag{1}$$

$$\begin{cases} \ell_{24} & : \quad X = U = Y + W = Z + V = 0, \\ \ell_{34} & : \quad Y = V = X + W = Z + U = 0, \\ \ell_{23} & : \quad Z = W = X + V = Y + U = 0. \end{cases} \tag{2}$$

$$\ell_{01} \quad : \quad X - U = Y - V = Z - W = X + Y + Z = 0. \tag{3}$$

Let Q be a quadratic polynomial in $\mathrm{H}^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2))$. For a monomial, say XY, we denote the coefficient of XY in Q by q_{XY}, etc.

4.2. We look at the condition that $Q = 0$ passes through the points $\widetilde{q}_{04}^{\pm}$. By the first row in (1) in 4.1 the line ℓ_{04} has the homogeneous coordinate $(X : V)$. This line has the intersection points with ℓ_{12}, ℓ_{23}, and ℓ_{13}, which are easily calculated to be $(1 : 0)$, $(-1 : 1)$, and $(0 : 1)$, respectively. By this one calculates the Hessian duad $\widetilde{q}_{04}^{\pm}$ to be $\{(\omega : 1), (\omega^2 : 1)\}$, that is, the zeros of $X^2 + XV + V^2$. Hence the condition in question is

$$q_{X^2} = q_{XV} = q_{V^2}.$$

We do the same for all the line listed in (1) in 4.1. Consequently, we get 12 equalities among the coefficients that stand cyclically

$$q_{X^2} = q_{XV} = q_{V^2} = \cdots = q_{W^2} = q_{WX} = q_{X^2}$$

with respect to the ordering

$$X \rightsquigarrow V \rightsquigarrow Z \rightsquigarrow U \rightsquigarrow Y \rightsquigarrow W \rightsquigarrow X.$$

We get, therefore, precisely 11 linearly independent relations among coefficients.

4.3. Next, we consider the points $\widetilde{q}_{ij}^{\pm}$ on the 3 lines in (2) in 4.1. The line ℓ_{23}, for example, has the homogeneous coordinate $(X : Y)$, and, in terms of it, the intersections with ℓ_{01}, ℓ_{14}, and ℓ_{40} are $(-1 : 1)$, $(0 : 1)$, and $(1 : 0)$, respectively. Hence the situation is parallel to the previous one. The condition is

$$Q(X, Y, 0, -Y, -X, 0) = \lambda(X^2 + XY + Y^2)$$

for some $\lambda \in K$. Since the coefficients of X^2 and Y^2 in the left-hand side are, as we say in the previous step, equal to each other, we have precisely one relation that gives q_{XY}. Similar observation on $\widetilde{q}_{24}^{\pm}$ and $\widetilde{q}_{34}^{\pm}$ gives relations involving q_{YZ} and q_{ZX}, and hence, we have so far $11+3=14$ linearly independent conditions on the coefficients.

4.4. Finally, we look at ℓ_{01} characterized by (3) in 4.1. By an argument similar to that in the previous steps, we deduce that the condition is

$$Q(X,Y,-X-Y,X,Y,-X-Y)=\lambda(X^2+XY+Y^2)$$

for some $\lambda \in K$. Now the coefficients of X^2 in the left-hand side is

$$q_{X^2}+q_{Z^2}-q_{XZ}+q_{U^2}+q_{W^2}-q_{UW}+q_{XU}-q_{XW}-q_{ZU}+q_{ZW}=2q_{X^2};$$

similarly, the coefficient of Y^2 is equal to $2q_{X^2}$, while that of XY is

$$\begin{aligned} q_{XY}-q_{XZ}+q_{XV}-q_{XW}-q_{YZ}+q_{YU}-q_{YW}+2q_{Z^2} & \\ -q_{ZU}-q_{ZV}+2q_{ZW}+q_{UV}-q_{UW}-q_{VW}+2q_{W^2} &= 2q_{X^2}. \end{aligned}$$

Hence the condition in question is already satisfied, and is superfluous. We conclude, therefore, that the condition of passing through the 20 points $\widetilde{q}_{ij}^{\pm}$ is exactly of rank 14, and hence that $\dim V=21-14=7$.

4.5. Since $\widetilde{\Pi}=\{G=0\}$, we have $G\in V$. Moreover, it is clear that $G\notin \mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$, since it cuts out the divisor $\widetilde{\Pi}$ on S. To prove that $F\in V$, it is enough to show that $F=0$ contains $\widetilde{q}_{01}^{\pm}$, for F is clearly $\mathfrak{S}_5$-invariant. This follows from

$$F|_{D(01)=0}=F(X,Y,-X-Y,X,Y,-X-Y)=4(X^2+XY+Y^2)$$

due to (3) in 4.1. Since G is not invariant under the $\mathfrak{S}_5$-action, F is not proportional to G. Since the quadratics $XX^*,YY^*,\ldots,WW^*$ do not contain the monomials $X^2,Y^2,\ldots,W^2$, G does not either, while one sees easily that F contains them. Since

$$XX^*-YY^*,\ YY^*-ZZ^*,\ ZZ^*+UU^*,\ UU^*-VV^*,\ VV^*-WW^*$$

give a basis of $\mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$, every element in $\mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$ does not contain the monomials $X^2,Y^2,\ldots,W^2$. Hence $F\notin \mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$. Since $G\notin \mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$, any linear combination of F and G but 0 does not belong to $\mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$, and hence, we have $V_0\cap \mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))=\{0\}$. Counting the dimension, we get $V=V_0\oplus \mathrm{H}^0(\mathbb{P}^5,\mathscr{I}_S(2))$, as desired.

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Daido Life Insurance Company, 1-2-1, Edobori, Nishi-ku, Osaka, 550-0002, Japan
E-mail: inoue.naoki@daido-life.co.jp

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail: kato@math.kyoto-u.ac.jp