Multiple Symmetric Periodic Solutions to the $2n$-body Problem with Equal Masses

Mitsuru Shibayama

November 30, 2005

Abstract

Using the variational method, Chenciner and Montgomery [2] proved the existence of an eight-shaped orbit of the planar three-body problem with equal masses. Since then a number of solutions to the N-body problem have been discovered. In particular, Ferrario and Terracini [4] proved the existence of symmetric periodic solutions under a quite general setting. In this paper we consider the $2n$-body problem with a certain symmetry and use the results of Ferrario and Terracini to prove the existence of multiple solutions for each n. Some of the solutions we find were already obtained by Ferrario-Terracini [4] and by Chen [1], but their argument does not allow us to distinguish the solutions obtained in this paper. By reducing the problem to the quotient space of the configuration space under the action of the group of symmetries and by observing boundary conditions of the solutions in the quotient space, we are able to distinguish these solutions and conclude that they are indeed distinct solutions. As a byproduct, we can also determine the sign of the angular momentum of masses.

1 Introduction and Main Theorem

This paper is concerned with the Newtonian N-body problem which is given by the following set of ODEs:

$$\ddot{x}_i = -\sum_{j\neq i} m_j \frac{x_i - x_j}{|x_i - x_j|^3} \qquad x_1, \cdots, x_N \in V \tag{1}$$

where $m_j > 0$ and $V = \mathbf{R}^2$ or $\mathbf{R}^3$. Using the variational method Chenciner and Montgomery [2] proved the existence of a new periodic solution of figure-eight shape to the planar ($V = \mathbf{R}^2$) three-body problem.

Since then, a number of periodic and quasi-periodic solutions have been found as minimizers of variational formulation of the N-body problem in various different settings. In particular, Ferrario and Terracini [4] introduced the rotating circle property and showed that a collision-less solution having a certain symmetry exists, provided the group action of the symmetry satisfies the rotating circle property. They proved this result by showing that , for a collision path, it is always possible to decrease the value of the action functional by slightly perturbing the path near the collision time, once the rotating circle property holds. Chen [1] obtained a similar result by estimating the infimum of paths with a collision and by finding a path which has lower value of the action functional than the infimum for collision paths.

In this paper, we consider the planar $2n$-body problem with equal masses:

$$\ddot{x}_i = -\sum_{j\neq i} \frac{x_i - x_j}{|x_i - x_j|^3} \qquad x_1, \cdots, x_{2n} \in V = \mathbf{R}^2 \tag{2}$$

and prove the existence of multiple solutions with a certain symmetry. This problem has already been studied by Ferrario-Terracini [4] and by Chen [1] where, for each n, at least one periodic solution with the symmetry was obtained by a variational method. In this paper, we shall show that the same problem indeed has $[n/2]$ periodic solutions. In order to distinguish these solutions by the variational method, we carefully study the action of symmetry group to the solutions and show that these solutions are a minimizer of a variational problem with different boundary conditions. Hence one can conclude that they are truly distinct. Our argument also allows us to determine the sign of the angular momentum of masses rigorously. The main result of this paper is as follows:

Main Theorem. *For any $T > 0$, each $n \geq 2$ and each $p = 1, \cdots, \left[\frac{n}{2}\right]$ (where $[s]$ is the greatest integer not exceeding s) there exists a T-periodic solution $x(t) = (x_1(t), \cdots, x_{2n}(t))$ of the $2n$-body problem (2) that satisfies the following symmetry property: There is a smooth closed curve $\alpha_1 : \mathbf{R}/T\mathbf{Z} \to \mathbf{R}^2 = \mathbf{C}$ and $\alpha_2, \cdots, \alpha_{2d}$ (with d being the greatest common divisor of n and p) are defined from α_1 by*

$$\begin{aligned} \alpha_2(t) &= \exp\left(\frac{2\pi i}{n}\right)\alpha_1(-t) \\ \alpha_3 = \omega\alpha_1,\ \alpha_5 &= \omega\alpha_3, \cdots, \alpha_{2d-1} = \omega\alpha_{2d-3} \\ \alpha_4 = \omega\alpha_2,\ \alpha_6 &= \omega\alpha_4, \cdots \quad, \alpha_{2d} = \omega\alpha_{2d-2}. \end{aligned} \tag{3}$$

with $\omega = \exp\left(\frac{2\pi i}{n}\right)$, such that the solution $x(t) = (x_1(t), \cdots, x_{2n}(t))$ is given by

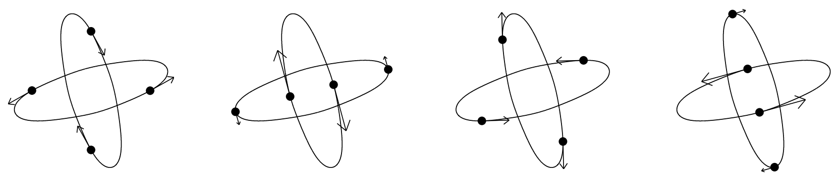
$$\begin{aligned} x_1(t) = \alpha_1(t), \qquad & x_2(t) = \alpha_2(t), \qquad \cdots \qquad, x_{2d}(t) = \alpha_{2d}(t) \\ x_{2d+1}(t) = \alpha_1\left(t + 2\overline{T}\right), & x_{2d+2}(t) = \alpha_2\left(t + 2\overline{T}\right), \cdots, x_{4d}(t) = \alpha_{2d}\left(t + 2\overline{T}\right) \\ & \vdots \\ x_{2d(l-1)+1}(t) = \alpha_1\left(t + 2(l-1)\overline{T}\right), & \qquad \cdots \qquad, x_{2n}(t) = \alpha_{2d}\left(t + 2(l-1)\overline{T}\right) \end{aligned} \tag{4}$$

where $\ell = \frac{n}{d}$, $\overline{T} = \frac{T}{2\ell}$. Furthermore, the closed curve α_1 is invariant under the rotation by $\frac{2\pi}{l}$, its angular momentum is always positive (i.e. $\alpha_1 \wedge \dot{\alpha}_1 > 0$), and it revolves around the origin $\frac{p}{d}$ times the period T.

We give several examples below.

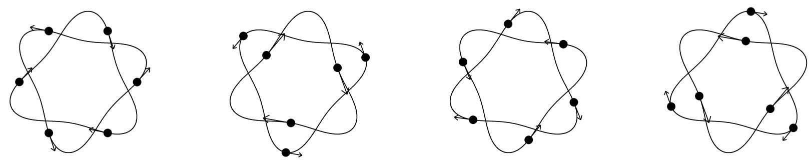
Example 1. *The case $n = 2$ (4-body), $p = 1$.*

$t = 0$ $\quad$ $t = \frac{T}{8}$ $\quad$ $t = \frac{T}{4}$ $\quad$ $t = \frac{3T}{8}$



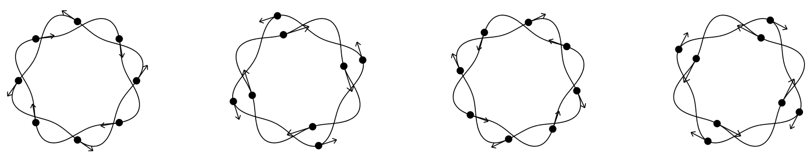
Example 2. *The case $n = 3$ (6-body), $p = 1$.*

$t = 0$ $\quad$ $t = \frac{T}{12}$ $\quad$ $t = \frac{T}{6}$ $\quad$ $t = \frac{T}{4}$



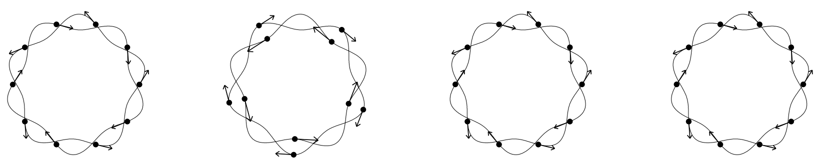
Example 3. *The case $n = 4$ (8-body), $p = 1$.*

$t = 0$ $\quad$ $t = \frac{T}{16}$ $\quad$ $t = \frac{T}{8}$ $\quad$ $t = \frac{3T}{16}$

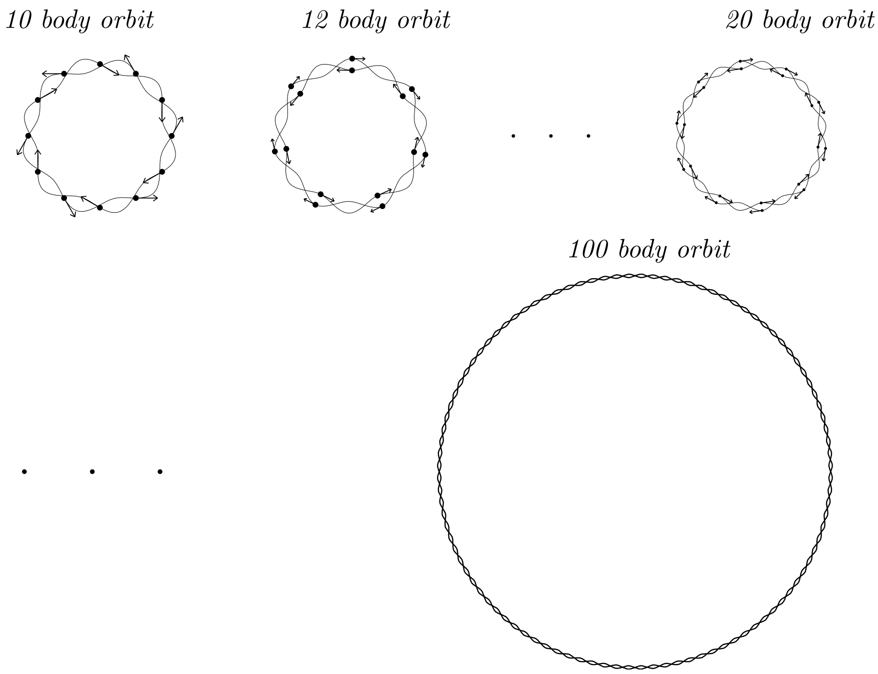


Example 4. *The case $n = 5$ (10-body), $p = 1$.*

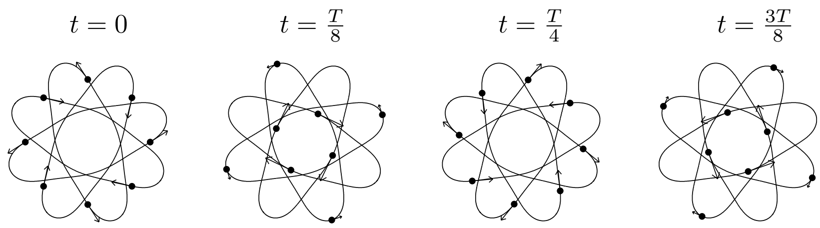
$t = 0$ $\quad$ $t = \frac{T}{20}$ $\quad$ $t = \frac{T}{10}$ $\quad$ $t = \frac{3T}{20}$



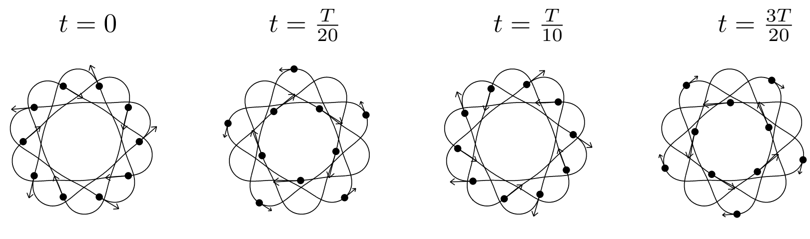
Example 5. *The case* $n = 6, 7, \ldots, 10, \ldots, 50$ *(*$12, 14, \ldots, 20, \ldots, 100$*-body),* $p = 1$*.*



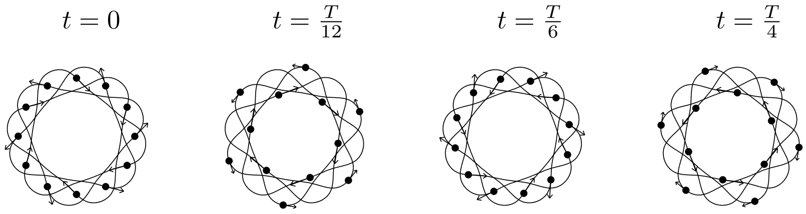
Example 6. *The case* $n = 4$ *(8-body),* $p = 2$*.*



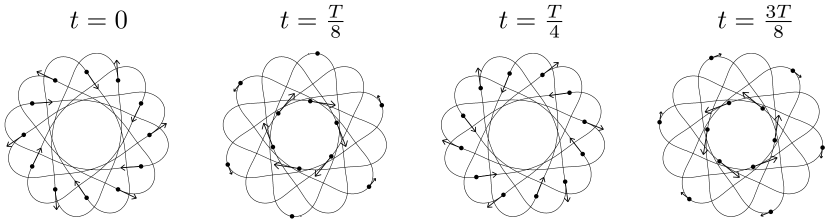
Example 7. *The case* $n = 5$ *(10-body),* $p = 2$*.*



Example 8. *The case* $n = 6$ *(12-body),* $p = 2$*.*



Example 9. *The case $n = 6$ (12-body), $p = 3$.*



For the proof of Main Theorem, we consider the action of groups $G_{n,p}$ ($p = 1, \ldots, [n/2]$) which makes certain closed curves invariant, and prove the existence of solutions to the $2n$-body problem which move along the closed curve and invariant under the group action. In this setting, masses always form a pair of regular n-gons, and p determines the angle of rotation of masses from a moment when they form a regular $2n$-gon to the next such moment. For each n there are $[n/2]$ choices for the groups $G_{n,p}$.

Our proof is based on the results of Ferrario and Terracini [4]. Notice, however, that the sets of invariant closed curves under the $G_{n,p}$-action for a fixed n and various $p = 1, \ldots, [n/2]$ may not be mutually disjoint, and therefore it is not obvious whether the solutions $x_{n,p}$ obtained as minimizers are mutually distinct. For example $x_{4,1}$ belongs to the symmetry classes for $p = 1$ and $p = 2$, so it might be possible that $x_{4,1}$ could be considered essentially the same as $x_{4,2}$. It is, however , not the case as one can see in Example 3 and Example 6.

Ferrario-Terracini [4] and Chen [1] both proved the existence of at least one solution for each n, but their argument did not allow to determine to which class of symmetry given by p the obtained solution belongs, because the invariant sets of the group action are not disjoint. Here we shall prove the existence of $[n/2]$ solutions for each n, which belong to distinct classes of symmetry for different p.

More precisely, we shall argue as follows. The kinetic energy and the self-potential energy are given by

$$K(\dot{x}) = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{x}_i|^2, \quad U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|}$$

and the Lagrangian is given by

$$L(x, \dot{x}) = K(\dot{x}) + U(x).$$

Equation (1) is equivalent to the variational problem with respect to the action functional

$$\mathcal{A}(x) = \int_0^T L(x, \dot{x})dt. \tag{5}$$

Here we consider the planar $2n$-body problem with equal mass $m_1 = \cdots = m_{2n} = 1$ restricted to the class Λ_T of T-periodic solutions which lie in the curves that are invariant under the group action given below: Let G be a group and let

$$\begin{aligned} \tau &: G \to O(2) \\ \rho &: G \to O(2) \\ \sigma &: G \to \mathfrak{S}_{2n}, \end{aligned}$$

be homomorphisms. We define the action of G to Λ_T by

$$g \cdot ((x_1, \dots, x_N)(t)) = (\rho(g)x_{\sigma(g^{-1})(1)}, \dots, \rho(g)x_{\sigma(g^{-1})(N)})(\tau(g^{-1})t)$$

for $g \in G$ and $x(t) = (x_1, \dots, x_N)(t) \in \Lambda_T$. More concretely, we take G as $G_{n,p} = \langle g_n, h_{n,p} \rangle$, where

$$\begin{aligned} \rho(g_n) &= \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \\ \sigma(g_n) &= (1, 2, \dots, 2n) \\ \tau(g_n) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \rho(h_{n,p}) &= 1 \\ \sigma(h_{n,p}) &= (1, 3, \dots, 2n-1)^{-p}(2, 4, \dots, 2n)^p \\ \tau(h_{n,p}) &= \begin{pmatrix} \cos\frac{2\pi d}{n} & -\sin\frac{2\pi d}{n} \\ \sin\frac{2\pi d}{n} & \cos\frac{2\pi d}{n} \end{pmatrix}. \end{aligned} \tag{6}$$

Here $\sigma(g_n)$ is a cyclic permutation and $\sigma(h_{n,p})$ is a product of cyclic permutations.

By the Palais principle, it follows that a critical point of the action functional restricted to Λ_T under the action of $G_{n,p}$ is that of the original variational problem. The existence of a minimizer of the restricted problem is proven by coercivity. It is not so obvious that the minimizer does not have a collision, but this can be proven by the Ferrario-Terracini method, since the group action of $G_{n,p}$ has the rotating circle property. Thus we have a non-collision solution for each n and $p = 1, \dots, [n/2]$, but we have to prove that they are indeed distinct. For that purpose, we take the quotient space of the configuration space under the group action, and reconsider the minimizing solutions in the quotient space. We can then show that these minimizing solutions have different boundary conditions for different p, and therefore we can conclude that these solutions are indeed dictinct. Moreover, we can also obtain information of the angular momentum of the solutions, and determine its sign rigorously.

We begin the proof of Main Theorem from the next section. Section 2 collects some known results about the existence of collision-free minimizers for symmetric curves (Palais, Ferrario-Terracini). In Section 3 we apply these results to the $2n$-body problem with equal masses to prove the existence of symmetric periodic solutions. In Section 4 we show that these solutions are essentially distinct. In Section 5 we estimate the angular momentum of each mass and determine the sign. Section 6 presents sketch of a proof of the existence of similar solutions in the spatial case ($V = \mathbf{R}^3$).

2 Symmetry and Collision-free Minimizer

In this section we discuss the planar or spatial N-body problem(1). Equation (1) is equivalent to the variational problem with respect to the action functional(5). Let $\mathcal{X}$ be defined by

$$\mathcal{X} = \left\{ x \in V^N \,\middle|\, \sum_{i=1}^{N} m_i x_i = 0 \right\}$$

and let

$$\Delta_{ij} = \{x \in \mathcal{X} | x_i = x_j\}, \quad \Delta = \cup_{i<j} \Delta_{ij}.$$

$\mathcal{X} \backslash \Delta$ is denoted by $\hat{\mathcal{X}}$. $\mathcal{A}$ is defined on the Sobolev space $\Lambda_T = H^1(\mathbf{T}, \mathcal{X})$ of $\mathcal{X}$-valued functions on $\mathbf{T} = \mathbf{R}/T\mathbf{Z}$. $\hat{\Lambda}_T = H^1(\mathbf{T}, \hat{\mathcal{X}})$ is the subspace of collision-free paths.

We consider an action of a finite group G to Λ_T which has the following property: there are representations

$$\begin{aligned}
\tau &: G \to O(2) \\
\rho &: G \to O(\dim V) \\
\sigma &: G \to \mathfrak{S}_N,
\end{aligned}$$

such that for $g \in G, x(t) = (x_1, \ldots, x_N)(t) \in \Lambda_T$

$$g \cdot ((x_1, \ldots, x_N)(t)) = (\rho(g) x_{\sigma(g^{-1})(1)}, \ldots, \rho(g) x_{\sigma(g^{-1})(N)})(\tau(g^{-1})t)$$

Let Λ_T^G and $\hat{\Lambda}_T^G$ be the set of loops fixed by G in Λ_T and $\hat{\Lambda}_T$ respectively, and let $\mathcal{A}^G = \mathcal{A}|_{\Lambda_T^G}$.

Proposition 1 (Palais principle [3]). *If $\mathcal{A}$ is invariant under the group action of G, then a critical point of $\mathcal{A}^G$ in $\hat{\Lambda}_T^G$ is that of $\mathcal{A}$ in $\hat{\Lambda}_T$.*

The group G acts on $\mathcal{X}$ by ρ and σ. $\mathcal{X}^G$ and $\hat{\mathcal{X}}^G$ are defined as the set of points fixed by G in $\mathcal{X}$ and $\hat{\mathcal{X}}$, respectively.

Proposition 2 ([4] Proposition 4.1). *If $\mathcal{X}^G = \{0\}$, then there exists a minimizer of $\mathcal{A}^G$ in Λ_T^G.*

For a fixed $t \in \mathbf{T}$, let G_t be the isotropy subgroup of G at t under the τ-action and for $i \in \{1, \dots, N\}$, G_t^i be the isotropy subgroup of G_t at i under the σ-action, namely,

$$G_t = \{g \in G | \tau(g)t = t\},$$
$$G_t^i = \{g \in G_t | \sigma(g)i = i\}.$$

Definition 1. *We say a finite group G acts on Λ_T with the rotating circle property (or G has the rotating circle property), if for any G_t and for at least $N-1$ indices i there exists a circle in V such that G_t acts on the circle by rotation and that the circle is contained in $(V)^{G_t^i}$.*

Proposition 3 ([4] Theorem 10.10). *Let G be a finite group acting on Λ_T. Suppose $m_i = m_{\sigma(g)(i)}$ for all $g \in G$ and all $i \in \{1, \dots, N\}$. If every G_t has the rotating circle property, then any local minimizer of $\mathcal{A}^G$ is collision-free.*

3 Symmetric solutions to the $2n$-Body Problem

We consider the case where $V = \mathbf{R}^2$ and N is even, say $N = 2n$, and let $m_i = 1$ for every i. Fix $p = 1, \dots, [n/2]$, where $[s]$ is the greatest integer no more than s. We take the group $G_{n,p}$ as introduced at (6) in Section 1. The fundamental domain is $[0, \overline{T}]$ where $\overline{T} = \frac{d}{2n}T$, i.e. $[0, \overline{T}]$ is the minimal closed interval such that the projection

$$\Lambda^G \to H^1([0, \overline{T}], \mathcal{X})$$

is injective. It is obvious that $\mathcal{A}$ is invariant under the group $G_{n,p}$, and hence we can apply Proposition 1. From $\mathcal{X}^{G_{n,p}} = \{0\}$, Proposition 2 leads to the exisitence of a minimizer in $\Lambda_T^{G_{n,p}}$, which is collision-free, because of the following proposition and Proposition 3:

Proposition 4. *For every n and p, the group $G_{n,p}$ has the rotating circle property.*

Proof. It is enough to show the claim for $t \in [0, \overline{T}]$, since $[0, \overline{T}]$ is the fundamental domain. Clearly we have

$$(G_{n,p})_t = \begin{cases} \langle g_n^2 \rangle & (t \in (0, \overline{T})) \\ \langle g_n \rangle & (t = 0) \\ \langle h_{n,p} g_n \rangle & (t = \overline{T}). \end{cases}$$

Thus $(G_{n,p})_t^i = \{1\}$ for $i \in \{1, \dots, 2n\}$ because in the case $t = \overline{T}$, $\sigma(h_{n,p}g_n)$ interchange odd particles with even particles and $\sigma(h_{n,p}g_n)^2 = (1, 2, \dots, 2n)^2$. The other cases are trivial. If we take a circle $\mathbf{S} \subset \mathbf{R}^2$ whose center is the origin, then g_n^2, g_n and $h_{n,p}g_n$ act on $\mathbf{S}$ by rotation and $\mathbf{S}$ is contained in $(\mathbf{R}^2)^{(G_{n,p})_t^i} = \mathbf{R}^2$. $\square$

Consequently we have shown that, for every n and p, a minimizer $x_{n,p}$ of $\mathcal{A}^{G_{n,p}}$ exists and it is collision-free. It turns out that $x_1, \ldots, x_{2n}$ move along different closed curves $\alpha_1, \ldots, \alpha_{2d}$, that these curves are related as (3), and that the other particles move along the curves as (4).

4 Multiplicity of the Solutions

If $p \equiv lq \mod n$, then $\Lambda_T^{G_{n,p}}$ contains $\Lambda_{lT}^{G_{n,q}}$. We show that $x_{n,p}$ and $x_{n,q}$ are essentially different. Since

$$\begin{aligned} \rho(g_n^2) &= \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \\ \sigma(g_n^2) &= (1,3,5,\ldots,2n-1)(2,4,6,\ldots,2n) \\ \tau(g_n^2) &= 1, \end{aligned}$$

the configuration always consists of two regular n-gons, namely, n particles $m_1, m_3, \ldots, m_{2n-1}$ always form a regular n-gon and n particles $m_2, m_4, \ldots, m_{2n}$ do so, too. Thus we can assume that

$$\begin{array}{llll} x_1 = \omega x_3, & x_5 = \omega x_3, & \cdots, & x_{2n-1} = \omega x_{2n-3}, \\ x_4 = \omega x_2, & x_6 = \omega x_4, & \cdots, & x_{2n} = \omega x_{2n-2}, \end{array} \tag{7}$$

where $\omega = \exp(\frac{2\pi i}{n})$. Under the restriction of (7), letting $q_1 = x_1, q_2 = x_{n+1}$, the Lagrangian becomes

$$L(q_1, q_2, \dot{q_1}, \dot{q_2}) = \frac{n}{2} K(\dot{q_1}, \dot{q_2}) + nU(q_1, q_2),$$

where

$$\begin{aligned} K(\dot{q_1}, \dot{q_2}) &= |\dot{q_1}|^2 + |\dot{q_2}|^2 \\ U(q_1, q_2) &= a_n \left(\frac{1}{|q_1|} + \frac{1}{|q_2|} \right) + \sum_{j=1}^{n} \frac{1}{|q_1 - \omega^j q_2|} \\ a_n &= \frac{1}{4} \sum_{j=1}^{n-1} \frac{1}{|\sin\frac{\pi j}{n}|}. \end{aligned}$$

It is possible to alter the above Lagrangian to $L = \frac{1}{2}K + U$.

We consider the group action of S^1 to $\mathbf{C}^2$ by

$$z \cdot (q_1, q_2) = (zq_1, zq_2), \quad z \in S^1, (q_1, q_2) \in \mathbf{C}^2.$$

The quotient space $(\mathbf{C}^2 - \{\mathbf{0}\})/S^1$ under the action is realized by the following projection:

$$\begin{aligned} \pi: \quad \mathbf{C^2} - \{\mathbf{0}\} &\longrightarrow \quad \mathbf{R}^3 - \{\mathbf{0}\} \left(\cong (\mathbf{C}^2 - \{\mathbf{0}\})/S^1\right) \\ (q_1, q_2) \quad &\longmapsto (u_1, u_2, u_3) := \left(|q_1|^2 - |q_2|^2, 2\mathrm{Re}\,(q_1\overline{q_2}), 2\mathrm{Im}\,(q_1\overline{q_2})\right). \end{aligned}$$

In the quotient space, collisions of the orginal particles correspond to

$$A_{\pm} := \{(\pm s, 0, 0) \,|\, s \in \mathbf{R}_0\},$$
$$B_{2j} := \left\{ \left(0, s\cos\left(\frac{2\pi j}{n}\right), s\sin\left(\frac{2\pi j}{n}\right)\right) \middle| \, s \in \mathbf{R}_{>0} \right\}, \ j \in \mathbf{Z},$$

and hence we have $\pi(\Delta) = A_+ \cup A_- \cup_{j\in\mathbf{Z}} B_{2j}$, where $A_{\pm}$ represent collisions where one of the regular n-gons collapses to the origin, and B_{2j} represent collisions where two regular n-gons coalesce.

$$B_{2j-1} := \left\{ \left(0, s\cos\left(\frac{(2j-1)\,\pi}{n}\right), s\sin\left(\frac{(2j-1)\,\pi}{n}\right)\right) \middle| \, s \in \mathbf{R}_{>0} \right\}, \ j \in \mathbf{Z}$$

represents configurations where $2n$ particles form a regular $2n$-gon.

The potential energy $U_{\rm red}$ is well-defined on $\mathbf{R}^3 - \pi(\Delta)$. In fact,

$$\begin{aligned} U_{\rm red}(u_1, u_2, u_3) = \sqrt{2}\, a_n \Bigg(&\frac{1}{\sqrt{\|\mathbf{u}\| + u_1}} + \frac{1}{\sqrt{\|\mathbf{u}\| - u_1}} \Bigg. \\ &\Bigg. + \sum_{j=1}^{n} \frac{1}{\sqrt{\|\mathbf{u}\| - u_2\cos\left(\frac{2\pi j}{n}\right) - u_3\sin\left(\frac{2\pi j}{n}\right)}} \Bigg), \\ &\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3. \end{aligned}$$

On the other hand, the total angular momentum is zero, since it is invariant under the action by g_n. Hence the kintetic energy can be defined on the quotient space. In fact,

$$K_{\rm red}(\mathbf{u}, \dot{\mathbf{u}}) = \frac{\|\dot{\mathbf{u}}\|^2}{4\|\mathbf{u}\|}, \qquad (\mathbf{u}, \dot{\mathbf{u}}) \in T(\mathbf{R}^3 \backslash \{0\}).$$

So the orginal action functional corresponds to $\mathcal{A}_{\rm red} = \int L_{\rm red} dt$, where $L_{\rm red} = \frac{1}{2} K_{\rm red} + U_{\rm red}$.

Here the invariance under g_n is associated with

$$\begin{pmatrix} u_1(-t) \\ u_2(-t) \\ u_3(-t) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ 0 & \sin\left(\frac{2\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}\right) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix},$$

which means that $u(t)$ and $u(-t)$ are symmetric with respect to B_{-1}. In particular $u(0) \in B_{-1}$. Similarly, the invariance under $h_{n,p}$ is associated with

$$\begin{pmatrix} u_1\left(t + \overline{T}\right) \\ u_2\left(t + \overline{T}\right) \\ u_3\left(t + \overline{T}\right) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\left(-\frac{2\pi p}{n}\right) & -\sin\left(-\frac{2\pi p}{n}\right) \\ 0 & \sin\left(-\frac{2\pi p}{n}\right) & \cos\left(-\frac{2\pi p}{n}\right) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix},$$

where $\overline{T} = \frac{d}{2n} T$. Since the fundamental domain is $[0, \overline{T}]$, $\hat{x}_{n,p} = \pi(x_{n,p})$ coincides with a minimizer of $\mathcal{A}_{\rm red}$ satisfying the boundary condition $\hat{x}_{n,p}(0) \in B_{-1}$ and $\hat{x}_{n,p}(\overline{T}) \in B_{-2p-1}$.

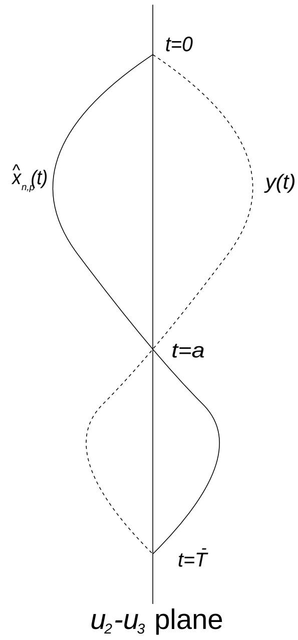


Figure 1: $\hat{x}_{n,p}$ and y

Proposition 5. *The u_1-component of $\hat{x}_{n,p}$ does not have zero on $(0, \overline{T})$. In particular, if $p \neq q$, $\hat{x}_{n,p}$ (or $x_{n,p}$) is essentially different from $\hat{x}_{n,q}$ (or $x_{n,q}$), i.e. if $p \equiv lq \mod n$ and $p \neq q$, then $x_{n,p} \in \Lambda_T^{G_{n,p}} \backslash \Lambda_{lT}^{G_{n,q}}$.*

Proof. Since $\hat{x}_{n,p}$ has no collision, $C = \{(u_1, u_2, u_3) | u_1 = 0\}$ does not contain $\hat{x}_{n,p}([0, \overline{T}])$. Since C is invariant, the u_1-component of $\frac{d\hat{x}_{n,p}}{dt}(0)$ is nonzero. Choose the smallest $a > 0$ such that the u_1-component of $\hat{x}_{n,p}$ is nonzero on $(0, a)$. By the symmetry, $y := Q\hat{x}_{n,p}$ is also a minimizer, where $Q = \mathrm{diag}(-1, 1, 1)$ (See figure 1).

Assume $a < \overline{T}$. Define z by

$$z(t) = \begin{cases} y(t) & t \in [0, a] \\ \hat{x}_{n,p}(t) & t \in (a, \overline{T}]. \end{cases}$$

This is also a minimizer but it cannot be smooth, which is a contradiction, since any minimizer must be smooth. □

5 Angular Momentum of the Each Particle

For each solution obtained above, we determine the sign of the angular momentum of each particle. More precisely,

Proposition 6. *Let $x = (x_1, \ldots, x_{2n})$ be a solution as above. Then the angular momentum of $x_1, x_3, x_5, \ldots, x_{2n-1}$ is always positive and that of $x_2, x_4, x_6, \ldots, x_{2n}$ is always negative.*

Proof. The angular momentum of x_k with k odd is the same as that of x_1. Let $\hat{x} = (u_1, u_2, u_3) = \pi(x)$, where π is the projection to the quotient space. From

$$\theta = \arg x_1, \qquad \phi = \arctan \frac{u_3}{u_2},$$

it is easy to show

$$\dot{\theta} = -\frac{1}{2}\left(1 - \frac{u_1}{||\mathbf{u}||}\right)\dot{\phi}.$$

Since $\hat{x}$ is the minimizer, $\dot{\phi} < 0$ and so $\dot{\theta} > 0$. It means that x_1 always has positive angular momentum. Since the total angular momentum is zero, the angular momentum of x_2, and hence that of $x_4, \ldots, x_{2n}$, is always negative. $\square$

It completes the proof of our Main Theorem.

6 Spatial case

We can apply an analogous argument to the spatial case ($V = \mathbf{R}^3$). We consider the spatial $2n$-body problem with equal masses, and take the group $G_{n,p}$ generated by the following g_n and $h_{n,p}$:

$$\begin{aligned}
\rho(g_n) &= \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\sigma(g_n) &= (1, 2, \ldots, 2n) \\
\tau(g_n) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\rho(h_{n,p}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\sigma(h_{n,p}) &= (1, 3, \ldots, 2n-1)^{-p}(2, 4, \ldots, 2n)^p \\
\tau(h_{n,p}) &= \begin{pmatrix} \cos\frac{2\pi d}{n} & -\sin\frac{2\pi d}{n} \\ \sin\frac{2\pi d}{n} & \cos\frac{2\pi d}{n} \end{pmatrix},
\end{aligned}$$

where d is the greatest common divisor of n and p. Then there exists a collision-free minimizer $x_{n,p}$ which is invariant under the group action. Similarly we can use the path corresponding to $x_{n,p}$ in $[\mathbf{C}^3 \backslash \{0\}]/SO(3)$ to show the existence of multiple solutions.

References

[1] K. -C. Chen, Binary decompositions for planar N-body problems and symmetric periodic solutions. *Arch. Rational Mech. Anal.* **170** (2003), 247-276.

[2] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in The accompanying figure shows the case of equal masses, *Ann. Math.* **152** (2000), 881-901.

[3] R. S. Palais, The principle of symmetric criticality. *Comm. Math. Phys.* **69** (1979), 19-30.

[4] D. Ferrario and S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invent. Math.* **155** (2004), no. 2, 305–362.