A CHARACTERIZATION OF SYMMETRIC SIEGEL DOMAINS BY CONVEXITY OF CAYLEY TRANSFORM IMAGES

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Abstract. In this paper we show that a homogeneous Siegel domain is symmetric if and only if its Cayley transform image is convex. Moreover, this convexity forces the parameter of the Cayley transform to be a specific one, so that the Cayley transform coincides with the inverse of the Cayley transform introduced by Korányi and Wolf.

1. Introduction

A homogeneous Siegel domain is a higher dimensional analogue of the right (or upper) half plane in $\mathbb{C}$, and is mapped to a bounded domain by the Cayley transforms introduced by [17]. Among homogeneous Siegel domains, we have an important subclass consisting of symmetric ones. In our previous paper [9], we gave a symmetry characterization for tube domains (homogeneous Siegel domains of type I) by convexity of the Cayley transform images, and in [7], for quasisymmetric Siegel domains. This article is the final step of these works and establishes the same type of symmetry characterization theorem for general homogeneous Siegel domains.

We mention here some of the works about characterizations of symmetric Siegel domains: a characterization by a certain norm equality related to the Cayley transform image [15], one by the commutativity of the Berezin transform and the Laplace-Beltrami operator [16], and one by the harmonicity of the Poisson-Hua kernel [18]. In the latter two, the geometric backgrounds of the symmetry characterizations are clarified through norm equalities involving the Cayley transforms. In [3], we can find several characterizations of symmetric Siegel domains concerning the isotropy representation and the action of the automorphism group of the domain. Differential geometric characterizations by means of the Bergman metric are given in [4] and [2], and an algebraic one in terms of the defining data of Siegel domains in [23, Theorem V.3.5] and [5, II, Sätze 3.3, 3.4].

Let us present the convexity of Cayley transform image of a symmetric Siegel domain. In the case of one complex variable, the Cayley transform

$$w \mapsto \frac{w-1}{w+1} \qquad (w \in \mathbb{C})$$

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maps the right half plane to the open unit disc, which is obviously a convex set. We have a similar situation for symmetric Siegel domains. Since a symmetric Siegel domain is a Hermitian symmetric space of non-compact type, it has a canonical bounded realization, namely, the Harish-Chandra realization. In [11], Korányi and Wolf defined in a Lie-theoretic way (the inverse of) the Cayley transform which maps a symmetric Siegel domain to its Harish-Chandra realization. It is known that the Harish-Chandra realization coincides with the open unit ball for the spectral norm defined for the Jordan triple system canonically associated with the domain (we refer the reader to [12, §10], [11] and [7] for details). Thus the Cayley transform image of a symmetric Siegel domain is a convex set. We shall show that this convexity characterizes symmetric Siegel domains among homogeneous ones. Before proceeding, we would like to mention that it is shown in [13] that the Harish-Chandra realization of a symmetric Siegel domain is characterized essentially among bounded realizations by its convexity. In other words, the Cayley transform is essentially the only bounded convex realizaion of a symmetric Siegel domain.

In this article we deal with the family of Cayley transforms defined by Nomura [17]. This family is parametrized by the admissible linear forms on the normal j-algebra associated with the Siegel domain. If the domain is quasisymmetric and the parameter is a specific one, the corresponding Cayley transform is the same as Dorfmeister's one given in [6] which we used in [7], and in particular if the domain is symmetric, our Cayley transform with the specific parameter coincides with Korányi-Wolf's one. Moreover, our family includes Penney's Cayley transform defined in [19] which is associated with Vinberg's $*$-map of the underlying cone of the domain, and Nomura's one associated with the Bergman kernel (resp. the Szegö kernel) of the domain appearing in [14], [15] and [16] (resp. [18]).

Let us fix the notation in order to present our results. Let Ω be a homogeneous convex cone in a real vector space V. We put $W := V_{\mathbb{C}}$, the complexification of V. Let U be another complex vector space and $Q : U \times U \to W$ an Ω-positive, Hermitian sesquilinear map. The Siegel domain D for these data is defined by

$$D := \{(u, w) \in U \times W \mid \operatorname{Re} w - \tfrac{1}{2}Q(u, u) \in \Omega\}.$$

In case $U = \{0\}$, the domain D is called a tube domain. We note that the tube domain $\Omega + iV$ is contained in D in such a way that $D \cap (\{0\} \times W) = \{0\} \times (\Omega + iV)$. We denote by $C_{\mathbf{s}}$ the Cayley transform for $\Omega + iV$, where $\mathbf{s}$ is the parameter of the family of Cayley transforms (see Section 3 for the definition). Using $C_{\mathbf{s}}$, we introduce the Cayley transform $\mathcal{C}_{\mathbf{s}}$ for D. If D is a tube domain, then $\mathcal{C}_{\mathbf{s}}$ reduces to $C_{\mathbf{s}}$.

Our first main theorem is a refinement of [9, Theorem 1]. Let $\Omega^{\mathbf{s}}$ be the dual cone of Ω. For the tube domain $\Omega^{\mathbf{s}} + iV$, the Cayley transform $C^*_{\mathbf{s}}$ is defined in a way similar to $C_{\mathbf{s}}$. In [9, Theorem 1], we characterized symmetric tube domains by requiring the convexity of both of $C_{\mathbf{s}}(\Omega + iV)$ and $C^*_{\mathbf{s}}(\Omega^{\mathbf{s}} + iV)$. In this paper we show that the condition concerning $C^*_{\mathbf{s}}$ can be removed:

Theorem 1.1. *$C_{\mathbf{s}}(\Omega+iV)$ is a convex set if and only if $\Omega+iV$ is symmetric and the parameter $\mathbf{s}$ is a specific one so that $C_{\mathbf{s}}$ coincides with the Cayley transform defined in terms of the Jordan algebra structure associated with Ω.*

Here we note that $\Omega + iV$ is symmetric if and only if Ω is a symmetric cone. Our second main theorem generalizes Theorem 1.1 to the case of homogeneous Siegel domains.

Theorem 1.2. *$\mathcal{C}_{\mathbf{s}}(D)$ is a convex set if and only if D is symmetric and the parameter $\mathbf{s}$ is a specific one so that $\mathcal{C}_{\mathbf{s}}$ coincides with Korányi-Wolf's Cayley transform.*

Our way of proving Theorem 1.2 is as follows. First, the convexity of $\mathcal{C}_{\mathbf{s}}(D)$ implies the convexity of $C_{\mathbf{s}}(\Omega + iV)$. By Theorem 1.1, $\Omega + iV$ is symmetric and the parameter $\mathbf{s}$ is a specific one. With this we show that D is quasisymmetric and $\mathcal{C}_{\mathbf{s}}$ is identical with Dorfmeister's Cayley transform which we used in [7]. Then by [7, Theorem 2.6] we conclude that D is symmetric.

The organization of this paper is as follows. In Section 2, we summarize the structure theory of normal j-algebras. In Section 3.1, we introduce the pseudoinverse maps and then in Section 3.2 the Cayley transforms of homogeneous Siegel domains. There we present the precise statement of Theorem 1.2 as Theorem 3.1. In Section 4, assuming that the domain is quasisymmetric, we compare our Cayley transform with Dorfmeister's one. We collect in Section 5 some facts which hold without any restrictions on the homogeneous Siegel domain for later use. The proof for Theorem 1.1 (the precise statement is Theorem 6.1) is given in Section 6. Theorem 1.2 is proved in Section 7.

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2. Homogeneous Siegel domains

The structure of a homogeneous Siegel domain is described in terms of a normal j-algebra. Our references are [20], [21] and [22]. A triple $(\mathfrak{g}, J, \omega)$ of a split solvable real Lie algebra $\mathfrak{g}$, a linear operator J on $\mathfrak{g}$ with $J^2 = -I$ and a linear form ω on $\mathfrak{g}$ is called a *normal j-algebra* if the following two conditions hold:

$$J([X,Y] - [JX,JY]) = [JX,Y] + [X,JY] \text{ for all } X, Y \in \mathfrak{g}, \tag{2.1}$$

$$\langle x|y\rangle_\omega := \langle [Jx,y], w\rangle \text{ defines a } J\text{-invariant inner product on } \mathfrak{g}. \tag{2.2}$$

Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. We put $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a} := \mathfrak{n}^\perp$, the orthogonal complement of $\mathfrak{n}$ with respect to the inner product $\langle\cdot|\cdot\rangle_\omega$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that $\operatorname{ad}\mathfrak{a}$ is a set of simultaneously diagonalizable operators on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{a} \oplus \sum_{\alpha\in\Delta} \mathfrak{n}_\alpha$ be the corresponding eigenspace decomposition of $\mathfrak{g}$, where Δ is a finite subset of $\mathfrak{a}^* \setminus \{0\}$, and for $\alpha \in \mathfrak{a}^*$, we have put

$$\mathfrak{n}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \langle H, \alpha\rangle X \text{ for all } H \in \mathfrak{a}\}.$$

The subspaces $\mathfrak{n}_\alpha$ ($\alpha \in \Delta$) are orthogonal to each other relative to the inner product $\langle\cdot|\cdot\rangle_\omega$. The number $r := \dim \mathfrak{a}$ is called the rank of $\mathfrak{g}$. We can choose a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ so that with $E_m := -JH_m$ ($m = 1, \ldots, r$), we have $[H_j, E_k] = \delta_{jk}E_k$ ($1 \le j \le k \le r$). Let $\alpha_1, \ldots, \alpha_r$ be the dual basis of $\mathfrak{a}^*$ with respect to

$H_1, \dots, H_r$. Then the elements of Δ are of the following forms (not all possibilities need to occur):

$$\tfrac{1}{2}(\alpha_k - \alpha_j) \quad (1 \le j < k \le r), \qquad \tfrac{1}{2}\alpha_m \quad (1 \le m \le r),$$
$$\alpha_m \quad (1 \le m \le r), \qquad \tfrac{1}{2}(\alpha_k + \alpha_j) \quad (1 \le j < k \le r),$$

and we have $\mathfrak{n}_{\alpha_m} = \mathbb{R}E_m$ $(m = 1, \dots, r)$. We set

$$\mathfrak{n}(0) := \sum_{j<k} \mathfrak{n}_{(\alpha_k - \alpha_j)/2}, \qquad \mathfrak{g}(0) := \mathfrak{a} \oplus \mathfrak{n}(0),$$
$$\mathfrak{g}(1/2) := \sum_{m=1}^{r} \mathfrak{n}_{\alpha_m/2}, \qquad \mathfrak{g}(1) := \sum_{m=1}^{r} \mathfrak{n}_{\alpha_m} \oplus \sum_{j<k} \mathfrak{n}_{(\alpha_k + \alpha_j)/2},$$

and put $H := H_1 + \cdots + H_r$, $E := E_1 + \cdots + E_r$. We see that the subspaces $\mathfrak{g}(0), \mathfrak{g}(1/2)$ and $\mathfrak{g}(1)$ are the $0, 1/2$ and 1-eigenspaces of $\operatorname{ad} H$ respectively. Moreover we have

$$[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subset \mathfrak{g}(\alpha + \beta), \tag{2.3}$$

where if $\alpha + \beta > 1$, then we put $\mathfrak{g}(\alpha + \beta) = \{0\}$. Also we have

$$J\mathfrak{n}_{(\alpha_k - \alpha_j)/2} = \mathfrak{n}_{(\alpha_k + \alpha_j)/2} \qquad (1 \le j < k \le r),$$
$$J\mathfrak{n}_{\alpha_m/2} = \mathfrak{n}_{\alpha_m/2} \qquad (1 \le m \le r),$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$, $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. We note that

$$JT = -[T, E] \qquad (T \in \mathfrak{g}(0)), \tag{2.4}$$
$$JT_{kj} = -[T_{kj}, E_j] \qquad (T_{kj} \in \mathfrak{n}_{(\alpha_k - \alpha_j)/2}). \tag{2.5}$$

The subspace $\mathfrak{n}(0)$ is a nilpotent Lie subalgebra. Let

$$N(0) := \exp \mathfrak{n}(0), \quad G(0) := \exp \mathfrak{g}(0), \quad A := \exp \mathfrak{a}.$$

Then $G(0) = A \ltimes N(0)$ and $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. We put $\Omega := G(0)E$, the $G(0)$-orbit through E. Then we know that Ω is a regular open convex cone in V on which $G(0)$ acts simply transitively. Since $\mathfrak{g}(1/2)$ is invariant under J, we can introduce a complex structure on $\mathfrak{g}(1/2)$ by $-J$. We denote by U this complex vector space. The Lie subalgebra $\mathfrak{g}(0)$ acts on U complex linearly by adjoint action. We put $W := V_{\mathbb{C}}$, the complexification of V and denote by $w \mapsto w^*$ the complex conjugation of W relative to the real form V. We define a sesquilinear map $Q : U \times U \to W$ by

$$Q(u, u') := [Ju, u'] - i[u, u'] \qquad (u, u' \in U). \tag{2.6}$$

Then we see that this map is Hermitian and Ω-positive:

$$Q(u, u') = Q(u', u)^* \qquad (u, u' \in U),$$
$$Q(u, u) \in \overline{\Omega} \setminus \{0\} \text{ for all } u \in U \setminus \{0\}. \tag{2.7}$$

The Siegel domain corresponding to these data is defined by

$$D := \{(u, w) \in U \times W \mid \operatorname{Re} w - \tfrac{1}{2}Q(u, u) \in \Omega\}. \tag{2.8}$$

We know that the Lie group $G := \exp\mathfrak{g}$ acts on D simply transitively on D by affine automorphisms. Every homogeneous Siegel domain is obtained from a normal j-algebra in this way. Throughout this paper, we always assume that D is irreducible. Hence the cone Ω is also irreducible by [10, Theorem 6.3].

Finally, we remark here that the Shilov boundary Σ of D is described as

$$\Sigma = \{(u,w) \in U \times W \mid \operatorname{Re} w - \tfrac{1}{2}Q(u,u) = 0\}.$$

3. Cayley transforms

3.1. **Pseudoinverse maps.** A linear form ω on $\mathfrak{g}$ satisfying (2.2) is said to be *admissible*. We know the set of admissible forms. To describe it we define $E_m^* \in \mathfrak{g}^*$ $(m = 1,\dots,r)$ by $\langle E_j, E_m^*\rangle = \delta_{jm}$ $(j = 1,\dots,r)$ and $E_m^* \equiv 0$ on $\mathfrak{g}(0) \oplus \mathfrak{g}(1/2) \oplus J\mathfrak{n}(0)$. For $\mathbf{s} = (s_1,\dots,s_r) \in \mathbb{R}^r$, we set $E_{\mathbf{s}}^* := \sum s_m E_m^*$ and $\langle v_1|v_2\rangle_{\mathbf{s}} := \langle v_1|v_2\rangle_{E_{\mathbf{s}}^*}$ for $v_1, v_2 \in V$. We say that $\mathbf{s} = (s_1,\dots,s_r) \in \mathbb{R}^r$ is positive and write $\mathbf{s} > 0$ if $s_1 > 0,\dots,s_r > 0$. Then [17, Proposition 3.4] says that the set of admissible linear forms on $\mathfrak{g}$ coincides with

$$\mathfrak{a}^* + \{E_{\mathbf{s}}^* \mid \mathbf{s} > 0\}.$$

Further we know by [17, Lemma 3.2] that the description of the structure of $\mathfrak{g}$ in Section 2 is independent of the choice of the admissible linear form ω.

For $\mathbf{s} = (s_1,\dots,s_r) \in \mathbb{R}^r$, we define a one-dimensional representation $\chi_{\mathbf{s}}$ of A by

$$\chi_{\mathbf{s}}\Big(\exp\Big(\sum t_m H_m\Big)\Big) := \exp\Big(\sum s_m t_m\Big) \qquad (t_m \in \mathbb{R}).$$

Since $G(0) = A \ltimes N(0)$, we can extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of $G(0)$ by putting $\chi_{\mathbf{s}}|_{N(0)} \equiv 1$. In what follows, we write hv for $h \in G(0)$ and $v \in V$ instead of $(\operatorname{Ad} h)v$ for simplicity. Recalling that $G(0)$ acts simply transitively on Ω by the adjoint action, we transfer $\chi_{\mathbf{s}}$ to a function $\Delta_{\mathbf{s}}$ on Ω:

$$\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h) \qquad (h \in G(0)).$$

We remark that by [17, (3.15)], we have for $\mathbf{s} > 0$,

$$\langle v_1|v_2\rangle_{\mathbf{s}} = D_{v_1} D_{v_2} \log \Delta_{-\mathbf{s}}(E) \qquad (v_1, v_2 \in V).$$

Let $\mathbf{s} > 0$. For $x \in \Omega$, we define the *pseudoinverse* $\mathcal{I}_{\mathbf{s}}(x)$ of x by

$$\langle \mathcal{I}_{\mathbf{s}}(x)|y\rangle_{\mathbf{s}} = -D_y \log \Delta_{-\mathbf{s}}(x) \qquad (y \in V). \tag{3.1}$$

We call $\mathcal{I}_{\mathbf{s}} : \Omega \to V$ the *pseudoinverse map*. We see that $\mathcal{I}_{\mathbf{s}}(E) = E$ and $\mathcal{I}_{\mathbf{s}}$ gives a diffeomorphism of Ω onto $\Omega^{\mathbf{s}}$, where $\Omega^{\mathbf{s}}$ is the dual cone of Ω realized in V by means of the inner product $\langle\cdot|\cdot\rangle_{\mathbf{s}}$:

$$\Omega^{\mathbf{s}} := \{x \in V \mid \langle x|y\rangle_{\mathbf{s}} > 0 \text{ for all } y \in \overline{\Omega} \setminus \{0\}\}.$$

Let $G(0)_{\mathbb{C}}$ be the complexification of $G(0)$. We extend $\langle\cdot|\cdot\rangle_{\mathbf{s}}$ to W by complex bilinearity and denote it by the same symbol. We know that $\mathcal{I}_{\mathbf{s}}$ is analytically continued to a rational map $W \to W$ which is $G(0)_{\mathbb{C}}$-equivariant: $\mathcal{I}_{\mathbf{s}}(hx) = {}^{\mathbf{s}}h^{-1}\mathcal{I}_{\mathbf{s}}(x)$ $(h \in G(0)_{\mathbb{C}})$, where for a linear operator T on W, ${}^{\mathbf{s}}T$ stands for the transpose of T relative to $\langle\cdot|\cdot\rangle_{\mathbf{s}}$.

Starting with the dual cone $\Omega^{\mathbf{s}}$, we get a similar map $\mathcal{I}_{\mathbf{s}}^{*} : \Omega^{\mathbf{s}} \to \Omega$. We see that $\mathcal{I}_{\mathbf{s}}^{*}$ is analytically continued to a rational map $W \to W$ which is $G(0)_{\mathbb{C}}$-equivariant: $\mathcal{I}_{\mathbf{s}}^{*}({}^{\mathbf{s}}h^{-1}x) = h\mathcal{I}_{\mathbf{s}}^{*}(x)$ ($h \in G(0)_{\mathbb{C}}$). Thus $\mathcal{I}_{\mathbf{s}}$ is a birational map with $\mathcal{I}_{\mathbf{s}}^{-1} = \mathcal{I}_{\mathbf{s}}^{*}$. Moreover, we see by [17, Theorem 3.19] that $\mathcal{I}_{\mathbf{s}}$ (resp. $\mathcal{I}_{\mathbf{s}}^{*}$) is holomorphic on $\Omega + iV$ (resp. $\Omega^{\mathbf{s}} + iV$) and $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ (resp. $\mathcal{I}_{\mathbf{s}}^{*}(\Omega^{\mathbf{s}} + iV)$) is contained in the holomorphic domain of $\mathcal{I}_{\mathbf{s}}^{*}$ (resp. $\mathcal{I}_{\mathbf{s}}$).

3.2. **Parametrized family of Cayley transforms.** We keep to the notation in Section 3.1 and continue to suppose that $\mathbf{s} > 0$. We define a sesquilinear form $(\cdot|\cdot)_{\mathbf{s}}$ on U by

$$(u_1|u_2)_{\mathbf{s}} := \langle Q(u_1, u_2)|E\rangle_{\mathbf{s}} = \langle Q(u_1, u_2), E_{\mathbf{s}}^{*}\rangle \qquad (u_1, u_2 \in U). \tag{3.2}$$

Then $(\cdot|\cdot)_{\mathbf{s}}$ is a positive definite Hermitian inner product on U. The subspaces $\mathfrak{n}_{\alpha_m/2}$ ($m = 1, \dots, r$) are orthogonal to each other with respect to $(\cdot|\cdot)_{\mathbf{s}}$. For $u \in U$, we set $\|u\|_{\mathbf{s}} := (u|u)_{\mathbf{s}}^{1/2}$. Let $u_m \in \mathfrak{n}_{\alpha_m/2}$. Then we see by (2.6) that $Q(u_m, u_m) \in \mathfrak{n}_{\alpha_m}$. Moreover we know by (3.2) that

$$Q(u_m, u_m) = s_m^{-1}\|u_m\|_{\mathbf{s}}^2 E_m. \tag{3.3}$$

For every $w \in W$, we define a complex linear operator $\varphi_{\mathbf{s}}(w)$ on U by

$$(\varphi_{\mathbf{s}}(w)u_1|u_2)_{\mathbf{s}} = \langle w|Q(u_1, u_2)\rangle_{\mathbf{s}} \qquad (u_1, u_2 \in U). \tag{3.4}$$

The assignment $w \mapsto \varphi_{\mathbf{s}}(w)$ is also complex linear and $\varphi_{\mathbf{s}}(E) = \mathrm{id}$.

We put

$$S := \{w \in W \mid w + E \in \Omega + iV\}, \quad \mathcal{S} := \{(u, w) \in U \times W \mid w \in S\}.$$

The *Cayley transform* $C_{\mathbf{s}} : S \to W$ for the tube domain $\Omega + iV$ is defined by

$$C_{\mathbf{s}}(w) := E - 2\mathcal{I}_{\mathbf{s}}(w + E) \qquad (w \in S).$$

Observe that the closure $\overline{\Omega} + iV$ is contained in S. Using $C_{\mathbf{s}}$, we introduce the Cayley transform $\mathcal{C}_{\mathbf{s}} : \mathcal{S} \to U \times W$ for D by

$$\mathcal{C}_{\mathbf{s}}(u, w) := (2\varphi_{\mathbf{s}}(\mathcal{I}_{\mathbf{s}}(w + E))u, C_{\mathbf{s}}(w)) \qquad ((u, w) \in \mathcal{S}). \tag{3.5}$$

By [17, Theorem 4.17], the Cayley transform image $\mathcal{C}_{\mathbf{s}}(D)$ of D is bounded.

Note that since the definition of the Hermitian map Q in [17] is different from ours (2.6) by the multiplication constant $1/2$, the Siegel domain dealt in [17] is expressed as $T(D)$, where $T(u, w) := (\sqrt{2}u, w)$ ($(u, w) \in U \times W$). This modification is made so that we have $\mathrm{Re}(u_1|u_2)_{\mathbf{s}} = \langle u_1|u_2\rangle_{E_{\mathbf{s}}^{*}}$ for $u_1, u_2 \in U$. However, the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$, the linear map $\varphi_{\mathbf{s}} : W \to \mathrm{End}_{\mathbb{C}} U$ and the Cayley transform $\mathcal{C}_{\mathbf{s}}$ are the same as those of [17].

We see that the inverse maps of $C_{\mathbf{s}}, \mathcal{C}_{\mathbf{s}}$ are given by

$$\begin{aligned} C_{\mathbf{s}}^{-1}(w) &= 2\mathcal{I}_{\mathbf{s}}^{*}(E - w) - E && (w \in S^{*}), \\ \mathcal{C}_{\mathbf{s}}^{-1}(u, w) &= (\varphi_{\mathbf{s}}(E - w)^{-1}u, C_{\mathbf{s}}^{-1}(w)) && ((u, w) \in \mathcal{S}^{*}), \end{aligned}$$

where we have put

$$S^{*} := \{w \in W \mid E - w \in \Omega^{\mathbf{s}} + iV\}, \qquad \mathcal{S}^{*} := \{(u, w) \in U \times W \mid w \in S^{*}\}.$$

We would like to remark that our Cayley transform $C_{\mathbf{s}}$ for the tube domain $\Omega + iV$ is identical with the Cayley transform given in [9]. In [9], we started with a homogeneous convex cone Ω and a split solvable Lie group H acting simply transitively on Ω. If we take $G(0)$ as H and E as the base point, $V = \mathfrak{g}(1)$ becomes a clan with the unit element E by the following product:

$$x \triangle y = [Jx, y] \qquad (x, y \in V).$$

Moreover the normal decomposition of the clan V is given by $V = \sum_{k \geq j} \mathfrak{n}_{(\alpha_k + \alpha_j)/2}$ (see also [4, Section 2]). Hence the inner product $\langle \cdot | \cdot \rangle_{\mathbf{s}}$ and the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$ defined in [9] coincide with ours.

Now we are in position to state our main theorem in its precise form:

Theorem 3.1. *Let D be an irreducible homogeneous Siegel domain. Suppose that the parameter $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$ is positive. Then $\mathcal{C}_{\mathbf{s}}(D)$ is a convex set if and only if D is symmetric and $s_1 = \cdots = s_r$.*

4. Quasisymmetric Siegel domains

Let D be the homogeneous Siegel domain defined by (2.8). Since D is holomorphically equivalent to a bounded domain, the Bergman space of D has the reproducing kernel called the *Bergman kernel*, which we denote by κ. By homogeneity we have an explicit expression for κ. Let

$$\begin{aligned} b_m &:= \tfrac{1}{2} \dim_{\mathbb{R}} \mathfrak{n}_{\alpha_m/2} \qquad (m = 1, \dots, r), \\ n_{kj} &:= \dim \mathfrak{n}_{(\alpha_k + \alpha_j)/2} \qquad (1 \leq j < k \leq r), \\ d_m &:= 1 + \tfrac{1}{2} \sum_{i<m} n_{mi} + \tfrac{1}{2} \sum_{i>m} n_{im} \qquad (m = 1, \dots, r), \\ \mathbf{b} &:= (b_1, \dots, b_r), \qquad \mathbf{d} := (d_1, \dots, d_r). \end{aligned}$$

Then by [15, 1.3], we have for $z_j = (u_j, w_j) \in D$ $(j = 1, 2)$,

$$\kappa(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2))$$

up to a positive constant multiple.

If the cone Ω is self-dual with respect to the inner product $\langle \cdot | \cdot \rangle_{2\mathbf{d}+\mathbf{b}}$, that is, $\Omega = \Omega^{2\mathbf{d}+\mathbf{b}}$, then D is said to be *quasisymmetric*. We quote here the following criterion due to D'Atri and Dotti:

Proposition 4.1 ([4, Proposition 3]). *The Siegel domain D is quasisymmetric if and only if*

(1) *n_{kj} are independent of k, j,*
(2) *b_m are independent of m.*

We assume that a paramter $\mathbf{s} = (s_1, \dots, s_r)$ is positive. We introduce a non-associative product on V by

$$\langle xy | z \rangle_{\mathbf{s}} = -\tfrac{1}{2} D_x D_y D_z \log \Delta_{-\mathbf{s}}(E) \qquad (x, y, z \in V). \tag{4.1}$$

Lemma 4.2. *Even if* $\mathbf{s}$ *is replaced by* $\mathbf{s}' := p\mathbf{s}$ $(p > 0)$*, the product defined by* (4.1)*, the pseudoinverse map* $\mathcal{I}_{\mathbf{s}}$*, the linear map* $\varphi_{\mathbf{s}}$ *and the Cayley transform* $\mathcal{C}_{\mathbf{s}}$ *all remain the same.*

Proof. By definition, we have $\langle\cdot|\cdot\rangle_{\mathbf{s}'} = p\langle\cdot|\cdot\rangle_{\mathbf{s}}$ and $\Delta_{-\mathbf{s}'} = \Delta^p_{-\mathbf{s}}$, so that the product defined by (4.1) will not change under the replacement of $\mathbf{s}$ by $\mathbf{s}'$, and $\mathcal{I}_{\mathbf{s}'} = \mathcal{I}_{\mathbf{s}}$ by (3.1). Moreover, since $(\cdot|\cdot)_{\mathbf{s}'} = p(\cdot|\cdot)_{\mathbf{s}}$, we know by (3.4) that $\varphi_{\mathbf{s}'} = \varphi_{\mathbf{s}}$. Therefore we have $\mathcal{C}_{\mathbf{s}'} = \mathcal{C}_{\mathbf{s}}$. $\square$

Let us suppose that D is quasisymmetric and $s_1 = \cdots = s_r > 0$. In view of Lemma 4.2, we simply write $\mathcal{I}, \varphi$ and $\mathcal{C}$ instead of $\mathcal{I}_{\mathbf{s}}, \varphi_{\mathbf{s}}$ and $\mathcal{C}_{\mathbf{s}}$ respectively in this section. Noting that $2d_1+b_1 = \cdots = 2d_r+b_r$ by Proposition 4.1, we know by Lemma 4.2 and [6, Theorem 2.1] that the vector space V equipped with the product defined by (4.1) is a Jordan algebra with the unit element E. This means that in addition to the commutativity $xy = yx$, we have the Jordan identity $x(x^2y) = x^2(xy)$ for all $x, y \in V$. The complexification W of V is a complex Jordan algebra in a natural way. The following proposition is due to Dorfmeister (see also [14, Section 4]).

Proposition 4.3 ([6, Theorem 2.1])**.** *The linear map* $\varphi : w \mapsto \varphi(w)$ *is a* $*$*-representation of the Jordan algebra* W*:*

$$\varphi(w^*) = \varphi(w)^* \qquad (w \in W),$$
$$\varphi(w_1w_2) = \tfrac{1}{2}(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1)) \qquad (w_1, w_2 \in W),$$

where for a linear operator T *on* U*, we denote by* T^* *the adjoint operator of* T *relative to* $(\cdot|\cdot)_{\mathbf{s}}$*.*

Moreover since $2d_j+b_j$ are all equal for $j = 1, \ldots, r$, we know by Lemma 4.2 and [14, Proposition 4.4] that $\mathcal{I}(w) = w^{-1}$ for invertible $w \in W$, where the right-hand side is the Jordan algebra inverse of w. Hence it holds that

$$\mathcal{C}(u, w) = (2\varphi((w+E)^{-1})u, (w-E)(w+E)^{-1}) \qquad ((u, w) \in \mathcal{S}).$$

Here we note that $\varphi((w+E)^{-1}) = \varphi(w+E)^{-1}$ by Proposition 4.3 and $\varphi(E) = \mathrm{id}$. Thus our Cayley transform $\mathcal{C}$ coincides with the Cayley transform treated in [7].

5. Basic facts

We collect here some of the facts that are true without any restrictions on the homogeneous Siegel domain D. In this section we always suppose that the positive integers j, k, l satisfy $1 \le j < k < l \le r$ and $w_{kj} \in (\mathfrak{n}_{(\alpha_k+\alpha_j)/2})_{\mathbb{C}}, w_{lj} \in (\mathfrak{n}_{(\alpha_l+\alpha_j)/2})_{\mathbb{C}}, w_{lk} \in (\mathfrak{n}_{(\alpha_l+\alpha_k)/2})_{\mathbb{C}}$.

We set

$$S_{lk} := \tfrac{1}{2}([Jw_{lj}, w_{kj}] + [Jw_{kj}, w_{lj}]) \in (\mathfrak{n}_{(\alpha_l+\alpha_k)/2})_{\mathbb{C}}.$$

We put $\nu[w] := \langle w|w\rangle_{\mathbf{s}}$ $(w \in W)$, where we note that $\nu[iw] = -\nu[w]$. Since the clan structure in V is introduced in a manner compatible with the normal j-algebra structure as we remarked at the end of Section 3, we can quote the following two propositions from [8], where we note that they are valid not only for real t_j, t_k, t_l but also for complex t_j, t_k, t_l.

Proposition 5.1 ([8, Proposition 4.2]). *Let $t_j, t_k, t_l \in \mathbb{C}$. Then one has*

$$\begin{aligned}
&\exp\left(Jw_{lj}+Jw_{kj}\right)\exp\left(Jw_{lk}\right)\exp\left(t_jH_j+t_kH_k+t_lH_l\right)E\\
&\quad=\sum_{m\neq j,k,l}E_m+e^{t_j}E_j+\left(e^{t_k}+(2s_k)^{-1}e^{t_j}\nu[w_{kj}]\right)E_k\\
&\qquad+\left(e^{t_l}+(2s_l)^{-1}e^{t_k}\nu[w_{lk}]+(2s_l)^{-1}e^{t_j}\nu[w_{lj}]\right)E_l\\
&\qquad+e^{t_j}w_{lj}+e^{t_j}w_{kj}+\left(e^{t_j}S_{lk}+e^{t_k}w_{lk}\right).
\end{aligned}$$

Proposition 5.2 ([8, Proposition 4.6]). *One has*

$$\begin{aligned}
&{}^{\mathbf{s}}\!\left(\exp(Jw_{lj}+Jw_{kj})\exp(Jw_{lk})\exp\left(t_jH_j+t_kH_k+t_lH_l\right)\right)^{-1}E\\
&=\sum_{m\neq j,k,l}E_m+\Big(e^{-t_j}+(2s_j)^{-1}\left(e^{-t_k}+(2s_k)^{-1}e^{-t_l}\nu[w_{lk}]\right)\nu[w_{kj}]\\
&\qquad\qquad+(2s_j)^{-1}e^{-t_l}\nu[w_{lj}]-s_j^{-1}e^{-t_l}\langle S_{lk}|w_{lk}\rangle_{\mathbf{s}}\Big)E_j\\
&\quad+\left(e^{-t_k}+(2s_k)^{-1}e^{-t_l}\nu[w_{lk}]\right)E_k+e^{-t_l}E_l\\
&\quad+\left(e^{-t_l}\,{}^{\mathbf{s}}(\operatorname{ad}Jw_{lj})w_{lk}-(e^{-t_k}+(2s_k)^{-1}e^{-t_l}\nu[w_{lk}])w_{kj}\right)\\
&\quad+e^{-t_l}\left({}^{\mathbf{s}}(\operatorname{ad}Jw_{kj})w_{lk}-w_{lj}\right)-e^{-t_l}w_{lk}.
\end{aligned}$$

We use also the following two lemmas to compute the Cayley transforms.

Lemma 5.3. (1) *For all $x \in V$, one has $\varphi_{\mathbf{s}}(x) = \operatorname{ad}_U Jx + (\operatorname{ad}_U Jx)^*$.*
(2) *The linear operators $\varphi_{\mathbf{s}}(E_m)$ $(m = 1, \ldots, r)$ are orthogonal projections onto $\mathfrak{n}_{\alpha_m/2}$.*

Proof. (1) Definition (2.6) of Q and the Jacobi identity together with the fact that $\operatorname{ad} Jx$ commutes with J gives

$$(\operatorname{ad}Jx)Q(u,u') = Q(u,(\operatorname{ad}Jx)u') + Q((\operatorname{ad}Jx)u,u').$$

Hence it follows that

$$\langle Q(u,u')|{}^{\mathbf{s}}(\operatorname{ad}Jx)E\rangle_{\mathbf{s}} = ((\operatorname{ad}Jx)^*u|u')_{\mathbf{s}} + ((\operatorname{ad}Jx)u|u')_{\mathbf{s}}. \tag{5.1}$$

Here we note that ${}^{\mathbf{s}}(\operatorname{ad} Jx)E = x$. In fact, we get by (2.4) that for any $v \in V$,

$$\langle v|{}^{\mathbf{s}}(\operatorname{ad}Jx)E\rangle_{\mathbf{s}} = \langle [J[Jx,v],E], E^*_{\mathbf{s}}\rangle = \langle v|x\rangle_{\mathbf{s}}.$$

Therefore, the left-hand side of (5.1) is equal to $(\varphi(x)u|u')_{\mathbf{s}}$, and the proof is completed.
(2) The linear operator $2\operatorname{ad}_U H_m$ is an orthogonal projection onto $\mathfrak{n}_{\alpha_m/2}$, and thus it is self-adjoint. Hence (1) yields $\varphi_{\mathbf{s}}(E_m) = 2\operatorname{ad}_U H_m$, and (2) follows. □

Lemma 5.4 ([14, Lemma 3.4]). $\varphi_{\mathbf{s}}({}^{\mathbf{s}}(\operatorname{Ad}_V h)x) = (\operatorname{Ad}_U h)^{\mathbf{s}}\varphi_{\mathbf{s}}(x)(\operatorname{Ad}_U h)$ *for all $h \in G(0)$ and $x \in V$.*

We have some inequalities concerning the dimensions of the root spaces of $\mathfrak{g}$.

Lemma 5.5 ([15, Corollary 4.4]). (1) *If $n_{lk} \neq 0$, then one has $n_{lj} \geq n_{kj}$.*

(2) *If $n_{kj} \neq 0$, then one has $n_{lj} \geq n_{lk}$.*

Proposition 5.6. *For $v_{kj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ and $u_j \in \mathfrak{n}_{\alpha_j/2}$, one has*

$$\|(\mathrm{ad}\, Jv_{kj})u_j\|_{\mathbf{s}}^2 = (2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2\|u_j\|_{\mathbf{s}}^2.$$

Proof. We have $[Ju_j, u_j] = Q(u_j, u_j) = s_j^{-1}\|u_j\|_{\mathbf{s}}^2 E_j$ by (2.6) and (3.3). Taking the commutator with Jv_{kj}, we see by the Jacobi identity and (2.5) that

$$[[Jv_{kj}, u_j], Ju_j] + [[Ju_j, Jv_{kj}], u_j] = -s_j^{-1}\|u_j\|_{\mathbf{s}}^2 v_{kj}.$$

Taking the commutator with Jv_{kj} once again, we obtain

$$[[[Jv_{kj}, u_j], Ju_j], Jv_{kj}] + [[[Ju_j, Jv_{kj}], u_j], Jv_{kj}] = s_j^{-1}\|u_j\|_{\mathbf{s}}^2[Jv_{kj}, v_{kj}]. \tag{5.2}$$

Since $Jv_{kj} \in \mathfrak{n}_{(\alpha_k-\alpha_j)/2}$ and $[Jv_{kj}, u_j], [Ju_j, Jv_{kj}] \in \mathfrak{n}_{\alpha_k/2}$, one has

$$[[Jv_{kj}, u_j], Jv_{kj}] = [[Ju_j, Jv_{kj}], Jv_{kj}] = 0.$$

It follows from the Jacobi identity that both terms of the left-hand side of (5.2) are equal to $[[Jv_{kj}, Ju_j], [Jv_{kj}, u_j]]$. Since the operator $\mathrm{ad}_U\, Jv_{kj}$ is complex linear, we have $[Jv_{kj}, Ju_j] = J[Jv_{kj}, u_j]$. Hence we see by (5.2) that

$$[J[Jv_{kj}, u_j], [Jv_{kj}, u_j]] = (2s_j)^{-1}\|u_j\|_{\mathbf{s}}^2[Jv_{kj}, v_{kj}].$$

Applying $E_{\mathbf{s}}^*$ to the both sides, we obtain the proposition by (2.6) and (3.2). □

If $n_{kj} \neq 0$, then Lemma 5.6 says that for a non-zero $v_{kj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$, the linear map $\mathfrak{n}_{\alpha_j/2} \ni u_j \mapsto (\mathrm{ad}\, Jv_{kj})u_j \in \mathfrak{n}_{\alpha_k/2}$ is injective. Hence we get the following lemma.

Lemma 5.7. *If $n_{kj} \neq 0$, one has $b_j \leq b_k$.*

Lemma 5.8 ([8, Lemma 7.5])**.** *Let $a_m \in \mathbb{R}$ $(m = 1, \dots, r)$ and $v_{kj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$. Then we have $\sum a_m E_m + v_{kj} \in \Omega$ if and only if $a_m > 0$ $(m = 1, \dots, r)$ and $a_j a_k - (2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 > 0$.*

Furthermore we have

Lemma 5.9. *Let $a_m \in \mathbb{R}$ $(m = 1, \dots, r), v_{lj} \in \mathfrak{n}_{(\alpha_l+\alpha_j)/2}$ and $v_{lk} \in \mathfrak{n}_{(\alpha_l+\alpha_k)/2}$. Then we have $\sum a_m E_m + v_{lj} + v_{lk} \in \Omega$ if and only if*

(i) $a_m > 0$ $(m = 1, \dots, r)$,
(ii) $a_j a_k a_l - a_k(2s_l)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 - a_j(2s_l)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 > 0$.

Remark 5.10. The conditions (i) and (ii) imply that

$$a_j a_l - (2s_l)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 > 0, \quad a_k a_l - (2s_l)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 > 0.$$

Proof of Lemma 5.9. For simplicity we set $v_1 := \sum a_m E_m + v_{lj} + v_{lk}$. Let us assume that $v_1 \in \Omega$. Then it is clear that $a_m > 0$ $(m = 1, \dots, r)$. It follows easily from (2.6) and (3.3) that $[Jw_{lj}, v_{lj}] = -(a_j s_l)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 E_l$. Hence we have by [8, Lemma 4.1],

$$v_2 := (\exp Jw_{lj})v_1 = \sum_{m \neq l} a_m E_m + \big(a_l - a_j^{-1}(2s_l)^{-1}\|v_{lj}\|_{\mathbf{s}}^2\big)E_l + v_{lk}.$$

Since $\exp Jw_{lj} \in G(0)$, we have $v_2 \in \Omega = G(0)E$. Therefore we obtain (ii) by Lemma 5.8.

Conversely we assume that (i) and (ii) hold. Then we have $v_2 \in \Omega$, so that $v_1 = (\exp Jw_{lj})^{-1} v_2 \in \Omega$. □

6. Refinement of the previous theorem

Before proving Theorem 3.1, we would like to present here a refinement of [9, Theorem 1].

Theorem 6.1. *Let* $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ *be positive. Then* $C_{\mathbf{s}}(\Omega + iV)$ *is a convex set if and only if* Ω *is a symmetric cone and* $s_1 = \cdots = s_r$.

In view of [9, Theorem 1], it is enough to prove the "only if" part of Theorem 6.1. More precisely, our only task is to prove Propositions 7 and 13 of [9] under the single assumption that $C_{\mathbf{s}}(\Omega + iV)$ is convex. Now we suppose that $C_{\mathbf{s}}(\Omega + iV)$ is a convex set. As in the previous section, we assume that the positive integers j, k, l satisfy $1 \leq j < k < l \leq r$.

6.1. **First step.** First we show that $s_1 = \cdots = s_r$.

Proposition 6.2. *If* $n_{kj} \neq 0$*, then one has* $s_k = s_j$.

Proof. Since the inequality $s_j \geq s_k$ is shown by [9, Lemma 5] under the same assumption as here, it suffices to show that $s_k \geq s_j$. Let us take any non-zero $\delta \in \mathbb{R}$ and non-zero $v_{kj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$. Let us compute the Cayley transform images $C_{\mathbf{s}}(z_1)$ and $C_{\mathbf{s}}(z_2)$ of the following two points of $\overline{\Omega} + iV$:

$$z_1 := i(\delta E_k + v_{kj}), \qquad z_2 := -i(\delta E_k + v_{kj}).$$

We set

$$p := \log\bigl((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 + 1 + i\delta\bigr).$$

If we put in Proposition 5.1

$$t_j = t_l = 0, \qquad t_k = p, \qquad w_{lj} = w_{lk} = 0, \qquad w_{kj} = iv_{kj},$$

then the formula in Proposition 5.1 becomes $\exp J(iv_{kj}) \exp(pH_k)E = z_1 + E$. Put $\eta := \exp J(iv_{kj}) \exp(pH_k)$. Since $\mathcal{I}_{\mathbf{s}}(z_1 + E) = \mathcal{I}_{\mathbf{s}}(\eta E) = {}^{\mathbf{s}}\eta^{-1}E$, we have by Proposition 5.2

$$C_{\mathbf{s}}(z_1) = -\sum_{m \neq j,k} E_m + \bigl(2e^{-p}(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 - 1\bigr)E_j + (1 - 2e^{-p})E_k + 2ie^{-p}v_{kj}.$$

Replacement of δ by $-\delta$ and v_{kj} by $-v_{kj}$ respectively gives

$$C_{\mathbf{s}}(z_2) = -\sum_{m \neq j,k} E_m + \bigl(2e^{-\overline{p}}(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 - 1\bigr)E_j + (1 - 2e^{-\overline{p}})E_k - 2ie^{-\overline{p}}v_{kj}.$$

Consider $\xi := \frac{1}{2}(C_{\mathbf{s}}(z_1) + C_{\mathbf{s}}(z_2))$, the midpoint of $C_{\mathbf{s}}(z_1)$ and $C_{\mathbf{s}}(z_2)$. We have

$$\xi = -\sum_{m \neq j,k} E_m + \bigl(2(\operatorname{Re} e^{-p})(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 - 1\bigr)E_j + (1 - 2\operatorname{Re} e^{-p})E_k - 2(\operatorname{Im} e^{-p})v_{kj}.$$

Since $C_{\mathbf{s}}(\overline{\Omega}+iV)$ is a convex set, too, we have $\xi \in C_{\mathbf{s}}(\overline{\Omega}+iV)$, so that $C_{\mathbf{s}}^{-1}(\xi) \in \overline{\Omega}+iV$.

We shall compute $C_{\mathbf{s}}^{-1}(\xi)$. We put in Proposition 5.2

$$\begin{aligned} t_j &= -\log\Big(1 - \big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big)^{-1}(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2\Big), \\ t_k &= -\log(\operatorname{Re} e^{-p}), \qquad t_l = 0, \\ w_{kj} &= \delta\big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big)^{-1}v_{kj}, \qquad w_{lk} = w_{lj} = 0, \end{aligned} \tag{6.1}$$

and set $\widetilde{\eta} := \exp(Jw_{kj})\exp(t_jH_j + t_kH_k)$. Here we note that the established inequality $s_j \geq s_k$ gurantees $t_j \in \mathbb{R}$ and that $\operatorname{Re} e^{-p} > 0$ implies $t_k \in \mathbb{R}$. Then the formula in Proposition 5.2 becomes ${}^{\mathbf{s}}\widetilde{\eta}^{-1}E = 2^{-1}(E-\xi)$. Since $2\mathcal{I}_{\mathbf{s}}^*(E-\xi) = \mathcal{I}_{\mathbf{s}}^*({}^{\mathbf{s}}\widetilde{\eta}^{-1}E) = \widetilde{\eta}E$, we have by Proposition 5.1

$$C_{\mathbf{s}}^{-1}(\xi) = (e^{t_j}-1)E_j + \big(e^{t_k} + e^{t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 - 1\big)E_k + e^{t_j}w_{kj}.$$

Since $C_{\mathbf{s}}^{-1}(\xi) \in \overline{\Omega}+iV$, we know by Lemma 5.8

$$\big(e^{t_j}-1\big)\big(e^{t_k} + e^{t_j}(2s_k)^{-1}\|w_{kj}\|_{\mathbf{s}}^2 - 1\big) - (2s_k)^{-1}\|e^{t_j}w_{kj}\|_{\mathbf{s}}^2 \geq 0. \tag{6.2}$$

Multiply the both sides by $e^{-t_j}\big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big)^2$. Then (6.1) and some simplification yield

$$\big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big)(2s_j)^{-1}\|v_{kj}\|_{\mathbf{s}}^2(e^{t_k}-1) - \delta^2(2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 \geq 0.$$

Further simplification using $e^{t_k} = (\operatorname{Re} e^p)^{-1}\big((\operatorname{Re} e^p)^2 + (\operatorname{Im} e^p)^2\big)$ and $\|v_{kj}\|_{\mathbf{s}} \neq 0$ gives

$$\big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big)^2 - \big((2s_k)^{-1}\|v_{kj}\|_{\mathbf{s}}^2+1\big) \geq \delta^2 s_k^{-1}(s_j - s_k).$$

Since the left-hand side is independent of δ, and since $s_j \geq s_k$, the arbitrariness of δ forces $s_j = s_k$. □

We now obtain the following proposition by Asano's theorem [1, Theorem 4] from Proposition 6.2, as we did in [9, Proposition 9].

Proposition 6.3. *The numbers s_m $(m = 1, \ldots, r)$ are independent of m.*

6.2. **Second step.** In view of Proposition 6.3, we put for simplicity $s = s_m$ $(m = 1, \ldots, r)$, independent of m.

Proposition 6.4. *If $n_{lk} \neq 0$, then one has $n_{lj} = n_{kj}$.*

Proof. Let us assume that $n_{lk} \neq 0$. By Lemma 5.5 (1) it is enough to show that $n_{lj} \leq n_{kj}$. Let us take any non-zero $v_{lk} \in \mathfrak{n}_{(\alpha_l+\alpha_k)/2}$. For every $v_{lj} \in \mathfrak{n}_{(\alpha_l+\alpha_j)/2}$, we have ${}^{\mathbf{s}}(\operatorname{ad} Jv_{lk})v_{lj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ by [8, Lemma 4.4] and [8, Lemma 7.7]. We shall prove that the linear map $\mathfrak{n}_{(\alpha_l+\alpha_j)/2} \ni v_{lj} \mapsto {}^{\mathbf{s}}(\operatorname{ad} Jv_{lk})v_{lj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ is injective, which implies $n_{lj} \leq n_{kj}$.

Let us suppose that $v_{lj} \in \mathfrak{n}_{(\alpha_l+\alpha_j)/2}$ and ${}^{\mathbf{s}}(\operatorname{ad} Jv_{lk})v_{lj} = 0$. Let $\delta \in \mathbb{R}$ and consider the following two points of $\overline{\Omega}+iV$:

$$z_1 := i(\delta E_l + v_{lk} + v_{lj}), \qquad z_2 := -i(\delta E_l + v_{lk} + v_{lj}).$$

We set in Proposition 5.1

$$t_j = t_k = 0, \qquad t_l = \log\bigl(1 + (2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 + (2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 + i\delta\bigr),$$
$$w_{kj} = 0, \qquad w_{lj} = iv_{lj}, \qquad w_{lk} = iv_{lk},$$

and $\eta := \exp Jw_{lj} \exp Jw_{lk} \exp(t_l H_l)$. Then the formula in Proposition 5.1 becomes $\eta E = z_1 + E$. Since $\mathcal{I}_{\mathbf{s}}(z_1 + E) = \mathcal{I}_{\mathbf{s}}(\eta E) = {}^{\mathbf{s}}\eta^{-1}E$, we know by Proposition 5.2 (note that ${}^{\mathbf{s}}(\operatorname{ad} Jv_{lj})v_{lk} = {}^{\mathbf{s}}(\operatorname{ad} Jv_{lk})v_{lj} = 0$ by [8, Lemma 7.7])

$$\begin{aligned} C_{\mathbf{s}}(z_1) = -\sum_{m\neq j,k,l} & E_m + \bigl(2q^{-1}(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 - 1\bigr)E_j \\ & + \bigl(2q^{-1}(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 - 1\bigr)E_k + (1 - 2q^{-1})E_l + 2iq^{-1}v_{lj} + 2iq^{-1}v_{lk}, \end{aligned}$$

where we have put

$$q := 1 + (2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 + (2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 + i\delta.$$

A similar argument gives

$$\begin{aligned} C_{\mathbf{s}}(z_2) = -\sum_{m\neq j,k,l} & E_m + \bigl(2\overline{q}^{-1}(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 - 1\bigr)E_j \\ & + \bigl(2\overline{q}^{-1}(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 - 1\bigr)E_k + (1 - 2\overline{q}^{-1})E_l - 2i\overline{q}^{-1}v_{lj} - 2i\overline{q}^{-1}v_{lk}. \end{aligned}$$

We set $\xi := \frac{1}{2}(C_{\mathbf{s}}(z_1) + C_{\mathbf{s}}(z_2))$, the midpoint of $C_{\mathbf{s}}(z_1)$ and $C_{\mathbf{s}}(z_2)$. Then

$$\begin{aligned} \xi = -\sum_{m\neq j,k,l} & E_m + \bigl(2\operatorname{Re}(q^{-1})(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 - 1\bigr)E_j \\ & + \bigl(2\operatorname{Re}(q^{-1})(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2 - 1\bigr)E_k + (1 - 2\operatorname{Re}(q^{-1}))E_l \\ & - 2\operatorname{Im}(q^{-1})v_{lj} - 2\operatorname{Im}(q^{-1})v_{lk}. \end{aligned}$$

By the convexity of $C_{\mathbf{s}}(\overline{\Omega} + iV)$, we have $\xi \in C_{\mathbf{s}}(\overline{\Omega} + iV)$, so that $C_{\mathbf{s}}^{-1}(\xi) \in \overline{\Omega} + iV$.

Let us compute $C_{\mathbf{s}}^{-1}(\xi)$. Put in Proposition 5.2

$$\begin{aligned} w_{kj} = 0, \qquad w_{lk} = (\operatorname{Re} q)^{-1}(\operatorname{Im} q)v_{lk}, \qquad & w_{lj} = (\operatorname{Re} q)^{-1}(\operatorname{Im} q)v_{lj}, \\ t_j = -\log\bigl(1 - (\operatorname{Re} q)^{-1}(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2\bigr), & \\ t_k = -\log\bigl(1 - (\operatorname{Re} q)^{-1}(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2\bigr), \qquad & t_l = -\log(\operatorname{Re}(q^{-1})), \end{aligned} \tag{6.3}$$

where we note that $t_j, t_k, t_l \in \mathbb{R}$. We set

$$\widetilde{\eta} := \exp Jw_{lj} \exp Jw_{lk} \exp(t_j H_j + t_k H_k + t_l H_l).$$

Then the formula in Proposition 5.2 becomes ${}^{\mathbf{s}}\widetilde{\eta}^{-1}E = 2^{-1}(E - \xi)$ by virtue of ${}^{\mathbf{s}}(\operatorname{ad} Jv_{lk})v_{lj} = 0$ and [8, Lemma 7.7] again. Since $2\mathcal{I}_{\mathbf{s}}^*(E - \xi) = \mathcal{I}_{\mathbf{s}}^*({}^{\mathbf{s}}\widetilde{\eta}^{-1}E) = \widetilde{\eta}E$, we get by Proposition 5.1

$$\begin{aligned} C_{\mathbf{s}}^{-1}(\xi) = & (e^{t_j} - 1)E_j + (e^{t_k} - 1)E_k \\ & + \bigl(e^{t_l} + e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 + e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2 - 1\bigr)E_l + e^{t_j}w_{lj} + e^{t_k}w_{lk}. \end{aligned}$$

Since $C_{\mathbf{s}}^{-1}(\xi) \in \overline{\Omega} + iV$, we know by Lemma 5.9 (ii) that

$$(e^{t_k}-1)\Big\{(e^{t_j}-1)\big(e^{t_l}+e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2+e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2-1\big) \\ -e^{2t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2\Big\}-(e^{t_j}-1)e^{2t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2 \geq 0.$$

After some simplification, we obtain

$$(e^{t_k}-1)\Big\{(e^{t_j}-1)\big(e^{t_l}-1\big)-e^{t_j}(2s)^{-1}\|w_{lj}\|_{\mathbf{s}}^2\Big\} \geq (e^{t_j}-1)e^{t_k}(2s)^{-1}\|w_{lk}\|_{\mathbf{s}}^2.$$

Multiplying both sides by $e^{-t_j}e^{-t_k}$, we see by using (6.3) that the above inequality becomes, after dividing by $(\operatorname{Re} q)^{-3}(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2$,

$$(\operatorname{Re} q)(e^{t_l}-1)(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2 \geq 2(\operatorname{Im} q)^2(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2. \tag{6.4}$$

By $e^{t_l}=(\operatorname{Re} q)^{-1}\big((\operatorname{Re} q)^2+(\operatorname{Im} q)^2\big)$, we arrive at

$$(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2(X^2-X) \geq \delta^2(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2,$$

where $X := 1+(2s)^{-1}\|v_{lk}\|_{\mathbf{s}}^2+(2s)^{-1}\|v_{lj}\|_{\mathbf{s}}^2$. Since $\delta \in \mathbb{R}$ is arbitrary and the left-hand side is independent of δ, we must have $v_{lj}=0$, which we had to show. □

Now that Propositions 7 and 13 in [9] are proven without using the convexity of $\mathcal{C}_{\mathbf{s}}^*(\Omega^{\mathbf{s}}+iV)$, the proof of Theorem 6.1 is completed.

7. Proof of the main theorem

Let D be the homogeneous Siegel domain defined by (2.8). If D is symmetric and the parameter $\mathbf{s}$ satisfies $s_1=\cdots=s_r>0$, then we know by Section 4 that $\mathcal{C}_{\mathbf{s}}$ is identical with the Cayley transform treated in [7], so that the Cayley transform image $\mathcal{C}_{\mathbf{s}}(D)$ is a convex set by [7, Theorem 2.6]. We now prove the "only if" part of Theorem 3.1.

Lemma 7.1. *One has*

$$\mathcal{C}_{\mathbf{s}}(D)\cap(\{0\}\times W)=\{0\}\times C_{\mathbf{s}}(\Omega+iV).$$

Proof. Since $D\cap(\{0\}\times W)=\{0\}\times(\Omega+iV)$, we have clearly by (3.5) that $\mathcal{C}_{\mathbf{s}}(D)\cap(\{0\}\times W)\supset\{0\}\times C_{\mathbf{s}}(\Omega+iV)$. For $(u,w)\in D$, we have $w\in\Omega+iV$ by (2.8) and (2.7). Hence we have $\mathcal{C}_{\mathbf{s}}(D)\cap(\{0\}\times W)\subset\{0\}\times C_{\mathbf{s}}(\Omega+iV)$. □

Lemma 7.2. *Let $1\leq j<k\leq r$. For $u_k\in\mathfrak{n}_{\alpha_k/2}$ and $v_{kj}\in\mathfrak{n}_{(\alpha_k+\alpha_j)/2}$, one has $(\operatorname{ad} Jv_{kj})^*u_k\in\mathfrak{n}_{\alpha_j/2}$.*

Proof. If $u'\in(\mathfrak{n}_{\alpha_j/2})^\perp=\sum_{m\neq j}\mathfrak{n}_{\alpha_m/2}$, then we have $(\operatorname{ad} Jv_{kj})u'=0$. Hence

$$((\operatorname{ad} Jv_{kj})^*u_k|u')_{\mathbf{s}}=(u_k|(\operatorname{ad} Jv_{kj})u')_{\mathbf{s}}=0.$$

This implies $(\operatorname{ad} Jv_{kj})^*u_k\in\mathfrak{n}_{\alpha_j/2}$. □

From now on, we assume that $\mathcal{C}_{\mathbf{s}}(D)$ is a convex set. By Lemma 7.1, $C_{\mathbf{s}}(\Omega+iV)$ is also convex, so that Ω is a symmetric cone and $s_1=\cdots=s_r$ by Theorem 6.1. Then we know by [24, Proposition 3] that n_{kj} $(1\leq j<k\leq r)$ are independent of j,k and they are all non-zero. We set $s:=s_m$ $(m=1,\dots,r)$, independent of m, as in the previous section.

Proposition 7.3. *Let $1 \le j < k \le r$. For any non-zero $v_{kj} \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$, the linear map $(\operatorname{ad} Jv_{kj})^*|_{\mathfrak{n}_{\alpha_k/2}} : \mathfrak{n}_{\alpha_k/2} \to \mathfrak{n}_{\alpha_j/2}$ is injective. Hence we obtain $b_k \le b_j$.*

Proof. Let us assume that $u_k \in \mathfrak{n}_{\alpha_k/2}$ and $(\operatorname{ad} Jv_{kj})^* u_k = 0$. Let $\delta > 0$ be arbitrary. We consider the following two points on the Shilov boundary of D:

$$z_1 = (u_{z_1}, w_{z_1}) := \big(u_k, \tfrac{1}{2}Q(u_k, u_k) + i(\delta E_k + v_{kj})\big),$$
$$z_2 = (u_{z_2}, w_{z_2}) := \big(-u_k, \tfrac{1}{2}Q(u_k, u_k) - i(\delta E_k + v_{kj})\big).$$

We know by (3.3) that

$$w_{z_1} = \big((2s)^{-1}\|u_k\|_{\mathbf{s}}^2 + i\delta\big)E_k + iv_{kj}, \qquad w_{z_2} = \big((2s)^{-1}\|u_k\|_{\mathbf{s}}^2 - i\delta\big)E_k - iv_{kj}.$$

Let us compute the Cayley transforms $\xi_1 := \mathcal{C}_{\mathbf{s}}(z_1)$ and $\xi_2 := \mathcal{C}_{\mathbf{s}}(z_2)$ of z_1, z_2. In what follows, we will write $\xi_j = (u_{\xi_j}, w_{\xi_j})$ $(j = 1, 2)$. We put

$$p := \log(1 + (2s)^{-1}\|u_k\|_{\mathbf{s}}^2 + (2s)^{-1}\|v_{kj}\|_{\mathbf{s}}^2 + i\delta).$$

We set in Proposition 5.1,

$$t_j = t_l = 0, \qquad t_k = p, \qquad w_{lj} = w_{lk} = 0, \qquad w_{kj} = iv_{kj},$$

and put $\eta := \exp Jw_{kj} \exp(t_k H_k)$. Then the formula in Proposition 5.1 becomes $\eta E = w_{z_1} + E$. Since $\mathcal{I}_{\mathbf{s}}(w_{z_1} + E) = \mathcal{I}_{\mathbf{s}}(\eta E) = {}^{\mathbf{s}}\eta^{-1}E$, we have by Proposition 5.2,

$$\mathcal{I}_{\mathbf{s}}(w_{z_1} + E) = \sum_{m \ne j,k} E_m + \big(1 - (2s)^{-1}e^{-p}\|v_{kj}\|_{\mathbf{s}}^2\big)E_j + e^{-p}E_k - ie^{-p}v_{kj}.$$

Hence we get by Lemma 5.3 and the assumption $(\operatorname{ad} Jv_{kj})^* u_k = 0$,

$$u_{\xi_1} = 2e^{-p}u_k,$$
$$w_{\xi_1} = -\sum_{m \ne j,k} E_m + \big(s^{-1}e^{-p}\|v_{kj}\|_{\mathbf{s}}^2 - 1\big)E_j + (1 - 2e^{-p})E_k + 2ie^{-p}v_{kj}.$$

Similarly, we have

$$u_{\xi_2} = -2e^{-\overline{p}}u_k,$$
$$w_{\xi_2} = -\sum_{m \ne j,k} E_m + \big(s^{-1}e^{-\overline{p}}\|v_{kj}\|_{\mathbf{s}}^2 - 1\big)E_j + (1 - 2e^{-\overline{p}})E_k - 2ie^{-\overline{p}}v_{kj}.$$

We set $\xi = (u_\xi, w_\xi) := \frac{1}{2}(\xi_1 + \xi_2)$, the midpoint of ξ_1 and ξ_2. Then

$$u_\xi = 2i(\operatorname{Im} e^{-p})u_k,$$
$$w_\xi = -\sum_{m \ne j,k} E_m + \big(s^{-1}(\operatorname{Re} e^{-p})\|v_{kj}\|_{\mathbf{s}}^2 - 1\big)E_j + (1 - 2(\operatorname{Re} e^{-p}))E_k - 2(\operatorname{Im} e^{-p})v_{kj}.$$

Since $\mathcal{C}_{\mathbf{s}}(\overline{D})$ is a convex set, one has $\xi \in \mathcal{C}_{\mathbf{s}}(\overline{D})$.

To compute the inverse Cayley transform $\mathcal{C}_{\mathbf{s}}^{-1}(\xi)$ of ξ, we put, in Proposition 5.2,

$$\begin{aligned} &t_j = -\log\big(1 - (2s\operatorname{Re} e^p)^{-1}\|v_{kj}\|_{\mathbf{s}}^2\big), \qquad t_k = -\log(\operatorname{Re} e^{-p}), \qquad t_l = 0, \\ &w_{lj} = w_{lk} = 0, \qquad w_{kj} = (\operatorname{Re} e^p)^{-1}(\operatorname{Im} e^p)v_{kj}, \end{aligned} \tag{7.1}$$

and set $\widetilde{\eta} := \exp Jw_{kj} \exp(t_j H_j + t_k H_k)$, where we note that $t_j, t_k \in \mathbb{R}$. Then the formula in Proposition 5.2 becomes ${}^{\mathbf{s}}\widetilde{\eta}^{-1} E = \frac{1}{2}(E - w_\xi)$. Since $2\mathcal{I}_{\mathbf{s}}^*(E - w_\xi) = \mathcal{I}_{\mathbf{s}}^*({}^{\mathbf{s}}\widetilde{\eta}^{-1} E) = \widetilde{\eta} E$, we have by Proposition 5.1

$$C_{\mathbf{s}}^{-1}(w_\xi) = (e^{t_j} - 1)E_j + \big(e^{t_k} + (2s)^{-1} e^{t_j} \|w_{kj}\|_{\mathbf{s}}^2 - 1\big) E_k + e^{t_j} w_{kj}.$$

On the other hand, we know by Lemma 5.4 that

$$\varphi_{\mathbf{s}}(E - w_\xi) = 2(\mathrm{Ad}_U \widetilde{\eta}^{-1})^* (\mathrm{Ad}_U \widetilde{\eta}^{-1}).$$

Since $\mathrm{ad}\, Jv_{kj}$ commutes with J, we also have $(\mathrm{ad}\, Jv_{kj})^* Ju_k = 0$. Since $u_\xi = 2i(\mathrm{Im}\, e^{-p}) u_k$, it holds that

$$\begin{aligned} \varphi_{\mathbf{s}}(E - w_\xi)^{-1} u_\xi &= \tfrac{1}{2} \mathrm{Ad}_U \widetilde{\eta} (\mathrm{Ad}_U \exp(t_j H_j + t_k H_k))^* u_\xi \\ &= \tfrac{1}{2} e^{t_k} (\mathrm{Ad}_U \exp Jw_{kj}) u_\xi \\ &= i(\mathrm{Im}\, e^{-p}) e^{t_k} u_k, \end{aligned}$$

where the last equality follows from $(\mathrm{ad}\, Jw_{kj}) u_k = 0$. Therefore we get

$$\mathcal{C}_{\mathbf{s}}^{-1}(\xi) = \big(i(\mathrm{Im}\, e^{-p}) e^{t_k} u_k, (e^{t_j} - 1)E_j + \big(e^{t_k} + (2s)^{-1} e^{t_j} \|w_{kj}\|_{\mathbf{s}}^2 - 1\big) E_k + e^{t_j} w_{kj}\big).$$

We put $\zeta = (u_\zeta, w_\zeta) := \mathcal{C}_{\mathbf{s}}^{-1}(\xi)$. Since $\mathcal{C}_{\mathbf{s}}(\overline{D})$ is convex, we get $\zeta \in \overline{D}$, so that we know by (2.8), $w_\zeta - \frac{1}{2} Q(u_\zeta, u_\zeta) \in \overline{\Omega}$. Hence it follows from (3.3) that

$$\begin{aligned} &(e^{t_j} - 1)E_j + \big(e^{t_k} + (2s)^{-1} e^{t_j} \|w_{kj}\|_{\mathbf{s}}^2 - 1 - (2s)^{-1} (\mathrm{Im}\, e^{-p})^2 e^{2t_k} \|u_k\|_{\mathbf{s}}^2\big) E_k \\ &\qquad + e^{t_j} w_{kj} \in \overline{\Omega}. \end{aligned}$$

Then by Proposition 5.8

$$\begin{aligned} &(e^{t_j} - 1)\big(e^{t_k} + (2s)^{-1} e^{t_j} \|w_{kj}\|_{\mathbf{s}}^2 - 1 - (2s)^{-1} (\mathrm{Im}\, e^{-p})^2 e^{2t_k} \|u_k\|_{\mathbf{s}}^2\big) \\ &\qquad - (2s)^{-1} e^{2t_j} \|w_{kj}\|_{\mathbf{s}}^2 \geq 0. \end{aligned}$$

A simplification gives

$$(e^{t_j} - 1)\big(e^{t_k} - 1 - (2s)^{-1} (\mathrm{Im}\, e^{-p})^2 e^{2t_k} \|u_k\|_{\mathbf{s}}^2\big) - (2s)^{-1} e^{t_j} \|w_{kj}\|_{\mathbf{s}}^2 \geq 0.$$

Multiplying both sides by e^{-t_j}, we obtain by (7.1) that

$$1 - (\mathrm{Re}\, e^p)^{-1} - \delta^2 (\mathrm{Re}\, e^p)^{-3} \|u_k\|_{\mathbf{s}}^2 \geq 0,$$

where we have divided the inequality by $(2s)^{-1} \|v_{kj}\|_{\mathbf{s}}^2 > 0$. This must be true for any $\delta \in \mathbb{R}$, so that we have $u_k = 0$. Therefore the linear map $(\mathrm{ad}\, Jv_{kj})^*|_{\mathfrak{n}_{\alpha_k/2}} : \mathfrak{n}_{\alpha_k/2} \to \mathfrak{n}_{\alpha_j/2}$ is injective. $\square$

We know by Lemma 5.7 and Proposition 7.3 that $\dim \mathfrak{n}_{\alpha_m/2}$ $(m = 1, \ldots, r)$ are independent of m. Now Proposition 4.1 tells us that D is quasisymmetric. Since $s_1 = \cdots = s_r$, we see by Section 4 that $\mathcal{C}_{\mathbf{s}}$ coincides with the Cayley transform defined in [7]. Therefore it follows from [7, Theorem 2.6] that D is symmetric.

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