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KOBAYASHI-HITCHIN CORRESPONDENCE FOR TAME HARMONIC BUNDLES AND AN APPLICATION

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2000 *Mathematics Subject Classification.* — 14J60, 53C07.

Key words and phrases. — Higgs bundle, harmonic bundle, Kobayashi-Hitchin correspondence, Hermitian-Einstein metric, Bogomolov-Gieseker inequality, flat bundle, variation of polarized Hodge structure, quasi projective variety.

This work was prepared with the partial support of Ministry of Education, Culture, Sports, Science and Technology.

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Abstract. —

We establish the correspondence between tame harmonic bundles and μ_L-stable parabolic Higgs bundles with trivial characteristic numbers. We also show the Bogomolov-Gieseker type inequality for μ_L-stable parabolic Higgs bundles.

Then we show that any local system on a smooth quasi projective variety can be deformed to a variation of polarized Hodge structure. As a consequence, we can conclude that some kind of discrete groups cannot be a split quotient of the fundamental group of a smooth quasi projective variety.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Background

1.1.1. Kobayashi-Hitchin correspondence. — We briefly recall some aspects of the so-called Kobayashi-Hitchin correspondence. (See the introduction of [**32**] for more about the history.) In 1960's, M. S. Narasimhan and C. S. Seshadri proved the correspondence between irreducible flat unitary bundles and stable vector bundles with degree 0, on a compact Riemann surface ([**41**]). Clearly, it was desired to extend their result to the higher dimensional case and the non-flat case.

In early 1980's, S. Kobayashi introduced the Einstein-Hermitian condition for holomorphic bundles on Kahler manifolds ([**24**], [**25**]). He and M. Lübke ([**31**]) proved that the existence of Einstein-Hermitian metric implies the polystability of the underlying holomorphic bundle. S. K. Donaldson pioneered the way for the inverse problem ([**9**] and [**10**]). He attributed the problem to Kobayashi and N. Hitchin. The definitive result was given by K. Uhlenbeck-S. T. Yau and Donaldson ([**57**] and [**11**]). We also remark that V. Mehta and A. Ramanathan ([**35**]) proved the correspondence in the case where the Chern class is trivial, i.e., the correspondence of flat unitary bundles and stable vector bundles with trivial Chern classes.

On the other hand, it was quite fruitful to consider the correspondences for vector bundles with some additional structures like Higgs fields, which was initiated by Hitchin ([**17**]). He studied the Higgs bundles on a compact Riemann surface and the moduli spaces. His work has influenced various fields of mathematics. It involves a lot of subjects and ideas, and one of his results is the correspondence of the stability and the existence of Hermitian-Einstein metrics for Higgs bundles on a compact Riemann surface.

1.1.2. A part of C. Simpson's work. — C. Simpson studied the Higgs bundles over higher dimensional complex manifolds, influenced by Hitchin, but motivated by his own subject: Variation of Polarized Hodge Structure. He made great innovations in various areas of algebraic geometry. Here, we recall just a part of his huge work.

Let X be a smooth projective variety over the complex number field, and E be an algebraic vector bundle on X. Let (E,θ) be a Higgs bundle, i.e., θ is a holomorphic section of $\mathrm{End}(E)\otimes\Omega_X^{1,0}$ satisfying $\theta^2=0$. The "stability" and the "Hermitian Einstein metric" are naturally defined for Higgs bundles, and Simpson proved that there exists a Hermitian-Einstein metric if and only if it is polystable. In the special case where the Chern class of the vector bundle is trivial, the Hermitian-Einstein metric gives the pluri-harmonic metric. Together with the result of K. Corlette who is also a great progenitor of the study of harmonic bundles ([**4**]), Simpson obtained the Trinity on a smooth projective variety:

(1)	**Algebraic Geometry** polystable Higgs bundle (trivial Chern class)	$\leftrightarrow$	**Differential Geometry** harmonic bundle	$\leftrightarrow$	**Topology** semisimple local system

If (E,θ) is a stable Higgs bundle, then $(E,\alpha\cdot\theta)$ is also a stable Higgs bundle. Hence we obtain the family of stable Higgs bundles $\bigl\{(E,\alpha\cdot\theta)\bigm|\alpha\in\boldsymbol{C}^*\bigr\}$. Correspondingly, we obtain the family of flat bundles $\bigl\{L_\alpha\bigm|\alpha\in\boldsymbol{C}^*\bigr\}$. Simpson showed that we obtain the variation of polarized Hodge structure as a limit $\lim_{\alpha\to 0}L_\alpha$. In particular, it can be concluded that any flat bundle can be deformed to a variation of polarized Hodge structure. As one of the applications, Simpson obtained the following remarkable result ([**47**]):

***Theorem 1.1* (Simpson)**. — *Let Γ be a rigid discrete subgroup of a real algebraic group which is not of Hodge type. Then Γ cannot be a split quotient of the fundamental group of a smooth projective variety.* □

There are classical known results on the rigidity of subgroups of Lie groups. The examples of rigid discrete subgroups can be found in 4.7.1–4.7.4 in the 53 page of [**47**]. The classification of real algebraic group of Hodge type was done by Simpson. The examples of real algebraic group which is not of Hodge type can be found in the 50 page of [**47**]. As a corollary, he obtained the following.

Corollary 1.2. — $SL(n,\mathbb{Z})$ $(n\geq 3)$ *cannot be a split quotient of the fundamental group of a smooth projective variety.* □

1.2. Our main purpose

1.2.1. Kobayashi-Hitchin correspondence for parabolic Higgs bundles. — It is an important and challenging problem to generalize the correspondence (1) to the quasi projective case from the projective case. As for the correspondence of harmonic bundles and semisimple local systems, an excellent result was obtained by J. Jost-K. Zuo [**23**], which says there exists a tame pluri-harmonic metric on any semisimple local system over a quasi projective variety. The metric is called the Jost-Zuo metric. (See also [**39**] for a minor refinement, and see Theorem 12.24 and Corollary 12.25 in this paper.)

In this paper, we restrict ourselves to the correspondence between Higgs bundles and harmonic bundles on a quasi projective variety Y. More precisely, we should consider not Higgs bundles on Y but *parabolic* Higgs bundles on (X,D), where (X,D) is a pair of a smooth projective variety and a normal crossing divisor such that $Y=X-D$. Such a generalization has been studied by several people. In the non-Higgs case, J. Li [**29**] and B. Steer-A. Wren [**55**] established the correspondence. In the Higgs case, Simpson established the correspondence in the one dimensional case [**46**], and O. Biquard established it in the case where D is smooth [**3**].

Remark 1.3. — Their results also include the correspondence in the non-flat case. □

For applications, however, it is desired that the correspondence for parabolic Higgs bundles should be given in the case where D is not necessarily smooth, which we would like to discuss in this paper.

We explain our result more precisely. Let X be a smooth projective variety over the complex number field provided an ample line bundle L. Let D be a simple normal crossing divisor of X. The main purpose of this paper is to establish the correspondence between tame harmonic bundles and μ_L-parabolic Higgs bundles whose characteristic numbers vanish. (See the section 3 for the meaning of the words.)

Theorem 1.4 **(Proposition 5.1–5.3, and Theorem 10.2)**
Let $\bigl(\boldsymbol{E}_*,\theta\bigr)$ be a regular filtered Higgs bundle on (X,D), and we put $E:=\boldsymbol{E}_{|X-D}$. It is μ_L-polystable with $\mathrm{par\text{-}deg}_L(\boldsymbol{E}_*)=\int_X \mathrm{par\text{-}ch}_{2,L}(\boldsymbol{E}_*)=0$, *if and only if there exists a pluri-harmonic metric h of (E,θ) on $X-D$ which is adapted to the parabolic structure. Such a metric is unique up to an obvious ambiguity.* □

Remark 1.5. — Regular Higgs bundles and parabolic Higgs bundles are equivalent. See the section 3. □

Remark 1.6. — More precisely on the existence result, we will show the existence of the adapted pluri-harmonic metric for μ_L-stable regular filetered Higgs bundle on (X,D) "in codimension two" with trivial characteristic numbers. (See the subsection 3.1.4 for the definition.) Then, due to our previous result in [**38**], it is regular filtered Higgs bundle on (X,D), in fact. But the reader does not have to care about it. □

We are mainly interested in the μ_L-stable parabolic Higgs bundles whose characteristic numbers vanish. But we also obtain the following theorem on more general μ_L-stable parabolic Higgs bundles.

Theorem 1.7 **(Bogomolov-Gieseker inequality, Theorem 6.10)**

Let X be a smooth projective variety of an arbitrary dimension, and D be a simple normal crossing divisor. Let L be an ample line bundle on X. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-stable regular Higgs bundle in codimension two on (X, D). Then the following inequality holds:

$$\int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) - \frac{\int_X \text{par-c}_{1,L}^2(\boldsymbol{E}_*)}{2 \operatorname{rank} E} \leq 0.$$

□

1.2.2. Strategy for the proof of Bogomolov-Gieseker inequality. — We would like to explain our strategy for the proof of the main theorems. First we describe an outline for Bogomolov-Gieseker inequality (Theorem 1.7), which is much easier. Essentially, it consists of the following two parts.

(1) The correspondence in the graded semisimple case :
We establish the Kobayashi-Hitchin correspondence for *graded semisimple* Higgs bundles. In particular, we obtain the Bogomolov-Gieseker inequality in this case.

(2) Perturbation of the parabolic structure and taking the limit :
Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a given $\boldsymbol{c}$-parabolic μ_L-stable Higgs bundle, which is not necessarily graded semisimple. For any small positive number ϵ, we take a perturbation $\boldsymbol{F}^{(\epsilon)}$ of $\boldsymbol{F}$ such that $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is a graded semisimple μ_L-stable parabolic Higgs bundle. Then the Bogomolov-Gieseker inequality holds for $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$. By taking a limit for $\epsilon \longrightarrow 0$, we obtain the Bogomolov-Gieseker inequality for the given $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

Let us describe for more detail.

(1) In [**47**], Simpson constructed a Hermitian-Einstein metric for Higgs bundle by the following process:

- **(i) :** Take an appropriate initial metric.
- **(ii) :** Deform it along the heat equation.
- **(iii) :** Take a limit, and then we obtain the Hermitian-Einstein metric.

If the base space is compact, the steps (ii) and (iii) are the main issues, and the step (i) is trivial. Actually, Simpson also discussed the non-compact case, and he showed the existence of a Hermitian-Einstein metric if we can take an initial metric satisfying some good conditions. (See the section 2.2 for more precise statements.) So, for a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ on (X, D), where X is a smooth projective surface and D is a simple normal crossing divisor, ideally, we would like to take an initial metric of $E := {}_{\boldsymbol{c}}E_{|X-D}$ adapted to the parabolic structure. But, it is rather difficult, and the author is not sure whether such a good metric can always be taken for any parabolic Higgs bundles. It seems one of the main obstacles to establish the Kobayashi-Hitchin correspondence for parabolic Higgs bundles.

However, we can easily take such a good initial metric, if we assume the vanishing of the nilpotent part of the residues of the Higgs field on the graduation of the parabolic filtration. Such a parabolic Higgs bundle will be called *graded semisimple* in this paper. We first establish the correspondence in this easy case. (Proposition 6.1).

Remark 1.8. — Precisely, we also have to take an appropriate metric for $X - D$. □

(2) Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X, D), where $\dim X = 2$. We take a perturbation of $\boldsymbol{F}^{(\epsilon)}$ as in the section 3.4. In particular, $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is a μ_L-stable graded semisimple $\boldsymbol{c}$-parabolic Higgs bundle, and the following holds:

$$\text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}) = \text{par-c}_1({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}),$$

$$\left| \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}) - \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) \right| \leq C \cdot \epsilon.$$

Then we obtain the inequality for $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ by applying the previous result to it. By taking the limit $\epsilon \to 0$, we obtain the inequality for the given $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

1.2.3. Strategy for the proof of Kobayashi-Hitchin correspondence. — Let X be a smooth projective surface, and D be a simple normal crossing divisor. Let L be an ample line bundle on X, and ω be the Kahler form representing $c_1(L)$. Roughly speaking, the correspondence on (X, D) as in Theorem 1.4 can be divided into the following two parts:

- For a given tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$, we obtain the μ_L-stable parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ with the trivial characteristic numbers.
- On the converse, we obtain a pluri-harmonic metric of $(E, \overline{\partial}_E, \theta)$ on $X - D$ for such $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

As for the first issue, most problem can be reduced to the one dimensional case, which was established by Simpson [**46**]. However, we have to show the vanishing of the characteristic numbers, for which our study of the asymptotic behaviour of tame harmonic bundles ([**38**]) is quite useful. It is also used for the uniqueness of the adapted pluri-harmonic metric.

As for the second issue, we use the perturbation method, again. Namely, let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). Take a perturbation $\boldsymbol{F}^{(\epsilon)}$ of the filtration $\boldsymbol{F}$ for a small positive number ϵ. We also take metrics ω_ϵ of $X - D$ such that $\lim_{\epsilon \to 0} \omega_\epsilon = \omega$, and then we obtain a Hermitian-Einstein metric h_ϵ for the Higgs bundle $(E, \overline{\partial}_E, \theta)$ on $X - D$ with respect to ω_ϵ, which is adapted to the parabolic structure. Ideally, we would like to consider the limit $\lim_{\epsilon \to 0} h_\epsilon$, and we expect that the limit gives the Hermitian-Einstein metric h for $(E, \overline{\partial}_E, \theta)$ with respect to ω, which is adapted to the given filtration $\boldsymbol{F}$. Perhaps, it may be correct, but it does not seem easy to show, in general.

We restrict ourselves to the simpler case where the characteristic numbers of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ are trivial. Under this assumption, we show such a convergence. More precisely, we show that there is a subsequence $\{\epsilon_i\}$ such that $\big\{(E,\overline{\partial}_E,h_{\epsilon_i},\theta)\big\}$ converges to a harmonic bundle $(E',\overline{\partial}'_E,\theta',h')$ on $X-D$, and we show that the given $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)$ is isomorphic to the parabolic Higgs bundles obtained from $(E',\overline{\partial}'_E,\theta',h')$. In our current understanding, we need a rather long argument for the proof. (See the sections 7–10.)

1.3. Additional results

1.3.1. The torus action and the deformation of a G-flat bundle. — Once Theorem 1.4 is established, we can use some of the arguments for the applications given in the projective case. For example, we can deform any flat bundle to the one which comes from a variation of polarized Hodge structure. We follow the well known framework given by Simpson with a minor modification. We briefly recall it, and we will mention the problem that we have to care about in the process.

Let X be a smooth projective variety, and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S}D_i$. Let x be a point of $X-D$. Let Γ denote the fundamental group $\pi_1(X-D,x)$. Any representation of Γ can be deformed to a semisimple representation, and hence we start with a semisimple one.

Let (E,∇) be a flat bundle over $X-D$ such that the induced representation $\rho:\Gamma\longrightarrow\mathrm{GL}(E_{|x})$ is semisimple. Recall we can take a Jost-Zuo metric of (E,∇), as is mentioned in the subsection 1.2.1. Hence we obtain a tame pure imaginary harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $X-D$, and the induced μ_L-polystable $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)$ on (X,D), where $\boldsymbol{c}$ denotes any element of $\boldsymbol{R}^S$. We have the canonical decomposition $({}_{\boldsymbol{c}}E,\boldsymbol{F},\theta)=\bigoplus_i({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,\theta_i)^{\oplus m_i}$, where each $({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,\theta_i)$ is μ_L-stable.

Let us consider the family of $\boldsymbol{c}$-parabolic Higgs bundles $\big({}_{\boldsymbol{c}}E,\boldsymbol{F},t\cdot\theta\big)$ for $t\in\boldsymbol{C}^*$, which are μ_L-polystable. Due to the standard Langton's trick [**27**], we have the semistable $\boldsymbol{c}$-parabolic Higgs sheaves $({}_{\boldsymbol{c}}\widetilde{E}_i,\widetilde{\boldsymbol{F}}_i,\widetilde{\theta}_i)$ which are limits of $({}_{\boldsymbol{c}}E_i,\boldsymbol{F}_i,t\cdot\theta_i)$ in $t\to 0$. On the other hand, we can take a pluri-harmonic metric h_t of the Higgs bundle $(E,\overline{\partial}_E,t\cdot\theta)$ on $X-D$ for each t, which is adapted to the parabolic structure. (Theorem 1.4). Then we obtain the family of flat bundles $(E,\mathbb{D}^1_t)$, and the associated family of the representations $\big\{\rho_t:\Gamma\longrightarrow\mathrm{GL}(E_{|x})\,\big|\,t\in\boldsymbol{C}^*\big\}$. Since $(E,\overline{\partial}_E,t\cdot\theta,h_t)$ is tame pure imaginary in the case $t\in\boldsymbol{R}_{>0}$, the representations ρ_t are semisimple. The family $\{\rho_t\,|\,t\in\boldsymbol{C}^*\}$ should be continuous with respect to t, and the limit $\lim_{t\to0}\rho_t$ should exist, ideally.

It is necessary to formulate the continuity of ρ_t with respect to t and the convergence of ρ_t in $t\to0$. Let V be a $\boldsymbol{C}$-vector space such that $\mathrm{rank}(V)=\mathrm{rank}(E)$. Let h_V denote the metric of V, and let $U(h_V)$ denote the unitary group for h_V. We put $R(\Gamma,V):=Hom(\Gamma,\mathrm{GL}(V))$. By the conjugate, $U(h_V)$ acts on the space $R(\Gamma,V)$. Let

$M(\Gamma, V, h_V)$ denote the usual quotient space. Let $\pi_{\mathrm{GL}(V)} : R(\Gamma, V) \longrightarrow M(\Gamma, V, h_V)$ denote the projection.

By taking any isometry $(E_{|x}, h_{t|x}) \simeq (V, h_V)$, we obtain the representation $\rho'_t : \Gamma \longrightarrow \mathrm{GL}(V)$. We put $\mathcal{P}(t) := \pi_{\mathrm{GL}(V)}(\rho'_t)$, and we obtain the map $\mathcal{P} : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$. It is well defined.

Proposition 1.9 **(Theorem 11.1, Lemma 11.3, Proposition 11.4)**

1. *The induced map $\mathcal{P}$ is continuous.*
2. *$\mathcal{P}\bigl(\{0 < t \leq 1\}\bigr)$ is relatively compact in $M(\Gamma, V, h_V)$.*
3. *If each $\bigl({}_{\boldsymbol{c}}\widetilde{E}_i, \widetilde{\boldsymbol{F}}_i, \widetilde{\theta}_i\bigr)$ is stable, then the limit $\lim_{t\to 0} \mathcal{P}(t)$ exists, and the limit flat bundle underlies the variation of polarized Hodge structure. As a result, we can deform any flat bundle to the one underlying a variation of polarized Hodge structure.* □

We would like to mention the point which we will care about. For simplicity, we assume $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ is μ_L-stable, and $\bigl({}_{\boldsymbol{c}}E, \boldsymbol{F}, t\cdot\theta\bigr)$ converges to the μ_L-stable parabolic Higgs bundle $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$. Let $\{t_i\}$ denote a sequence converging to 0. By taking an appropriate subsequence, we may assume that the sequence $\{(E, \overline{\partial}_E, h_{t_i}, t_i\cdot\theta_i)\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, h', \theta')$ weakly in L_2^p locally over $X - D$, which is due to Uhlenbeck's compactness theorem and the estimate for the Higgs fields. Then we obtain the induced parabolic Higgs bundle $({}_{\boldsymbol{c}}E', \boldsymbol{F}', \theta')$. We would like to show that $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$ and $({}_{\boldsymbol{c}}E', \boldsymbol{F}', \theta')$ are isomorphic. Once we have known the existence of a non-trivial map $G : {}_{\boldsymbol{c}}E' \longrightarrow {}_{\boldsymbol{c}}\widetilde{E}$ which is compatible with the parabolic structure and the Higgs field, it is isomorphic due to the stability of $({}_{\boldsymbol{c}}\widetilde{E}, \widetilde{\boldsymbol{F}}, \widetilde{\theta})$. Hence the existence of such G is the main issue for this argument. We remark that the problem does not appear if D is empty.

Remark 1.10. — Even if $({}_{\boldsymbol{c}}\widetilde{E}_i, \widetilde{\boldsymbol{F}}_i, \widetilde{\theta}_i)$ are not μ_L-stable, the conclusion in the third claim of Proposition 1.9 should be true. In fact, Simpson gave a detailed argument to show it, in the case where D is empty ([**50**], [**51**]). More strongly, he obtained the homeomorphism of the coarse moduli spaces of semistable flat bundles and semistable Higgs bundles.

In this paper, we do not discuss the moduli spaces, and hence we omit to discuss the general case. Instead, we use an elementary inductive argument on the rank of local systems, which is sufficient to obtain a deformation to a variation of polarized Hodge structure. □

Remark 1.11. — For an application, we have to care about the relation between the deformation and the monodromy groups. We will discuss only a rough relation in the section 11.2. More precise relation will be studied elsewhere. □

Once we can deform any local system on a smooth quasi projective variety to a variation of polarized Hodge structure, preserving some compatibility with the monodromy group, we obtain the following corollary. It is a natural generalization of Theorem 1.1.

Corollary 1.12. — *Let Γ be a rigid discrete subgroup of a real algebraic group, which is not of Hodge type. Then Γ cannot be a split quotient of the fundamental groups of any smooth quasi projective variety.* □

Remark 1.13. — Such a deformation of flat bundles on a quasi projective variety was also discussed in [**22**] in a different way. □

1.3.2. Tame pure imaginary pluri-harmonic reduction (Appendix). — Let G be a linear algebraic group defined over $\boldsymbol{C}$ or $\boldsymbol{R}$. We will give a characterization of reductive representations $\pi_1(X-D,x) \longrightarrow G$ by the existence of tame pure imaginary pluri-harmonic reduction (Theorem 12.24 and Corollary 12.25). Here a representation is called reductive if the Zariski closure of the image is reductive. The author thinks that it is of independent interest. We have already known such a characterization for $GL(n,\boldsymbol{C})$-principal bundles (see [**39**] for example). In the differential geometric term, it means that if we are given a semisimple homomorphism $\pi_1(X-D) \longrightarrow G$, then we have the "tame pure imaginary" twisted pluri-harmonic map $X-D \longrightarrow G/K$ where K denotes a maximal compact group of G, which is unique up to some equivalence. Without the "pure imaginary" property, the existence theorem was proved by Jost-Zuo, directly for G. (However, their definition of reductivity looks different from ours.) On the contrary, if we impose the "pure imaginary" property, then we obtain some uniqueness, and it admits us to use a Tannakian consideration, as in [**47**]. Hence the existence theorem can also be reduced to the $\mathrm{GL}(n,\boldsymbol{C})$-case.

1.4. Remark

This is the second version of the paper about the correspondence of tame harmonic bundles and parabolic Higgs bundles. Compared to the first version [**40**], the results and the methods are significantly improved.

- In the first version, we discussed the Higgs bundles which are *of Hodge type in codimension two*. The condition is a kind of compatibility of the residues of the Higgs field, which holds for the parabolic Higgs bundles obtained from tame harmonic bundles. The previous result was sufficient for the applications discussed in this paper.

  However, when we consider the moduli spaces of parabolic Higgs bundles, it was not clear whether the subset determined by the condition had algebraically nice property (for example, openness or closedness). Moreover, the condition is defined over $\boldsymbol{C}$. So even if the condition is satisfied for $(\boldsymbol{E}_*,\theta)\otimes_{k,\iota}\boldsymbol{C}$, where

$(\boldsymbol{E}_*, \theta)$ is a μ_L-stable regular filtered Higgs bundle defined over $\iota : k \subset \boldsymbol{C}$, it is not clear whether the condition holds for $(\boldsymbol{E}_*, \theta) \otimes_{k,\iota'} \boldsymbol{C}$ where ι' is other embedding $k \subset \boldsymbol{C}$. Hence it is important and significant to remove the "Hodge type in codimension two" condition.

– In the previous version, we didn't use the argument to take the perturbation of the filtration. And hence, we need the Hodge type in codimension two condition, and we construct an initial metric for such a parabolic Higgs bundle. It was a rather delicate and bothering task. Moreover, we had to use the nilpotent orbit in the theory of mixed twistor structure, which is not unfamiliar for the most readers.

  Now we use the perturbation method. We have only to construct an initial metric for graded semisimple Higgs bundle, which is easy to construct. Although we need a rather long argument for the convergence, the author believes that the argument in the second version is easier to understand.

1.5. Outline of the paper

The chapter 2 is an elementary preparation for the discussion in the later chapters. The reader can skip this chapter. The chapter 3 is preparation about parabolic Higgs bundles. We recall some definitions in the sections 3.1–3.3. We discuss the perturbation of a given filtration in the section 3.4, which is one of the keys in this paper.

In the chapter 4, an ordinary metric for parabolic Higgs bundle is given. The construction is standard. Our purpose is to establish the relation between the parabolic characteristic numbers and some integrations, in the case of graded semisimple parabolic Higgs bundles.

In the chapter 5, we show the fundamental properties of the parabolic Higgs bundles obtained from tame harmonic bundles. Namely, we show the μ_L-stability and the vanishing of the characteristic numbers. In the chapter 6, we show the preliminary Kobayashi-Hitchin correspondence for graded semisimple parabolic Higgs bundles. Bogomolov-Gieseker inequality can be obtained as an easy corollary of this preliminary correspondence and the perturbation argument of the parabolic structure.

The chapters 7–9 are technical preparation for the proof of the main part of Theorem 1.4, which will be completed in the chapter 10.

Once the Kobayashi-Hitchin correspondence for tame harmonic bundles is established, we can apply Simpson's argument of the tours action, and we can obtain some topological consequence of quasi projective varieties. It is explained in the chapter 11. The chapter 12 is regarded as an appendix, in which we recall something related to pluri-harmonic metrics of G-flat bundles.

1.6. Acknowledgement

The author owes much thanks to C. Simpson. The paper is a result of an effort to understand his works, in particular, [**45**] and [**47**], where the reader can find the framework and many ideas used in this paper. The problem of Bogomolov-Gieseker inequality was passed to the author from him a few years ago. The author thanks Y. Tsuchimoto and A. Ishii for their constant encouragement. He is grateful to the colleagues of Department of Mathematics at Kyoto University for their help.

CHAPTER 2

PRELIMINARY

This chapter is a preparation for the later discussions. The sections are independent. The readers can skip here, but we will often use the notation given in the sections 2.1–2.2, especially.

2.1. Notation and Words

We will use the following notation:

$\mathbb{Z}$: the set of the integers, $\boldsymbol{Q}$: the set of the rational numbers,

$\boldsymbol{R}$: the set of the real numbers, $\boldsymbol{C}$: the set of the complex numbers.

For real numbers a, b, we put as follows:

$$[a,b] := \{x \in \boldsymbol{R} \,|\, a \leq x \leq b\} \quad [a,b[:= \{x \in \boldsymbol{R} \,|\, a \leq x < b\}$$
$$]a,b] := \{x \in \boldsymbol{R} \,|\, a < x \leq b\} \quad]a,b[:= \{x \in \boldsymbol{R} \,|\, a < x < b\}$$

For any positive number $C > 0$ and $z_0 \in \boldsymbol{C}$, the open disc $\big\{z \in \boldsymbol{C} \,\big|\, |z - z_0| < C\big\}$ is denoted by $\Delta(z_0, C)$, and the punctured disc $\Delta(z_0, C) - \{z_0\}$ is denoted by $\Delta^*(z_0, C)$. When $z_0 = 0$, $\Delta(0, C)$ and $\Delta^*(0, C)$ are often denoted by $\Delta(C)$ and $\Delta^*(C)$. Moreover, if $C = 1$, $\Delta(1)$ and $\Delta^*(1)$ are often denoted by Δ and Δ^*. If we emphasize the variable, it is denoted as the subscript like Δ_z. Unfortunately, the notation Δ is also used to denote the Laplacian. The author hopes that there will be no confusion.

For sets S and Y, $q_s : Y^S \longrightarrow Y$ $(s \in S)$ often denotes the projection onto the s-th component.

We say as follows:

- Let X be a complex manifold and D be a normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. For simplicity, we restrict ourselves to the case where each irreducible component of D is smooth in this case. Recall that such a divisor is called *simple* normal crossing divisor.

– Let Y be a manifold, E be a vector bundle on Y, and $\{f_i\}$ be a sequence of sections of E. We say that $\{f_i\}$ is *bounded* in L_l^p locally on Y, if the restriction $\{f_{i\,|\,K}\}$ is bounded in $L_l^p(K)$ for each compact subset $K \subset Y$. We say $\{f_i\}$ converges to f weakly in L_l^p locally on Y, if the restriction $\{f_{i\,|\,K}\}$ converges to $f_{|K}$ weakly in $L_l^p(K)$.

Finally, we recall the definition of differential geometric (local) convergence of Higgs bundles.

Definition 2.1. — Let Y be a complex manifold, and let $\bigl\{(E^{(i)}), \overline{\partial}^{(i)}, \theta^{(i)}\bigr\}$ be a sequence of Higgs bundles on Y. We say that the sequence $\bigl\{(E^{(i)}), \overline{\partial}^{(i)}, \theta^{(i)}\bigr\}$ weakly converges to $(E^{(\infty)}, \overline{\partial}^{(\infty)}, \theta^{\infty)})$ in L_2^p locally on Y, if there exist locally L_2^p-isomorphisms $\Phi^{(i)} : E^{(i)} \longrightarrow E^{(\infty)}$ on Y such that the sequences $\{\Phi^{(i)}\bigl(\overline{\partial}^{(i)}\bigr)\}$ and $\{\Phi^{(i)}(\theta^{(i)})\}$ weakly converge to $\overline{\partial}^{(\infty)}$ and $\theta^{(\infty)}$ respectively in L_1^p locally on Y. □

2.2. Review of a result of Simpson on Kobayashi-Hitchin correspondence

2.2.1. Analytic stability and the Hermitian-Einstein metric. — Let Y be an n-dimensional complex manifold which is not necessarily compact. Let ω be a Kahler form of Y. The adjoint for the multiplication of ω is denoted by Λ_ω, or simply by Λ if there are no confusion. The Laplacian for ω is denoted by Δ_ω.

Condition 2.2. —

1. The volume of Y with respect to ω is finite.
2. There exists a real valued function ϕ on Y satisfying the following:
   – $0 \leq \sqrt{-1}\partial\overline{\partial}\phi \leq C \cdot \omega$ for some positive constant C.
   – $\bigl\{x \in Y \bigm| \phi(x) \leq A\bigr\}$ is compact for any $A \in \boldsymbol{R}$.
3. There exists an increasing function $\boldsymbol{R}_{\geq 0} \longrightarrow \boldsymbol{R}_{\geq 0}$ such that $a(0) = 0$ and $a(x) = x$ for $x \geq 1$, and the following holds:
   – Let f be a positive bounded function on Y such that $\Delta_\omega f \leq B$ for some positive number B, Then $\sup_Y |f| \leq C(B) \cdot a\Bigl(\int_Y f\Bigr)$ for some positive constant $C(B)$ depending on B. Moreover $\Delta_\omega f \leq 0$ implies $\Delta_\omega f = 0$. □

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on Y. Let h be a hermitian metric of E. Then we have the $(1,0)$-operator ∂_E determined by $\overline{\partial}h(u,v) = h\bigl(\overline{\partial}_E u, v\bigr) + h\bigl(u, \partial_E v\bigr)$. We also have the adjoint $\theta^\dagger$. If we emphasize the dependence on h, we use the notation $\partial_{E,h}$ and $\theta_h^\dagger$. We obtain the connections $D_h := \overline{\partial}_E + \partial_E$ and $\mathbb{D}^1 := D_h + \theta + \theta^\dagger$. The curvatures of D_h and $\mathbb{D}^1$ are denoted by $R(h)$ and $F(h)$ respectively. When we emphasize the dependence on $\overline{\partial}_E$, they are denoted by $R(\overline{\partial}_E, h)$ and $F(\overline{\partial}_E, h)$.

Condition 2.3. — $\Lambda_\omega F(h)$ is bounded with respect to h, and $F(h)$ is L^2 with respect to h and ω. □

When Condition 2.3 is satisfied, we put as follows:

$$\deg_\omega(V,h) := \frac{\sqrt{-1}}{2\pi}\int_Y \operatorname{tr}\bigl(F(h)\bigr)\cdot\omega^{n-1}.$$

For any saturated Higgs subsheaf $V \subset E$, there is a Zariski closed subset Z of codimension two such that $V_{|Y-Z}$ gives a subbundle of $E_{|Y-Z}$, on which the metric h_V of $V_{|Y-Z}$ is induced. Let π_V denote the orthogonal projection of $E_{|Y-Z}$ onto $V_{|Y-Z}$. Let tr_V denotes the trace for endomorphisms of V.

***Proposition 2.4* ([45] Lemma 3.2)**. — *When the conditions* 2.2 *and* 2.3 *are satisfied, the integral*

$$\deg_\omega(V,K) := \frac{\sqrt{-1}}{2\pi}\int_Y \operatorname{tr}_V\bigl(F(h_V)\bigr)\cdot\omega^{n-1}$$

is well defined, and it takes the value in $\boldsymbol{R}\cup\{-\infty\}$. The Chern-Weil formula holds as follows, for some positive number C:

$$\deg_\omega(V,h_V) = \frac{\sqrt{-1}}{2\pi}\int_Y \operatorname{tr}\Bigl(\pi_V\circ\Lambda_\omega F(h)\Bigr) - C\int_Y \bigl|D''\pi_V\bigr|_h^2\cdot \mathrm{dvol}_\omega\,.$$

Here we put $D'' = \overline{\partial}_E + \theta$. In particular, if the value $\deg_\omega(V,K_V)$ is finite, $\overline{\partial}_E(\pi_V)$ and $[\theta,\pi_V]$ are L^2. □

For any $V\subset E$, we put $\mu_\omega(V,h_V) := \deg_\omega(V,h_V)/\operatorname{rank} V$.

***Definition 2.5* ([45])**. — A metrized Higgs bundle $(E,\overline{\partial}_E,\theta,h)$ is called analytic stable, if the inequalities

$$\mu_\omega(V,h_V) < \mu_\omega(E,h)$$

hold for any non-trivial Higgs subsheaves $(V,\theta_V)\subsetneq(E,\theta)$. □

***Proposition 2.6* (Simpson)**. — *Let (Y,ω) be a Kahler manifold satisfying Condition* 2.2*, and let $(E,\overline{\partial}_E,\theta,h_0)$ be a metrized Higgs bundle satisfying Condition* 2.3*. Then there exists a hermitian metric $h = h_0\cdot s$ satisfying the following conditions:*

- *h and h_0 are mutually bounded.*
- $\det(h) = \det(h_0)$
- *$D''(s)$ is L^2 with respect to h_0 and ω.*
- *It satisfies the Hermitian Einstein condition $\Lambda_\omega F(h)^\perp = 0$, where $F(h)^\perp$ denotes the trace free part of $F(h)$.*
- *The following equalities hold:*

$$\int_Y \operatorname{tr}\Bigl(F(h)^2\Bigr)\cdot\omega^{n-2} = \int_Y \operatorname{tr}\Bigl(F(h_0)^2\Bigr)\cdot\omega^{n-2},$$
$$\int_Y \operatorname{tr}\Bigl(F(h)^{\perp\,2}\Bigr)\cdot\omega^{n-2} = \int_Y \operatorname{tr}\Bigl(F(h_0)^{\perp\,2}\Bigr)\cdot\omega^{n-2}.$$

Proof See Theorem 1, Proposition 3.5 and Lemma 7.4 (with the remark just before the lemma) in [**45**]. □

2.2.2. The uniqueness. — Although the following proposition does not seem to be clearly stated in [**45**], it can be proved by the methods contained in [**45**].

Proposition 2.7. — *Let (Y, ω) be a Kahler manifold as above, and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on Y. Let h_i $(i = 1, 2)$ be hermitian metrics of E such that $\Lambda_\omega F(h_i) = 0$. We assume that h_1 and h_2 are mutually bounded. Then the following holds:*

- *We have the decomposition $(E, \theta) = \bigoplus (E_a, \theta_a)$ which is orthogonal with respect to both of h_i.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then there exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.*

Proof We take the endomorphism s_1 determined by $h_2 = h_1 \cdot s_1$. Then we have the following inequality due to Lemma 3.1 (d) in [**45**] on $X - D$:

$$\Delta_\omega \log \operatorname{tr}\bigl(s_1\bigr) \leq \bigl|\Lambda_\omega F(h_1)\bigr| + \bigl|\Lambda_\omega F(h_2)\bigr| = 0.$$

Here we have used $\Lambda_\omega F(h_i) = 0$. Then we obtain $\Delta_\omega \operatorname{tr}\bigl(s_1\bigr) \leq 0$. Since the function $\operatorname{tr}(s_1)$ is bounded on Y, we obtain the harmonicity $\Delta_\omega \operatorname{tr}(s_1) = 0$.

We put $D'' = \overline{\partial} + \theta$ and $D' := \partial_{E,h_1} + \theta^\dagger_{h_1}$, where $\theta^\dagger_{h_1}$ denotes the adjoint of θ with respect to the metric h_1. Then we also have the following equality:

$$0 = F\bigl(h_2\bigr) - F\bigl(h_1\bigr) = D''\bigl(s_1^{-1} D' s_1\bigr) = -s_1^{-1} D'' s_1 \cdot s_1^{-1} \cdot D' s_1 + s_1^{-1} D'' D' s_1.$$

Hence we obtain $D'' D' s_1 = D'' s_1 \cdot s_1^{-1} \cdot D' s_1$. As a result, we obtain the following equality:

$$\int \bigl|s_1^{-1/2} D'' s_1\bigr|^2_{h_1} \cdot \mathrm{dvol}_\omega = -\sqrt{-1} \int \Lambda_\omega \operatorname{tr}\bigl(D'' D' s_1\bigr) \cdot \mathrm{dvol}_\omega = -\int \Delta_\omega \operatorname{tr}(s_1) \cdot \mathrm{dvol}_\omega = 0.$$

Hence we obtain $D'' s_1 = 0$, i.e., $\overline{\partial} s_1 = \bigl[\theta, s_1\bigr] = 0$. Since s_1 is self-adjoint with respect to h_1, we obtain the flatness $\bigl(\overline{\partial} + \partial_{E,h_1}\bigr) s_1 = 0$. Hence we obtain the decomposition $E = \bigoplus_{a \in S} E_a$ such that $s_a = \bigoplus b_a \cdot \mathrm{id}_{E_a}$ for some positive constants b_a. Let π_{E_a} denote the orthogonal projection onto E_a. Then we have $\overline{\partial} \pi_{E_a} = 0$. Hence the decomposition $E = \bigoplus_{a \in S} E_a$ is holomorphic. It is also compatible with the Higgs field. Hence we obtain the decomposition as the Higgs bundles. Then the claim of Proposition 2.7 is clear. □

Remark 2.8. — We have only to impose $\Lambda_\omega F(h_1) = \Lambda_\omega F(h_2)$ instead of $\Lambda_\omega F(h_i) = 0$, which can be shown by a minor refinement of the argument. □

2.2.3. The one dimensional case. — In the one dimensional case, he established the Kobayashi-Hitchin correspondence for parabolic Higgs bundle. Here we restrict ourselves to the special case. See the chapter 3 of this paper for some definitions.

***Proposition 2.9* (Simpson).** — *Let X be a smooth projective curve, and D be a divisor of X. Let $\bigl(\boldsymbol{E}_*, \theta\bigr)$ be a filtered regular Higgs bundle on (X, D). We put $E = {}_{\boldsymbol{c}}E_{|X-D}$. The following conditions are equivalent:*

- *$(\boldsymbol{E}_*, \theta)$ is poly-stable with* $\text{par-deg}(\boldsymbol{E}_*) = 0$.
- *There exists a harmonic metric h of (E, θ), which is adapted to the parabolic structure of $\boldsymbol{E}_*$.*

Moreover, such a metric is unique up to obvious ambiguity. Namely, let h_i $(i = 1, 2)$ be two harmonic metrics. Then we have the decomposition of Higgs bundles $(E, \theta) = \bigoplus (E_a, \theta_a)$ satisfying the following:

- *The decomposition is orthogonal with respect to both of h_i.*
- *The restrictions of h_i to E_a are denoted by $h_{i,a}$. Then there exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.*

Proof See [**46**]. We give only a remark on the uniqueness. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X - D$, and h_i $(i = 1, 2)$ be harmonic metrics on it. Assume that the induced prolongments ${}_{\boldsymbol{c}}E(h_i)$ are isomorphic. (See the section 3.3 for prolongment.) Recall the norm estimate for tame harmonic bundles in the one dimensional case ([**46**]), which says that the harmonic metrics are determined up to boundedness by the parabolic filtration and the weight filtration. Hence we obtain the mutually boundedness of h_1 and h_2. Then the uniqueness follows from Proposition 2.7. □

2.3. Weitzenbeck formula

Let (Y, ω) be a Kahler manifold. Let h be a Hermitian-Einstein metric for a Higgs bundle $(E, \overline{\partial}_E, \theta)$ on Y. More strongly, we assume $\Lambda_\omega F(h) = 0$. The following lemma is a minor modification of Weitzenbeck formula for harmonic bundles by Simpson ([**46**]).

Lemma 2.10. — *Let s be any holomorphic section of E such that $\theta s = 0$. Then we have $\Delta_\omega \log |s|_h^2 \leq 0$, where Δ_ω denotes the Laplacian for ω.*

Proof We have $\partial\overline{\partial}|s|_h^2 = \partial\bigl(s, \partial_E s\bigr) = (\partial_E s, \partial_E s) + (s, \overline{\partial}_E \partial_E s) = (\partial_E s, \partial_E s) + (s, R(h)s)$. Then we obtain the following:

$$\partial\overline{\partial} \log |s|_h^2 = \frac{\partial\overline{\partial}|s|^2}{|s|^2} - \frac{\partial|s|^2 \cdot \overline{\partial}|s|^2}{|s|^4} = \frac{(s, R(h)s)}{|s|^2} + \frac{(\partial_E s, \partial_E s)}{|s|^2} - \frac{\partial|s|^2 \cdot \overline{\partial}|s|^2}{|s|^4}.$$

We have $R(h) = -(\theta^{\dagger}\theta + \theta\theta^{\dagger}) + F(h)^{(1,1)}$, where $F(h)^{(1,1)}$ denotes the $(1,1)$-part of $F(h)$. Hence we have the following:

$$
\begin{aligned}
(2) \quad \Lambda_{\omega}\big(s, R(h)s\big) &= \Lambda_{\omega}\Big(s,\ (-\theta\theta^{\dagger} - \theta^{\dagger}\theta)s\Big) + \Lambda_{\omega}\big(s, F(h)^{(1,1)}s\big) \\
&= -\Lambda_{\omega}\big(\theta^{\dagger}s, \theta^{\dagger}s\big) - \Lambda_{\omega}\big(\theta s, \theta s\big) + \Lambda_{\omega}\big(s, F(h)^{1,1}s\big) = -\Lambda_{\omega}(\theta^{\dagger}s,\ \theta^{\dagger}s).
\end{aligned}
$$

Here we have used $\Lambda_{\omega}F(h) = \Lambda_{\omega}F(h)^{(1,1)} = 0$. Therefore we obtain the following:

$$
-\sqrt{-1}\Lambda_{\omega}\big(s, R(h)s\big) = \sqrt{-1}\Lambda_{\omega}(\theta^{\dagger}s, \theta^{\dagger}s) = -\big|\theta^{\dagger}s\big|_h^2.
$$

On the other hand, we also have the following:

$$
-\sqrt{-1}\Lambda_{\omega}\left(\frac{(\partial s, \partial s)}{|s|^2} - \frac{\partial|s|^2\overline{\partial}|s|^2}{|s|^4}\right) \leq 0.
$$

Hence we obtain $\Delta_{\omega}\log|s|^2 \leq 0$. □

2.4. Preliminary from linear algebra

Let V be a $\boldsymbol{C}$-vector space with a hermitian metric h. Assume that we have the orthogonal decomposition $V = \bigoplus_{i=1}^{r} V_i$. We put $S := \bigoplus_{i>j} Hom(V_i, V_j)$. Let π_S denote the orthogonal projection $\mathrm{End}(V) \longrightarrow S$. For $f \in \mathrm{End}(V)$, the element $G_f \in \mathrm{End}(S)$ is given by $G_f(g) := \pi_S\big([f, g]\big)$.

Let us consider an element $f \in \bigoplus_{i\leq j} Hom(V_i, V_j)$. We have the decomposition $f = \sum_{i\leq j} f_{j\,i}$, where $f_{j\,i} \in Hom(V_i, V_j)$. We assume $f_{i,i} = \alpha_i \cdot \mathrm{id}_{V_i} + N_i$, where N_i are nilpotent for any i. We also assume that $\alpha_i \neq \alpha_j$ for $i \neq j$.

Lemma 2.11. — *The norm of G_f is dominated by the norm of f. G_f is invertible. The norm of G_f^{-1} is dominated by a polynomial of the norm of f.*

Proof Let us take an orthonormal frame $\boldsymbol{v}_i = (v_{i,1}, \ldots, v_{i,r_i})$ of V_i for which f_i is represented by a lower triangular matrix $A_{i,i}$, i.e., $f_{i,i}\boldsymbol{v}_i = \boldsymbol{v}_i \cdot A_{i,i}$. All the diagonal components of $A_{i,i}$ are α_i. We also have the matrices $A_{j,i}$ given by $f_{j\,i}\boldsymbol{v}_i = \boldsymbol{v}_j \cdot A_{j,i}$ for $i < j$. Let $\boldsymbol{v}$ be a frame of V obtained from $\boldsymbol{v}_i$ $(i = 1, \ldots, r)$. Then f is represented by a lower triangular matrix with respect to the frame $\boldsymbol{v}$, which is obtained from the matrices $A_{j\,i}$.

The element $E_{(j,l),(i,k)} \in \mathrm{End}(V)$ is given as follows:

$$
E_{(j,l),(i,k)}v_{(p,q)} := \begin{cases} v_{(j,l)} & \big((i,k) = (p,q)\big), \\ 0 & (\text{otherwise}) \end{cases}
$$

The tuple $\big\{E_{(j,l),(i,k)} \,\big|\, i < j\big\}$ gives the orthonormal frame of S. We give the lexicographic order to the set $\big\{(i.k) \,\big|\, i = 1, \ldots, r,\ k = 1, \ldots, r_i\big\}$. We have the expression:

$$
f = \sum_{(i,k)\leq(j,l)} \alpha_{(j,l),(i,k)} \cdot E_{(j,l),(i,k)}.
$$

We have $\alpha_{(i,k),(i,k)} = \alpha_i$. We remark $E_{(j,l),(i,k)} \circ E_{(p,q),(r,s)} = E_{(j,l),(r,s)} \cdot \delta_{(i,k),(p,q)}$, where $\delta_{(i,k),(p,q)}$ denotes 1 if $(i,k) = (p,q)$, or 0 if $(i,k) \neq (p,q)$. Hence we have the following:

$$f \circ E_{(p,q),(r,s)} = \sum_{(i,k)\leq(j,l)} \alpha_{(j,l),(i,k)} \cdot E_{(j,l),(i,k)} \circ E_{(p,q),(r,s)} = \sum_{(p,q)\leq(j,l)} \alpha_{(j,l),(p,q)} \cdot E_{(j,l),(r,s)}.$$

We also have the following:

$$E_{(p,q),(r,s)} \circ f = \sum_{(i,k)\leq(j,l)} \alpha_{(j,l),(i.k)} \cdot E_{(p,q),(r,s)} \circ E_{(j,l),(i,k)} = \sum_{(i,k)\leq(r,s)} \alpha_{(r,s),(i,k)} \cdot E_{(p,q),(i,k)}.$$

The element $E_{(p,q),(r,s)}$ is contained in S if and only if $p < r$. Hence we have the following:

$$\pi_S\big(f \circ E_{(p,q),(r,s)}\big) = \sum_{\substack{(p,q)\leq(j,l)\\ j<r}} \alpha_{(j,l),(p,q)} \cdot E_{(j,l),(r,s)},$$

$$\pi_S\big(E_{(p,q),(r,s)} \circ f\big) = \sum_{\substack{(i,k)\leq(r,s)\\ p<i}} \alpha_{(r,s),(i,k)} \cdot E_{(p,q),(i,k)}.$$

Therefore G_f is expressed by a lower triangular matrix with respect to the frame $\big\{E_{(p,q),(r,s)} \,\big|\, p < r\big\}$, and the diagonal components are given by $\alpha_p - \alpha_r$ $(p \neq r)$. Then the first two claims immediately follow. We also obtain the estimate of the norm of G_f^{-1} due to the formula for the inverse matrix. $\square$

Let f be an element of $\bigoplus_{i\geq j} Hom(V_i, V_j)$. We have the decomposition $f = \sum_{i\geq j} f_{j\,i}$. We assume $f_{i,i} - \alpha_i \cdot \mathrm{id}_{V_i}$ are nilpotent and $\alpha_i \neq \alpha_j$ $(i \neq j)$. The endomorphism $F_f \in \mathrm{End}(S)$ is given by $F_f(g) = [f, g]$. The next lemma can be proved similarly.

Lemma 2.12. — *The endomorphism F_f is invertible. The norm of F_f is dominated by the norm of f. The inverse of F_f^{-1} is dominated by a polynomial of the norm of f.* $\square$

2.5. Preliminary from elementary calculus

2.5.1. The estimate of a solution v of $\Delta(v) = f$. — Take $\epsilon > 0$ and $N > 1$. In this section, we use the following volume form $\mathrm{dvol}_{\epsilon,N}$ of a punctured disc Δ^*:

$$\mathrm{dvol}_{\epsilon,N} := \big(\epsilon^{N+2} \cdot |z|^{2\epsilon} + |z|^2\big)^{-1} \cdot \frac{\sqrt{-1}dz \cdot d\bar{z}}{|z|^2} < \infty.$$

Let f be a function on a punctured disc Δ^* such that $\|f\|_{L^2}^2 := \int_{\Delta^*} |f|^2 \cdot \mathrm{dvol}_{\epsilon,N} < \infty$. We use the polar coordinate $z = r \cdot e^{\sqrt{-1}\theta}$. For the decomposition $f = \sum f_n(r) \cdot e^{\sqrt{-1}n\theta}$, we have $\|f\|_{L^2}^2 = 2\pi \sum_n \|f_n\|_{L^2}^2$, where $\|f_n\|_{L^2}^2$ are given as follows:

$$\|f_n\|_{L^2}^2 := \int_0^1 |f_n(\rho)|^2 \cdot \big(\epsilon^{N+2}\rho^{2\epsilon} + \rho^2\big)^{-1} \frac{d\rho}{\rho}.$$

Proposition 2.13. — *Let f be as above. Then we have a function v satisfying the following:*

$$\overline{\partial}\partial v = f \cdot \frac{d\bar{z}\cdot dz}{|z|^2}, \quad \big|v(z)\big| \leq C \cdot \Big(|z|^{\epsilon}\epsilon^{(N-1)/2} + |z|^{1/2}\Big) \cdot \|f\|_{L^2}.$$

The constant C can be independent of ϵ, N and f.

Proof We use the argument in [**59**]. First let us consider the equation $\overline{\partial}u = f \cdot d\bar{z}/\bar{z}$. For the decomposition $u = \sum u_n(\rho)\cdot e^{\sqrt{-1}n\theta}$, it is equivalent to the following equations:

$$\frac{1}{2}\left(r\frac{\partial}{\partial r}u_n - n\cdot u_n\right) = f_n, \quad (n \in \mathbb{Z}).$$

We put as follows:

$$u_n := \begin{cases} 2r^n \int_0^r \rho^{-n-1} f_n(\rho)\cdot d\rho & (n \leq 0), \\ \\ 2r^n \int_A^r \rho^{-n-1} f_n(\rho)\cdot d\rho & (n > 0). \end{cases}$$

Then $u = \sum u_n \cdot e^{\sqrt{-1}n\theta}$ satisfies the equation $\overline{\partial}u = f\cdot d\bar{z}/\bar{z}$.

Lemma 2.14. — *There exists $C_1 > 0$ such that*

$$|u_n(r)| \leq C_1 \cdot \|f_n\|_{L^2} \cdot \left(\frac{\epsilon^{(N+2)/2}\cdot r^{\epsilon}}{|2\epsilon - 2n|^{1/2}} + \frac{r^{1/2}}{(1+|n|)^{1/2}}\right).$$

The constant C_1 is independent of n, ϵ, N and f.

Proof In the case $n \leq 0$, we have the following:

(3) $|u_n(r)|$

$$\leq \left|2r^n \int_0^r f_n(\rho)(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)^{-1/2}\rho^{-1/2}\cdot\rho^{-n-1}(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)^{1/2}\cdot\rho^{1/2}\cdot d\rho\right|$$

$$\leq 2r^n\left(\int_0^r |f_n(\rho)|^2(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)^{-1}\frac{d\rho}{\rho}\right)^{1/2}\cdot\left(\int_0^r \rho^{-2n-1}(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)\cdot d\rho\right)^{1/2}.$$

We have the following:

$$\int_0^r \rho^{-2n-1}(\epsilon^{N+2}\rho^{2\epsilon}+\rho^2)d\rho = \frac{\epsilon^{N+2}\cdot r^{2\epsilon-2n}}{2\epsilon-2n} + \frac{r^{-2n+2}}{-2n+2}.$$

Hence we obtain the following:

$$|u_n(r)| \leq 2\|f_n\|_{L^2}\cdot\left(\frac{\epsilon^{(N+2)/2}\cdot r^{\epsilon}}{|2\epsilon-2n|^{1/2}} + \frac{r}{|2-2n|^{1/2}}\right).$$

In the case $n > 0$, we also have the following:

$$(4) \quad |u_n(r)| \leq 2r^n \cdot \left(\int_A^r |f_n(\rho)|^2 (\epsilon^{N+2}\rho^{2\epsilon} + \rho^2)^{-1} \frac{d\rho}{\rho} \right)^{1/2} \times \left(\int_A^r \rho^{-2n-1} (\epsilon^{N+2}\rho^{2\epsilon} + \rho^2) d\rho \right)^{1/2}.$$

We have the following:

$$\left| \int_A^r \rho^{-2n-1} \epsilon^{N+2} \cdot \rho^{2\epsilon} \cdot d\rho \right| \leq \frac{\epsilon^{N+2}}{|-2n+2\epsilon|} r^{-2n+2\epsilon}.$$

We also have the following:

$$\int_A^r \rho^{-2n+1} d\rho = \begin{cases} \log r - \log A & (n=1) \\ (-2n+2)^{-1}(r^{-2n+2} - A^{-2n+2}) & (n \geq 2) \end{cases}$$

Therefore we obtain the following:

$$|u_n(r)| \leq C \cdot \|f_n\|_{L^2} \left(\frac{\epsilon^{(N+2)/2} \cdot r^\epsilon}{|2\epsilon - 2n|^{1/2}} + \frac{r^{1/2}}{(1+|n|)^{1/2}} \right)$$

Thus we are done. □

Then let us consider the equation $\partial v = u \cdot dz/z$. For the decomposition $v = \sum v_n \cdot e^{\sqrt{-1}n\theta}$, it is equivalent to the following equations:

$$\frac{1}{2}\left(r\frac{\partial v_n}{\partial r} + n \cdot v_n \right) = u_n, \quad (n \in \mathbb{Z}).$$

We put as follows:

$$v_n(r) := \begin{cases} 2r^{-n} \cdot \int_0^r \rho^{n-1} u_n(\rho) \cdot d\rho & (n \geq 0) \\ 2r^{-n} \cdot \int_A^r \rho^{n-1} u_n(\rho) \cdot d\rho & (n < 0). \end{cases}$$

Then we have $\partial v = u \cdot dz/z$ for $v := \sum v_n \cdot e^{\sqrt{-1}n\theta}$. From Lemma 2.14, we obtain the following in the case $n > 0$:

$$(5) \quad |v_n(r)| \leq 2r^{-n} \int_0^r \rho^{n-1} \left(\frac{\epsilon^{(N+1)/2} \cdot \rho^\epsilon}{|2\epsilon - 2n|^{1/2}} + \frac{\rho^{1/2}}{(1+|n|)^{1/2}} \right) d\rho \cdot \|f_n\|_{L^2}$$
$$\leq C_2 \cdot \|f_n\|_{L^2} \cdot \left(\frac{\epsilon^{(N+2)/2}}{|2\epsilon - 2n|^{1/2}} \frac{r^\epsilon}{|n+\epsilon|} + \frac{1}{(1+|n|)^{1/2}} \frac{r^{1/2}}{n+1/2} \right).$$

We have a similar estimate in the case $n < 0$. Hence we obtain the following:

$$|v(r)| \leq \sum_n |v_n(r)| \leq C_4 \cdot (\epsilon^{(N-1)/2} r^\epsilon + r^{1/2}) \cdot \|f\|_{L^2}.$$

Thus the proof of Proposition 2.13 is finished. □

2.5.2. The estimate of the integrals on the subdomain. — In this section, we put $I := [0,1]$. We have the embedding $I^2 \subset \boldsymbol{C}$ given by $(x,y) \longmapsto x + \sqrt{-1}y$, which gives the complex structure of I^2. We will use the standard measure of I^2, which is omitted to denote.

Let δ_m $(m = 1,2,\ldots)$ be any positive numbers such that $\sum \delta_m < \infty$. Let f_m $(m = 1,2,\ldots)$ be functions on I^2 such that $\int_{I^2} f_m < \delta_m$. We put as follows:

$$L_{m,n} := \min\bigl\{p \in \mathbb{Z} \bigm| p \geq \epsilon_m^{-(N+2)/2} \cdot e^{n\epsilon_n}\bigr\},$$

$$\widehat{L}_{m,n} := \min\Bigl\{p \in \mathbb{Z} \Bigm| p \geq n^{-1} \cdot e_m^{-(N+2)/2} \cdot e^{n\epsilon_m}\Bigr\}.$$

We divide I into $L_{m,n}$ segments: $I = \bigcup_k \bigl[k/L_{m,n},\ (k+1)/L_{m,n}\bigr]$. Then we divide I^2 into $L^2_{m,n}$ squares:

$$I^2 = \bigcup_{l=1}^{L^2_{m,n}} D_{m,n,l}.$$

Similarly, we divide I^2 into $\widehat{L}^2_{m,n}$ squares $\widehat{D}_{m,n,l}$ $(l = 1,2,\ldots,\widehat{L}^2_{m,n})$. Let $D^1_{m,n,l}$ denote the union of the squares $D_{m,n,k}$ which intersect with $D_{m,n,l}$. Similarly, $\widehat{D}^1_{m,n,l}$ are given.

Lemma 2.15. — *If M is sufficiently large, then there exists a set $Z \subset I^2$ with the positive measure satisfying the following property.*

– *Let P be any point of Z. For $\widehat{D}_{m,n,k} \ni P$ and $D_{m,n,l} \ni P$, the following inequalities hold:*

$$\int_{D^1_{m,n,l}} f_m \leq M \cdot \epsilon_m^{N+2} e^{-2n\epsilon_m}, \qquad \int_{\widehat{D}^1_{m,n,k}} f_m \leq M \cdot n^2 \cdot \epsilon_m^{N+2} e^{-2n\epsilon_m}. \tag{6}$$

Moreover we may assume $f(P) < M$ for any $P \in Z$.

Proof The measure of the set $T_1(M) := \bigl\{P \bigm| f_m(P) > M\bigr\}$ is less than δ_m/M, which can be small if M is sufficiently large. Take any large integer n_0, and we put as follows:

$$S_m(n_0) := \left\{ l \;\middle|\; \int_{D_{m,n,l}} f_m \geq M \cdot \epsilon_m^{N+2} \cdot e^{-2n_0\epsilon_m} \right\}, \qquad X_{m,n_0} := \bigcup_{l \in S_m(n_0)} D_{m,n_0,l}.$$

Inductively, we put as follows:

$$S_m(n) := \left\{ l \;\middle|\; \int_{D_{m,n,l}} f_m \geq M \cdot \epsilon_m^{N+2} \cdot e^{-2n\epsilon_m},\ D_{m,n,l} \cap X_{m,n-1} = \emptyset \right\},$$

$$X_{m,n} := X_{m,n-1} \cup \bigcup_{l \in S_m(n)} D_{m,n,l}.$$

We put $X_m := \bigcup_n X_{m,n}$. By our construction, we have the following:

(7)

$$|X_m| = \sum_n L_{m,n}^{-2} \cdot \big|S_m(n)\big| \le \sum_n L_{m,n}^{-2} \sum_{l\in S_m(n)} \Big(\int_{D_{m,n,l}} f_m\Big) \cdot M^{-1} \cdot \epsilon_m^{-(N+2)} \cdot e^{2n\epsilon_m}$$
$$\le \frac{1}{M} \sum_n \sum_{l\in S_m(n)} \int_{D_{m,n,l}} f_m \le \frac{1}{M} \int_{I^2} f_m \le \frac{\delta_m}{M}$$

Let $D^2_{m,n,l}$ denote the union of the squares $D_{m,n,k}$ which have the intersection $D^1_{m,n,l}$. We put $Y_{m,n} := \bigcup_{l\in S_m(n)} D^2_{m,n,l}$.

Lemma 2.16. — *If we have $\int_{D_{m,n,k}} f_m \ge M \cdot \epsilon_m^{N+2} e^{-2n\epsilon_m}$ and $D_{m,n,k} \subset D^1_{m,n,l}$ for some k and l, then we also have $D_{m,n,l} \subset \bigcup_{j\le n} Y_{m,j}$.*

Proof If $D_{m,n,k} \cap X_{m,n-1} = \emptyset$, then $k \in S_m(n)$, and hence $D_{m,n,l} \subset Y_{m,n}$. If $D_{m,n,k} \cap X_{m,n-1} \neq \emptyset$, then there exists $(n',k') \in S_m(n')$ such that $n' \le n-1$ and $D_{m,n,k} \cap D_{m,n',k'} \neq \emptyset$. Then we have $D_{m,n,k} \subset D^1_{m,n',k'}$ and hence $D_{m,n,l} \subset D^1_{m,n,k} \subset D^2_{m,n',k'}$. □

We put $\widetilde{X}_m := \bigcup_n Y_{m,n}$, and $T_2(M) := \bigcup_m \widetilde{X}_m$, and then we have the following: $\big|\widetilde{X}_m\big| \le 25\big|X_m\big| \le 25 \cdot \delta_m/M$. Hence we obtain $|T_2(M)| \le 25 \cdot M^{-1} \sum_m \delta_m$. Let P be any point of $I^2 \setminus \bigcup_m \widetilde{X}_m$. For (m,n,l) such that $P \in D_{m,n,l}$, we have the first inequality in (6):

$$\int_{D^1_{m,n,l}} f_m \le 9 \cdot M \cdot \epsilon_m^{N+2} e^{-2n\epsilon_m}.$$

Similarly, we can show the existence of a set $T_3(M)$ such that the measure $|T_3(M)| < C \cdot M^{-1}$ for some constant C and that the second inequality in (6) holds for any $P \in I^2 \setminus T_3$ and for any $\widehat{D}^2_{m,n,k} \ni P$.

If M is sufficiently large, the measure of $T_1(M) \cup T_2(M) \cup T_3(M)$ is small, and $Z = I^2 \setminus \bigcup T_i(M)$ gives the desired set. □

2.6. The moduli spaces of representations

Let Γ be a finitely presented group, and V be a finite dimensional vector space over $\boldsymbol{C}$. The space of homomorphisms $R(\Gamma, V) := Hom(\Gamma, \mathrm{GL}(V))$ is naturally an affine variety over $\boldsymbol{C}$. We regard it as a Hausdorff topological space with the usual topology, not the Zariski space. We have the natural adjoint action of $\mathrm{GL}(V)$ on $R(\Gamma, V)$. Let h_V be a hermitian metric of V, and let $U(h_V)$ denote the unitary group of V with respect to h_V. The usual quotient space $R(\Gamma, V)\big/U(h_V)$ is denoted by $M(\Gamma, V, h_V)$. Let $\pi_{\mathrm{GL}(V)}$ denote the projection $R(\Gamma, V) \longrightarrow M(\Gamma, V, h_V)$.

More generally, we consider the moduli spaces of representations to a complex reductive subgroup G of $\mathrm{GL}(V)$. We put $R(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$, which we regard as a Hausdorff topological space with the usual topology. It is the closed subspace of $R(\Gamma, V)$.

Let K be a maximal compact subgroup of G. Assume that the hermitian metric h_V of V is K-invariant. We put $N_G\big(h_V\big) := \big\{u \in U(h_V) \,\big|\, \mathrm{ad}(u)(G) = G\big\}$ which is compact. We have the natural adjoint action of $N_G\big(h_V\big)$ on G, which induces the action on $R(\Gamma, G)$. The usual quotient space is denoted by $M(\Gamma, G, h_V)$. Let π_G denote the projection $R(\Gamma, G) \longrightarrow M(\Gamma, G, h_V)$. We have the naturally defined map $\Phi : M(\Gamma, G, h_V) \longrightarrow M(\Gamma, V, h_V)$. The map Φ is clearly proper in the sense that the inverse image of any compact subset via Φ is also compact.

A representation $\rho \in R(\Gamma, G)$ is called Zariski dense, if the image of ρ is Zariski dense in G. Let $\mathcal{U}$ be the subset of $R(\Gamma, G)$, which consists of Zariski dense representations. Then the restriction of Φ to $\mathcal{U}$ is injective.

Let ρ and ρ' be elements of $R(\Gamma, G)$. We say that ρ and ρ' are isomorphic in G, if there is an element $g \in G$ such that $\mathrm{ad}(g) \circ \rho = \rho'$. We say ρ' is a deformation of ρ in G, if there is a continuous family of representations $\rho_t : [0,1] \times \Gamma \longrightarrow G$ such that $\rho_0 = \rho$ and $\rho_1 = \rho'$. We say ρ' is a deformation of ρ in G modulo $N_G(h_V)$, if there is an element $u \in N_G(h_V)$ such that ρ can be deformed to $\mathrm{ad}(u) \circ \rho'$ in G. We remark that the two notions are different if $N_G(h_V)$ is not connected, in general. We also remark that ρ can be deformed to ρ' in G modulo $N_G(h_V)$, if and only if $\pi_G(\rho)$ and $\pi_G(\rho')$ are contained in the same connected component of $M(\Gamma, G, h_V)$.

We recall some deformation invariance from [**47**]. A representation $\rho \in R(\Gamma, G)$ is called rigid, if the orbit $G \cdot \rho$ is open in $R(\Gamma, G)$.

Lemma 2.17. — *Let $\rho \in R(\Gamma, G)$ be a rigid and Zariski dense representation. Then any deformation ρ' of ρ in G is isomorphic to ρ in G.*

Proof If ρ is Zariski dense, then $G \cdot \rho$ is closed in $R(\Gamma, G)$. Hence it is a connected component. □

Lemma 2.18. — *Assume that there exist an element $\rho \in R(\Gamma, G)$ and a subgroup $\Gamma_0 \subset \Gamma$ such that $\rho_{|\Gamma_0}$ is Zariski dense and rigid. Let B be a connected component of $M(\Gamma, G, h_V)$ which contains $\pi_G(\rho)$. Then any element $\rho' \in R(\Gamma, G)$ is Zariski dense if $\pi_G(\rho') \in B$. In particular, the map $\Phi_{|B} : B \longrightarrow M(\Gamma, V, h_V)$ is injective.*

Proof Let ρ' be any element of $R(\Gamma, G)$ such that $\pi_G(\rho') \in B$. Then there exists $u \in N_G(h_V)$ and $\rho'' \in R(\Gamma, G)$ such that $\rho' = \mathrm{ad}(u) \circ \rho''$ and that ρ'' can be deformed to ρ. Since $\rho''_{|\Gamma_0}$ is a deformation of $\rho_{|\Gamma_0}$, they are isomorphic in G, due to Lemma 2.17. In particular, ρ'' is Zariski dense. Hence ρ' is also Zariski dense. □

CHAPTER 3

PARABOLIC HIGGS BUNDLE AND REGULAR FILTERED HIGGS BUNDLE

We recall the notion of parabolic structure, and then we give some detail about the characteristic numbers for parabolic sheaves. In the section 3.4, a perturbation of the filtration is given, which will be useful in our later argument.

3.1. Parabolic Higgs bundle

3.1.1. $\boldsymbol{c}$-parabolic Higgs sheaf. — Let us recall the notion of parabolic structure and the Chern characteristic numbers of parabolic bundles following [**33**], [**45**], [**29**], [**55**] and [**58**]. Our convention is slightly different from theirs.

Let X be a complex manifold and D be a simple normal crossing divisor with the irreducible decomposition $D=\bigcup_{i\in S}D_i$. Let $\boldsymbol{c}=(c_i\,\big|\,i\in S)$ be an element of $\boldsymbol{R}^S$. Let $\mathcal{E}$ be a torsion-free coherent $\mathcal{O}_X$-module. Let us consider a collection of the increasing filtrations ${}^i\mathcal{F}$ $(i\in S)$ indexed by $]c_i-1,c_i]$ such that ${}^i\mathcal{F}_a(\mathcal{E})\supset\mathcal{E}(-D)$ for any $a\in]c_i-1,c_i]$. We put ${}^i\operatorname{Gr}^{\mathcal{F}}_a\mathcal{E}:={}^i\mathcal{F}_a(\mathcal{E})\big/{}^i\mathcal{F}_{<a}(\mathcal{E})$. We assume that the sets $\bigl\{a\,\big|\,{}^i\operatorname{Gr}^{\mathcal{F}}_a\mathcal{E}\neq 0\bigr\}$ are finite for any i. Such tuples of filtrations are called the $\boldsymbol{c}$-parabolic structure of $\mathcal{E}$ at D, and the tuple $\bigl(\mathcal{E},\{{}^i\mathcal{F}\,|\,i\in S\}\bigr)$ is called a $\boldsymbol{c}$-parabolic sheaf on (X,D). We will sometimes omit $\boldsymbol{c}$.

Remark 3.1. — We will use the notation $\mathcal{E}_*$ instead of $\bigl(\mathcal{E},\{{}^i\mathcal{F}\}\bigr)$ for simplicity of the notation. When we emphasize $\boldsymbol{c}$, we will often use the notation ${}_{\boldsymbol{c}}\mathcal{E}$ and ${}_{\boldsymbol{c}}\mathcal{E}_*$ instead of $\mathcal{E}$ and $\mathcal{E}_*$. In the case $\boldsymbol{c}=(0,\ldots,0)$, the notation ${}^\diamond\mathcal{E}_*$ is used. □

We will use the following notation.

$$\mathcal{P}ar\bigl(\mathcal{E}_*,i\bigr):=\bigl\{a\,\big|\,{}^i\operatorname{Gr}^{\mathcal{F}}_a(\mathcal{E})\neq 0\bigr\},\quad \mathcal{P}ar'\bigl(\mathcal{E}_*,i\bigr):=\mathcal{P}ar\bigl(\mathcal{E}_*,i\bigr)\cup\{c_i,c_i-1\},\tag{8}$$

$$\operatorname{gap}(\mathcal{E}_*,i):=\min\bigl\{|a-b|\,\big|\,a,b\in\mathcal{P}ar'(\mathcal{E}_*,i)\ a\neq b\bigr\},\quad \operatorname{gap}(\mathcal{E}_*):=\min_{i\in S}\operatorname{gap}\bigl(\mathcal{E}_*,i\bigr).\tag{9}$$

Let us recall a Higgs field ([**58**]) of a $\boldsymbol{c}$-parabolic sheaf on (X, D). A holomorphic homomorphism $\theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X^{1,0}(\log D)$ is called a Higgs field of $\mathcal{E}_*$, if the following holds:

– The naturally defined composite $\theta^2 = \theta \wedge \theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X^{2,0}(\log D)$ vanishes.
– $\theta\big({}^i\mathcal{F}_a\big) \subset {}^i\mathcal{F}_a \otimes \Omega_X^{1,0}(\log D)$

The tuple $(\mathcal{E}_*, \theta)$ is called a Higgs sheaf on (X, D).

3.1.2. The parabolic first Chern class and the degree. — For a $\boldsymbol{c}$-parabolic sheaf $\mathcal{E}_*$ on (X, D), we put as follows:

$$\mathrm{wt}(\mathcal{E}_*, i) := \sum_{a \in]c_i - 1, c_i]} a \cdot \mathrm{rank}_{D_i}\, {}^i\mathrm{Gr}_a^{\mathcal{F}}(\mathcal{E}).$$

The parabolic first Chern class of $\mathcal{E}_*$ is defined as follows:

$$\text{par-c}_1(\mathcal{E}_*) := c_1(\mathcal{E}) - \sum_{i \in S} \mathrm{wt}(\mathcal{E}_*, i) \cdot [D_i] \in H^2(X, \boldsymbol{R}).$$

Here $[D_i]$ denotes the cohomology class given by D_i. If X is a compact Kahler manifold with a Kahler form ω, we put as follows:

$$\text{par-deg}_\omega(\mathcal{E}_*) := \int_X \text{par-c}_1(\mathcal{E}_*) \cdot \omega^{\dim X - 1}, \quad \mu_\omega(\mathcal{E}_*) := \frac{\text{par-deg}_\omega(\mathcal{E}_*)}{\mathrm{rank}\, \mathcal{E}}.$$

If ω is the first Chern class of an ample line bundle L, we also use the notation $\text{par-deg}_L(\mathcal{E}_*)$ and $\mu_L(\mathcal{E}_*)$.

3.1.3. μ_L-stability. — Let X be a smooth projective variety with an ample line bundle L, and D be a simple normal crossing divisor of X. The μ_L-stability of torsion-free parabolic Higgs sheaves is defined as usual. Namely, a parabolic Higgs sheaf $\big(\mathcal{E}_*, \theta\big)$ is called μ_L-stable, if the inequality $\text{par-deg}_L(\mathcal{E}'_*) < \text{par-deg}_L(\mathcal{E}_*)$ holds for any saturated non-trivial subsheaf $\mathcal{E}' \subsetneq \mathcal{E}$ such that $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega^{1,0}(\log D)$. Here the parabolic structure of $\mathcal{E}'_*$ is the naturally induced one from the parabolic structure of $\mathcal{E}_*$. Similarly, μ_L-semistability and μ_L-polystability are also defined in the obvious way.

The following lemma can be proved by a standard argument.

Lemma 3.2. — *Let $\big(\mathcal{E}_*^{(i)}, \theta^{(i)}\big)$ $(i = 1, 2)$ be μ_L-stable parabolic Higgs sheaves such that $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. If there is a non-trivial map $f : \mathcal{E}^{(1)} \longrightarrow \mathcal{E}^{(2)}$ compatible with the parabolic structure and the Higgs fields, then f is isomorphic.* □

Corollary 3.3. — *Let $(\mathcal{E}_*^{(i)}, \theta^{(i)})$ be μ_L-semistable Higgs bundles with $\mu_L(\mathcal{E}_*^{(1)}) = \mu_L(\mathcal{E}_*^{(2)})$. Assume that one of them is known to be μ_L-stable. Let $f : \mathcal{E}^{(1)} \longrightarrow \mathcal{E}^{(2)}$ be a non-trivial homomorphism compatible with the parabolic structure and the Higgs fields. Then f is isomorphic.* □

Corollary 3.4. — *Let $(\mathcal{E}_*,\theta)$ be a μ_L-polystable Higgs sheaf. Then we have the unique decomposition:*

$$(\mathcal{E}_*,\theta)=\bigoplus_j(\mathcal{E}_*^{(j)},\theta^{(j)})\otimes \boldsymbol{C}^{m(j)}.$$

Here $(\mathcal{E}_*^{(j)},\theta_j)$ are μ_L-stable, and they are mutually non-isomorphic. It is called the canonical decomposition in the rest of the paper. □

3.1.4. $\boldsymbol{c}$-parabolic Higgs bundle in codimension k. — We will often use the notation ${}_{\boldsymbol{c}}E$ instead of $\mathcal{E}$. When ${}_{\boldsymbol{c}}E$ is locally free, we put as follows, for each $i\in S$:

$${}^iF_a\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr):=\mathrm{Im}\Bigl({}^i\mathcal{F}_a({}_{\boldsymbol{c}}E)_{|D_i}\longrightarrow {}_{\boldsymbol{c}}E_{|D_i}\Bigr).$$

The tuples $\bigl({}^i\mathcal{F}\,\big|\,i\in S\bigr)$ can clearly be reconstructed from the tuple of the filtrations $\boldsymbol{F}:=\bigl({}^iF\,\big|\,i\in S\bigr)$. Hence we will often consider $\bigl({}_{\boldsymbol{c}}E,\boldsymbol{F}\bigr)$ instead of $\bigl({}_{\boldsymbol{c}}E,\{{}^i\mathcal{F}\,|\,i\in S\}\bigr)$, when ${}_{\boldsymbol{c}}E_*$ is locally free.

Definition 3.5. — Let ${}_{\boldsymbol{c}}E_*=({}_{\boldsymbol{c}}E,\boldsymbol{F})$ be a $\boldsymbol{c}$-parabolic sheaf such that ${}_{\boldsymbol{c}}E$ is locally free. When the following conditions are satisfied, ${}_{\boldsymbol{c}}E_*$ is called a $\boldsymbol{c}$-parabolic bundle.

- Each ${}^iF$ of ${}_{\boldsymbol{c}}E_{|D_i}$ is the filtration in the category of vector bundles on D_i. Namely, ${}^i\,\mathrm{Gr}_a^F({}_{\boldsymbol{c}}E_{|D_i})={}^iF_a\big/{}^iF_{<a}$ are locally free $\mathcal{O}_{|D_i}$-modules.
- The tuple of the filtrations $\boldsymbol{F}$ is compatible in the sense of Definition 4.37 in [**38**]. (In this case, the decompositions are trivial.)

We remark that the second condition is trivial in the case $\dim X=2$. □

The notion of $\boldsymbol{c}$-parabolic Higgs bundle is too restrictive in the case $\dim X>2$. Hence we will also use the following notion in the case $k=2$.

Definition 3.6. — Let ${}_{\boldsymbol{c}}E_*$ be a parabolic sheaf on (X,D). It is called a parabolic $\boldsymbol{c}$-parabolic Higgs bundle in codimension k, if the following condition is satisfied:

- There is a Zariski closed subset $Z\subset D$ such that $\dim X-\dim Z>k$ such that the restriction of ${}_{\boldsymbol{c}}E_*$ to $(X-Z,D-Z)$ is $\boldsymbol{c}$-parabolic bundle. □

Remark 3.7. — Actually, the compatibility of the filtration will not be important in the later argument. So it is not necessary for the reader to care about it. We include it just to make the statements precise. □

3.1.5. The characteristic number for $\boldsymbol{c}$-parabolic bundle in codimension two. — For any parabolic bundle ${}_{\boldsymbol{c}}E_*$ in codimension two, the parabolic second

Chern character $\text{par-ch}_2({}_{\boldsymbol{c}}E_*) \in H^4(X, \boldsymbol{R})$ is defined as follows:

$$\text{(10)} \quad \text{par-ch}_2({}_{\boldsymbol{c}}E_*) := \mathrm{ch}_2({}_{\boldsymbol{c}}E) - \sum_{\substack{i \in S \\ a \in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)}} a \cdot \iota_{i\,*}\Big(c_1\big({}^i\,\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\big)\Big) + \frac{1}{2} \sum_{\substack{i \in S \\ a \in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, i)}} a^2 \cdot \mathrm{rank}\big({}^i\,\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\big) \cdot [D_i]^2 + \frac{1}{2} \sum_{\substack{(i,j) \in S^2 \\ i \neq j}} \sum_{\substack{P \in \mathrm{Irr}(D_i \cap D_j) \\ (a_i, a_j) \in \mathcal{P}ar({}_{\boldsymbol{c}}E_*, P)}} a_i \cdot a_j \cdot \mathrm{rank}\,{}^P\,\mathrm{Gr}^F_{(a_i,a_j)}({}_{\boldsymbol{c}}E) \cdot [P].$$

Let us explain some of the notation:

- $\mathrm{ch}_2({}_{\boldsymbol{c}}E)$ denotes the second Chern character of ${}_{\boldsymbol{c}}E$.
- ι_i denotes the closed immersion $D_i \longrightarrow X$, and $\iota_{i\,*} : H^2(D_i) \longrightarrow H^4(X)$ denotes the associated Gysin map.
- $\mathrm{Irr}(D_i \cap D_j)$ denotes the set of the irreducible components of $D_i \cap D_j$.
- Let P be an element of $\mathrm{Irr}(D_i \cap D_j)$. The generic point of the component is also denoted by P. We put ${}^P F_{(a,b)} := {}^i F_{a\,|\,P} \cap {}^j F_{b\,|\,P}$ and ${}^P\,\mathrm{Gr}^F_{\boldsymbol{a}} := {}^P F_{\boldsymbol{a}} \big/ \sum_{\boldsymbol{a}' \lneq \boldsymbol{a}} {}^P F_{\boldsymbol{a}'}$. Then $\mathrm{rank}\,{}^P\,\mathrm{Gr}^F_{\boldsymbol{a}}$ denotes the rank of ${}^P\,\mathrm{Gr}^F_{\boldsymbol{a}}$ as an $\mathcal{O}_P$-module.
- We put $\mathcal{P}ar({}_{\boldsymbol{c}}E_*, P) := \big\{\boldsymbol{a} \,\big|\, {}^P\,\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E) \neq 0\big\}$.
- $[D_i] \in H^2(X, \boldsymbol{R})$ and $[P] \in H^4(X, \boldsymbol{R})$ denote the cohomology classes given by D_i and P respectively.

If X is a compact Kahler manifold with a Kahler form ω, we put as follows:

$$\text{par-ch}_{2,\omega}({}_{\boldsymbol{c}}E_*) := \text{par-ch}_2({}_{\boldsymbol{c}}E_*) \cdot \omega^{\dim X - 2}, \quad \text{par-c}^2_{2,\omega}({}_{\boldsymbol{c}}E_*) := \text{par-c}_1({}_{\boldsymbol{c}}E_*)^2 \cdot \omega^{\dim X - 2}.$$

If ω is the first Chern class of an ample line bundle L, we use the notation $\text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*)$ and $\text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*)$. In the case $\dim X = 2$, we have the obvious equalities $\text{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*) = \text{par-c}^2_1({}_{\boldsymbol{c}}E_*)$ and $\text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \text{par-ch}_2({}_{\boldsymbol{c}}E_*)$.

3.2. Filtered sheaf

3.2.1. Definitions. — Let X be a complex manifold, and D be a simple normal crossing divisor. A filtered sheaf on (X, D) is defined to be a data $\boldsymbol{E}_* = \big(\boldsymbol{E}, \big\{{}_{\boldsymbol{c}}E \,\big|\, \boldsymbol{c} \in \boldsymbol{R}^S\big\}\big)$ as follows:

- $\boldsymbol{E}$ is a quasi coherent $\mathcal{O}_X$-module. We put $E := \boldsymbol{E}_{|X-D}$.
- ${}_{\boldsymbol{c}}E$ is a coherent $\mathcal{O}_X$-submodule of $\boldsymbol{E}$ for each $\boldsymbol{c} \in \boldsymbol{R}^S$ such that ${}_{\boldsymbol{c}}E_{|X-D} = E$.
- In the case $\boldsymbol{a} \leq \boldsymbol{b}$, i.e., $q_i(\boldsymbol{a}) \leq q_i(\boldsymbol{b})$ for each $i \in S$, we have ${}_{\boldsymbol{a}}E \subset {}_{\boldsymbol{b}}E$, where q_i denotes the projection onto the i-th component. We also have $\bigcup_{\boldsymbol{a} \in \boldsymbol{R}^S} {}_{\boldsymbol{a}}E = \boldsymbol{E}$.
- We have ${}_{\boldsymbol{a}'}E = {}_{\boldsymbol{a}}E \otimes \mathcal{O}(-\sum n_j \cdot D_j)$ as submodules of $\boldsymbol{E}$, where $\boldsymbol{a}' = \boldsymbol{a} - (n_j \,\big|\, j \in S)$ for some integers n_j.

– For each $\boldsymbol{c} \in \boldsymbol{R}^S$, the filtration ${}^i\mathcal{F}$ of ${}_{\boldsymbol{c}}E$ is given as follows:

$$ {}^i\mathcal{F}_d({}_{\boldsymbol{c}}E) := \bigcup_{\substack{q_i(\boldsymbol{c})\le d \\ \boldsymbol{c}\le \boldsymbol{a}}} {}_{\boldsymbol{a}}E. $$

Then the tuple $\big({}_{\boldsymbol{c}}E, \{{}^i\mathcal{F} \,|\, i \in S\}\big)$ is a $\boldsymbol{c}$-parabolic sheaf, i.e., the sets $\big\{a \in]c_i - 1, c_i] \,\big|\, {}^i\operatorname{Gr}^{\mathcal{F}}_a({}_{\boldsymbol{c}}E)\big\}$ are finite.

Remark 3.8. — By definition, we obtain the $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}E_*$ obtained from filtered sheaf $\boldsymbol{E}_*$ for any $\boldsymbol{c} \in \boldsymbol{R}^S$, which is called the $\boldsymbol{c}$-truncation of $\boldsymbol{E}_*$. On the other hand, a filtered sheaf $\boldsymbol{E}_*$ can be reconstructed from any $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}E_*$. So we can identify them. □

Definition 3.9. — A filtered sheaf $\boldsymbol{E}_*$ is called a filtered bundle in codimension k, if any $\boldsymbol{c}$-truncations are $\boldsymbol{c}$-parabolic Higgs bundle in codimension k. □

Remark 3.10. — In the definition, "any $\boldsymbol{c}$" can be replaced with "some $\boldsymbol{c}$". □

A Higgs field of $\boldsymbol{E}_*$ is defined to be a holomorphic homomorphism $\theta : \boldsymbol{E} \longrightarrow \boldsymbol{E} \otimes \Omega^{1,0}(\log D)$ satisfying $\theta({}_{\boldsymbol{c}}E) \subset {}_{\boldsymbol{c}}E \otimes \Omega^{1,0}_X(\log D)$.

Let $\boldsymbol{E}^{(i)}_*$ $(i = 1, 2)$ be a filtered bundle on (X, D). We put as follows:

$$ \widetilde{\boldsymbol{E}} := Hom(\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}), \qquad {}_{\boldsymbol{a}}\widetilde{E} := \big\{ f \in \widetilde{\boldsymbol{E}} \,\big|\, f\big({}_{\boldsymbol{c}}E^{(1)}\big) \subset {}_{\boldsymbol{c}+\boldsymbol{a}}E^{(2)},\ \forall \boldsymbol{c} \big\}. $$

$$ \widehat{\boldsymbol{E}} := \boldsymbol{E}^{(1)} \otimes \boldsymbol{E}^{(2)}, \qquad {}_{\boldsymbol{a}}\widehat{E} := \sum_{\boldsymbol{a}_1+\boldsymbol{a}_2\le \boldsymbol{a}} {}_{\boldsymbol{a}_1}E^{(1)} \otimes {}_{\boldsymbol{a}_2}E^{(2)}. $$

Then $\big(\widetilde{\boldsymbol{E}}, \{{}_{\boldsymbol{a}}\widetilde{E}\}\big)$ and $\big(\widehat{\boldsymbol{E}}, \{{}_{\boldsymbol{a}}\widehat{E}\}\big)$ are also filtered bundles. They are denoted by $Hom\big(\boldsymbol{E}^{(1)}_*, \boldsymbol{E}^{(2)}_*\big)$ and $\boldsymbol{E}^{(1)}_* \otimes \boldsymbol{E}^{(2)}_*$.

Let $(\boldsymbol{E}_*, \theta)$ be a regular filtered Higgs bundle. Let a and b be non-negative integers. Applying the above construction, we obtain the parabolic structures and the Higgs fields on $T^{a,b}(\boldsymbol{E}) := Hom\big(\boldsymbol{E}^{\otimes a}, \boldsymbol{E}^{\otimes b}\big)$. We denote it by $(T^{a,b}\boldsymbol{E}_*, \theta)$.

3.2.2. The characteristic number of filtered bundles in codimension two.

— Let X be a smooth projective variety, and D be a simple normal crossing divisor. Let $\boldsymbol{E}_*$ be a filtered bundle in codimension two on (X, D). Let L be an ample line bundle of X.

Lemma 3.11. — *For any $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, we have $\operatorname{par-c}_1({}_{\boldsymbol{c}}E_*) = \operatorname{par-c}_1({}_{\boldsymbol{c}'}E_*)$ in $H^2(X, \boldsymbol{R})$.*

Proof The j-th components of $\boldsymbol{c}$ and $\boldsymbol{c}'$ are denoted by c_i and c'_i for any $j \in S$. Take an element $i \in S$. We have only to consider the case where the j-th components of $\boldsymbol{c}$ and $\boldsymbol{c}'$ are same if $j \neq i$. We may also assume $c'_i \in \mathcal{P}ar\big(\boldsymbol{E}_*, i\big)$ and $c_i < c'_i$.

Moreover it can be assumed that c_i is sufficiently close to c_i'. Then we have the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \longrightarrow {}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}'}E \longrightarrow {}^i\operatorname{Gr}^F_{c_i'}\big({}_{\boldsymbol{c}'}E_{|D_i}\big) \longrightarrow 0.$$

We put $c := c_i' - 1$. Then we have the following:

$$(11)\quad {}^i\operatorname{Gr}^F_c({}_{\boldsymbol{c}}E)\otimes\mathcal{O}(D_i) \simeq {}^i\operatorname{Gr}^F_{c_i'}({}_{\boldsymbol{c}'}E), \qquad {}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E) \simeq {}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}'}E),\ (c < a < c_i).$$

Therefore we have $\operatorname{wt}({}_{\boldsymbol{c}}E_*, i) = \operatorname{wt}({}_{\boldsymbol{c}'}E_*, i) - \operatorname{rank}\operatorname{Gr}^F_c({}_{\boldsymbol{c}}E)$. On the other hand, we have the following:

$$c_1\big({}_{\boldsymbol{c}'}E\big) = c_1\big({}_{\boldsymbol{c}}E\big) + c_1\big(\iota_*\operatorname{Gr}^F_{c'}({}_{\boldsymbol{c}'}E)\big).$$

There is a subset $W \subsetneq D_i$ such that $\operatorname{Gr}^F_{c'}({}_{\boldsymbol{c}'}E)_{D_i - W}$ is isomorphic to a direct sum of $\mathcal{O}_{D_i - W}$. We remark that $H^2(X, \boldsymbol{R}) \simeq H^2(X \backslash W, \boldsymbol{R})$, because the codimension of W in X is larger than two. Then it is easy to check $c_1\big(\iota_*\operatorname{Gr}^F_{c'}({}_{\boldsymbol{c}'}E)\big) = \operatorname{rank}\operatorname{Gr}^F_c({}_{\boldsymbol{c}}E)\cdot[D_i]$. Then the claim of the lemma immediately follows. □

Corollary 3.12. — *For any $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, we have the following:*

$$\operatorname{par-deg}_L({}_{\boldsymbol{c}}E_*) = \operatorname{par-deg}_L({}_{\boldsymbol{c}'}E_*), \qquad \int_X \operatorname{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*) = \int_X \operatorname{par-c}^2_{1,L}({}_{\boldsymbol{c}'}E_*).$$

In particular, the characteristic numbers $\operatorname{par-deg}_L(\boldsymbol{E}_*) := \operatorname{par-deg}_L({}_{\boldsymbol{c}}E_*)$ *and* $\int_X \operatorname{par-c}^2_{1,L}(\boldsymbol{E}_*) := \int_X \operatorname{par-c}^2_{1,L}({}_{\boldsymbol{c}}E_*)$ *are well defined.* □

Remark 3.13. — The μ_L-stability of a regular filtered Higgs bundle is defined, which is equivalent to the stability of any $\boldsymbol{c}$-truncation. Due to Corollary 3.12, it is independent of a choice of $\boldsymbol{c}$. □

Proposition 3.14. — *For any $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{R}^S$, we have the following:*

$$\int_X \operatorname{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \int_X \operatorname{par-ch}_{2,L}({}_{\boldsymbol{c}'}E_*).$$

In particular, $\int_X \operatorname{par-ch}_{2,\omega}(\boldsymbol{E}_*) := \int_X \operatorname{par-ch}_{2,\omega}({}_{\boldsymbol{c}}E_*)$ *is well defined.*

Proof We have only to show the case $\dim X = 2$. We use the following lemma.

Lemma 3.15. — *Let Y be a smooth projective surface, and D be a smooth divisor of Y. Let $\mathcal{F}$ be an $\mathcal{O}_D$-coherent module. Then we have the following:*

$$\int_X \operatorname{ch}_2(\iota_*\mathcal{F}) = \deg_D \mathcal{F} - \frac{1}{2}\operatorname{rank}(\mathcal{F})\cdot(D, D).$$

Proof By considering the blow up of $D \times \{0\}$ in $Y \times \boldsymbol{C}$ as in [**14**], we can reduce the problem in the case Y is a projective space bundle over D. We can also reduce the problem to the case $\mathcal{F}$ is a locally free sheaf on D. Then, in particular, we may assume that there is a locally free sheaf $\widetilde{\mathcal{F}}$ such that $\widetilde{\mathcal{F}}_{|D} = \mathcal{F}$. In the case, we have the K-theoretic equality $\iota_*\mathcal{F} = \widetilde{\mathcal{F}}\cdot\big(\mathcal{O} - \mathcal{O}(-D)\big)$. Therefore we have the following:

$$\operatorname{ch}(\iota_*\mathcal{F}) = \operatorname{ch}(\widetilde{\mathcal{F}})\cdot\big(D - D^2/2\big) = \operatorname{rank}\widetilde{\mathcal{F}}\cdot D + \left(-\frac{1}{2}\operatorname{rank}\widetilde{\mathcal{F}}\cdot D^2 + c_1(\widetilde{\mathcal{F}})\cdot D\right).$$

Then the claim of the lemma is clear. □

Let us return to the proof of Lemma 3.14. We use the notation in the proof of Lemma 3.11. We have the following equalities:

$$
\begin{aligned}
(12)\qquad \int_X \mathrm{ch}_2({}_{\boldsymbol{c}'}E) &= \int_X \mathrm{ch}_2({}_{\boldsymbol{c}}E) + \deg_{D_i,\omega}\big({}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)\big) - \frac{1}{2}\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)\cdot D_i^2 \\
&= \int_X \mathrm{ch}_2({}_{\boldsymbol{c}}E) + \deg_{D_i,\omega}\big({}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\big) + \frac{1}{2}\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2.
\end{aligned}
$$

Here we have used (11). We also have the following:

(13)

$$
\begin{aligned}
c'_i\cdot\deg_{D_i,L}\big({}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)\big) &= (c+1)\cdot\Big(\deg_{D_i,L}\big({}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\big)+\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2\Big) \\
&= c\cdot\deg_{D_i,L}{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E) + \deg{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E) + (c+1)\,\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2.
\end{aligned}
$$

We remark the isomorphism ${}^P\mathrm{Gr}^F_{(a,c'_i)}({}_{\boldsymbol{c}'}E)\simeq{}^P\mathrm{Gr}^F_{(a,c)}({}_{\boldsymbol{c}}E)$ and the following exact sequence:

$$
0\longrightarrow{}^j\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\longrightarrow{}^j\mathrm{Gr}^F_a({}_{\boldsymbol{c}'}E)\longrightarrow\bigoplus_{P\in D_i\cap D_j}{}^P\mathrm{Gr}^F_{(a,c'_i)}({}_{\boldsymbol{c}'}E)\longrightarrow 0.
$$

Hence we obtain the following equality:

$$
a\cdot\deg_{D_j,L}\big({}^j\mathrm{Gr}^F_a({}_{\boldsymbol{c}'}E)\big) = a\cdot\deg_{D_j,L}\big({}^j\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E)\big) + a\cdot\sum_{P\in D_i\cap D_j}\mathrm{rank}\,{}^P\mathrm{Gr}^F_{(c,a)}({}_{\boldsymbol{c}}E).
$$

We have the following equalities:

(14)

$$
\frac{1}{2}c'^{\,2}\cdot\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)\cdot D_i^2 = \frac{1}{2}c^2\,\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2 + \left(c+\frac{1}{2}\right)\cdot\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c'_i}({}_{\boldsymbol{c}'}E)\cdot D_i^2.
$$

$$
(15)\qquad c'_i\cdot a\cdot\mathrm{rank}\,{}^P\mathrm{Gr}^F_{(c'_i,a)}\big({}_{\boldsymbol{c}'}E\big) = c\cdot a\cdot\mathrm{rank}\,{}^P\mathrm{Gr}^F_{(c,a)}({}_{\boldsymbol{c}}E) + a\cdot\mathrm{rank}\,{}^P\mathrm{Gr}^F_{(c,a)}({}_{\boldsymbol{c}}E).
$$

Then we obtain the following:

(16)

$$
\begin{aligned}
&\text{par-ch}_{2,L}({}_{\boldsymbol{c}'}E_*) - \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = \deg_{D_i,L}\big({}^i\mathrm{Gr}^F_{c({}_{\boldsymbol{c}}E)}\big) + \frac{1}{2}\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\cdot D_i^2 \\
&-\deg_{D_i,L}\big({}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)\big) - (c+1)\cdot\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)D_i^2 - \sum_{j\neq i}\sum_{P\in D_i\cap D_j}\sum_a a\cdot\mathrm{rank}\,{}^P\mathrm{Gr}^F_{a,c}({}_{\boldsymbol{c}}E) \\
&\qquad + \left(c+\frac{1}{2}\right)\mathrm{rank}\,{}^i\mathrm{Gr}^F_{c}({}_{\boldsymbol{c}}E)D_i^2 + \sum_{j\neq i}\sum_{P\in D_i\cap D_j}\sum_a a\cdot\mathrm{rank}\,{}^P\mathrm{Gr}^F_{(c,a)}({}_{\boldsymbol{c}}E) = 0.
\end{aligned}
$$

Thus we are done. □

3.3. Adapted metric

We recall a 'typical' example of filtered sheaf. Let E be a holomorphic vector bundle on $X - D$. If we are given a hermitian metric h of E, we obtain the $\mathcal{O}_X$-module ${}_{\boldsymbol{c}}E(h)$ for any $\boldsymbol{c} \in \boldsymbol{R}^S$, as is explained in the following. Let us take hermitian metrics h_i of $\mathcal{O}(D_i)$. Let $\sigma_i : \mathcal{O} \longrightarrow \mathcal{O}(D_i)$ denote the canonical section. We denote the norm of σ_i with respect to h_i by $|\sigma_i|_{h_i}$. For any open set $U \subset X$, we put as follows:

$$\Gamma\big(U, {}_{\boldsymbol{c}}E(h)\big) := \Big\{ f \in \Gamma(U \setminus D, E) \,\Big|\, |f|_h = O\Big(\prod |\sigma_i|_{h_i}^{-c_i-\epsilon}\Big) \ \forall \epsilon > 0 \Big\}.$$

Thus we obtain the $\mathcal{O}_X$-module ${}_{\boldsymbol{c}}E(h)$. We also put $\boldsymbol{E}(h) := \bigcup_{\boldsymbol{c}} {}_{\boldsymbol{c}}E(h)$.

Remark 3.16. — In general, ${}_{\boldsymbol{c}}E(h)$ are not coherent, and $\boldsymbol{E}(h)$ is not quasi coherent. □

Definition 3.17. — Let $\widetilde{\boldsymbol{E}}_*$ be a filtered vector bundle in codimension k. We put $E := \widetilde{E} = \widetilde{\boldsymbol{E}}_{|X-D}$. A hermitian metric h of E is called adapted to the parabolic structure of $\widetilde{\boldsymbol{E}}_*$, if the isomorphism $E \simeq \widetilde{E}$ is extended to the isomorphisms ${}_{\boldsymbol{c}}E(h) \simeq {}_{\boldsymbol{c}}\widetilde{E}$ for any $\boldsymbol{c} \in \boldsymbol{R}^S$. □

3.4. Perturbation of parabolic structure

Let X be a smooth projective *surface*, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle over (X, D). Due to the projectivity of D, the eigenvalues of $\mathrm{Res}_i(\theta) \in \mathrm{End}\big({}_{\boldsymbol{c}}E_{|D_i}\big)$ are constant. Hence we obtain the generalized eigen decomposition with respect to $\mathrm{Res}_i(\theta)$:

$${}^i\,\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E_{|D_i}\big) = \bigoplus_{\alpha \in \boldsymbol{C}} {}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}\big({}_{\boldsymbol{c}}E_{|D_i}\big).$$

Let $\mathcal{N}_i$ denote the nilpotent part of the induced endomorphism $\mathrm{Gr}^F \mathrm{Res}_i(\theta)$ on ${}^i\,\mathrm{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})$.

Definition 3.18. — The $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ is called graded semisimple, if the nilpotent parts $\mathcal{N}_i$ are 0 for any $i \in S$. □

For simplicity, we assume $c_i \notin \mathcal{P}ar\big({}_{\boldsymbol{c}}E_*, i\big)$ for any i, where $\boldsymbol{c} = (c_i \,|\, i \in S)$.

Proposition 3.19. — *Let ϵ be any sufficiently small positive number. There exists a tuple of the parabolic structure $\boldsymbol{F}^{(\epsilon)} = \big({}^iF^{(\epsilon)} \,\big|\, i \in S\big)$ such that the following holds:*

- $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)})$ *is a graded semisimple $\boldsymbol{c}$-parabolic Higgs bundle.*
- *We have* $\text{par-deg}_\omega({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \text{par-deg}_\omega({}_{\boldsymbol{c}}E, \boldsymbol{F})$.

– *There is a constant C, which is independent of ϵ, such that the following holds:*

$$\left|\int_X \text{par-ch}_{2,\omega}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) - \int_X \text{par-ch}_{2,\omega}({}_{\boldsymbol{c}}E, \boldsymbol{F})\right| \leq C \cdot \epsilon,$$

$$\int_X \text{par-c}^2_{1,\omega}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) = \int_X \text{par-c}^2_{1,\omega}({}_{\boldsymbol{c}}E, \boldsymbol{F}).$$

– $\operatorname{gap}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) \geq \epsilon/r$.

Such $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is called an ϵ-perturbation of $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$.

Proof To take a refinement of the filtration ${}^iF$, we see the weight filtration induced on ${}^i\operatorname{Gr}^F$. Let η be a generic point of D_i. We have the weight filtration W_η of the nilpotent map $\mathcal{N}_{i,\eta}$ on ${}^i\operatorname{Gr}^F\big({}_{\boldsymbol{c}}E_{|D_i}\big)_\eta$, which is indexed by $\mathbb{Z}$. Then we can extend it to the filtration W of ${}^i\operatorname{Gr}^F\big({}_{\boldsymbol{c}}E_{|D_i}\big)$ in the category of vector bundles on D_i due to $\dim D_i = 1$. By our construction, $\mathcal{N}_i(W_k) \subset W_{k-2}$. The endomorphism $\operatorname{Res}_i(\theta)$ preserves the filtration W on ${}^i\operatorname{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$, and the nilpotent part of the induced endomorphisms on $\operatorname{Gr}^W {}^i\operatorname{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ are trivial.

Recall that the Higgs field θ induces the Higgs field ${}^i\theta$ of the vector bundle ${}_{\boldsymbol{c}}E_{|D_i}$ on D_i with the induced parabolic structure at $\bigcup_{j\neq i} D_i \cap D_j$. To explain it in terms of local coordinate, let us take a holomorphic coordinate neighbourhood (U_P, z_1, z_2) around $P \in D_i^\circ$ such that $U_P \cap D_i = \{z_1 = 0\}$. Then we have the expression $\theta = f_1(z_1,z_2)\cdot dz_1/z_1 + f_2(z_1,z_2)\cdot dz_2$. Then ${}^i\theta$ is given by $f_2(0,z_2)\cdot dz_2$. The well-definedness can be checked easily. The Higgs field ${}^i\theta$ preserves the parabolic filtration ${}^iF$ on ${}_{\boldsymbol{c}}E_{|D_i}$. Hence the Higgs field ${}^i\operatorname{Gr}^F({}^i\theta)$ of a parabolic bundle ${}^i\operatorname{Gr}^F({}_{\boldsymbol{c}}E_{|D_i})$ is induced. Since ${}^i\operatorname{Gr}^F({}^i\theta)$ commutes with $\operatorname{Res}_i\theta$, it preserves the filtration W.

Let us take the refinement of the filtration ${}^iF$. For any $a \in]c_i-1, c_i]$, we have the surjection: $\pi_a : {}^iF_a({}_{\boldsymbol{c}}E_{|D_i}) \longrightarrow {}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})$. We put ${}^i\widetilde{F}_{a,k} := \pi_a^{-1}(W_k)$. We use the lexicographic order on $]c_i-1, c_i]\times\mathbb{Z}$. Thus we obtain the increasing filtration ${}^i\widetilde{F}$ indexed by $]c_i-1,c_i]\times\mathbb{Z}$. Obviously, the set $\widetilde{S}_i := \big\{(a,k)\in]c_i-1,c_i]\times\mathbb{Z} \,\big|\, {}^i\operatorname{Gr}^{\widetilde{F}}_{(a,k)} \neq 0\big\}$ is finite.

Next, we explain the perturbation of the weight for the parabolic structure. We put as follows:

$${}^ir_a := \operatorname{rank} {}^i\operatorname{Gr}^F_a, \quad {}^ir_{a,k} := \operatorname{rank} {}^i\operatorname{Gr}^{\widetilde{F}}_{(a,k)}, \quad r := \operatorname{rank} E.$$

We assume $c_i \notin \mathcal{P}ar(\boldsymbol{E}_*)$ for each $i \in S$. Let ϵ be a small positive number such that $0 < \epsilon < \operatorname{gap}({}_{\boldsymbol{c}}E_*)$. Let us take an increasing map $\varphi_i : \widetilde{S}_i \longrightarrow]c_i-1, c_i]$ satisfying the following:

– $\big|\varphi_i(a,k) - a\big| \leq \epsilon$ and $\big|\varphi_i(a,k) - \varphi_i(a,k')\big| \geq \epsilon/r$ if $k \neq k'$.

– The following equality holds:

$$\sum_{(a,k)\in\widetilde{S}_i} \varphi_i(a,k)\cdot {}^ir_{a,k} = \sum_{a\in S_i} a\cdot r_a.$$

Then ${}^i\widetilde{F}$ and φ_i give the $\boldsymbol{c}$-parabolic filtration $\boldsymbol{F}'_\epsilon = \big({}^iF^{(\epsilon)}\,\big|\, i\in S\big)$. We call it an ϵ-perturbation of $\boldsymbol{F}$. Thus we obtain the $\boldsymbol{c}$-parabolic Higgs bundle $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta\big)$, which has the desired properties clearly due to our construction. □

The following proposition is standard.

Proposition 3.20. — *Assume that $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta\big)$ is μ_L-stable. If ϵ is sufficiently small, then any ϵ-perturbation $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta\big)$ is also μ_L-stable.*

Proof Let ${}_{\boldsymbol{c}}\widehat{E} \subset {}_{\boldsymbol{c}}E$ be a subsheaf such that $\theta\big({}_{\boldsymbol{c}}\widehat{E}\big) \subset {}_{\boldsymbol{c}}\widehat{E}\otimes\Omega^{1,0}(\log D)$. Let $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{F}}^{(\epsilon)}$ be the tuples of the filtrations of ${}_{\boldsymbol{c}}\widehat{E}$ induced by $\boldsymbol{F}$ and $\boldsymbol{F}^{(\epsilon)}$ respectively. There is a constant C, which is independent of choices of ${}_{\boldsymbol{c}}\widehat{E}$ and small $\epsilon>0$, such that the following holds:

$$\big|\mu_\omega({}_{\boldsymbol{c}}\widehat{E}, \widehat{\boldsymbol{F}}) - \mu_\omega({}_{\boldsymbol{c}}\widehat{E}, \widehat{\boldsymbol{F}}^{(\epsilon)})\big| \leq C\cdot\epsilon.$$

Therefore, we have only to show the existence of a positive number η satisfying the inequalities $\mu_\omega({}_{\boldsymbol{c}}\widehat{E}, \boldsymbol{F}) + \eta < \mu_\omega({}_{\boldsymbol{c}}E, \boldsymbol{F})$, for any saturated Higgs subsheaf $0 \neq {}_{\boldsymbol{c}}\widehat{E} \subsetneq {}_{\boldsymbol{c}}E$ under the assumption μ_L-stability of $\big({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta\big)$. It is standard, so we give only a brief outline. Due to a lemma of Grothendieck (see Lemma 2.6 in the paper of Huybrechts and Lehn [**20**]), we know the boundedness of the family $\mathcal{G}(A)$ of saturated Higgs subsheaves ${}_{\boldsymbol{c}}\widehat{E} \subsetneq {}_{\boldsymbol{c}}E$ such that $\deg_\omega({}_{\boldsymbol{c}}\widehat{E}) \geq -A$ for any fixed number A.

Let us consider the case where A is sufficiently large. Then $\mu_\omega({}_{\boldsymbol{c}}\widehat{E}_*)$ is sufficiently small for any ${}_{\boldsymbol{c}}\widehat{E} \notin \mathcal{G}(A)$. On the other hand, since the family $\mathcal{G}(A)$ is bounded, the function μ_ω on $\mathcal{G}(A)$ have the maximum, which is strictly smaller than $\mu_\omega({}_{\boldsymbol{c}}E_*)$ due to the stability. Thus we are done. □

3.5. Convergence

We give the definition of convergence of a sequence of parabolic Higgs bundles. Although we need such a notion only in the case where the base complex manifold is a curve, the definition is given generally. Let X be a complex manifold, and $D = \bigcup_{j\in S} D_j$ be a simple normal crossing divisor of X.

Definition 3.21. — Let b be any integer larger than 1. Let $\big(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)}\big)$ be a sequence of $\boldsymbol{c}$-parabolic Higgs bundle on (X,D). We say that the sequence $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)})\big\}$ weakly converges to $\big(E^{(\infty)}, \overline{\partial}^{(\infty)}, \boldsymbol{F}^{(\infty)}, \theta^{(\infty)}\big)$ in L^p_b on X, if there exist locally L^p_b-isomorphisms $\Phi^{(i)} : E^{(i)} \longrightarrow E^{(\infty)}$ on X satisfying the following conditions:

- The sequence $\{\Phi^{(i)}(\overline{\partial}^{(i)}) - \overline{\partial}^{(\infty)}\}$ weakly converges to 0 locally in L^p_{b-1} on X.
- The sequence $\{\Phi^{(i)}(\theta^{(i)}) - \theta^{(\infty)}\}$ weakly converges to 0 locally in L^p_{b-1} on X, as sections of $\mathrm{End}(E^{(\infty)})\otimes\Omega^{1,0}(\log D)$.
- For simplicity, we assume that $\Phi^{(i)}$ are C^∞ around D.

– The sequence $\{\Phi^{(i)}\big({}^jF^{(i)}\big)\}$ converges to ${}^jF^{(\infty)}$ in an obvious sense. More precisely, for any $\delta > 0$, $j \in S$ and $a \in]c_j - 1, c_j]$, there exists m_0 such that $\operatorname{rank}{}^jF^{(\infty)}_a = \operatorname{rank}{}^jF^{(i)}_{a+\delta}$ and that ${}^jF^{(\infty)}_a$ and $\Phi^{(i)}\big({}^jF^{(i)}_{a+\delta}\big)$ are sufficiently close in the Grassmaniann varieties, for any $i > m_0$. □

The following lemma is standard.

Lemma 3.22. — *Let X be a smooth projective variety, and D be a simple normal crossing divisor of X. Assume that a sequence of $\boldsymbol{c}$-parabolic Higgs bundles $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)})\big\}$ on (X, D) converges to $(E^{(\infty)}, \overline{\partial}^{(\infty)}, \boldsymbol{F}^{(\infty)}, \theta^{(\infty)})$ weakly in L^p_b on X. Assume that there exist non-zero holomorphic sections $s^{(i)}$ of $(E^{(i)}, \overline{\partial}^{(i)})$ such that $\theta^{(i)}(s^{(i)}) = 0$ and that $s^{(i)}_{|P} \in {}^jF_0\big(E^{(i)}_{|P}\big)$ for any $P \in D_j$ and $j \in S$.*

Then there exists a non-zero holomorphic section $s^{(\infty)}$ of $\big(E^{(\infty)}, \overline{\partial}^{(\infty)}\big)$ such that $\theta^{(\infty)}(s^{(\infty)}) = 0$ and that $s^{(\infty)}_{|P} \in {}^jF_0\big(E^{(\infty)}_{|P}\big)$ for any $P \in D$ and $j \in S$.

Proof Let us take a C^∞-metric $\widetilde{h}$ of $E^{(\infty)}$ on X. We put $t^{(i)} := \Phi^{(i)}(s^{(i)})$. Since p is large, we remark that $\Phi^{(i)}$ are C^0. Hence we have $\max_{P\in X} |t^{(i)}(P)|_{\widetilde{h}}$. We may assume that $\max_{P\in X} |t^{(i)}(P)|_{\widetilde{h}} = 1$.

We have $\Phi^{(i)}(\overline{\partial}^{(i)}) = \overline{\partial}^{(\infty)} + a_i$, and hence $\overline{\partial}^{(\infty)} t^{(i)} = -a_i(t^{(i)})$. Due to $|t^{(i)}| \leq 1$ and $a_i \longrightarrow 0$ in L^p_{b-1}, the L^p_b-norm of $t^{(i)}$ are bounded. Hence we can take an appropriate subsequence $\{t^{(i)} \,\big|\, i \in I\}$ which weakly converges to $t^{(\infty)}$ in L^p_b on X. In particular, $t^{(i)}$ converges to a section $t^{(\infty)}$ in C^0. Due to $\max_P |t^{(\infty)}(P)|_{\widetilde{h}} = 1$, the section $t^{(\infty)}$ is non-trivial. We also have $\overline{\partial}^{(\infty)} t^{(\infty)} = 0$ in L^p_{b-1}, and hence $t^{(\infty)}$ is a non-trivial holomorphic section of $(E^{(\infty)}, \overline{\partial}^{(\infty)})$. □

Corollary 3.23. — *Let (X, D) be as in Lemma 3.22. Assume that a sequence of $\boldsymbol{c}$-parabolic Higgs bundles $\big\{(E^{(i)}, \overline{\partial}^{(i)}, \boldsymbol{F}^{(i)}, \theta^{(i)})\big\}$ on (X, D) weakly converges to both $(E, \overline{\partial}_E, \boldsymbol{F}, \theta)$ and $(E', \overline{\partial}_{E'}, \boldsymbol{F}', \theta')$ in L^p_b on X. Then there exists a non-trivial holomorphic map $f : (E, \overline{\partial}_E) \longrightarrow (E', \overline{\partial}_{E'})$ on X which is compatible with the parabolic structures and the Higgs fields.* □

CHAPTER 4

AN ORDINARY METRIC FOR A PARABOLIC HIGGS BUNDLE

In this chapter, we would like to explain about an ordinary metric for parabolic Higgs bundles, which is a metric adapted to the parabolic structure. Such a metric has been standard in the study of parabolic bundles (for example, see [**29**]). It gives a rather good metric when the parabolic Higgs bundle is *graded semisimple*. The global construction is given in the subsection 4.3.2. For later argument, we need the estimates of the curvatures and the connection forms around the divisor, which are given in the sections 4.1–4.2. Although the estimates looks tiresome, they are quite standard and easy. We include it just for completeness, and the reader can skip them.

4.1. Around the intersection $D_i \cap D_j$

4.1.1. The construction of a metric. — We put $X := \{(z_1, z_2) \in \boldsymbol{C}^2 \,\big|\, |z_i| < 1\}$, $D_i := \{z_i = 0\}$ and $D = D_1 \cup D_2$. Take a positive number ϵ, and let ω_ϵ denote the following metric, for some positive number N:

$$\sum \epsilon^{N+2} \cdot |z_i|^{2\epsilon} \cdot \frac{dz_i \cdot d\bar{z}_i}{|z|_i^2}.$$

Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). We put $E := {}_{\boldsymbol{c}}E_{|X-D}$. We take $0 < \epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E_*)/2$. We have the description:

$$\theta = f_1 \cdot \frac{dz_1}{z_1} + f_2 \cdot \frac{dz_2}{z_2}, \quad f_i \in End({}_{\boldsymbol{c}}E).$$

We have $\mathrm{Res}_i(\theta) = f_{i\,|\,D_i}$.

Assumption 4.1. —

- The eigenvalues of $\mathrm{Res}_i(\theta)$ are constant. The set of the eigenvalues of $\mathrm{Res}_i(\theta)$ is denoted by S_i.

– We have the decomposition:

$$ {}_{\boldsymbol{c}}E = \bigoplus_{\boldsymbol{\alpha}\in S_1\times S_2} {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}} \quad \text{such that} \quad f_i({}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}) \subset {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}. $$

– There are some positive constants C and η such that any eigenvalue β of $f_{i\,|\,E_{\boldsymbol{\alpha}}}$ satisfies $|\beta-\alpha_i| \leq C\cdot|z_i|^{\eta}$ for $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$.

Remark 4.2. — The first condition is satisfied, when we are given a projective surface X' with a simple normal crossing divisor D' and a $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}'}E'_*,\theta')$ on (X',D'), such that $(X,D)\subset(X',D')$ and $({}_{\boldsymbol{c}}E_*,\theta)=({}_{\boldsymbol{c}'}E'_*,\theta')_{|X}$. The second condition is also satisfied if we replace X with a smaller open subset around the origin $O=(0,0)$. □

Let us take a holomorphic decomposition ${}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}} = \bigoplus_{\boldsymbol{a}\in\boldsymbol{R}^2} U_{\boldsymbol{\alpha},\boldsymbol{a}}$ satisfying the following conditions, where q_i denotes the projection onto the i-th component:

$$ \bigoplus_{\boldsymbol{b}\leq\boldsymbol{a}} U_{\boldsymbol{\alpha},\boldsymbol{b}\,|\,O} = {}^1F_{a_1\,|\,O}\cap {}^2F_{a_2\,|\,O}\cap {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}\,|\,O}, \qquad \bigoplus_{q_i(\boldsymbol{b})\leq a} U_{\boldsymbol{\alpha},\boldsymbol{b}\,|\,D_i} = {}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}\,|\,D_i}\cap {}^iF_a. $$

Let us take C^∞-hermitian metrics $h'_{\boldsymbol{\alpha},\boldsymbol{a}}$ of $U_{\boldsymbol{\alpha},\boldsymbol{a}}$ over X, and we put as follows:

$$ h_{\boldsymbol{\alpha},\boldsymbol{a}} := |z_1|^{-2a_1}\cdot|z_2|^{-2a_2}\cdot h'_{\boldsymbol{\alpha},\boldsymbol{a}}. $$

Then we obtain the metrics $h_0=\bigoplus h_{\boldsymbol{\alpha},\boldsymbol{a}}$ of E on $X-D$ which is adapted to the parabolic structure of ${}_{\boldsymbol{c}}E_*$. The metric h_0 is called an ordinary metric for $({}_{\boldsymbol{c}}E_*,\theta)$. We also obtain the C^∞-metric $h'_0=\bigoplus h'_{\boldsymbol{\alpha},\boldsymbol{a}}$ of ${}_{\boldsymbol{c}}E$ on X.

4.1.2. Claim. — In the rest of this section, we will explain the following proposition. (See the subsection 2.2.1 for $F(h_0)$ and $R(h_0)$).

Proposition 4.3. —

– *$R(h_0)$ is bounded with respect to ω_ϵ and h_0.*
– *If $({}_{\boldsymbol{c}}E_*,\theta)$ is graded semisimple in the sense of Definition* 3.18*, then $F(h_0)$ is bounded with respect to ω_ϵ and h_0.*

We have $F(h_0)=R(h_0)+[\theta,\theta^\dagger]+\partial_{h_0}\theta+\overline{\partial}\theta^\dagger$. We have only to estimate $R(h_0)$, $[\theta,\theta^\dagger]$, $\partial_{h_0}\theta$ and $\overline{\partial}\theta^\dagger$. We also see an estimate $\partial_0-\partial'_0$. The reader can skip the rest of this section, if he is not interested in the proof.

4.1.3. The connections and the curvatures. — From the metrics $h_{\boldsymbol{\alpha},\boldsymbol{a}}$ and h_0, we obtain the $(1,0)$-operators $\partial_{\boldsymbol{\alpha},\boldsymbol{a}}$ and ∂_0 over $X-D$, respectively. We also have $\partial'_{\boldsymbol{\alpha},\boldsymbol{a}}$ and ∂'_0 obtained from $h'_{\boldsymbol{\alpha},\boldsymbol{a}}$ and h'_0 over X, respectively. We have the following relations:

$$ \partial_{\boldsymbol{\alpha},\boldsymbol{a}} = \partial'_{\boldsymbol{\alpha},\boldsymbol{a}} - \sum a_i\cdot\frac{dz_i}{z_i}\cdot \mathrm{id}_{U_{\boldsymbol{\alpha},\boldsymbol{a}}}, \quad \partial_0=\bigoplus\partial_{\boldsymbol{\alpha},\boldsymbol{a}}, \quad \partial'_0=\bigoplus\partial'_{\boldsymbol{\alpha},\boldsymbol{a}}. $$

We put as follows:

$$A = A_1 + A_2, \quad A_i = \bigoplus \left(-a_i \frac{dz_i}{z_i}\right) \cdot \mathrm{id}_{U_{\boldsymbol{\alpha},\boldsymbol{a}}}.$$

Then we have $\partial_0 = \partial_0' + A$. As for the curvatures, we have $R(h_0) = R(h_0')$, which is bounded with respect to h_0 and ω_ϵ.

4.1.4. The estimate related to the Higgs field in the graded semisimple case. — In this subsection, we will assume that $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple. We have the natural decompositions $f_i = \bigoplus f_{i\,\boldsymbol{\alpha}}$ for $i = 1, 2$, where $f_{i\,\boldsymbol{\alpha}} \in \mathrm{End}({}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}})$. We also have $f_{i\,\boldsymbol{\alpha}} = \alpha_i \cdot \mathrm{id}_{{}_{\boldsymbol{c}}E_{\boldsymbol{\alpha}}} + N_{i\,\boldsymbol{\alpha}}$ for $i = 1, 2$, where $N_{i,\boldsymbol{\alpha}}$ are nilpotent maps.

Lemma 4.4. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, we have $\left|N_{i,\boldsymbol{\alpha}}\right|_{h_0} \leq C \cdot |z_i|^{2\epsilon}$ for some positive constant C.*

Proof Since $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, we have $N_{1\,\boldsymbol{\alpha}\,|\,D_1}\left({}^1F_a\right) \subset {}^1F_{<a}$. We also have $N_{1\,\boldsymbol{\alpha}\,|\,D_2}\left({}^2F_a\right) \subset {}^2F_a$. Then we obtain the estimate in the case $i = 1$. Similarly we can obtain the estimate in the case $i = 2$. □

Since the decomposition of $E = \bigoplus E_{\boldsymbol{\alpha}}$ is orthogonal with respect to h_0, the adjoint $f_i^\dagger$ of f_i with respect to h_0 preserves the decomposition. Hence we have the decomposition $f_i^\dagger = \bigoplus f_{i\,\boldsymbol{\alpha}}^\dagger$, and $f_{i\,\boldsymbol{\alpha}}^\dagger$ is adjoint of $f_{i\,\boldsymbol{\alpha}}$ with respect to $h_{\boldsymbol{\alpha},\boldsymbol{a}}$.

Lemma 4.5. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, then $\left[\theta, \theta^\dagger\right]$ is bounded with respect to h_0 and ω_ϵ.*

Proof It immediately follows from $\left[f_i, f_j^\dagger\right] = \bigoplus_{\boldsymbol{\alpha}} \left[N_{i,\boldsymbol{\alpha}}, N_{j,\boldsymbol{\alpha}}^\dagger\right]$ and Lemma 4.4. □

Next we would like to estimate $\partial_0\theta$.

Lemma 4.6. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, then $\partial_0\left(N_{i,\boldsymbol{\alpha}} \cdot dz_i/z_i\right)$ are bounded with respect to ω_ϵ and h_0.*

Proof Let us show the claim in the case $i = 1$. The case $i = 2$ is similar. We have the decomposition:

$$N_{1,\boldsymbol{\alpha}} = \sum_{\boldsymbol{a},\boldsymbol{b}} N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}, \qquad N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}} : U_{\boldsymbol{\alpha},\boldsymbol{a}} \longrightarrow U_{\boldsymbol{\alpha},\boldsymbol{b}}.$$

Since we have $N_{1\,\boldsymbol{\alpha}\,|\,D_1}\left({}^1F_a\right) \subset {}^1F_{<a}$ and $N_{1\,\boldsymbol{\alpha}\,|\,D_2}\left({}^2F_a\right) \subset {}^2F_a$, we obtain the following vanishings, where q_i denotes the projection onto the i-th component:

$$\begin{cases} N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}\,|\,D_1} = 0 \quad (\text{if } q_1(\boldsymbol{a}) \leq q_1(\boldsymbol{b})), \\ N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}\,|\,D_2} = 0 \quad (\text{if } q_2(\boldsymbol{a}) < q_2(\boldsymbol{b})). \end{cases} \tag{17}$$

Due to $\partial_0 = \partial_0' + A$, we have the following:

$$\partial_0\left(N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}\frac{dz_1}{z_1}\right) = \partial_0'\left(N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}\frac{dz_1}{z_1}\right) + \left[A_2,\ N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}\frac{dz_1}{z_1}\right].$$

Then the first term $G(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}) := \partial_0'\big(N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}\cdot dz_1/z_1\big)$ is the C^∞-section of $\mathrm{Hom}(U_{\boldsymbol{\alpha},\boldsymbol{a}},U_{\boldsymbol{\alpha},\boldsymbol{b}})\otimes dz_2\otimes dz_1/z_1$. Due to (17), the restriction of $G(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b})$ to D_1 is 0 if $q_1(\boldsymbol{a})\leq q_1(\boldsymbol{b})$, and the restriction to D_2 is 0 if $q_2(\boldsymbol{a})<q_2(\boldsymbol{b})$. Therefore, we obtain the boundedness of $G(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b})$ with respect to ω_ϵ and h_0. Let us estimate the second term $F(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}) := \big[A_2,\ N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}\cdot dz_1/z_1\big]$, which is a C^∞-section of the following:

$$Hom(U_{\boldsymbol{\alpha},\boldsymbol{a}},U_{\boldsymbol{\alpha},\boldsymbol{b}})\otimes\frac{dz_2}{z_2}\otimes\frac{dz_1}{z_1}.$$

Due to (17), the restriction of $F(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b})$ to D_1 is 0 if $q_1(\boldsymbol{a})\leq q_1(\boldsymbol{b})$, and the restriction to D_2 is 0 if $q_2(\boldsymbol{a})<q_2(\boldsymbol{b})$. Moreover, we obtain the vanishing of the restriction to D_2 in the case $q_2(\boldsymbol{a})=q_2(\boldsymbol{b})$ from the commutativity of $N_{1,\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b}}$ and A_2 in the case. Therefore we obtain the boundedness of $F(\boldsymbol{\alpha},\boldsymbol{a},\boldsymbol{b})$ with respect to h_0 and ω_ϵ. In all, we obtain the desired boundedness. □

Lemma 4.7. — *If $({}_{\boldsymbol{c}}E_*,\theta)$ is graded semisimple, then $\partial_0\theta$ and $\overline{\partial}\theta^\dagger$ are bounded with respect to ω_ϵ and h_0.*

Proof Since we have $\partial_0\big(\alpha_1\cdot\mathrm{id}_{E_{\boldsymbol{\alpha}}}\cdot dz_1/z_1\big)=0$, the boundedness of $\partial_0\theta$ immediately follows from Lemma 4.6. Since $\overline{\partial}\theta^\dagger$ is adjoint of ∂_0, we also obtain the boundedness of $\overline{\partial}\theta^\dagger$. □

Now Proposition 4.3 immediately follows from the result in the subsection 4.1.3, Lemma 4.5 and Lemma 4.7. □

4.2. Around a smooth point of the divisor

4.2.1. The setting. — Let Y be a complex manifold, and L be a line bundle on Y. Let $\pi: L\longrightarrow Y$ denote the projection, and σ denote the canonical section $L\longrightarrow\pi^*L$. Let $\big|\cdot\big|$ denote the hermitian metric of L. We use the same notation for the pull back of $|\cdot|$ via π. We put $DL(1) := \big\{(x,v)\,\big|\,x\in Y,\ v\in L_x,\ |v|<1\big\}$. We have the natural complex structure of $DL(1)$ as a submanifold of L, which is denoted by $J_{DL(1)}$. Let J denote any other integrable complex structure of $DL(1)$ such that $J-J_{DL(1)}=O(|\sigma|)$. Let $\overline{\partial}$ and $\widehat{\overline{\partial}}$ denote the $(0,1)$-operator determined by $J_{DL(1)}$ and J respectively. Similarly we use the notation ∂ and $\widehat{\partial}$.

Let $\omega_{DL(1)}$ be a Kahler form of $\big(DL(1),J\big)$. Take a small positive number ϵ and C, and take a large real number N. Then we put as follows:

$$\omega_\epsilon := \omega_{DL(1)} + C\cdot\epsilon^N\sqrt{-1}\partial\overline{\partial}|\sigma|^{2\epsilon}.$$

We remark the following obvious lemma.

Lemma 4.8. — *We put* $s_Y := \partial - \widehat{\partial} \in \Omega^1_{DL(1)}$. *Then we have* $s_Y = O(|\sigma|)$, *and* $\overline{\partial} s_Y$ *is bounded with respect to* $\omega_{DL(1)}$. *We also have* $\overline{\partial} s_Y = O(|\sigma|^{2\epsilon})$ *with respect to* ω_ϵ $(\epsilon > 0)$. □

Let $(E, \widehat{\overline{\partial}})$ be a holomorphic vector bundle on $(DL(1), J)$. We put $E_Y := E_{|Y}$, and let F be a filtration of E_Y in the category of holomorphic vector bundles, indexed by $]c-1, c]$. In other words, $E_* = (E, F)$ is a c-parabolic bundle on $(DL(1), Y)$. Let us take a positive number ϵ such that $2\epsilon < \mathrm{gap}(E_*)$.

Remark 4.9. — For the consistency of the notation, it may probably be better to use ${}_cE$ instead of E. But we omit to denote the subscript c for the simplicity of the notation. □

Let θ be a Higgs field of the c-parabolic bundle E_*. We put $f := \mathrm{Res}(\theta) \in End(E_Y)$.

Assumption 4.10. — The eigenvalues of f are assumed to be constant on Y. □

Remark 4.11. — The condition is satisfied if there exists a smooth projective variety Y' with a normal crossing divisor D' and a $\boldsymbol{c}$-parabolic Higgs bundle (E'_*, θ') on (Y', D') such that $(Y, D) \subset (Y', D')$ and $(E'_*, \theta')_{|(Y,D)} = (E_*, \theta)$. □

We have the generalized eigen decomposition $E_Y = \bigoplus \mathrm{Gr}^{\mathbb{E}}_\alpha(E_Y)$ with respect to f. We also have the generalized eigen decomposition $\mathrm{Gr}^F_a(E_Y) = \bigoplus_\alpha \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_Y)$ of $\mathrm{Gr}^F(E_Y)$ with respect to $\mathrm{Gr}^F(f)$. Then we put as follows, for $u \in \boldsymbol{R} \times \boldsymbol{C}$:

$$\widehat{E}_{Y,u} := \mathrm{Gr}^{F,\mathbb{E}}_u(E_Y), \qquad \widehat{E}_Y := \bigoplus \widehat{E}_{Y,u}.$$

We use the C^∞-identifications of E, $\pi^* E_Y$ and $\pi^* \widehat{E}_Y$, which is taken as follows:

- We take a C^∞-isomorphism $\Phi : E \simeq \pi^* E_Y$ for which we have $\Phi(\widehat{\overline{\partial}}_E) - \overline{\partial}_{\pi^* E_Y} = O(|\sigma|)$ with respect to $\omega_{DL(1)}$ and any C^∞-metric of E.
- On the other hand, we take C^∞-splittings of the surjections $\mathrm{Gr}^{\mathbb{E}}_\alpha(E_Y) \cap F_a \longrightarrow \mathrm{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_Y)$. The image of the splittings are denoted by $\mathcal{G}_u$. Then we have the C^∞-decomposition $E_Y = \bigoplus \mathcal{G}_u$, and we naturally obtain the C^∞-isomorphism $E_Y \simeq \widehat{E}_Y$. It induces the C^∞-isomorphism $\pi^* E_Y \simeq \pi^* \widehat{E}_Y$.

Via the identifications, we obtain the C^∞-decomposition $E = \bigoplus E_u$.

4.2.2. The construction of a metric. — Let $h_{Y,u}$ be a C^∞-metric of $\widehat{E}_{Y,u}$, and we put $h_Y := \bigoplus h_{Y,u}$. It gives the C^∞-metric of $\widehat{E}_Y = E_Y$. We put as follows:

$$h'_0 := \pi^* h_Y = \bigoplus \pi^* h_{Y,(a,\alpha)}, \qquad h_0 := \bigoplus \pi^* h_{Y,(a,\alpha)} \cdot |\sigma|^{-2a}.$$

The metric h'_0 is the C^∞-metric of $E = \pi^* E_Y = \pi^* \widehat{E}_Y$. The metric h_0 is the C^∞-metric of $E_{|DL(1)-Y}$, which is adapted to the parabolic structure. It is called an ordinary metric for (E_*, θ).

In the following, let $\partial_{\pi^*\widehat{E}_Y,h_0}$ denote the $(1,0)$-operators obtained from $\overline{\partial}_{\pi^*\widehat{E}_Y}$, the hermitian metric h_0 and the complex structure $J_{DL(1)}$ of $DL(1)$, for example. The curvature is denoted by $R(h_0,\overline{\partial}_{\pi^*\widehat{E}_Y})$. We will use similar notation for the other operators. We have the following relation:

$$\partial_{\pi^*\widehat{E}_Y,h_0}=\partial_{\pi^*\widehat{E}_Y,h_0'}+\gamma,\qquad \gamma:=\bigoplus\bigl(-a\cdot\partial\log|\sigma|^2\,\mathrm{id}_{E_{(a,\alpha)}}\bigr).$$

4.2.3. Claim. — In the rest of this section, we will give estimates of $R(h_0)$ and $F(h_0)$.

Lemma 4.12. — *$R(h_0,\widehat{\overline{\partial}}_E)$ is bounded with respect to ω_ϵ and h_0. More strongly, we have the following estimate, with respect to h_0 and ω_ϵ:*

$$R(h_0,\widehat{\overline{\partial}}_E)-\bigoplus\Bigl(\pi^*R(h_{Y,(a,\alpha)},\overline{\partial}_{E_{Y,(a,\alpha)}})+\overline{\partial}\partial\log|\sigma|^{-2a}\Bigr)=O\bigl(|\sigma|^{\epsilon}\bigr).$$

Proposition 4.13. — *If (E_*,θ) is graded semisimple, $F(E,\widehat{\overline{\partial}}_E,h_0,\theta)$ is bounded.*

The reader, who is uninterested in the proof, can skip the rest of this section.

4.2.4. The difference of $\overline{\partial}$-operators. — We put $S:=\overline{\partial}_{E_Y}-\overline{\partial}_{\widehat{E}_Y}$. We have the decomposition $S=\sum S_{u,u'}$, where $S_{u,u'}\in Hom(\mathcal{G}_u,\mathcal{G}_{u'})\otimes\Omega_Y^{0,1}$. Obviously, we have $\overline{\partial}_{\pi^*E_Y}=\overline{\partial}_{\pi^*\widehat{E}_Y}+\pi^*S$. Due to our identification given in the subsection 4.2.1. We also have the following vanishing, for $u=(a,\alpha)$ and $u'=(a',\alpha')$:

$$S_{u,u'}=0\qquad\text{unless } \alpha=\alpha' \text{ and } a>a'. \tag{18}$$

Lemma 4.14. —

- *$\pi^*S=O\bigl(|\sigma|^{2\epsilon}\bigr)$ with respect to h_0 and ω_ϵ.*
- *$\partial_{\pi^*\widehat{E}_Y,h_0'}(\pi^*S)$, $\bigl[\pi^*S,\gamma\bigr]$ and $\partial_{\pi^*\widehat{E}_Y,h_0}(\pi^*S)$ are $O(|\sigma|^{\epsilon})$ with respect to h_0 and ω_ϵ.*

Proof The first claim is clear from (18). The estimate for $\partial_{\pi^*\widehat{E}_Y,h_0'}(\pi^*S)$ is obtained from the following equality and (18):

$$\partial_{\pi^*\widehat{E}_Y,h_0'}(\pi^*S)=\sum\pi^*\bigl(\partial_{\widehat{E}_Y,h_Y}S_{u,u'}\bigr).$$

The estimate for $\bigl[\pi^*S,\gamma\bigr]$ follows from the first claim. The estimate for $\partial_{\pi^*\widehat{E}_Y,h_0}\pi^*S$ follows from those for $\partial_{\pi^*\widehat{E}_Y,h_0'}\pi^*S$ and $\bigl[\pi^*S,\gamma\bigr]$. □

The next lemma is clear.

Lemma 4.15. — *π^*S is bounded with respect to h_0' and $\omega_{DL(1)}$. $\partial_{\pi^*\widehat{E}_Y,h_0'}(\pi^*S)$ is bounded with respect to h_0' and $\omega_{DL(1)}$.* □

We put $T := \widehat{\overline{\partial}}_E - \overline{\partial}_{\pi^* E_Y}$. We have the decomposition:

$$T = \sum T_{u,u'}, \qquad T_{u,u'} \in Hom(E_u, E_{u'}) \otimes \Omega^1_{DL(1)}.$$

We have $T = O(|\sigma|)$ with respect to h_0' and $\omega_{DL(1)}$.

Lemma 4.16. —

- *$\partial_{\pi^*\widehat{E}_Y,h_0'}(T)$, $\big[\gamma, T\big]$ and $\partial_{\pi^*\widehat{E}_Y,h_0}(T)$ are bounded with respect h_0' and $\partial\overline{\partial}|\sigma|^2 + |\sigma| \cdot \omega_{|DL(1)}$.*
- *We have $T = O(|\sigma|^{2\epsilon})$ with respect to h_0 and $\omega_{DL(1)}$.*
- *$\partial_{\pi^*\widehat{E}_Y,h_0'}T$, $\big[\gamma, T\big]$ and $\partial_{\pi^*\widehat{E}_Y,h_0}T$ are $O(|\sigma|^{2\epsilon})$ with respect to h_0 and ω_ϵ.*

Proof The first two claims are clear. The third claim follows from the first claim. □

We put $Q = \pi^* S + T$, and then we have $\widehat{\overline{\partial}}_E = \overline{\partial}_{\pi^*\widehat{E}_Y} + Q$.

Corollary 4.17. —

- *$Q = O(|\sigma|^{2\epsilon})$ with respect to ω_ϵ and h_0.*
- *$\partial_{\pi^*\widehat{E}_Y,h_0'}Q$, $\big[\gamma, Q\big]$ and $\partial_{\pi^*\widehat{E}_Y,h_0'}Q$ are $O(|\sigma|^{\epsilon})$ with respect to ω_ϵ and h_0.* □

4.2.5. The connection and the curvature. — Let $\widehat{\partial}_{E,h_0}$ denote the $(1,0)$-operator obtained from $\widehat{\overline{\partial}}_E$, the metric h_0 and the complex structure J of $DL(1)$. We would like to estimate $\widehat{\partial}_{E,h_0} - \partial_{\pi^*\widehat{E}_Y,h_0}$ and the curvature $R(h_0, \widehat{\overline{\partial}}_E)$.

Lemma 4.18. — *We have $\widehat{\partial}_{E,h_0} = \partial_{\pi^*\widehat{E}_Y,h_0} - Q^\dagger_{h_0} + s_Y \cdot \mathrm{id}_E$. Here $Q^\dagger_{h_0}$ denotes the adjoint of Q with respect to h_0.*

Proof We have the following equalities, for C^∞-sections u and v of E:

$$\partial h_0(u,v) = h_0\big(\partial_{\pi^*\widehat{E}_Y,h_0} u, v\big) + h_0\big(u, \overline{\partial}_{\pi^*\widehat{E}_Y} v\big)$$

$$\widehat{\partial} h_0(u,v) = h_0\big(\widehat{\partial}_{E,h_0} u, v\big) + h_0\big(u, \widehat{\overline{\partial}}_E v\big).$$

Since we have $\widehat{\partial} - \partial = s_Y$, the claim immediately follows. □

Lemma 4.19. — *We have $Q^\dagger_{h_0} = O(|\sigma|^{2\epsilon})$ and $\overline{\partial}_{\pi^*\widehat{E}_Y} Q^\dagger_{h_0} = O(|\sigma|^{\epsilon})$ with respect to ω_ϵ and h_0.*

Proof It immediately follows from Corollary 4.17. □

Similarly, we also have the following.

Lemma 4.20. — *We have the formula $\widehat{\partial}_{E,h_0'} = \partial_{\pi^*\widehat{E}_Y,h_0'} - Q^\dagger_{h_0'} + s_Y \cdot id_E$ and the boundedness of $Q^\dagger_{h_0'}$ and $\overline{\partial}_{\pi^*\widehat{E}_Y,h_0} Q^\dagger_{h_0'}$.* □

Now we give the proof of Lemma 4.12. We have the following equalities:

$$(19)\quad R(h_0,\widehat{\overline{\partial}}_E)=\big[\widehat{\overline{\partial}}_E,\widehat{\partial}_{E,h_0}\big]=\Big[\overline{\partial}_{\pi^*\widehat{E}_Y}+Q,\ \partial_{\pi^*\widehat{E}_Y,h_0}-Q^{\dagger}_{h_0}+s_Y\cdot\mathrm{id}_E\Big]$$
$$=R(h_0,\overline{\partial}_{\pi^*\widehat{E}_Y})-\overline{\partial}_{\pi^*\widehat{E}_Y}Q^{\dagger}_{h_0}+\partial_{\pi^*\widehat{E}_Y,h_0}Q+(\overline{\partial}s_Y)\cdot\mathrm{id}_E-\big[Q,Q^{\dagger}_{h_0}\big].$$

The first term in the right hand side is as follows:

$$R(h_0,\overline{\partial}_{\pi^*\widehat{E}_Y})=\bigoplus\left(\pi^*R(h_{Y,(a,\alpha)},\overline{\partial}_{E_{Y,(a,\alpha)}})+\overline{\partial}\partial\log|\sigma|^{-2a}\right).$$

Then the proof of Lemma 4.12 is immediately obtained. □

4.2.6. The estimate for the Higgs field. — In this subsection, we assume that (E_*,θ) is graded semisimple. We would like to estimate $F(h_0)$. We put $\rho_0:=\bigoplus\alpha\cdot\mathrm{id}_{E_{(a,\alpha)}}$. Let P be any point of Y. Let (U,z_1,z_2) be a holomorphic coordinate neighbourhood of $\big(DL(1),J\big)$ around P such that $U\cap Y=\{z_1=0\}$. We are given the Higgs field:

$$\theta=f_1\cdot\frac{dz_1}{z_1}+f_2\cdot dz_2.$$

Since $f_{2\,|\,Y}$ preserves the filtration F, f_2 is bounded with respect to h_0. We have the decomposition:

$$f_2=\sum f_{2,u,u'},\qquad f_{2,u,u'}\in Hom(E_u,E_{u'}).$$

Then we have $f_{2,u,u'\,|\,Y}=0$ for $u=(a,\alpha)$ and $u=(a',\alpha')$ unless $\alpha=\alpha'$ and $a\geq a'$.

Lemma 4.21. — *If (E_*,θ) is graded semisimple, then $[\bar\rho_0,f_2]$ is estimated as $O(|\sigma|)$ with respect to h_0' and as $O(|\sigma|^{2\epsilon})$ with respect to h_0.*

Proof We have the following:

$$[\bar\rho_0,f_2]=\sum_{\substack{u=(a,\alpha)\\u'=(a',\alpha')}}\big(\bar\alpha'-\bar\alpha\big)f_{2,u,u'}.$$

Hence we have $[\bar\rho_0,f_2]_{|Y}=0$. Then the claim immediately follows. □

Let us see f_1. Due to the graded semisimplicity of (E_*,θ), we have $(f_1-\rho_0)_{|Y}\big(F_a\big)\subset F_{<a}$. Hence $f_1-\rho_0$ is $O(|\sigma|^{\epsilon})$ with respect to h_0.

Lemma 4.22. — *If (E_*,θ) is graded semisimple, then $f_1\cdot dz_1/z_1-\rho_0\cdot\partial\log|\sigma|^2$ is $O(|\sigma|^{\epsilon})$ with respect to ω_ϵ and h_0.*

Proof Let ζ be a function on U, which is holomorphic with respect to $J_{DL(1)}$, such that ζ is a defining equation of $U\cap Y$. Then we have $|\sigma|^2=|\zeta|^2\cdot\tau$ for some positive function τ. Since we have $J-J_{DL(1)}=O(|\sigma|)$, there is a function $a:U\cap Y\longrightarrow\boldsymbol{C}$ such that $\zeta=a(z_1)\cdot z_1+O(|z_1|^2)$. Then we have $d\zeta/\zeta=dz_1/z_1+O(1)$ with respect to $\omega_{DL(1)}$. Therefore, we obtain $\partial \log|\sigma|^2 - dz_1/z_1 = O(1)$ with respect to $\omega_{DL(1)}$. Hence we have the following estimate with respect to ω_ϵ and h_0:

$$f_1 \frac{dz_1}{z_1} - \rho_0 \cdot \partial \log|\sigma|^2 = (f_1 - \rho_0) \cdot \frac{dz_1}{z_1} + f_0 \left(\frac{dz_1}{z_1} - \partial \log|\sigma|^2 \right) = O(|\sigma|^\epsilon).$$

Thus we are done. □

Lemma 4.23. — *If (E_*, θ) is graded semisimple, then $[\theta, \theta^\dagger]$ is bounded with respect to ω_ϵ and h_0.*

Proof We put $\theta_1 = f_1 \cdot dz_1/z_1$, $\theta_2 = f_2 \cdot dz_2$ and $\theta_0 = \rho_0 \cdot dz_1/z_1$. We have $\big[\theta_1, \theta_1^\dagger\big] = \big[\theta_1 - \theta_0, \theta_0^\dagger\big] + \big[\theta_0, (\theta_1 - \theta_0)^\dagger\big] + \big[\theta_1 - \theta_0, (\theta_1 - \theta_0)^\dagger\big]$. Therefore it is bounded with respect to ω_ϵ and h_0, due to Lemma 4.22. We have $\big[\theta_2, \theta_1^\dagger\big] = \big[\theta_2, \theta_0^\dagger\big] + \big[\theta_2, \theta_1^\dagger - \theta_0^\dagger\big] = O(|\zeta|^\epsilon)$ with respect to ω_ϵ and h_0, due to Lemma 4.21. The boundedness of $[\theta_2, \theta_2^\dagger]$ with respect to ω_ϵ and h_0 is clear. □

Lemma 4.24. — *If (E_*, θ) is graded semisimple, then $\widehat{\partial}_{E,h_0}\theta$ and $\widehat{\overline{\partial}}_E \theta^\dagger$ are bounded with respect to h_0 and ω_ϵ.*

Proof We have $\widehat{\partial}_{E,h_0}\theta = \partial_{\pi^*\widehat{E}_Y, h_0}\theta + [\gamma, \theta] + [-Q^\dagger_{h_0}, \theta]$. We use the notation in the proof of Lemma 4.23. We have $\theta = \theta_1 + \theta_2$.

First, let us see the estimate related to θ_2. Since $\partial_{\pi^*\widehat{E}_Y, h_0'}\theta_2$ is a C^∞-section of $\mathrm{End}(E) \otimes \Omega^2$, it is easy to obtain the boundedness with respect to ω_ϵ and h_0. The boundedness of $[\theta_2, Q^\dagger]$ easily follows from the estimates for θ_2 and $Q^\dagger$. We have

$$\big[\theta_2, \gamma\big] = -\sum_{\substack{u=(a,\alpha)\\ u'=(a',\alpha')}} f_{2,u,u'} \cdot (a - a') \cdot \partial \log|\sigma|^2 \cdot dz_2.$$

We have $f_{2,u,u'} \cdot (a-a')_{|Y} = 0$ unless $a > a'$, and hence we obtain the boundedness of $\big[\theta_2, \gamma\big]$ with respect to h_0 and ω_ϵ.

Next we see the estimate related to $\theta_1 = f_1 \cdot dz_1/z_1$. We have the following equality:

$$\partial_{\pi^*\widehat{E}_Y, h_0'}\left(f_1 \cdot \frac{dz_1}{z_1} \right) = \partial_{\pi^*\widehat{E}_Y, h_0'}(f_1 - \rho_0) \cdot \frac{dz_1}{z_1} + f_1 \cdot (\partial - \widehat{\partial}) \frac{dz_1}{z_1}.$$

Here we have used $\partial_{\pi^*\widehat{E}_Y, h_0'}(\rho_0) = 0$ and $\widehat{\partial}(dz_1/z_1) = 0$. Since we have $\partial_{\widehat{E}_Y, h_0'}\big((f_1 - \rho_0)_{|Y}\big)(F_a) \subset F_{<a} \otimes \Omega^1_Y$, it is easy to see that the term $\partial_{\pi^*\widehat{E}_Y, h_0'}(f_1 - \rho_0) \cdot \frac{dz_1}{z_1}$ is bounded with respect to ω_ϵ and h_0. Since f_1 is bounded with respect to h_0, and since $(\partial - \widehat{\partial})(dz_1/z_1) = s_Y(dz_1/z_1)$ is bounded with respect to ω_ϵ, the term $f_1 \cdot (\partial - \widehat{\partial})\frac{dz_1}{z_1}$ is also bounded with respect to h_0 and ω_ϵ. Thus we obtain the boundedness of $\partial_{\pi^*\widehat{E}_Y, h_0'}(\theta_1)$. The boundedness of $[\theta_1, Q^\dagger]$ follows from the estimate of θ_1 and $Q^\dagger = O(|\sigma|^\epsilon)$ with respect to h_0 and ω_ϵ. Finally, we have

$$[\theta_1, \gamma] = \left[(f_1 - \rho_0) \frac{dz_1}{z_1}, \gamma \right],$$

and $\partial \log |\sigma|^2 - dz_1/z_1 = O(1)$ with respect to $\omega_{DL(1)}$. Thus it is easy to check the boundedness of $[\theta_1, \gamma]$ with respect to ω_ϵ and h_0. $\square$

Now, Proposition 4.13 immediately follows from Lemma 4.12, Lemma 4.23 and Lemma 4.24. $\square$

4.3. Global ordinary metric

4.3.1. Decomposition and metric of a base space. — Let X be a smooth projective surface, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let L be an ample line bundle on X, and ω be a Kahler form which represents $c_1(L)$.

For any point $P \in D_i \cap D_j$, we take a holomorphic coordinate (U_P, z_i, z_j) around P such that $U_P \cap D_k = \{z_k = 0\}$ $(k = i, j)$ and $U_P \simeq \Delta^2$ by the coordinate.

Let us take a hermitian metric g_i of $\mathcal{O}(D_i)$ and the canonical section $\mathcal{O} \longrightarrow \mathcal{O}(D_i)$ is denoted by σ_i. We may assume $|\sigma_k|^2_{g_k} = |z_k|^2$ $(k = i, j)$ on U_P for $P \in D_i \cap D_j$.

Let us take a hermitian metric g of the tangent bundle TX such that $g = dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$ on U_P. It is not necessarily same as ω, and not necessarily Kahler. The metric g induces the exponential map $\exp : TX \longrightarrow X$.

Let $N_{D_i}X$ denote the normal bundle of D_i in X. We can take a sufficiently small neighbourhood U'_i of D_i in $N_{D_i}X$ such that the restriction of $\exp_{|U'_i}$ gives the diffeomorphism of U'_i and the neighbourhood U_i of D_i in X. We may assume $U_i \cap U_j = \coprod_{P \in D_i \cap D_j} U_P$ and $U_i = \big\{|\sigma_i|_{g_i} < 1\big\}$.

Let p_i denote the diffeomorphism $\exp_{|U_i} : U_i \longrightarrow U'_i$. Let π_i denote the natural projection $U'_i \longrightarrow D_i$. Via the diffeomorphism p_i, we also have the C^∞-map $U_i \longrightarrow D_i$, which is also denoted by π_i. On U_P, π_i is same as the natural projection $(z_i, z_j) \longmapsto z_j$.

Via p_i, we have two complex structure $\overline{\partial}_{U'_i}$ and $\overline{\partial}_{U_i}$ on U_i. Due to our choice of the hermitian metric g, p_i preserves the holomorphic structure (i.e., $\overline{\partial}_{U'_i} - \overline{\partial}_{U_i} = 0$) on U_P. The derivative of p_i gives the isomorphism of the complex bundles $T(N_{D_i}(X))_{|D_i} \simeq TD_i \oplus N_{D_i}X \simeq TX_{|D_i}$ on D_i. Hence we have the estimate $\overline{\partial}'_{U_i} - \overline{\partial}_{U_i} = O\big(|\sigma_i|\big)$.

Let ϵ be any number such that $0 < \epsilon < 1/2$. Let us fix a real number N, which is sufficiently large, say $N > 10$. We put as follows, for some positive number $C > 0$:

$$\omega_\epsilon := \omega + \sum_i C \cdot \epsilon^N \cdot \sqrt{-1}\partial\overline{\partial}|\sigma_i|^{2\epsilon}_{g_i}.$$

Proposition 4.25. — *If C is sufficiently small, then ω_ϵ are Kahler metrics of $X - D$ for any $0 < \epsilon < 1/2$.*

Proof We put $\phi_i := |\sigma_i|^2_{g_i}$. We have $\sqrt{-1}\cdot\partial\overline{\partial}\phi_i^{\epsilon} = \sqrt{-1}\cdot\epsilon^2\cdot\phi_i^{\epsilon}\cdot\partial\log\phi_i\cdot\overline{\partial}\log\phi_i + \sqrt{-1}\cdot\epsilon\cdot\phi_i^{\epsilon}\cdot\partial\overline{\partial}\log\phi_i$. Hence the claim of Proposition 4.25 immediately follows from the next lemma. □

Lemma 4.26. — *We put $f_t(\epsilon) := \epsilon^l\cdot t^{2\epsilon}$ for $0<\epsilon\leq 1/2$ and for $l\geq 1$. The following inequality holds:*

$$f_t(\epsilon) \leq \left(\frac{l}{-\log t^2}\right)^l \cdot e^{-l} \qquad (0<t<e^{-l}) \tag{20}$$

$$f_t(\epsilon) \leq \left(\frac{1}{2}\right)^l \cdot t \qquad (t\geq e^{-l}) \tag{21}$$

Proof We have $f_t'(\epsilon) = \epsilon^{l-1}t^{2\epsilon}\cdot\big(l+\epsilon\log t^2\big)$. If $t<e^{-l}$, we have $\epsilon_0 := l\times(-\log t^2)^{-1} < 1/2$ and $f_t'(\epsilon_0)=0$. Hence f_t takes the maximum at $\epsilon=\epsilon_0$, and we obtain (20). If $t\geq e^{-1}$, we have $f_t'(\epsilon)>0$ for any $0<\epsilon<1/2$, and thus $f_t(\epsilon)$ takes the maximum at $\epsilon = 1/2$. Thus we obtain (21). □

Lemma 4.27. — *Let τ be a closed 2-form on $X-D$ which is bounded with respect to ω_ϵ. Then the following formula holds:*

$$\int_{X-D}\omega\cdot\tau = \int_{X-D}\omega_\epsilon\cdot\tau.$$

In particular, we also have $\int_X\omega^2 = \int_{X-D}\omega_\epsilon^2$.

Proof The first claim follows from Stokes formula. We remark that $\partial|\sigma_i|^{2\epsilon}_{g_i}$ are bounded with respect to ω_ϵ, and hence it is easy to check that the contributions of $\partial|\sigma_i|^{2\epsilon}_{g_i}\cdot\tau$ vanish. The second claim immediately follows from the first one. □

The Kahler forms ω_ϵ behave well around any point of D in the following sense, which is clear from the construction.

Lemma 4.28. — *Let P be any point of $D_i\cap D_j$. Then there exist positive constants $C_i(\epsilon)$ $(i=1,2)$ such that the following holds on U_P, for any $0<\epsilon<1/2$*

$$C_1\cdot\omega_\epsilon \leq \sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dz_i\cdot d\bar{z}_i}{|z_i|^{2-2\epsilon}}+\frac{dz_j\cdot d\bar{z}_j}{|z_j|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dz_i\cdot d\bar{z}_i + dz_j\cdot d\bar{z}_j\big) \leq C_2\cdot\omega_\epsilon.$$

Let Q be any point of D_i°, and (U,w_1,w_2) be a holomorphic coordinate around Q such that $U\cap D_i=\{w_1=0\}$. Then there exist positive constants C_i $(i=1,2)$ such that the following holds for any $0<\epsilon<1/2$ on U:

$$C_1\cdot\omega_\epsilon\leq\sqrt{-1}\cdot\epsilon^{N+2}\cdot\left(\frac{dw_1\cdot d\bar{w}_1}{|w_1|^{2-2\epsilon}}\right)+\sqrt{-1}\big(dw_1\cdot d\bar{w}_1+dw_2\cdot d\bar{w}_2\big)\leq C_2\cdot\omega_\epsilon.$$

□

***Lemma 4.29* (Simpson [45], Li [29])**. — *Let us consider the case $\epsilon=1/m$ for some positive integer m. Then the metric ω_ϵ satisfies Condition 2.2.*

Proof We use the argument of Simpson in [**45**]. The first condition is easy to check. Since we have assumed that D is ample, we can take a C^∞-metric $|\cdot|$ of $\mathcal{O}(D)$ with the non-negative curvature. We put $\phi := -\log|1|$. Then $\sqrt{-1}\partial\overline{\partial}\phi$ is non-negative C^∞-two form, and it is easy to check that the second condition is satisfied.

To check the condition 3, we give the following remark: Let P be a point of $D_i \cap D_j$. For simplicity, let us consider the case $(i,j) = (1,2)$. We put $V_P := \big\{(\zeta_1,\zeta_2) \,\big|\, |\zeta_i| < 1\big\}$. Let us take the ramified covering $\varphi : V_P \longrightarrow U_P$ given by $(\zeta_1,\zeta_2) \longmapsto (\zeta_1^m, \zeta_2^m)$. Then it is easy to check that $\varphi^{-1}\omega_\epsilon$ naturally gives the C^∞-form on V_P, which is Kahler.

If f is a bounded positive function on $U_P \setminus D$ satisfying $\Delta_{\omega_\epsilon}(f) \leq B$ for some constant B, we obtain $\Delta_{\widetilde{\omega}}\big(\varphi^* f\big) \leq B$ on $V_P - \varphi^{-1}(D \cap U_P)$. Since $\widetilde{\omega}$ is C^∞ on V_P, we may apply the argument of Proposition 2.2 in [**45**]. Hence $\Delta_{\widetilde{\omega}}\big(\varphi^* f\big) \leq B$ holds weakly on V_P. Then we can apply the arguments of Proposition 2.1 in [**45**], and we obtain an appropriate estimate for the sup norm of f. By a similar argument, we obtain such an estimate around any smooth points of D. Thus we are done. □

4.3.2. A construction of an ordinary metric of the bundle. — Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). In the following, we shrink the open sets U_i without mentioning, if it is necessary. For each point $P \in D_i \cap D_j$, we may assume that there is a decomposition, as in the section 4.1.

$$(22) \qquad {}_{\boldsymbol{c}}E_{|U_P} = \bigoplus {}^{P}U_{\boldsymbol{a},\boldsymbol{\alpha}}.$$

We can take a C^∞-isomorphism ${}^i\Phi : \pi_i^*\big({}_{\boldsymbol{c}}E_{|D_i}\big) \simeq {}_{\boldsymbol{c}}E$ on U_i, satisfying the following:

- ${}^i\Phi(\overline{\partial}_{\pi_i^*({}_{\boldsymbol{c}}E_{|D_i})}) - \overline{\partial}_{{}_{\boldsymbol{c}}E} = O(|\sigma_i|_{g_i})$.
- The restriction of ${}^i\Phi$ to D_i is the identity.
- The restriction of ${}^i\Phi$ to U_P is holomorphic.
- The decomposition (22) induces the decompositions of ${}_{\boldsymbol{c}}E_{|U_P}$ and $\pi_i^*\big({}_{\boldsymbol{c}}E_{|D_i}\big)_{|U_P}$. The restriction of ${}^i\Phi$ to U_P preserves the decompositions.

We take the C^∞-decomposition ${}_{\boldsymbol{c}}E_{|U_i} = \bigoplus {}^i_{\boldsymbol{c}}E_u$, as in the section 4.2. We may assume that the following holds on U_P:

$${}^i_{\boldsymbol{c}}E_{u\,|\,U_P} = \bigoplus_{q_i(\boldsymbol{a},\boldsymbol{\alpha})=u} {}^{P}U_{\boldsymbol{a},\boldsymbol{\alpha}}.$$

Here $(\boldsymbol{a},\boldsymbol{\alpha})$ denotes an element $(a_i, a_j, \alpha_i, \alpha_j) \in \boldsymbol{R}^2 \times \boldsymbol{C}^2$, and $q_i(\boldsymbol{a},\boldsymbol{\alpha})$ denotes (a_i, α_i). We will use a similar notation. We can take a hermitian metric h_0' of ${}_{\boldsymbol{c}}E$ satisfying the following:

- The decompositions ${}_{\boldsymbol{c}}E_{|U_P} = \bigoplus {}^{P}U_{\boldsymbol{a},\boldsymbol{\alpha}}$ and ${}_{\boldsymbol{c}}E_{|U_i} = \bigoplus {}^i_{\boldsymbol{c}}E_u$ are orthogonal with respect to h_0'. Thus we have the decompositions $h_0' = \bigoplus {}^{P}h'_{\boldsymbol{a},\boldsymbol{\alpha}}$ on U_P, and $h_0' = \bigoplus {}^i h_u'$ on U_i.
- We put $h'_{0\,D_i} := h'_{0\,|\,D_i}$. Then we have ${}^i\Phi(\pi_i^* h'_{0\,D_i}) = h_0'$ on U_i. We have the decomposition $h_{0\,D_i} = \bigoplus h'_{u\,D_i}$.

We put $D_i^\circ := D_i \setminus \bigcup_{j\neq i} D_j$. By modifying $h'_{0\,D_i}$, we take a C^∞-hermitian metric $h_{0\,D_i}$ of ${}_{\boldsymbol{c}}E_{|D_i^\circ}$ satisfying the following:

- The decomposition ${}_{\boldsymbol{c}}E_{|D_i^\circ} = \bigoplus {}^iE_{\boldsymbol{u}\,|\,D_i^\circ}$ are orthogonal. Hence we have the decomposition $h_{0\,D_i} = \bigoplus h_{u\,D_i}$.
- Recall we have the decomposition on $U_P \cap D_i$ for $P \in D_i \cap D_j$, which induces the following:
$$ {}^i_{\boldsymbol{c}}E_{u\,|\,D_i^\circ\cap U_P} = \bigoplus_{q_i(\boldsymbol{a},\boldsymbol{\alpha})=u} U_{\boldsymbol{a},\boldsymbol{\alpha}\,|\,D_i^\circ\cap U_P}. $$
On $U_P \cap D_i^\circ$, $h_{u\,D_i}$ are assumed to be same as
$$ \bigoplus_{q_i(\boldsymbol{a},\boldsymbol{\alpha})=u} h'_{\boldsymbol{a},\boldsymbol{\alpha}\,|\,D_i} \cdot |z_j|^{-2a_j}. $$

Then we can take a C^∞-metric h_0 of E on $X - D$ satisfying the following:

- The decompositions $E_{|U_P\setminus D} = \bigoplus {}^PU_{\boldsymbol{a},\boldsymbol{\alpha}\,|\,U_P\setminus D}$ and $E_{|U_i\setminus D} = \bigoplus {}^iU_{a,\alpha\,|\,U_i\setminus D}$ are orthogonal with respect to h_0. Thus we have the decomposition $h_0 = \bigoplus {}^Ph_{\boldsymbol{a},\boldsymbol{\alpha}}$ on $U_P \setminus D$ and $h_0 = \bigoplus {}^ih_u$ on $U_i \setminus D$.
- ${}^Ph_{\boldsymbol{a},\boldsymbol{\alpha}} = |z_i|^{-2a_i} \cdot |z_j|^{-2a_j} \cdot {}^Ph'_{\boldsymbol{a},\boldsymbol{\alpha}}$.
- ${}^ih_u = \pi_i^* h_{u,D_i} \cdot |\sigma_i|_{g_i}^{-2a_i}$.

Such a hermitian metric h_0 is called an ordinary metric of $({}_{\boldsymbol{c}}E_*, \theta)$.

We can apply the results in the section 4.1 to $h_{0\,|\,U_P}$, and the results in the section 4.2 to $h_{0\,|\,U_i}$. In particular, we can show the following lemma.

Lemma 4.30. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, then $F(h_0)$ is bounded with respect to h_0 and ω_ϵ.*

Proof It follows from Proposition 4.3 and Proposition 4.13. □

We will see that the integrations of the characteristic classes obtained from h_0 has nice properties in the rest of this section. (Corollary 4.34, Proposition 4.35 and Corollary 4.42.)

4.3.3. Preliminary for the calculus of the integrations. — We put $A := \partial_{E,h_0} - \partial_{E,h'_0}$. On U_P, we have the following:

$$ A - \bigoplus_{\boldsymbol{a},\boldsymbol{\alpha}} \left(\sum -a_i \frac{dz_i}{z_i} \cdot \mathrm{id}_{{}^PU_{\boldsymbol{a},\boldsymbol{\alpha}}} \right) = 0, \quad R(h_0) = R(h'_0) = \bigoplus R({}^Ph_{\boldsymbol{u}}) = \bigoplus R({}^Ph'_{\boldsymbol{u}}). $$

Lemma 4.31. — *On U_P, we have the following formula with respect to h_0 and ω_ϵ:*

$$ \mathrm{tr}\big(A \cdot R(h_0)\big) = \mathrm{tr}\big(A \cdot R(h'_0)\big) = \sum \Big(-a_i \frac{dz_i}{z_i} - a_j \frac{dz_j}{z_j} \Big) \cdot \mathrm{tr}\, R({}^Ph_{\boldsymbol{a},\boldsymbol{\alpha}}) \\ = \sum \Big(-a_i \frac{dz_i}{z_i} - a_j \frac{dz_j}{z_j} \Big) \cdot \mathrm{tr}\, R({}^Ph'_{\boldsymbol{a},\boldsymbol{\alpha}}). \tag{23} $$

□

We put $U_i^{\star} := U_i \setminus \bigl(D \cup \coprod_{P \in D_i \cap D_j} U_P\bigr)$. On $U_i^{\star}$, we have the following estimate with respect to h_0 and ω_ϵ:

$$A - \bigoplus_{(a,\alpha)} \bigl(-a \cdot \partial \log |\sigma_i|^2\bigr) \cdot \mathrm{id}_{{}^iE_{a,\alpha}} = O(|\sigma_i|^{\epsilon}).$$

We also have the following estimate with respect to ω_ϵ and h_0 on $U_i^{\star}$:

$$R(h_0) - \bigoplus_{(a,\alpha)} \Bigl(\pi_i^* R\bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{(a,\alpha)}(E_{|D_i^{\circ}}),\ h_{a,\alpha\,D_i}\bigr) - a \cdot \overline{\partial}\partial \log |\sigma_i|^2 \cdot \mathrm{id}_{{}^iE_{a,\alpha}}\Bigr) = O(|\sigma_i|^{\epsilon}).$$

Here we have used the natural identification of ${}^i_{\boldsymbol{c}}E_{u\,|\,D_i^{\circ}}$ and ${}^i\operatorname{Gr}^{F,\mathbb{E}}_{\boldsymbol{u}}({}_{\boldsymbol{c}}E_{|D_i})$.

Lemma 4.32. — *On $U_i^{\star}$, we have the following estimate with respect to ω_ϵ:*

$$\operatorname{Tr}(A \cdot R(h_0)) = -\sum_{a,\alpha} \pi_i^* \operatorname{Tr}\Bigl(R\bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr), h_{u\,D_i}\bigr)\Bigr) \cdot a \cdot \partial \log |\sigma_i|^2 + \sum_{a,\alpha} \operatorname{rank} {}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}\bigl({}_{\boldsymbol{c}}E_{|D_i}\bigr) \cdot a^2 \cdot \overline{\partial}\partial \log |\sigma_i|^2 \cdot \partial \log |\sigma_i|^2 + O(|\sigma_i|^{\epsilon}). \tag{24}$$

We also have the following estimate, with respect to ω_ϵ:

$$\operatorname{Tr}(A \cdot R(h_0')) = -\sum \pi_i^* \operatorname{Tr}\Bigl(R\bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{\boldsymbol{u}}({}_{\boldsymbol{c}}E_{|D_i}), h'_{u\,D_i}\bigr)\Bigr) \cdot a \cdot \partial \log |\sigma_i|^2 + O(|\sigma_i|^{\epsilon}).$$

□

4.3.4. par-$\mathrm{c}_1^2({}_{\boldsymbol{c}}E)$. —

Lemma 4.33. —

$$\Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \int_{X-D} \bigl(\operatorname{tr} R(h_0)\bigr)^2 = \int_X \text{par-}\mathrm{c}_1^2({}_{\boldsymbol{c}}E_*).$$

Proof We have the following, where $A := \partial_{E,h_0} - \partial_{E,h_0'}$:

$$\bigl(\operatorname{tr} R(h_0)\bigr)^2 = \bigl(\operatorname{tr} R(h_0')\bigr)^2 + \operatorname{tr} R(h_0') \cdot \overline{\partial} \operatorname{tr} A + \operatorname{tr} R(h_0) \cdot \overline{\partial} \operatorname{tr} A.$$

We have the following equality:

$$\Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \cdot \int_{X-D} \bigl(\operatorname{tr} R(h_0')\bigr)^2 = \int_X c_1({}_{\boldsymbol{c}}E)^2,$$

By using the estimates in the subsection 4.3.3, we obtain the following:

$$\Bigl(\frac{\sqrt{-1}}{2\pi}\Bigr)^2 \int_{X-D} \operatorname{tr} R(h_0') \cdot \overline{\partial} \operatorname{tr} A = \sum_i \frac{\sqrt{-1}}{2\pi} \int_{D_i} \operatorname{tr} R(h'_{0\,D_i}) \cdot (-\operatorname{wt}({}_{\boldsymbol{c}}E_*, i)) = \sum_i -\operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \cdot \deg_{D_i}({}_{\boldsymbol{c}}E_{|D_i}) = -\sum_i \operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \int_X c_1({}_{\boldsymbol{c}}E) \cdot [D_i] \tag{25}$$

We also have the following:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr} R(h_0) \cdot \overline{\partial} \operatorname{tr} A \tag{26}$$
$$= \sum_i \bigl(-\operatorname{wt}({}_{\boldsymbol{c}}E_*, i)\bigr) \frac{\sqrt{-1}}{2\pi} \int_{D_i} \operatorname{tr} R(h_{0\,D_i}) + \bigl(-\operatorname{wt}({}_{\boldsymbol{c}}E_*, i)\bigr)^2 \frac{\sqrt{-1}}{2\pi} \int_{D_i} \overline{\partial}\partial \log |\sigma_i|^2$$

We have the canonically induced parabolic structure of ${}_{\boldsymbol{c}}E_{|D_i}$ at $D_i \cap \bigcup_{j\neq i} D_j$, which is denoted by ${}_{\boldsymbol{c}}E_{|D_i\,*}$. Then we have the following equality:

$$\frac{\sqrt{-1}}{2\pi} \int_{D_i} \operatorname{tr} R(h_{0\,D_i}) = \text{par-deg}_{D_i}({}_{\boldsymbol{c}}E_{|D_i\,*}) = \deg_{D_i}({}_{\boldsymbol{c}}E) - \sum_{j\neq i} \operatorname{wt}({}_{\boldsymbol{c}}E_*, j) \cdot \int_X [D_i]\cdot[D_j].$$

We also have the following:

$$\frac{\sqrt{-1}}{2\pi} \int_{D_i} \overline{\partial}\partial \log |\sigma_i|^2 = \int_X [D_i]^2.$$

Thus we obtain the following:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr} R(h_0) \cdot \overline{\partial} \operatorname{tr} A \tag{27}$$
$$= -\sum_i \operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \int_X c_1({}_{\boldsymbol{c}}E) \cdot [D_i] + \sum_i \sum_{j\neq i} \operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \cdot \operatorname{wt}({}_{\boldsymbol{c}}E_*, j) \int_X [D_i] \cdot [D_j]$$
$$+ \sum_i \operatorname{wt}({}_{\boldsymbol{c}}E_*, i)^2 \cdot \int_X [D_i]^2$$
$$= -\sum_i \operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \int_X c_1({}_{\boldsymbol{c}}E) \cdot [D_i] + \sum_i \sum_j \operatorname{wt}({}_{\boldsymbol{c}}E_*, i) \cdot \operatorname{wt}({}_{\boldsymbol{c}}E_*, j) \int_X [D_i] \cdot [D_j].$$

Then the claim of the lemma immediately follows. □

Corollary 4.34. —

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \bigl(\operatorname{tr} F(h_0)\bigr)^2 = \int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*).$$

Proof Since we have $\bigl(\operatorname{tr} F(h_0)\bigr)^2 = \bigl(\operatorname{tr} R(h_0)\bigr)^2$, the claim immediately follows from the previous lemma. □

4.3.5. par-ch$_2({}_{\boldsymbol{c}}E_*)$**.** —

Proposition 4.35. — *If $(\boldsymbol{E}_*, \theta)$ is graded semisimple, then the following equality holds:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Bigl(F(h_0)^2\Bigr) = 2 \int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*).$$

Proof We have only to show the following three equalities:

$$\int_{X-D} \operatorname{tr}\Bigl(F(h_0)^2\Bigr) = \int_{X-D} \operatorname{tr}\Bigl(R(h_0)^2\Bigr). \tag{28}$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Bigl(R(h_0)^2\Bigr) = 2\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*). \tag{29}$$

We use the following lemma.

Lemma 4.36. — *Let ∇ be a connection of a vector bundle V, and $R(\nabla)$ denote the curvature. Let P be a section of $\operatorname{End}(V)\otimes\Omega^1$, then we obtain the connection $\nabla+P$ and the curvature $R(\nabla+P)$. Then we have the following formula:*

$$\operatorname{tr}\bigl(R(\nabla+P)^2\bigr) = \operatorname{tr}\bigl(R(\nabla)^2\bigr) + d\Bigl(\operatorname{tr}\bigl(R(\nabla)\cdot P\bigr) + \operatorname{tr}\bigl(R(\nabla+P)\cdot P\bigr) - \frac{1}{3}\operatorname{tr}(P^3)\Bigr) \tag{30}$$

We also have the following:

$$\operatorname{tr}\bigl(R(\nabla+P)^2\bigr) = \operatorname{tr}\bigl(R(\nabla)^2\bigr) + 2\operatorname{tr}\bigl(\nabla P\cdot R(\nabla)\bigr) + d\left(\operatorname{tr}\bigl(P\cdot\nabla P\bigr) + \frac{2}{3}\operatorname{tr}P^3\right). \tag{31}$$

Proof We have the following:

$$\begin{aligned}\operatorname{tr}\Bigl(R(\nabla+P)\cdot R(\nabla+P)\Bigr) &= \operatorname{tr}\Bigl(\bigl(R(\nabla) + \bigl[\nabla+P,\, P\bigr] - P^2\bigr)\cdot R(\nabla+P)\Bigr)\\ &= \operatorname{tr}\Bigl(R(\nabla)\cdot R(\nabla+P)\Bigr) + d\Bigl(\operatorname{tr}\bigl(P\cdot R(\nabla+P)\bigr)\Bigr) - \operatorname{tr}\bigl(P^2\cdot R(\nabla+P)\bigr).\end{aligned} \tag{32}$$

We have the following:

$$\begin{aligned}\operatorname{tr}\Bigl(R(\nabla)\cdot R(\nabla+P)\Bigr) &= \operatorname{tr}\Bigl(R(\nabla)\cdot\bigl(R(\nabla)+\nabla P+P^2\bigr)\Bigr)\\ &= \operatorname{tr}\Bigl(R(\nabla)^2\Bigr) + d\operatorname{tr}\Bigl(R(\nabla)\cdot P\Bigr) + \operatorname{tr}\Bigl(R(\nabla)\cdot P^2\Bigr).\end{aligned} \tag{33}$$

We have the following:

$$\operatorname{tr}\Bigl(P^2\cdot\bigl(R(\nabla)-R(\nabla+P)\bigr)\Bigr) = \operatorname{tr}\Bigl(P^2\cdot\nabla P + P^4\Bigr) = d\left(\frac{1}{3}\operatorname{tr}\bigl(P^3\bigr)\right) + \operatorname{tr}P^4. \tag{34}$$

Since we have $\operatorname{tr}P^4 = -\operatorname{tr}P^4$, the equality $\operatorname{tr}P^4 = 0$ holds. The formula (30) follows from (32), (33) and (34). The formula (31) immediately follows from (30) and $d\operatorname{tr}\bigl(P\cdot R(\nabla)\bigr) = \operatorname{tr}\Bigl(\nabla P\cdot R(\nabla)\Bigr)$. Thus the proof of Lemma 4.36 is finished. □

Let us return to the proof of Proposition 4.35. Let us show (28). The following is obtained from (31):

(35)

$$\begin{aligned}\operatorname{tr}\bigl(F(h_0)^2\bigr) = \operatorname{tr}\bigl(R(h_0)^2\bigr) + \operatorname{tr}\Bigl(\bigl(\partial_{h_0}\theta+\overline{\partial}\theta^\dagger\bigr)\cdot R(h_0)\Bigr) + d\Bigl(\operatorname{tr}\bigl((\theta+\theta^\dagger)\cdot(\partial_{h_0}\theta+\overline{\partial}\theta^\dagger)\bigr)\\ + \operatorname{tr}\bigl(\theta+\theta^\dagger\bigr)^3\Bigr).\end{aligned}$$

We remark that $R(h_0)$, $\partial_{h_0}\theta$ and $\overline{\partial}\theta^\dagger$ are a $(1,1)$-form, a $(2,0)$-form and a $(0,2)$-form respectively. Therefore we obtain the vanishing of the second term in the right hand side. It is easy to obtain $\operatorname{tr}\bigl(\theta+\theta^\dagger\bigr)^3=0$ from $\theta^2=\theta^{\dagger\,2}=0$.

We put $Y_i(\delta):=\Bigl\{x\in X\Bigm| |\sigma_i(x)|=\min_j|\sigma_j(x)|=\delta\Bigr\}$ and $Y(\delta):=\bigcup_i Y_i(\delta)$. The formula (28) immediately follows from the next lemma.

Lemma 4.37. — *We can neglect the contribution of* $\operatorname{tr}(\theta\cdot\overline{\partial}\theta^\dagger)$*, Namely we have*

$$\lim_{\delta\to 0}\int_{Y(\delta)}\operatorname{tr}(\theta\cdot\overline{\partial}\theta^\dagger)=0.$$

Similarly, the contribution of $\operatorname{tr}(\theta^\dagger\cdot\partial_{h_0}\theta)$ *can be neglected.*

Proof Let P be a point of D_i°, and U_P be an appropriately small neighbourhood of P in X. Then we have the following estimate which follows from the boundedness of $\overline{\partial}\theta^\dagger$ with respect to ω_ϵ and h_0:

$$\int_{U_P\cap Y(\delta_i)}\operatorname{tr}\bigl(\theta\cdot\overline{\partial}\theta^\dagger\bigr)=O\bigl(\delta^\epsilon\bigr).$$

Let P be a point of $D_i\cap D_j$, and U_P be an appropriately small neighbourhood of P in X. We remark that $\partial\theta$ and $\overline{\partial}\theta^\dagger$ are a $(2,0)$-form and a $(0,2)$-form respectively. Then we obtain the following estimate:

$$\int_{U_P\cap Y(\delta_i)}\operatorname{tr}\bigl(\theta\cdot\overline{\partial}\theta^\dagger\bigr)=O\left(\delta^\epsilon\cdot\int_{|z|\geq\delta}|z|^\epsilon\frac{dz\cdot d\bar{z}}{|z|^2}\right)=O\bigl(\delta^\epsilon\bigr).$$

Hence it is easy to check that $\int_{Y(\delta)}\operatorname{tr}(\theta\cdot\overline{\partial}\theta^\dagger)$ converges to 0 in $\delta\to 0$. $\square$

Finally let us see (29). We put $A:=\partial_{h_0}-\partial_{h_0'}$. Then we have the following:

$$\operatorname{tr}\bigl(R(h_0)^2\bigr)=\operatorname{tr}\bigl(R(h_0')^2\bigr)+d\operatorname{tr}\bigl(A\cdot R(h_0)\bigr)+d\operatorname{tr}\bigl(A\cdot R(h_0')\bigr).$$

The contribution of the first term is as follows:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}\operatorname{tr}\bigl(R(h_0')^2\bigr)=2\operatorname{ch}_2({}_{\boldsymbol{c}}E).$$

Due to (23) and Lemma 4.32, the contribution of the third term is as follows:

$$\begin{aligned}(36)\quad &\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X-D}d\operatorname{tr}\bigl(A\cdot R(h_0')\bigr)\\ &=\sum_i\sum_{a,\alpha}(-a)\left(\frac{\sqrt{-1}}{2\pi}\right)\cdot\int_{D_i}\operatorname{tr}\Bigl(R\bigl({}^i\operatorname{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i}),h'_{u\,D_i}\bigr)\Bigr)\\ &=-\sum_i\sum a\cdot\deg_{D_i}\Bigl({}^i\operatorname{Gr}^F_a({}_{\boldsymbol{c}}E_{|D_i})\Bigr).\end{aligned}$$

We obtain the following formula from (23) and Lemma 4.32:

$$
\begin{aligned}
(37)\quad & \Big(\frac{\sqrt{-1}}{2\pi}\Big)^2 \int_{X-D} d\,\mathrm{tr}\Big(A\cdot R(h_0)\Big) \\
&\qquad = \sum_i \sum_{a,\alpha} (-a)\Big(\frac{\sqrt{-1}}{2\pi}\Big)\int_{D_i} \mathrm{tr}\Big(R\big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big), h_{a,\alpha\,D_i}\big)\Big) \\
&\qquad + \sum_i \sum_{a,\alpha} a^2 \Big(\frac{\sqrt{-1}}{2\pi}\Big)\mathrm{rank}\big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})\big)\cdot \int_{D_i}\overline{\partial}\partial\log|\sigma_i|^2 \\
&= -\sum_{i,a,\alpha} a\cdot \text{par-deg}_{D_i}\big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}({}_{\boldsymbol{c}}E_{|D_i})_*\big) + \sum_{i,a,\alpha} a^2\,\mathrm{rank}\,{}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)\int_X [D_i]^2.
\end{aligned}
$$

Here ${}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*$ is the parabolic bundle on $\big(D_i, D_i\cap\bigcup_{j\neq i} D_j\big)$ with the canonically induced parabolic structure. We have the following:

$$
\begin{aligned}
(38)\quad & \sum_\alpha \text{par-deg}_{D_i}\Big({}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*\Big) = \text{par-deg}_{D_i}\Big({}^i\,\mathrm{Gr}^{F}_{a}\big({}_{\boldsymbol{c}}E_{|D_i}\big)_*\Big) \\
&\qquad = \deg_{D_i}\big(\mathrm{Gr}^F_a\big({}_{\boldsymbol{c}}E_{|D_i}\big)\big) - \sum_{\substack{j\neq i,\\ P\in D_i\cap D_j}} \sum_{\substack{\boldsymbol{a}\in\mathcal{P}ar({}_{\boldsymbol{c}}E,P)\\ q_i(\boldsymbol{a})=a}} a_j\cdot\mathrm{rank}\big({}^P\,\mathrm{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E_{|O})\big).
\end{aligned}
$$

Then (29) immediately follows. □

4.3.6. The degree of subsheaves. — Let V be a saturated coherent $\mathcal{O}_{X-D}$-submodule of E. Let π_V denote the orthogonal projection of E onto V with respect to h_{in}, which is defined outside a Zariski closed subset of codimension two. It is also the orthogonal projection with respect to h_0. Let $h_{0,V}$ and $h_{in,V}$ be the metric of V induced by h_0 and h_{in}. J. Li proved the following lemma [**29**] based on the result of Y. T. Siu [**54**].

Lemma 4.38. — *$\overline{\partial}\pi_V$ is L^2 with respect to h_0 and ω_ϵ if and only if there exists a coherent subsheaf ${}_{\boldsymbol{c}}V\subset{}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D}=V$.* □

***Lemma 4.39* (Li).** — *Let h_V denotes the metric of V induced by h_0 or h_{in}. Then the following holds:*

$$\deg_\omega(V,h_V) = \text{par-deg}_\omega({}_{\boldsymbol{c}}V_*).$$

Proof Since we have $\mathrm{tr}\big(R(h_{0,V})\big)-\mathrm{tr}\big(R(h_{in,V})\big)=\mathrm{rank}\,V\cdot\partial\overline{\partial}g$, we have only to check the claim in the case where h_V is induced by h_0. We have only to show the following:

$$\frac{\sqrt{-1}}{2\pi}\int_{X-D}\mathrm{tr}_V\,R(h_V)\cdot\omega = \text{par-deg}_\omega({}_{\boldsymbol{c}}V_*).$$

Let Z denote the finite subset of X where V is not a subbundle of E. Let d_ω denote the distance induced by ω, and we put $T_Z(\delta):=\big\{x\in X\,\big|\,d_\omega(x,Z)=\delta\big\}$. Let $Y_i(\delta)$ be as in the proof of Proposition 4.35.

Let h'_V be the metric induced by h'_0. Due to the Stokes formula, we have the following:

$$(39)\quad \frac{\sqrt{-1}}{2\pi}\int_{X-D}\mathrm{tr}_V\,R(h_V)\cdot\omega=\frac{\sqrt{-1}}{2\pi}\int_{X-D}\mathrm{tr}_V\,R(h'_V)\cdot\omega$$
$$+\lim_{\delta\to 0}\frac{\sqrt{-1}}{2\pi}\int_{Y(\delta)}\mathrm{tr}_V(\partial_{V,h_V}-\partial_{V,h'_V})\cdot\omega+\lim_{\delta\to 0}\int_{T(\delta)}\frac{\sqrt{-1}}{2\pi}\mathrm{tr}_V(\partial_{V,h_V}-\partial_{V,h'_V})\cdot\omega.$$

The first term in the right hand side is $\deg_\omega(V)$.

The endomorphism $\partial_{V,h_V}-\partial_{V,h'_V}$ is the restriction of $\pi_V^{h_0}\circ\partial_{h_0}-\pi_V^{h'_0}\circ\partial_{h'_0}$ to V, and we have the following:

$$(40)\qquad \pi_V^{h_0}\circ\partial_{h_0}-\pi_V^{h'_0}\circ\partial_{h'_0}=\pi_V^{h_0}\circ\big(\partial_{h_0}-\partial'_{h_0}\big)+\big(\pi_V^{h_0}-\pi_V^{h'_0}\big)\circ\partial_{h'_0}.$$

Let us estimate the second term. Around $P\in D_i\cap D_j$, let us take a holomorphic frame $\boldsymbol{v}$. The form B is determined by $\partial_{h'_0}\circ\pi_V^{h'_0}\boldsymbol{v}=\boldsymbol{v}\cdot B$. Let F_B be the section of $\mathrm{End}({}_{\boldsymbol{c}}E)\otimes\Omega^{0,1}$, determined by $F_B(\boldsymbol{v})=\boldsymbol{v}\cdot B$. Then F_B is bounded with respect to h'_0 and ω. We put $Q:=\pi_V^{h_0}-\pi_V^{h'_0}$.

Lemma 4.40. — *We have $Q\circ\partial_{h'_0}\circ\pi_V^{h'_0}=Q\circ F_B$.*

Proof We have $Q\circ\pi_V^{h'_0}=0$. Hence $Q\circ\partial_{h'_0}\circ\pi_V^{h'_0}$ is a 1-form, i.e. it does not contain derivative. By the construction, $Q\circ\partial_{h'_0}\circ\pi_V^{h'_0}(v_i)=Q\circ F_B(v_i)$. Hence they are same. □

Due to the following lemma, we have $\big|Q\big|_{h'_0}\leq C\cdot|z_1|^{-1+\epsilon}\cdot|z_2|^{-1+\epsilon}$.

Lemma 4.41. — *Let U be a vector space with hermitian metrics h_1 and h_2. Let s be the endomorphism of U such that $h_1=h_2\cdot s$. Let U' be a subspace. Let π_1 be the orthogonal projection of U onto U' with respect to h_1. Then $\big|\pi_1\big|_{h_2}\leq C|s|_{h_1}^{1/2}\cdot|s^{-1}|_{h_2}^{1/2}$, where C depends only on $\dim V$.*

Proof Let v be any element of U.

Let $v=v_1+v_2$ be the orthogonal decomposition with respect to h_1 such that $v_1\in U'$. We have $|v_1|_{h_1}\leq|v|_{h_1}$. We also have $|v_1|_{h_2}\leq|s^{-1}|_{h_1}^{1/2}\cdot|v_1|_{h_1}$. We also have $|v|_{h_1}\leq|s|_{h_2}^{1/2}\cdot|v|_{h_2}$. □

We have $\big(Q\circ\partial_{h'_0}\big)_{|V}=\big(Q\circ\partial_{h'_0}\circ\pi_V^{h'_0}\big)_{|V}=\big(Q\circ F_B\big)_{|V}$. Hence, we obtain the following estimate with respect to h'_0:

$$\big(Q\circ\partial_{h'_0}\big)_{|V}=O\Big(\big(|z_1|^{-1+\epsilon}\cdot|z_2|^{-1+\epsilon}\big)\cdot(dz_1+dz_2)\Big).$$

Thus, we obtain the following estimate around $P\in D_i\cap D_j$:

$$\mathrm{tr}_V\big((Q\circ\partial_{h'_0})_{|V}\big)\cdot\omega=O\Big(\big(|z_1|^{-1+\epsilon}\cdot|z_2|^{-1+\epsilon}\big)\cdot(dz_1+dz_2)\Big)\cdot\omega.$$

Similarly, we obtain the following estimate around $P \in D_i^\circ$ on an appropriate coordinate neighbourhood (U_P, z_1, z_2) such that $U_P \cap D_i = \{z_1 = 0\}$:

$$\mathrm{tr}_V\big((Q \circ \partial_{h_0'})_{|V}\big) \cdot \omega = O\Big(\big(|z_1|^{-1+\epsilon}\big) \cdot (dz_1 + dz_2)\Big) \cdot \omega.$$

Hence the contribution of the $Q \circ \partial_{h_0'}$ to $\sqrt{-1} \cdot (2\pi)^{-1} \lim_{\delta \to 0} \int_{Y(\delta)}$ in (39) is 0. Similarly, the contribution to $\sqrt{-1} \cdot (2\pi)^{-1} \lim_{\delta \to 0} \int_{T(\delta)}$ is also 0.

Let us see the first term in (40). We have the following around $P \in D_i \cap D_j$:

$$\mathrm{tr}\big(\pi_V^{h_0} \circ (\partial_{h_0} - \partial_{h_0'})\big) = \sum_{k=i,j} \big(-\mathrm{wt}({}_{\boldsymbol{c}}E_*, k)\big) \cdot \frac{dz_k}{z_k} + O(\omega_\epsilon).$$

We also have a similar estimate around $P \in D_i^\circ$. Hence we obtain

$$\frac{\sqrt{-1}}{2\pi} \lim_{\delta \to 0} \int_{Y(\delta)} \mathrm{tr}_V\big(\partial_{V,h_V} - \partial_{V,h_V'}\big) \cdot \omega = -\sum_i \mathrm{wt}({}_{\boldsymbol{c}}V_*, i) \cdot (D_i, \omega).$$

We can also obtain $\sqrt{-1} \cdot (2\pi)^{-1} \lim_{\delta \to 0} \int_{T(\delta)} = 0$ in (39). □

Corollary 4.42. — *$(E, \overline{\partial}_E, \theta, h_{in})$ is analytic stable with respect to ω_ϵ if and only if $({}_{\boldsymbol{c}}E, \theta)$ is stable with ω.* □

CHAPTER 5

PARABOLIC HIGGS BUNDLE ASSOCIATED TO TAME HARMONIC BUNDLE

In this chapter, we show the fundamental property of the parabolic Higgs bundle associated to tame harmonic bundle, such as μ_L-polystability and the vanishing of characteristic numbers. We also see the uniqueness of the adapted pluri-harmonic metric for parabolic Higgs bundles. These results give the half of Theorem 1.4. Although the arguments are rather standard, we remark that our results in [**38**] are crucial in the proof.

5.1. The fundamental property

Let X be a smooth projective variety, and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. Let L be any ample line bundle of X.

Proposition 5.1. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$, and $({}_{\boldsymbol{c}}E_*, \theta)$ denotes the $\boldsymbol{c}$-parabolic Higgs bundle, for any $\boldsymbol{c} \in \boldsymbol{R}^S$.*

- *$({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-polystable, and* $\text{par-deg}_L({}_{\boldsymbol{c}}E_*) = 0$.
- *Let $({}_{\boldsymbol{c}}E_*, \theta) = \bigoplus_i ({}_{\boldsymbol{c}}E_{i*}, \theta_i) \otimes \boldsymbol{C}^{p(i)}$ be the canonical decomposition (Corollary* 3.4*). Then we have the orthogonal decomposition $h = \bigoplus_i h_i \otimes g_i$. Here h_i are pluri-harmonic metrics for $(E_i, \overline{\partial}_{E_i}, \theta_i)$, and g_i are hermitian metrics of $\boldsymbol{C}^{p(i)}$.*

Proof The equality $\text{par-deg}_L({}_{\boldsymbol{c}}E_*) = 0$ can be easily reduced to the curve case (Proposition 2.9). It also follows from the curve case that $({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-semistable.

Let us show $({}_{\boldsymbol{c}}E_*, \theta)$ is μ-polystable. Let $({}_{\boldsymbol{c}}V_*, \theta_V)$ be a non-trivial Higgs subsheaf of $({}_{\boldsymbol{c}}E_*, \theta)$ such that $\mu_L({}_{\boldsymbol{c}}V_*) = \mu_L({}_{\boldsymbol{c}}E_*) = 0$ and $\text{rank}(V) < \text{rank}(E)$. Recall that we have the closed subset $Z \subset X$ such that ${}_{\boldsymbol{c}}V_{|X-Z}$ is the subbundle of ${}_{\boldsymbol{c}}E_{|X-Z}$. The codimension of Z is larger than 2. We have the orthogonal projection $\pi_V : E \longrightarrow V$ on the open set $X - (Z \cup D)$. Due to Lemma 4.39 and Proposition 2.4, we obtain the following equality:

$$0 = \text{par-deg}({}_{\boldsymbol{c}}V_*) = \deg(V, h_V) = -C \cdot \int_X \left(\left|\overline{\partial}\pi_V\right|^2 + \left|[\theta, \pi_V]\right|^2\right).$$

Hence we obtain $\overline{\partial}\pi_V = 0$ and $[\theta, V] = 0$ holds. In particular, π_V gives the homomorphism $E_{|X-(D\cup Z)} \longrightarrow V_{|X-(D\cup Z)}$. Since the codimension of Z is larger than two, π_V naturally gives the holomorphic map $E \longrightarrow E$ on $X - D$, which is also denoted by π_V. It is easy to see $\pi_V^2 = \pi_V$, and that the restriction of π_V to V is the identity. Hence we obtain the decomposition $E = V \oplus V'$, where we put $V' = \operatorname{Ker} \pi_V$. We can conclude that V and V' are vector subbundles of E, and the decomposition is orthogonal with respect to the metric h. Since we have $[\pi_V, \theta] = 0$, the decomposition is also compatible with the Higgs field. Hence we obtain the decomposition of $(E, \overline{\partial}_E, \theta, h)$ into $(V, \overline{\partial}_V, \theta_V, h_V) \oplus (V', \overline{\partial}_{V'}, \theta_{V'}, h_{V'})$ as harmonic bundles. Then it is easy that $({}_{\boldsymbol{c}}E_*, \theta)$ is also decomposed into $({}_{\boldsymbol{c}}V_*, \theta_V) \oplus ({}_{\boldsymbol{c}}V'_*, \theta_{V'})$. Since both of $({}_{\boldsymbol{c}}V_*, \theta_V)$ $({}_{\boldsymbol{c}}V'_*, \theta_{V'})$ are obtained from tame harmonic bundles, they are μ_L-semistable. And we have $\operatorname{rank}(V) < \operatorname{rank}(E)$ and $\operatorname{rank}(V') < \operatorname{rank}(E)$. Hence the polystability of $({}_{\boldsymbol{c}}E, \theta)$ can be shown by an easy induction on the rank.

From the argument above, the second claim is also clear. □

Proposition 5.2. — *Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D). We put $E := {}_{\boldsymbol{c}}E_{|X-D}$. Assume that we have pluri-harmonic metrics h_i of $(E, \overline{\partial}_E, \theta)$ $(i = 1, 2)$, which are adapted to the parabolic structures. Then we have the decomposition of Higgs bundles $(E, \theta) = \bigoplus_a (E_a, \theta_a)$ satisfying the following conditions:*

- *The decomposition is orthogonal with respect to both of h_i. The restrictions of h_i to E_a are denoted by $h_{i,a}$.*
- *There exist positive numbers b_a such that $h_{1,a} = b_a \cdot h_{2,a}$.*

We remark that the decomposition $(E, \theta) = \bigoplus (E_a, \theta_a)$ induces the decomposition of the $\boldsymbol{c}$-parabolic Higgs bundles:

$$\left({}_{\boldsymbol{c}}E_*, \theta\right) = \bigoplus ({}_{\boldsymbol{c}}E_{a*}, \theta_a).$$

Proof Recall the norm estimate for tame harmonic bundles ([**38**]) which says that the harmonic metrics are determined up to boundedness by the parabolic filtration and the weight filtration. Hence we obtain the mutually boundedness of h_1 and h_2. Then the uniqueness follows from Proposition 2.7. (The Kahler metric of $X - D$ is given by the restriction of a Kahler metric of X. It satisfies Condition 2.2, according to [**45**].) □

Proposition 5.3. — *Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$, and $({}_{\boldsymbol{c}}E, \theta)$ be the induced $\boldsymbol{c}$-parabolic Higgs bundle. Then the following characteristic numbers vanish.*

$$\int_X \text{par-ch}_{2,L}({}_{\boldsymbol{c}}E_*) = 0, \qquad \int_X \text{par-c}_{1,L}^2({}_{\boldsymbol{c}}E_*) = 0.$$

(See the subsection 3.1.5 *for the characteristic numbers.)*

Proof We will reduce Proposition 5.3 to Lemma 5.4 below, and the proof of Lemma 5.4 will be given in the next section. Let h_0 be an ordinary metric as in the section 4.3.2. Due to Lemma 4.33 and 29, we have only to show the following equalities:

$$\int \operatorname{tr}\bigl(F(h)^2\bigr) = \int \operatorname{tr}\bigl(R(h_0)^2\bigr) \tag{41}$$

$$\int \bigl(\operatorname{tr} F(h)\bigr)^2 = \int \bigl(\operatorname{tr} R(h_0)\bigr)^2 \tag{42}$$

We use a Kahler metric $\widetilde{\omega}$ of $X - D$, which is Poincaré like around D. We put $A := \partial_h - \partial_{h_0}$. The following lemma will be proved in the next section.

Lemma 5.4. — *A is L^2 with respect to $\widetilde{\omega}$ and h.*

Let us show (42) by admitting Lemma 5.4. We have $\operatorname{tr} F(h) = \operatorname{tr} R(h)$. We also have the following:

$$\begin{aligned}\bigl(\operatorname{tr} R(h)\bigr)^2 &= \bigl(\operatorname{tr} R(h_0)\bigr)^2 + \operatorname{tr} R(h_0)\cdot\overline{\partial}\operatorname{tr} A + \operatorname{tr} R(h)\cdot\overline{\partial}\operatorname{tr} A\\ &= \bigl(\operatorname{tr} R(h_0)\bigr)^2 + d\Bigl(\bigl(\operatorname{tr} R(h) + \operatorname{tr} R(h_0)\bigr)\cdot\operatorname{tr} A\Bigr).\end{aligned} \tag{43}$$

We know that $\operatorname{tr} R(h_0)$ and $\operatorname{tr} R(h)$ are bounded with respect to $\widetilde{\omega}$, and $\operatorname{tr} A$ is L^2 with respect to $\widetilde{\omega}$. Therefore, $\operatorname{tr} A\cdot\bigl(\operatorname{tr} R(h) + \operatorname{tr} R(h_0)\bigr)$ is L^2 with respect to $\widetilde{\omega}$. We also know that $d\Bigl(\bigl(\operatorname{tr} R(h) + \operatorname{tr} R(h_0)\bigr)\cdot\operatorname{tr} A\Bigr)$ is integrable. Then we obtain the vanishing, due to Lemma 5.2 in [**45**]. (It is not difficult to check the claim directly):

$$\int d\Bigl(\operatorname{tr} R(h) + \operatorname{tr} R(h_0)\Bigr)\cdot\operatorname{tr} A = 0.$$

Let us show (41) by admitting Lemma 5.4.

Lemma 5.5. — *We have the following:*

$$\begin{aligned}\operatorname{tr}\bigl(F(h)^2\bigr) &= \operatorname{tr}\bigl(R(h)^2\bigr) + \operatorname{tr}\Bigl(\bigl(\partial_h\theta + \overline{\partial}_h\theta^{\dagger}\bigr)R(h)\Bigr)\\ &\quad + d\Bigl(\operatorname{tr}(\theta+\theta^{\dagger})\cdot\bigl(\partial_h\theta + \overline{\partial}\theta^{\dagger}\bigr) + (\theta+\theta^{\dagger})^3\Bigr) = \operatorname{tr}\bigl(R(h)^2\bigr).\end{aligned} \tag{44}$$

Proof We have used $\partial_h\theta = 0$ due to pluri-harmonicity and $\operatorname{tr}(\theta+\theta^{\dagger})^3 = 0$ which is obtained from $\theta^2 = 0$. □

Then we obtain the following:

$$\operatorname{tr}\bigl(R(h)^2\bigr) = \operatorname{tr}\bigl(R(h_0)^2\bigr) + \overline{\partial}\Bigl(\operatorname{tr}\bigl(A\cdot R(h_0)\bigr) + \operatorname{tr}\bigl(A\cdot R(h)\bigr)\Bigr).$$

Since A is L^2 with respect to h and $\widetilde{\omega}$, and $R(h)$ and $R(h_0)$ are bounded with respect to $\widetilde{\omega}$, we obtain that $\operatorname{tr}\bigl(A\cdot R(h_0)\bigr)+\operatorname{tr}\bigl(A\cdot R(h)\bigr)$ is L^2 with respect to $\widetilde{\omega}$ and h. We also have $\overline{\partial}\Big(\mathrm{tr}\big(A\cdot R(h_0)\big)+\mathrm{tr}\big(A\cdot R(h)\big)\Big)=d\Big(\mathrm{tr}\big(A\cdot R(h_0)\big)+\mathrm{tr}\big(A\cdot R(h)\big)\Big)$ is integrable. Thus we obtain the vanishing, due to Lemma 5.2 in [**45**], again:

$$\int\overline{\partial}\Big(\mathrm{tr}\big(A\cdot R(h_0)\big)+\mathrm{tr}\big(A\cdot R(h)\big)\Big)=0.$$

Thus Proposition 5.3 is reduced to Lemma 5.4. □

5.2. Proof of Lemma 5.4

Let us prove Lemma 5.4. We have only to estimate A around D. Hence the lemma will immediately follow from Proposition 5.8 and Lemma 5.10 below.

5.2.1. Preliminary. — Let U_0 be an open subset of $\boldsymbol{C}$, and $U_1\Subset U_0$ be a relatively compact open subset. Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on U_0. Let h_0 be a metric of E, then we have the endomorphism s_0 such that $h=h_0\cdot s_0$, which is self-adjoint with respect to both of h and h_0. We remark $\sqrt{-1}\big(s_0^{-1}\partial_{h_0}s_0,\,s_0^{-1}\partial_{h_0}s_0\big)_h\leq\sqrt{-1}C_1\cdot\big(s_0^{-1}\partial_{h_0}s_0,\,\partial_{h_0}s_0\big)$ for some constants C_1 which depends only on C_0.

Lemma 5.6. — *The following formula holds:*

$$\int_{U_0}\big(s_0^{-1}\partial_{h_0}(\chi\cdot s_0),\,\partial_{h_0}(\chi\cdot s_0)\big)_{h_0}=\int_{U_0}\big(\chi\cdot\overline{\partial}(s_0^{-1}\partial_{h_0}s_0),\,\chi\cdot s_0\big)+\int\partial\chi\cdot\overline{\partial}\chi\cdot\mathrm{tr}(s_0).$$

Proof We have the following:

$$\int_{U_0}\big(s_0^{-1}\partial_{h_0}(\chi\cdot s_0),\,\partial_{h_0}(\chi\cdot s_0)\big)_{h_0}=\int_{U_0}\Big(\overline{\partial}\Big(s_0^{-1}\cdot\partial_{h_0}(\chi\cdot s_0)\Big),\ \chi\cdot s_0\Big)_{h_0}\tag{45}$$
$$=\int_{U_0}\big(\overline{\partial}\partial\chi,\ \chi\cdot s_0\big)_{h_0}+\int_{U_0}\Big(\chi\cdot\overline{\partial}\big(s_0^{-1}\partial_{h_0}s_0\big),\ \chi\cdot s_0\Big)_{h_0}+\int_{U_0}\big(\overline{\partial}\chi\cdot s_0^{-1}\partial_{h_0}s_0,\ \chi\cdot s_0\big)_{h_0}.$$

Moreover, we have the following:

(46)
$$\big(\overline{\partial}\partial\chi,\chi\cdot s_0\big)_{h_0}+\big(\overline{\partial}\chi\wedge s_0^{-1}\partial_{h_0}s_0,\ \chi\cdot s_0\big)_{h_0}=\mathrm{tr}\big(\overline{\partial}\partial\chi\cdot\chi\cdot s_0\big)+\mathrm{tr}\big(\overline{\partial}\chi\cdot\partial_{h_0}s\cdot\chi\big)$$
$$=-\partial\Big(\mathrm{tr}\big(\overline{\partial}\chi\cdot\chi\cdot s_0\big)\Big)-\mathrm{tr}\big(\overline{\partial}\chi\partial\chi\cdot s_0\big).$$

Thus we are done. □

Let us consider the case U_0 is a punctured disc. Let $(E,\overline{\partial}_E,\theta,h)$ be a tame harmonic bundle on $\Delta^*(T)=\{z\in\boldsymbol{C}\,|\,|z|<T\}$. We obtain the holomorphic bundle ${}_cE$ with the parabolic filtration F, and the weight filtration W of $\mathrm{Gr}^F({}_cE_{|O})$. Let $\boldsymbol{v}$ be a holomorphic frame of ${}_cE$ which is compatible with F and W. We put

$a(v_i) := \deg^F(v_i)$ and $k(v_i) := \deg^W(v_i)$. The metric h_1 is give as follows:

$$h_1(v_i, v_j) := \begin{cases} |z|^{-2a(v_i)} \cdot (-\log|z|)^{k(v_i)} & (i = j) \\ 0 & (i \neq j). \end{cases}$$

We know that h and h_1 are mutually bounded, i.e. $C_0 \cdot h_1 \leq h \leq C_0^{-1} \cdot h_1$ for some $C_0 > 0$ [**46**]. Both of the curvatures $|R(h)|_h$ and $|R(h_1)|_{h_1}$ are dominated by $|z|^{-2} \cdot \bigl(-\log|z|\bigr)^{-2} \cdot dz \cdot d\bar{z}$. Let s_1 denote the endomorphism such that $h = h_1 \cdot s_1$. Hence we have the following, for some constant C_2:

$$\bigl|\overline{\partial}\bigl(s_1^{-1}\partial s_1\bigr)\bigr|_h = \bigl|R(h) - R(h_1)\bigr|_h \leq C_2 \cdot \frac{dz \cdot d\bar{z}}{|z|^2\bigl(-\log|z|^2\bigr)^2}.$$

Lemma 5.7. — *There exists a constant C_3 such that the following holds:*

$$0 \leq \sqrt{-1}\int_{\Delta^*(3/4)} \bigl(s_1^{-1}\partial_{h_1}s_1,\, s_1^{-1}\partial_{h_1}s_1\bigr)_{h_1} \leq C_3$$

The constant C_3 depends only on the constants C_i $(i = 0, 2)$.

Proof We take C^∞-functions χ_1 and χ_2 satisfying the following:

$$\chi_1(z) = \begin{cases} 1 & (|z| \leq 3/4) \\ 0 & (|z| \geq 5/6) \end{cases} \qquad \chi_2(z) = \begin{cases} 1 & (|z| \geq 2) \\ 0 & (|z| \leq 1) \end{cases}$$

We put $\chi_{2,N}(z) := \chi_2\bigl(N \cdot |z|\bigr)$ and $\rho_N := \chi_1 \cdot \chi_{2,N}$. Then we have the following:

$$\left|\int_{\Delta^*} \bigl(s_1^{-1}\partial_{h_1}(\rho_N \cdot s_1),\, s_1^{-1}\partial_{h_1}(\rho_N \cdot s_1)\bigr)_{h_1}\right| \\ \leq C_4\left|\int_{\Delta^*}\bigl(\rho_N \cdot \overline{\partial}\bigl(s_1^{-1}\partial_{h_1}s_1\bigr),\ \rho_N \cdot s_1\bigr)\right| + C_4\left|\int (\partial\rho_N \cdot \overline{\partial}\rho_N) \cdot \operatorname{tr} s_1\right|. \tag{47}$$

The constant C_4 depends only on C_0. There exist constants C_i $(i = 5.6.7)$ such that the following holds:

$$\left|\int \bigl(\partial\rho_N \cdot \overline{\partial}\rho_N\bigr) \cdot \operatorname{tr} s_1\right| \leq C_5\left|\int \partial\bigl(\chi_2(N|z|)\bigr) \cdot \overline{\partial}\bigl(\chi_2(N \cdot |z|)\bigr)\right| + C_6 \leq C_7.$$

The constants C_i depends only on C_0 and C_2. Then we obtain the desired estimate easily. □

5.2.2. Around the intersection point of D. — We put $X := \{(z_1, z_2) \,\big|\, |z_i| < 2\}$, $D_i = \{z_i = 0\}$ and $D = D_1 \cup D_2$. Let $\widetilde{\omega}$ denote the Poincaré metric of $X - D$:

$$\widetilde{\omega} := \sum \frac{dz_i \cdot d\bar{z}_i}{|z_i|^2 \cdot \bigl(-\log|z_i|\bigr)^2}.$$

Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$. Let $\bigl({}_{\boldsymbol{c}}E, {}^1F, {}^2F\bigr)$ be the induced parabolic bundle. We take the decomposition ${}_{\boldsymbol{c}}E = \bigoplus U_{(\boldsymbol{a},\boldsymbol{\alpha})}$ as in the section 4.1. The metric h_0 of E on $X-D$ is given $h_0=\bigoplus h'_{\boldsymbol{a},\boldsymbol{\alpha}}\cdot|z_1|^{-2a_1}\cdot|z_2|^{-2a_2}$, where $h'_{\boldsymbol{a},\boldsymbol{\alpha}}$ are C^{∞}-metrics of $U_{(\boldsymbol{a},\boldsymbol{\alpha})}$. We put $A:=\partial_h-\partial_{h_0}$. We will prove the following lemma in the rest of this subsection.

Proposition 5.8. — *A is L^2 with respect to $\widetilde{\omega}$ and h.*

The $\mathrm{Res}_i(\theta)$ induce the endomorphisms of ${}^{i}\,\mathrm{Gr}^F({}_{\boldsymbol{c}}E)$. The nilpotent part of them are denoted by N_i. Then we have the weight filtration $W(\underline{1})$ of ${}^{1}\,\mathrm{Gr}^F({}_{\boldsymbol{c}}E)$ induced by N_1. We also have the weight filtration $W(\underline{2})$ of ${}^{\underline{2}}\,\mathrm{Gr}^F({}_{\boldsymbol{c}}E)={}^{1}\,\mathrm{Gr}^{F}\,{}^{2}\,\mathrm{Gr}^F({}_{\boldsymbol{c}}E_{|O})$. Let $\boldsymbol{v}$ be a frame of ${}_{\boldsymbol{c}}E$ compatible with the decomposition $\bigoplus U_{\boldsymbol{a},\boldsymbol{\alpha}}$. Moreover $\boldsymbol{v}$ is assumed to be compatible with the filtrations $W(\underline{1})$ and $W(\underline{2})$. We put as follows:

$$a_l(v_i):={}^{l}\deg^F(v_i)\ \ (l=1,2),$$

$$k_1(v_i):=\deg^{W(\underline{1})}(v_i),\quad k_2(v_i):=\deg^{W(\underline{2})}(v_i)-\deg^{W(\underline{1})}(v_i).$$

The metric h_1 is given as follows:

$$h_1(v_i,v_j):=\begin{cases}|z_1|^{-2a_1(v_i)}|z_2|^{-2a_2(v_i)}\big(-\log|z_1|\big)^{k_1(v_i)}\big(-\log|z_2|\big)^{k_2(v_i)} & (i=j)\\ 0 & (i\neq j).\end{cases}$$

We put $A':=\partial_{h_1}-\partial_{h_0}$, then we have the following:

$$A'(v_i)=\left(\frac{k_1(v_i)}{\log|z_1|}\frac{dz_1}{z_1}+\frac{k_2(v_i)}{\log|z_2|}\frac{dz_2}{z_2}\right)\cdot v_i,$$

$$\big|A'\big|_{h_1}=\big|A'\big|_{h_0}=O\left(\frac{1}{-\log|z_1|}\frac{dz_1}{z_1}+\frac{1}{-\log|z_2|}\frac{dz_2}{z_2}\right).$$

In particular, A' is L^2 with respect to h and $\widetilde{\omega}$. Hence we have only to estimate the L^2-norm of $\partial_h-\partial_{h_1}=s_1^{-1}\partial_{h_1}s_1$. Recall the following:

***Lemma 5.9* ([38])**. — *The metrics h and h_1 are mutually bounded on the region $Z_1:=\big\{|z_1|\leq 3\cdot|z_2|/2\big\}$.* □

We may and will restrict ourselves to the integrals on the region Z_1.

We put $Z_1':=\big\{|z_1|\leq 2\cdot|z_2|\big\}$. Let P be any point of $D_1-\{O\}$. We put $C_P:=\pi_1^{-1}(P)\cap Z_1$ and $C_P':=\pi_1^{-1}(P)\cap Z_1'$. Let us consider the following integral:

$$\int_{C_P}\big(s_1^{-1}\partial_{h_1}s_1,\,s_1^{-1}\partial_{h_1}s_1\big).$$

We put $\widetilde{X}:=\{(\zeta_1,\zeta_2)\,\big|\,|\zeta_i|<1\}$, $\widetilde{D}_i:=\{\zeta_i=0\}$ and $\widetilde{D}:=\widetilde{D}_1\cup\widetilde{D}_2$. We use the map $\varphi:\widetilde{X}\longrightarrow X$ given by $(\zeta_1,\zeta_2)\longmapsto(2\zeta_1\zeta_2,\zeta_2)$. It identifies $\widetilde{D}_1$ and D_1. We have $\varphi^{-1}(Z_1)=\big\{(\zeta_1,\zeta_2)\in\widetilde{X}\,\big|\,|\zeta_1|\leq 3/4\big\}$ and $\varphi^{-1}(Z_1')=\widetilde{X}$. We identify $\widetilde{D}_1$

and D_1. And $\widetilde{C}_P, \widetilde{C}'_P \subset \widetilde{X}$ denote curves corresponding to $C_P, C'_P \subset X$. We put $\big(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{h}, \widetilde{\theta}\big) := \varphi^{-1}(E, \overline{\partial}_E, h, \theta)$ and $\widetilde{h}_1 = \varphi^{-1} h_1$. We have the following:

$$\int_{C_P} \big(s_1^{-1}\partial_{h_1} s_1,\ s_1^{-1}\partial_{h_1} s_1\big) = \int_{\widetilde{C}_P} \big(\widetilde{s}_1^{-1}\partial_{\widetilde{h}_1}\widetilde{s}_1,\ \widetilde{s}_1^{-1}\partial_{\widetilde{h}_1}\widetilde{s}_1\big). \tag{48}$$

Due to Lemma 5.7, we know that the right hand side of (48) is bounded independently of P.

Let Q be any point of $D_2 - \{O\}$. We put $C_Q := \pi_2^{-1}(Q) \cap Z_1$ and $C'_Q := \pi_2^{-1}(Q) \cap Z'_1$. Let us consider the following integral:

$$\int_{C_Q} \big(s_1^{-1}\partial_{h_1} s_1,\ s_1^{-1}\partial_{h_1} s_1\big).$$

Let us take C^∞-functions χ_1 and χ_2 satisfying the following:

$$\chi_1(z) := \begin{cases} 1 & (|z| < 1) \\ 0 & (|z| \geq 5/4 \end{cases} \qquad \chi_2(z) := \begin{cases} 1 & (|z| \geq 2/3), \\ 0 & (|z| \leq 1/2). \end{cases}$$

We put $r_Q := |z_1(Q)|$ and $\chi_Q(z) := \chi_1(z) \cdot \chi_2(r_Q^{-1} \cdot z)$. Then we have the following:

$$\begin{aligned} &\int_{C'_Q} \big(z_1^{-1}\partial_{h_1}\big(\chi_Q \cdot s_1\big),\ s_1^{-1}\partial_{h_1}\big(\chi_Q \cdot s_1\big)\big) \\ &\qquad = \int_{C'_Q} \big(\chi_Q \cdot \overline{\partial}(s_1^{-1}\partial_{h_1} s_1),\ \chi_Q \cdot s_1\big) + \int_{C'_Q} \partial\chi_Q \cdot \overline{\partial}\chi_Q \cdot \mathrm{tr}(s_1). \end{aligned} \tag{49}$$

It can be shown that the integral $\int_{C'_Q} \partial\chi_Q \cdot \overline{\partial}\chi_Q$ is bounded independently of Q. Hence $\int_{C_Q} (s_1^{-1}\partial_{h_1} s_1,\ s_1^{-1}\partial_{h_1} s_1)$ is dominated, independently of Q. □

Due to the above results, $s_1^{-1}\partial_{h_1} s_1$ is L^2 with respect to h and $\widetilde{\omega}$. Hence A is also L^2. Namely the proof of Proposition 5.8 is accomplished. □

5.2.3. Around a smooth point of D. — Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $\Delta \times \Delta^*$. We obtain the holomorphic vector bundle ${}_cE$ on $\Delta \times \Delta$, with the parabolic filtration F of ${}_cE_{|\Delta \times \{0\}}$ and the weight filtration W of $\mathrm{Gr}^F({}_cE)$. We take C^∞-decomposition ${}_cE = \bigoplus {}_cE_{(a,\alpha)}$ as in the section 4.2. The C^∞-metric h_0 is given as $h_0 = \bigoplus |\sigma|^{-2a} \cdot h'_{(a,\alpha)}$, where $h'_{(a,\alpha)}$ are C^∞-metrics of ${}_cE_{(a,\alpha)}$. We put $A = \partial_h - \partial_{h_0}$.

Lemma 5.10. — *A is L^2 with respect to h and $\widetilde{\omega}$, where $\widetilde{\omega}$ is the Poincaré like metric:*

$$\widetilde{\omega} := dz_1 \cdot d\bar{z}_1 + \frac{dz_2 \cdot d\bar{z}_2}{|z_2|^2\big(-\log|z_2|\big)^2}$$

Proof Due to the estimates given in the section 4.2, a choice of C^{∞}-splitting does not matter. Hence we may assume that the decomposition is holomorphic. In that case, it can be discussed as in the previous subsection. $\square$

Thus the proof of Lemma 5.4 is also accomplished. $\square$

CHAPTER 6

PRELIMINARY CORRESPONDENCE AND BOGOMOLOV-GIESEKER INEQUALITY

In this chapter, we show the existence of the adapted pluri-harmonic metric for *graded semisimple* parabolic Higgs bundles on surface (Proposition 6.1). We will use it together with the perturbation of the parabolic structure (the section 3.4) to derive more interesting results. One of the immediate consequence, given in this chapter, is Bogomolov-Gieseker inequality (Theorem 6.10).

6.1. Graded semisimple parabolic Higgs bundles on surface

6.1.1. The statement. — Our purpose in this section to prove the next proposition, which will immediately follow from Lemma 6.4 and Lemma 6.8 below.

Proposition 6.1. — *Let X be a smooth projective surface, and D be a simple normal crossing divisor. Let ω be a Kahler form of X. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a $\boldsymbol{c}$-parabolic Higgs bundle on (X, D), which is μ_ω-stable and graded semisimple. Let us take a positive number ϵ satisfying the following:*

- $2\epsilon < \mathrm{gap}({}_{\boldsymbol{c}}E_*)$, *and* $\epsilon = m^{-1}$ *for some positive integer* m.

We take a Kahler form ω_ϵ of $X - D$, as in the subsection 4.3.1 We put $E = {}_{\boldsymbol{c}}E_{|X-D}$, and the restriction of θ to $X - D$ is denoted by the same notation. Then there exists a hermitian metric h of E satisfying the following conditions:

- *Hermitian-Einstein condition $\Lambda_{\omega_\epsilon} F(h) = a \cdot \mathrm{id}_E$ for some constant a determined by the following condition:*

$$a \cdot \frac{\sqrt{-1}}{2\pi} \int_{X-D} \omega_\epsilon^2 = a \cdot \frac{\sqrt{-1}}{2\pi} \int_X \omega^2 = \text{par-deg}_\omega({}_{\boldsymbol{c}}E_*). \tag{50}$$

- *h is adapted to the parabolic structure of ${}_{\boldsymbol{c}}E_*$.*
- $\deg_{\omega_\epsilon}(E, h) = \text{par-deg}_\omega({}_{\boldsymbol{c}}E_*)$.

– *We have the following equalities:*

$$\int_X 2\,\text{par-ch}_2({}_{\boldsymbol{c}}E_*) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(F(h)^2\Big),$$

$$\int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\big(F(h)\big)^2.$$

Remark 6.2. — The graded semisimple condition makes the problem much easier. Later, we will discuss the case where the graded semisimplicity is not assumed. □

6.1.2. Reduction to the construction of an initial metric. — We start the proof of Proposition 6.1 by reducing the problem to the construction of an "initial metric". We will use the notation in the section 4.3.1. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a graded semisimple $\boldsymbol{c}$-parabolic Higgs bundle. We put $E := {}_{\boldsymbol{c}}E_{|X-D}$.

Definition 6.3. — In this paper, a hermitian metric h_{in} of E is called an initial metric for $({}_{\boldsymbol{c}}E_*, \theta)$ with respect to ω_ϵ, if the following conditions are satisfied:

- h_{in} is adapted to the parabolic structure of ${}_{\boldsymbol{c}}E_*$.
- $F(h_{in})$ is bounded with respect to h_{in} and ω_ϵ.
- Let V be any coherent subsheaves E, and let π_V denote the orthogonal projection of E onto V, which is defined outside a Zariski closed subset of codimension 2. Then $\overline{\partial}\pi_V$ is L^2 with respect to h_{in} and ω_ϵ, if and only if there exists a coherent subsheaf ${}_{\boldsymbol{c}}V$ of ${}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D} = V$. Moreover we have $\text{par-deg}_\omega({}_{\boldsymbol{c}}V_*) = \deg_{\omega_\epsilon}(V, h_{in,V})$.
- $\operatorname{tr} F(h_{in}) \cdot \omega_\epsilon = a \cdot \omega_\epsilon^2$ for some constant a. The constant a is determined by the condition (50).
- The following equalities hold:

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(F(h_{in})^2\Big) = \int_X 2\,\text{par-ch}_2({}_{\boldsymbol{c}}E_*),$$

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \operatorname{tr}\Big(F(h_{in})\Big)^2 = \int_X \text{par-c}_1^2({}_{\boldsymbol{c}}E_*).$$

Due to the third condition, (E, h_{in}, θ) is analytic stable with respect to ω_ϵ, if and only if $({}_{\boldsymbol{c}}E_*, \theta)$ is μ_L-stable. □

Lemma 6.4. — *To show Proposition* 6.1*, we have only to show the existence of an initial metric with respect to* ω_ϵ *for* $({}_{\boldsymbol{c}}E_*, \theta)$.

Proof Assume that we are given an initial metric h_{in} for $({}_{\boldsymbol{c}}E_*, \theta)$ with respect to ω_ϵ. Then we may apply Proposition 2.6, and it is easy to see that the obtained Hermitian-Einstein metric satisfies the conditions in Proposition 6.1. □

Remark 6.5. — In the graded semisimple case, the construction of an initial metric is quite simple. In fact, we have only to modify the determinant part of an ordinary metric as in the next subsection. Hence we can say that the essential argument was already given by Li [**29**], based on Siu's result. But we give some detail, especially about the calculation of the characteristic numbers. (See the chapter 4.)

If we do not impose the graded semisimple condition, an ordinary metric does not give an initial metric, and the construction is much more difficult. Perhaps, such an metric does not exist without any assumption on the residues of Higgs fields. In the previous version of this paper, we discussed the problem under the "codimension of Hodge type two" condition. □

6.1.3. Construction of an initial metric. — Let us take an ordinary metric h_0 for the parabolic bundle $({}_{\boldsymbol{c}}E_*, \theta)$ as in the section 4.3. By modifying it, we will construct an initial metric. Note we have $\Lambda_{\omega_\epsilon} \operatorname{tr} R(h_0) = \Lambda_{\omega_\epsilon} \operatorname{tr} F(h_0)$. For simplicity, we put $\gamma_i := \mathrm{wt}({}_{\boldsymbol{c}}E_*, i)$.

Lemma 6.6. — *$\operatorname{tr} R(h_0)$ is C^∞ on X.*

Proof Let us see the induced metric $\det(h_0)$ of $\det(E)$. Due to our construction, $\det(h_0)$ is of the form $\tau \cdot |z_i|^{-2\gamma_i} \cdot |z_j|^{-2\gamma_j}$ around $P \in D_i \cap D_j$, where τ denotes a positive C^∞-metric of $\det\big({}^\diamond E\big)_{|U_P}$. If P is a smooth point of D_i. then the metric $\det(h_0)$ is of the form $\tau \cdot |\sigma_i|_{g_i}^{-2\gamma_i}$, where τ and γ_i are as above.

Since we have $\operatorname{tr} R(h_0) = R\big(\det(h_0)\big)$, we are done. □

Lemma 6.7. — *We can take a bounded C^∞-function g on $X - D$ satisfying the following:*

- $\Delta_{\omega_\epsilon} g = \Lambda_{\omega_\epsilon} \operatorname{tr}\big(F(h_0)\big) - C$, *where C is determined as* $\int \big(\Lambda_{\omega_\epsilon} \operatorname{tr}\big(F(h_0)\big) - C\big) \cdot \omega_\epsilon^2 = 0$.
- *∂g, $\overline{\partial} g$, and $\partial\overline{\partial} g$ are bounded with respect to ω_ϵ.*

Proof Recall $\epsilon = m^{-1}$ for some positive integer m. Since it looks standard in the theory of orbifolds, we give only a brief outline. For any point Q of D, we have an appropriate neighbourhood W_Q and a ramified covering $\varphi_Q : \widetilde{W}_Q \longrightarrow W_Q$ such that $\varphi_Q^* \omega_\epsilon$ is a C^∞-metric of $\widetilde{W}_Q$. Let $C^\infty(X, \omega_\epsilon)$ denote the space of C^∞-functions f on $X - D$ such that $\varphi_Q^* f$ is C^∞ on $\widetilde{W}_Q$. We remark $\Lambda_{\omega_\epsilon} \operatorname{tr} F(h_0)$ is contained in $C^\infty(X, \omega_\epsilon)$. We also remark that if f is an element of $C^\infty(X, \omega_\epsilon)$, then ∂f, $\overline{\partial} f$ and $\partial\overline{\partial} f$ are bounded with respect to ω_ϵ. It is easy to see that $\Delta_{\omega_\epsilon} : C^\infty(X, \omega_\epsilon) \longrightarrow C^\infty(X, \omega_\epsilon)$ is self adjoint with respect to the pairing $(f, g) := \int f \cdot \bar{g} \cdot \omega_\epsilon^2$.

Let $L^2(X, \omega_\epsilon)$ denote the L^2-space with respect to the norm $\int_X |f|^2 \cdot \omega_\epsilon^2$. We put $\mathrm{Dom}(\Delta_{\omega_\epsilon}) := \big\{ f \in L^2(X, \omega_\epsilon) \,\big|\, \Delta_{\omega_\epsilon} f \in L^2(X, \omega_\epsilon) \big\}$, and thus we obtain the operator $\Delta_{\omega_\epsilon} : \mathrm{Dom}(\Delta_{\omega_\epsilon}) \longrightarrow L^2(X, \omega_\epsilon)$.

Let us check that Δ_{ω_ϵ} is a closed operator. Let $\{f_i\}$ be a sequence of $\mathrm{Dom}(\Delta_{\omega_\epsilon})$ such that $f_i \to f$ and $\Delta_{\omega_\epsilon} f_i \to g$ in $L^2(X,\omega_\epsilon)$. Due to the standard result on the Laplacian for the smooth metrics, we have $\varphi_Q^*\Delta_{\omega_\epsilon} f = \varphi_Q^* g$ on $\widetilde{W}_Q$ for any $Q \in D$. Hence we obtain $f \in \mathrm{Dom}(\Delta_{\omega_\epsilon})$ and $\Delta_{\omega_\epsilon} f = g$ in $L^2(X,\omega_\epsilon)$. It means Δ_{ω_ϵ} is closed.

The inclusions $C^\infty(X,\omega_\epsilon) \subset \mathrm{Dom}(\Delta_{\omega_\epsilon}) \subset L^2(X,\omega_\epsilon)$ are dense, and Δ_{ω_ϵ} is formally self-adjoint. Hence we obtain that Δ_{ω_ϵ} is self adjoint. Then the image of Δ_{ω_ϵ} is the orthogonal complement of the kernel of Δ_{ω_ϵ}, and the kernel is the space of the constant functions. Hence we can take some function $g \in \mathrm{Dom}(\Delta_{\omega_\epsilon})$ such that $\Delta_{\omega_\epsilon} g = \Lambda_{\omega_\epsilon} \operatorname{tr}\bigl(R(h_0)\bigr) - C$. Due to the classical elliptic regularity, we obtain $g \in C^\infty(X,\omega_\epsilon)$, and thus we are done. □

We put $g' := g/\operatorname{rank} E$ and $h_{in} := h_0 \cdot \exp(-g')$. Then the function $\Lambda_{\omega_\alpha} \operatorname{tr}\bigl(R(h_{in})\bigr)$ is constant due to the construction. We remark that the adjoints θ for h_0 and h_{in} are same. We also remark that $\partial_{h_{in}} - \partial_{h_0}$ and $R(h_{in}) - R(h_0)$ are just multiplications $-\partial g' \cdot \mathrm{id}_E$ and $\partial\overline{\partial} g' \cdot \mathrm{id}_E$ respectively. They are bounded with respect to ω_ϵ.

Lemma 6.8. — *If $({}_{\boldsymbol{c}}E_*, \theta)$ is graded semisimple, then the above metric h_{in} satisfies the conditions in Definition* 6.3.

Proof Since g' is bounded and since h_0 is adapted to the parabolic structure, h_{in} is also adapted to the parabolic structure. We have $F(h_{in}) = F(h_0) + \partial\overline{\partial} g' \cdot \mathrm{id}_E$. Hence the boundedness of $F(h_{in})$ with respect to ω_ϵ and h_0 follows from those of $F(h_0)$ and $\partial\overline{\partial} g'$.

For any saturated subsheaf $V \subset E$, the orthogonal decomposition $\pi_V^{h_0}$ and $\pi_V^{h_{in}}$ are same. Hence $\overline{\partial}\pi_V^{h_{in}}$ is L^2, if and only if there exists a coherent subsheaf ${}_{\boldsymbol{c}}V \subset {}_{\boldsymbol{c}}E$ such that ${}_{\boldsymbol{c}}V_{|X-D} = V$, due to Lemma 4.38 (Li). Let $h_{0,V}$ and $h_{in,V}$ denote the metric of V induced by h_0 and $h_{in,V}$. We have $\operatorname{tr} F(h_{in,V}) = \operatorname{tr} F(h_0) + \operatorname{rank}(V) \cdot \partial\overline{\partial} g'$. Then we obtain $\deg_{\omega_\epsilon}(V, h_{0,V}) = \deg_{\omega_\epsilon}(V, h_{in,V})$ from the boundedness of $\partial\overline{\partial} g$ and ∂g with respect to ω_ϵ. Therefore the third condition is satisfied. The fourth condition is satisfied by our construction.

Let us show the fifth condition. We have the following:

$$\bigl(\operatorname{tr} F(h_{in})\bigr)^2 = \bigl(\operatorname{tr} F(h_0)\bigr)^2 + 2 \operatorname{tr} F(h_0) \cdot \frac{\partial\overline{\partial} g}{\operatorname{rank} E} + \frac{(\partial\overline{\partial} g)^2}{\operatorname{rank} E}.$$

Due to the boundedness of ∂g, $\partial\overline{\partial} g$ and $\operatorname{tr}(R(h_0))$, we obtain $\int_{X-D}\bigl(\operatorname{tr} F(h_{in})\bigr)^2 = \int_{X-D}\bigl(\operatorname{tr} F(h_0)\bigr)^2$. Then we have $\operatorname{tr}\bigl(F(h_{in})^2\bigr) = \operatorname{tr}\bigl(F(h_0)^2\bigr) + 2 \operatorname{tr} F(h_0) \cdot \partial\overline{\partial} g' + \operatorname{rank}(E) \cdot (\partial\overline{\partial} g')^2$. Hence we obtain $\int_{X-D} \operatorname{tr}\bigl(F(h_{in})^2\bigr) = \int_{X-D} \operatorname{tr}\bigl(F(h_0)^2\bigr)$ from the boundedness of $F(h_{in})$, $F(h_0)$, $\overline{\partial}\partial g$ and $\overline{\partial} g'$. □

Now Proposition 6.1 immediately follows from Lemma 6.4 and Lemma 6.8. □

6.2. Bogomolov's inequality

6.2.1. The graded semisimple case. — We have an immediate and standard corollary of Proposition 6.1, as in [**45**].

Corollary 6.9. — *Let X be a smooth projective surface and D be a simple normal crossing divisor of X. Let $({}_{\boldsymbol{c}}E_*, \theta)$ be a μ-stable $\boldsymbol{c}$-parabolic graded semisimple Higgs bundle on (X, D). Then we have the following inequality:*

$$\int_X \mathrm{par}\text{-}\mathrm{ch}_2({}_{\boldsymbol{c}}E_*) - \frac{\int_X \mathrm{par}\text{-}\mathrm{c}_1^2({}_{\boldsymbol{c}}E_*)}{2\operatorname{rank} E} \leq 0.$$

Proof Let h be the metric of E as in Proposition 6.1. Then we have the following:

$$\int_X \mathrm{par}\text{-}\mathrm{ch}_2({}_{\boldsymbol{c}}E_*) - \frac{\int_X \mathrm{par}\text{-}\mathrm{c}_1^2({}_{\boldsymbol{c}}E_*)}{2\operatorname{rank} E} = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X-D} \mathrm{tr}\Big(F(h)^{\perp\, 2}\Big).$$

Then the claim follows from $\mathrm{tr}\Big(F(h)^{\perp\, 2}\Big) \geq 0$. (See the pages 878–879 in [**45**].) □

6.2.2. The general case. — By using the perturbation of the parabolic structure, we can remove the assumption of graded semisimplicity. We can also remove the assumption $\dim X = 2$ by using Mehta-Ramanathan type theorem.

***Theorem 6.10* (Bogomolov's inequality)**. — *Let X be a smooth projective variety of an arbitrary dimension, and D be a simple normal crossing divisor. Let L be an ample line bundle on X. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-stable regular Higgs bundle in codimension two on (X, D). Then the following inequality holds:*

$$\int_X \mathrm{par}\text{-}\mathrm{ch}_{2,L}(\boldsymbol{E}_*) - \frac{\int_X \mathrm{par}\text{-}\mathrm{c}_{1,L}^2(\boldsymbol{E}_*)}{2\operatorname{rank} E} \leq 0.$$

(See the subsection 3.1.5 *for the characteristic numbers.)*

Proof If a smooth subvariety Y of X is transversal to D, then the regular filtered Higgs bundle on $(Y, D \cap Y)$ is naturally induced. We denote it by $(\boldsymbol{E}_*, \theta)_{|Y}$. The following Mehta-Ramanathan type theorem holds for parabolic Higgs bundles.

Lemma 6.11. — *Let X and D be as above. Let L be an ample line bundle on X. Let $\big(\boldsymbol{E}_*, \theta\big)$ be a parabolic Higgs bundle over (X, D). Assume that $\big(\boldsymbol{E}_*, \theta\big)$ is μ_L-stable. There exists an integer m_0 with the following property:*

- *Let l be any integer larger than m_0. Let s be an integer such that $0 \leq s \leq \dim X - 1$. Let Y be a generic complete intersection of $\bigoplus^s L^l$. Then $(\boldsymbol{E}_*, \theta)_{|Y}$ is also μ_L-stable.*

Proof It can be shown by the arguments of Mehta-Ramanathan and Simpson ([**34**], [**35**] and [**47**]) with obvious modification for parabolic structure. □

Due to the Mehta-Ramanathan type theorem, the problem can be reduced to the case where X is a surface. Take a real number $c_i \notin \mathcal{P}ar(\boldsymbol{E}_*, i)$ for each i, and let us consider the $\boldsymbol{c}$-truncation $({}_{\boldsymbol{c}}E_*, \theta)$. Let $\boldsymbol{F}$ denote the induced $\boldsymbol{c}$-parabolic structure of ${}_{\boldsymbol{c}}E$. Let ϵ be any sufficiently small positive number, and let us take an ϵ-perturbation $\boldsymbol{F}^{(\epsilon)}$ of $\boldsymbol{F}$ as in the section 3.4. Since $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}, \theta)$ is μ_L-stable and graded semisimple, we obtain the following inequality due to Corollary 6.9:

$$\int_X \operatorname{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)}) - \frac{\int_X \operatorname{par-c}_1^2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon)})}{2 \operatorname{rank} E} \leq 0.$$

By taking the limit in $\epsilon \to 0$, we obtain the desired inequality. □

Remark 6.12. — Narasimhan posed it to Simpson how to get a Bogomolov-Gieseker inequality for parabolic Higgs bundle in higher dimensional case, and it was passed to the author. ([**52**]). □

Corollary 6.13. — *Let X be a smooth projective surface, and D be a simple normal crossing divisor. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-stable parabolic Higgs bundle on (X, D). Assume* $\int_X \operatorname{par-ch}_2(\boldsymbol{E}_*) = \operatorname{par-deg}_\omega(\boldsymbol{E}_*) = 0$. *Then we have* $\operatorname{par-c}_1(\boldsymbol{E}_*) = 0$.

Proof $\operatorname{par-deg}_\omega(\boldsymbol{E}_*) = 0$ implies $\int_X \operatorname{par-c}_1(\boldsymbol{E}_*) \cdot \omega = 0$. Due to the Hodge index theorem, it implies $-\int \operatorname{par-c}_1^2(\boldsymbol{E}_*) \geq 0$, and if the equality holds then $\operatorname{par-c}_1(\boldsymbol{E}_*) = 0$. On the other hand, we have the following inequality, due to Theorem 6.10:

$$-\frac{\int_X \operatorname{par-c}_1^2(\boldsymbol{E}_*)}{2 \operatorname{rank} E} \leq -\int_X \operatorname{par-ch}_2(\boldsymbol{E}_*) = 0.$$

Thus the claim follows. □

CHAPTER 7

A PRIORI ESTIMATE OF HIGGS FIELDS

We give a priori estimates of Higgs fields in some situations. It is a preparation for the proof of the convergence result (Theorem 9.2), which is crucial in our proof of Kobayashi-Hitchin correspondence in the surface case (Theorem 10.1). Let us mention briefly why we need the estimate of Higgs fields. Let $(E_m, \overline{\partial}_m, \theta_m)$ be a sequence of Higgs bundles with the hermitian metrics h_m. We obtain the curvatures $R(h_m)$ and $F(h_m)$. (See the subsection 2.2.1.) In our discussion of the convergence, we may assume that $F(h_m)$ are dominated in some sense. But we have to show that $R(h_m)$ are dominated. Since we have $R(h_m) = F(h_m)^{(1,1)} - [\theta_m, \theta_m^{\dagger}]$, where $F(h_m)^{(1,1)}$ denotes the $(1,1)$-part of $F(h_m)$, we clearly need the estimate of the Higgs field θ_m.

7.1. A priori estimate of Higgs fields on a disc

In this section, we put $X(T) := \big\{ z \in \boldsymbol{C} \,\big|\, |z| < T \big\}$ for any positive number T. In the case $T = 1$, $X(1)$ is denoted by X. We will use the usual Euclidean metric $dz \cdot d\bar{z}$ in this section. Let Δ denote the Laplacian $-\partial_z \overline{\partial}_z$. By the standard theory of Dirichlet problem, there exists a constant C' such that the following holds:

– We have the solution ψ of the equation $\Delta\psi = \kappa$ such that $|\psi(P)| \leq C' \cdot \|\kappa\|_{L^2}$ for any L^2-function κ and for any $P \in X$.

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X with a hermitian metric h. Let $F(h)$ and $R(h)$ denote the curvatures of the connections $\partial_E + \overline{\partial}_E + \theta + \theta^{\dagger}$ and $\partial_E + \overline{\partial}_E$ respectively. We have the expression $\theta = g \cdot dz$. We would like to estimate $\big|g\big|_h$ by the eigenvalues of g and the L^2-norm $\big\|F(h)\big\|_{L^2}$ as in the following proposition.

Proposition 7.1. — *Let t be any positive number such that $t < 1$. There exist constants C and C' such that the following inequality holds on $X(t)$:*

$$|g|_h^2 \leq C \cdot e^{2C'||F(h)||_{L^2}}.$$

The constant C' is as above, and the constant C depends only on t, the rank of E and the eigenvalues of g.

Proof Let us begin with the following lemma, which is just a minor modification of the fundamental inequality in the theory of harmonic bundles.

Lemma 7.2. — *We have the inequality:*

$$\Delta \log |g|_h^2 \leq -\frac{\big|[g, g^\dagger]\big|_h^2}{|g|_h^2} + |F(h)|_h.$$

Proof By a general formula, we have the following inequality:

$$-\sqrt{-1}\Lambda\partial\overline{\partial} \log |g|_h^2 \leq -\sqrt{-1}\Lambda\frac{\big(g,\, [R(h), g]\big)}{|g|_h^2}.$$

Then we obtain the desired inequality from $R(h) = F(h)-[\theta, \theta^\dagger] = F(h)-[g, g^\dagger]\cdot dz\cdot d\bar{z}$. □

Let us take a function A satisfying $\Delta A = |F(h)|_h$ and $|A| \leq C'||F(h)||_{L^2}$. Then we obtain the following:

$$\Delta\big(\log |g|_h^2 - A\big) = \Delta \log\big(|g|_h^2 \cdot e^{-A}\big) \leq -\frac{\big|[g, g^\dagger]\big|_h^2}{|g|_h^2}.$$

We take the decomposition $g(Q) = \rho_Q + N_Q$ as follows:

- There exists an orthonormal basis $\boldsymbol{v}$ of $E_{|Q}$, for which $g(Q)$ can be represented by a triangular matrix, and ρ_Q corresponds to the diagonal part, and N_Q corresponds to the off-diagonal part.

Then there exists a constant C_1 which depends only on the rank of E, such that the following inequality holds:

$$\Delta \log\big(e^{-A} \cdot |g|_h^2\big)(Q) \leq -C_1 \cdot \frac{|N_Q|_h^4}{|g|_h^2}.$$

We also have a constant C_2 which depends only on the eigenvalues of g, such that $\big|\rho_Q\big|_h^2 \leq C_2$ holds.

Let T be a number such that $0 < T < 1$, and $\phi_T : X(T) \longrightarrow \boldsymbol{R}$ is given by the following:

$$\phi_T(z) = \frac{2T}{(T - |z|)^2}.$$

Then we have $\Delta\phi_T = -\phi_T$ and $\phi_T \geq 2$. In particular, we have $\big|\rho_Q\big|^2 \leq 2^{-1} \cdot C_2 \cdot \phi_T$. The following lemma is clear.

Lemma 7.3. — *Either one of $|g|_h^2 \leq C_2 \cdot \phi_T$ or $|g|_h^2 \leq 2 \cdot |N_Q|_h^2$ holds.* □

We take a constant $\widehat{C}_3 > 0$ satisfying $\widehat{C}_3 > C_2$ and $\widehat{C}_3 > 4 \cdot C_1^{-1}$, and we put $C_3 := \widehat{C}_3 \cdot e^{C'\|F(h)\|_{L^2}}$. We put $S_T := \bigl\{P \in X(T) \bigm| \bigl(e^{-A} \cdot |g|^2\bigr)(P) > C_3 \cdot \phi_T(P)\bigr\}$. For any point $P \in S_T$, we have the inequality:

$$|g(P)|_h^2 > C_3 \cdot e^{A(P)} \cdot \phi_T(P) > C_2 \cdot \phi_T(P).$$

Due to Lemma 7.3, we obtain the following:

$$\begin{aligned}(51)\quad \Delta \log\bigl(e^{-A} \cdot |g|_h^2\bigr)(P) &\leq -\frac{C_1}{4} \cdot |g(P)|_h^2 \\ &= \left(-\frac{C_1}{4} \cdot e^{A(P)}\right) \cdot \bigl(e^{-A(P)} \cdot |g(P)|_h^2\bigr) \leq -\frac{1}{C_3}\bigl(e^{-A} \cdot |g|_h^2\bigr)(P).\end{aligned}$$

On the other hand, we have the following:

$$\Delta \log(C_3 \cdot \phi_R) = -\frac{1}{C_3}(C_3 \cdot \phi_R).$$

Moreover, it is easy to see $\partial S_T \cap \{|z| = T\} = \emptyset$. Hence, we obtain $S_T = \emptyset$ by a standard argument. (See [**1**], [**46**] or the proof of Proposition 7.2 in [**38**].) Namely, we obtain the inequality $e^{-A}|g|_h^2 \leq \widehat{C}_3 \cdot e^{\|F(h)\|_{L^2}} \cdot \phi_T$ on $X(T)$. Taking a limit for $T \to 1$, we obtain $|g|_h^2 \leq e^{2C'||F||_{L^2}} \cdot \widehat{C}_3 \cdot (1 - |z|^2)^{-1}$. Hence there exists a constant C, which depends only on t, the rank of E and the eigenvalues of g, such that the following inequality holds on $|z| < t < 1$:

$$|g|_h^2 \leq C \cdot e^{2C'||F(h)||_{L^2}}.$$

Thus the proof of Proposition 7.1 is accomplished. $\square$

7.2. A priori estimate of Higgs field on a punctured disc

7.2.1. Statement. — We use the notation in the previous section. Namely, we put $X(T) := \{z \in \boldsymbol{C} \,|\, |z| < T\}$ and $X := X(1)$. Let D denote the origin of X. Take positive numbers ϵ and N such that $\epsilon < 1/2$ and $N > 10$. The metric $g(\epsilon, N)$ of $X - D$ is given as follows:

$$g(\epsilon, N) := (\epsilon^{N+2}|z|^{2\epsilon} + |z|^2)\frac{dz \cdot d\bar{z}}{|z|^2}.$$

Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X(1 + \epsilon') - D$ ($\epsilon' > 0$), and let h be a hermitian metric of E. Let $F(h)$ and $R(h)$ denote the curvatures of the connections $\overline{\partial}_E + \partial_E + \theta + \theta^\dagger$ and $\overline{\partial}_E + \partial_E$ respectively. We have $F(h) = R(h) + [\theta, \theta^\dagger]$. We have the expression:

$$F(h) = F \cdot \frac{dz \cdot d\bar{z}}{|z|^2}.$$

In this section, $\|F(h)\|_{L^2}$ denote the L^2-norm of $F(h)$ with respect to h and $g(\epsilon, N)$:

$$\|F(h)\|_{L^2} := \int |F(h)|^2_{h,g(\epsilon,N)} \cdot \mathrm{dvol}_{g(\epsilon,N)} = \int |F|_h^2 \cdot \mathrm{dvol}_{\epsilon,N}$$

Here $\mathrm{dvol}_{\epsilon,N}$ is the volume form given in the subsection 2.5.1.

Assumption 7.4. — We assume $\|F(h)\|_{L^2} < \infty$. □

We have the expression $\theta = f_0 \cdot dz/z$.

Assumption 7.5. — We assume that the coefficients $a_j(z)$ of $P(z,t) := \det(t - f_0(z)) = \sum a_j(z) \cdot t^j$ are holomorphic on X. □

The set of the the polynomial $P(0,t)$ is denoted by S_0.

Assumption 7.6. — We assume the following:

- We have the decomposition $E = \bigoplus_{a \in S_0} E_a$, such that $f_0(E_a) \subset E_a$. In particular, we have the decomposition $f_0 = \bigoplus f_{0\,a}$.
- There exist some positive numbers C_0 and ϵ_0 such that $|b - a| < C_0 \cdot |z(Q)|^{\epsilon_0}$ holds for any eigenvalue b of $f_{a\,|\,Q}$ $(Q \in X - D)$.
- We put $\xi := \sum_{a \in S_0} \operatorname{rank}(E_0) \cdot |a|^2 \neq 0$. We assume $\xi \neq 0$, for simplicity. □

Remark 7.7. — The first two assumptions are always satisfied, if we replace X by a smaller open set. The third condition is minor, for we may perturb ξ by taking tensor products with an appropriate Higgs bundle of rank one. □

We take a total order $\leq_1$ on S_0, and we put $F_a E := \bigoplus_{b \leq_1 a} E_b$ and $F_{<a} E := \bigoplus_{b <_1 a} E_b$. Let E'_a denote the orthogonal complement of $F_{<a}(E)$ in $F_a(E)$. We put $\rho := \bigoplus_{a \in S_0} a \cdot \mathrm{id}_{E_a}$ and $\rho' := \bigoplus_{a \in S_0} a \cdot \mathrm{id}_{E'_a}$. We have $|\rho'|_h^2 = \xi$. It is our purpose to give the estimate of f_0 and ρ as in the next proposition.

Proposition 7.8. —

(I) : *Let T_1 be a positive number such that $T_1 < 1$. There exist positive constants C_1 and $\widehat{C}_1$ satisfying the following inequality on $\Delta^*(T_1)$:*

$$|f_0 - \rho'|_h \leq C_1 \cdot \frac{\exp\big(\widehat{C}_1 \cdot \|F(h)\|_{L^2}\big)}{-\log|z|} \tag{52}$$

(II) : *There exist positive constants C_2, $\widehat{C}_2$ and T_2, satisfying the following inequality on $\Delta^*(T_2)$:*

$$\big|\rho - \rho'\big|_h \leq C_2 \cdot \frac{\exp\big(\widehat{C}_2 \cdot \|F(h)\|_{L^2}\big)}{(-\log|z|)^2} \tag{53}$$

Here C_i, $\widehat{C}_i$ $(i = 1, 2)$ and T_2 depends only on the constants C_0, ϵ_0, T_1, S_0 and $\operatorname{rank}(E)$.

We will prove the proposition in the rest of this section. In the proof, a constant will be called good, if it depends only on the constants C_0, ϵ_0, T_1, S_0 and $\operatorname{rank}(E)$, for simplicity. We will also use the real coordinate $z = r \cdot \exp(\sqrt{-1}\beta)$.

Remark 7.9. — Similar estimates for harmonic bundles on a punctured disc are given by Simpson ([**46**]. See also Proposition 7.2 in [**38**]). The proof for Proposition 7.8 is a minor modification. However, we need a slightly different argument for the claim (II).

It is one of the points in Proposition 7.8 that the constants can be taken independently of ϵ and N. The estimate in (II) is weaker than that given in loc. cit., because of the uniformness for ϵ and N. It is possible to obtain shaper estimate by the argument given in loc. cit. if ϵ and N are fixed. □

7.2.2. Preliminary estimate for f_0. — We put $f = f_0/z$. As in Lemma 7.2, we have the following inequality:

$$\Delta \log |f|_h^2 \leq \frac{\big|[f, f^{\dagger}]\big|_h^2}{|f|_h^2} + \frac{|F|_h}{|z|^2}.$$

Due Proposition 2.13, we have the function v satisfying the following:

$$\Delta v = \frac{|F|_h}{|z|^2}, \qquad |v| \leq C \cdot \|F(h)\|_{L^2} \cdot \Big(\epsilon^{(N-1)/2} r^{\epsilon} + r^{1/2}\Big) \leq C' \cdot \|F(h)\|_{L^2}. \tag{54}$$

Here C and C' are good constants. Then we obtain the following:

$$\Delta \log\big(|f|_h^2 \cdot e^{-v}\big) \leq -\frac{\big|[f, f^{\dagger}]\big|_h^2}{|f|_h^2}.$$

For any point $Q \in X - D$, we have the generalized eigen decomposition with respect to $f_{0\,a\,|\,Q}$:

$$E_{a\,|\,Q} = \bigoplus_{\alpha \in \mathcal{S}p(f_{0,a\,|\,Q})} \mathbb{E}(f_{0\,a\,|\,Q}, \alpha).$$

We have the natural bijection $\mathcal{S}p(f_{0\,|\,Q}) \simeq \big\{(a,\alpha) \,\big|\, a \in S_0,\ \alpha \in \mathcal{S}p(f_{0\,a\,|\,Q})\big\}$. We pick a total order $\leq_2$ on $\mathcal{S}p(f_{0\,a\,|\,Q})$ on each a. Then we obtain the total order $\leq_3$ on $\mathcal{S}p(f_{0\,|\,Q})$, which is given by the lexicographic order of $\leq_1$ and $\leq_2$. We obtain the filtration $F^{(1)}$ on $E_{|Q}$ defined as follows:

$$F^{(1)}_{(a,\alpha)}(E_{|Q}) = \bigoplus_{(b,\beta) \leq_3 (a,\alpha)} \mathbb{E}(f_{0\,b\,|\,Q}, \beta).$$

Let $H_{(a,\alpha)}$ denote the orthogonal complement of $F^{(1)}_{<(a,\alpha)}$ in $F^{(1)}_{(a,\alpha)}$. We put as follows:

$$\tilde{\rho}_Q := \bigoplus_{(a,\alpha) \in \mathcal{S}p(f_{0\,|\,Q})} \alpha \cdot id_{H_{(a,\alpha)}}. \tag{55}$$

There exists good constants C'' and ϵ_0 satisfying the following:

$$\big|\tilde{\rho}_Q - \rho'_{|Q}\big|_h \leq C'' \cdot |z(Q)|^{\epsilon_0}.$$

As a result, we have a good constant A_2 such that $|\widetilde{\rho}_Q|_h^2 \leq A_2$.

We put $g_Q := f_{|Q} - \tilde{\rho}_Q$. Then we have the equality $|f_{|Q}|_h^2 = |\tilde{\rho}_Q|_h^2 \cdot |z(Q)|^{-2} + |g_Q|_h^2$. There exists a good constant A_1 satisfying $\big|\big[f_{|Q},\ f_{|Q}^{\dagger}\big]\big|_h \geq A_1 \cdot |g_Q|^2$. Then it is easy to see the existence of good constants A_3 and A_4 satisfying the following:

> Either one of $|f(Q)|_h^2 \leq A_3 \cdot |z(Q)|^{-2}$ or $\Delta \log\big(|f|_h^2 \cdot e^{-v}\big)(Q) \leq -A_4 \cdot |f|_h^2(Q)$ holds for any point $Q \in X - D$:

Hence we obtain the following lemma.

Lemma 7.10. — *For any point $Q \in X - D$, one of the following holds:*
- $|f(Q)|_h^2 \leq A_3 \cdot |z(Q)|^{-2}$.
- $\Delta \log\big(|f|_h^2 \cdot e^{-v}\big)(Q) \leq -A_4 \cdot e^{-C'\|F(h)\|_{L^2}} \cdot \big(|f|_h^2 \cdot e^{-v}\big)(Q)$. □

Let η be any positive number. Let us take a positive number B satisfying $B \geq 4 \cdot {A_4}^{-1}$ and $B > A_3$. In particular, B is a good constant. We put $\widetilde{B} := B \cdot e^{C'\|F(h)\|_{L^2}}$, and we put $m_{\eta,\widetilde{B}} := \widetilde{B} \cdot (|z| - \eta)^{-2} \cdot (|z| - 1)^{-2}$ which is a C^∞-function on the region $\big\{z \,\big|\, \eta < |z| < 1\big\}$. The following inequalities can be shown by a direct calculation:

$$\Delta \log m_{\eta,\widetilde{B}} \geq -A_4 \cdot e^{-C'\|F(h)\|_{L^2}} \cdot m_{\eta,\widetilde{B}}, \quad m_{\eta,\widetilde{B}} \geq A_3 \cdot |z|^{-2}. \tag{56}$$

We put $S_1 := \big\{z \in X - D \,\big|\, \eta < |z| < 1,\ |f(z)|_h^2 \cdot e^{-v(z)} > m_{\eta,\widetilde{B}}(z)\big\}$. It is easy to see that S_1 is relatively compact in $\{\eta < |z| < 1\}$. Then we can obtain $S_1 = \emptyset$ from Lemma 7.10 and (56) by a standard argument. (See [**45**] or the proof of Proposition 7.2 in [**38**].) In other words, we have the inequality $|f(z)|_h^2 \cdot e^{-v(z)} \leq m_{\eta,\widetilde{B}}(z)$ for any point $z \in X - D$ such that $|z| > \eta$. Hence we obtain the following inequality:

$$|f(z)|_h^2 \leq \frac{e^{v(z)} \cdot B \cdot e^{C'\|F(h)\|_{L^2}}}{|z|^2\big(|z| - 1\big)^2}.$$

Let T_3 be any positive number such that $T_1 < T_3 < 1$. Therefore there exists a good constant A_5, such that the following holds on $X(T_3) - D$:

$$|f(z)|_h^2 \leq \frac{A_5 \cdot e^{2C'\|F(h)\|_{L^2}}}{|z|^2}.$$

7.2.3. Estimate for f_0. — We put $k := \log |f|_h^2 - \log\big(\xi \cdot |z|^{-2}\big)$. Then we have the following:

$$k(Q) = \log\Big(\frac{|z(Q)|^2}{\xi} \cdot |f_{|Q}|^2\Big) = \log\Big(\frac{|z(Q)|^2}{\xi} \cdot \big(|\tilde{\rho}_Q|_h^2 \cdot |z(Q)|^{-2} + |g_Q|_h^2\big)\Big).$$

We put $b_Q := |\tilde{\rho}_Q|^2 - \xi$. We have a good constant A_6 such that $|b_Q| \leq A_6 \cdot |z(Q)|^{\epsilon_0}$.

Lemma 7.11. — *There exist good constants A_7 and $\widehat{A}_7$ satisfying the following inequality for any point $Q \in \Delta^*(T_3)$:*

$$A_7 \cdot e^{-\widehat{A}_7\|F(h)\|_{L^2}}\Big(\xi^{-1} \cdot b_Q + \frac{|z(Q)|^2}{\xi} \cdot |g_Q|_h^2\Big) \leq k(Q) \leq \xi^{-1} \cdot b_Q + \frac{|z(Q)|^2}{\xi} \cdot |g_Q|_h^2. \tag{57}$$

Proof We have the following description:

$$k(Q)=\log\Big(1+\xi^{-1}\cdot b_Q+\frac{|z(Q)|^2}{\xi}\cdot|g_Q|_h^2\Big).$$

Then the right inequality is obvious. We have only to obtain the left inequality. Recall we have obtained $|g_Q|_h^2\leq|f_{|Q}|_h^2\leq A_5\cdot e^{2C'\|F(h)\|_{L^2}}\cdot|z(Q)|^{-2}$ on $\Delta^*(T_3)$. Hence we obtain $\xi^{-1}\big(\xi+|z(Q)|^2\cdot|g_Q|_h^2\big)\leq A_5'\cdot e^{2C'\|F(h)\|_{L^2}}$ for some good constant A_5'. Thus we obtain the following inequality:

$$\frac{\log(A_5'\cdot e^{2C'\|F(h)\|_{L^2}})}{A_5'\cdot e^{2C'\|F(h)\|_{L^2}}}\left(\frac{b_Q}{\xi}+\frac{|z(Q)|^2}{\xi}|g_Q|^2\right)\leq k(Q).$$

We can take good constants A_7 and $\widehat{A}_7$ satisfying the following:

$$A_7\cdot e^{-\widehat{A}_7\|F(h)\|_{L^2}}\leq\frac{\log\big(A_5'\cdot e^{2C'\|F(h)\|_{L^2}}\big)}{A_5'\cdot e^{2C'\|F(h)\|_{L^2}}}.$$

Thus the desired inequality (57) is obtained. □

Recall the estimate of v as in (54). We have assumed that N is sufficiently large. Hence there exists a good constant A_{10} satisfying the following:

$$\xi^{-1}\cdot A_6\cdot|z|^{\epsilon_0}\leq\frac{1}{2}A_{10}\cdot\left(-\log\frac{|z|}{T_3}\right)^{-2},\qquad |v|\leq\frac{1}{3}A_{10}\cdot\|F(h)\|_{L^2}\cdot\left(-\log\frac{|z|}{T_3}\right)^{-2}.$$

Lemma 7.12. — *Let Q be a point of $X(T_3)-D$ satisfying the following:*

$$k(Q)-v(Q)\geq A_{10}\cdot(1+\|F(h)\|_{L^2})\cdot\left(-\log\frac{|z(Q)|}{T_3}\right)^{-2}.$$

Then the following holds:

$$\Delta(k-v)(Q)\leq-\frac{A_1^2\cdot\xi^2}{16\cdot A_5}\frac{\big|(k-v)(Q)\big|^2}{|z(Q)|^2}\cdot e^{-2C'\|F(h)\|_{L^2}}.$$

Proof We obtain the inequality: $k(Q)-v(Q)\geq 2\xi^{-1}A_6\cdot|z(Q)|^{\epsilon_0}\geq 2b_Q\cdot\xi^{-1}$. Thus we obtain the following:

$$\frac{2b_Q}{\xi}\leq k(Q)\leq\frac{b_Q}{\xi}+\frac{|z(Q)|^2}{\xi}|g_Q|^2.$$

Hence we have $\xi^{-1}\cdot b_Q\leq\xi^{-1}\cdot|z(Q)|^2\cdot|g_Q|^2$ and $k(Q)\leq 2\xi^{-1}\cdot|z(Q)|^2\cdot|g_Q|_h^2$. We also have the inequality $k(Q)-v(Q)\geq 3|v|(Q)$, and hence $k(Q)\geq 2|v(Q)|$. It implies $|k(Q)-v(Q)|^2\geq k(Q)^2/4$. Therefore we obtain the following:

$$\begin{aligned}(58)\quad \Delta(k-v)(Q)&\leq-\frac{\big|[f,f^\dagger]\big|_h^2}{|f|_h^2}(Q)\leq-A_1^2\cdot\frac{|g_Q|_h^4}{|f(Q)|_h^2}\leq-\frac{A_1^2}{|f(Q)|_h^2}\cdot\frac{\xi^2}{4\cdot|z(Q)|^4}k(Q)^2\\&\leq-\frac{A_1^2\cdot\xi^2}{4A_5}\frac{k(Q)^2}{|z(Q)|^2}\cdot e^{-2C'\|F(h)\|_{L^2}}\leq-\frac{A_1^2\cdot\xi^2}{16A_5}\frac{\big|(k-v)(Q)\big|^2}{|z(Q)|^2}\cdot e^{-2C'\|F(h)\|_{L^2}}.\end{aligned}$$

Thus we are done. □

Let us take a good constant A_{11} satisfying $A_{11} \leq \xi^2 \cdot A_1^2 \cdot (16 \cdot A_5)^{-1}$ and $A_{11} < 6 \cdot A_{10}^{-1}$. Due to the previous lemma, either one of the following holds:

$$(k-v)(Q) < A_{10} \cdot \bigl(1 + \|F(h)\|_{L^2}\bigr) \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2} \tag{59}$$

or

$$\Delta(k-v)(Q) \leq -A_{11} \cdot e^{-2C'\|F(h)\|_{L^2}} \frac{\bigl|k(Q)-v(Q)\bigr|^2}{|z(Q)|^2}. \tag{60}$$

We put $B = 6 \cdot A_{11}^{-1} \cdot e^{2C'\|F(h)\|_{L^2}}$. Let ϵ_1 be any positive number, and we put as follows:

$$p_{B,\epsilon_1} = B \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2} + \epsilon_1 \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr).$$

By a direct calculation, we have the following inequalities:

$$\Delta p_{B,\epsilon_1} \geq -A_{11} \cdot e^{2C'\|F(h)\|_{L^2}} \cdot \frac{p_{B,\epsilon_1}^2}{|z|^2},$$

$$p_{B,\epsilon_1} > A_{10} \cdot e^{C'\|F(h)\|_{L^2}} \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2} \geq A_{10} \cdot \bigl(1 + \|F(h)\|_{L^2}\bigr) \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2}.$$

We put $S_2 := \bigl\{Q \bigm| (k-v)(Q) > p_{B,\epsilon_1}(Q)\bigr\}$.

Lemma 7.13. — *The set S_2 is empty. In other words, we have $k(Q) \leq p_{B,\epsilon_1}(Q) + v(Q)$ for any point $Q \in \Delta^*$.*

Proof For any point $Q \in S$, the inequality (60) holds. Hence we obtain the following subharmonicity:

$$\Delta(k - v - p_{B,\epsilon_1})(Q) \leq -A_{11} \cdot e^{-2C'\|F(h)\|_{L^2}} \cdot \frac{(k(Q)-v(Q))^2 - p_{B,\epsilon_1}^2}{|z(Q)|^2} \leq 0.$$

On the other hand, for any $Q \in \partial\overline{S}$ we have $k(Q) - v(Q) = p_{B,\epsilon_1}(Q)$. Hence we obtain $k - v \leq p_{B,\epsilon_1}$ on S, and hence we arrive at the contradiction. □

As a result, we obtain the following inequalities:

$$k(Q) \leq \frac{1}{3} A_{10} \cdot \|F(h)\|_{L^2} \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2} + 6A_{11}^{-1} \cdot e^{2C'\|F(h)\|_{L^2}} \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2} + \epsilon_1 \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr). \tag{61}$$

By taking a limit for $\epsilon_1 \to 0$, we obtain the following inequality for some good constants A_{12} and $\widehat{A}_{12}$:

$$k(Q) \leq A_{12} \cdot e^{\widehat{A}_{12}\|F(h)\|_{L^2}} \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2}.$$

Therefore we obtain the following, for some good constants A_{13} and A_{14}:

$$|z(Q)|^2 \cdot |g_Q|^2 \leq A_{13} \cdot e^{A_{14}\|F(h)\|_{L^2}} \cdot \Bigl(-\log \frac{|z|}{T_3}\Bigr)^{-2}.$$

Then the claim (I) follows immediately.

7.2.4. Estimate for ρ. — Let us show the claim (II). We put $q := \rho - \rho'$, which is an element of $\bigoplus_{a<_1 b} Hom(E'_b, E'_a)$. We use the notation in the section 2.4.

Lemma 7.14. — *There exist good constants A_{15} and A_{16} such that $|q|_h \leq A_{15} \cdot e^{A_{16}\|F(h)\|_{L^2}} \cdot \bigl(-\log|z|\bigr)^{-1}$ on $X(T_1) - D$.*

Proof We have $0 = [f_0, \rho] = [f_0, \rho' + q] = F_{f_0}(q) + [\rho' + \widetilde{g}, \rho'] = F_{f_0}(q) + [\widetilde{g}, \rho']$. We know $\bigl|[\widetilde{g}, \rho']\bigr|_h \leq A_{17} \cdot e^{A_{18}\|F(h)\|_{L^2}} \cdot \bigl(-\log|z|\bigr)^{-1}$. Then we have only to apply Lemma 2.12. □

Lemma 7.15. — *The following inequality holds:*

$$\bigl|[\rho, f_0^\dagger]\bigr|_h^2 \geq A_{19} \frac{(-\log|z|)^2}{\bigl(-\log|z| + e^{A_{20}\|F(h)\|_{L^2}}\bigr)^2} \cdot |q|_h^2.$$

Proof We have $[\rho, f_0^\dagger] = [\rho - \rho', f_0^\dagger] + [\rho', f_0^\dagger]$. We put as follows:

$$S = \bigoplus_{a<_1 b} Hom(E'_b, E'_a).$$

We obtain $\bigl|[\rho, f_0^\dagger]\bigr|_h^2 \geq \bigl|\pi_S\bigl([\rho - \rho', f_0^\dagger]\bigr)\bigr|_h^2 = \bigl|G_{f_0^\dagger}(\rho - \rho')\bigr|_h^2$. We know that $G_{f_0^\dagger}$ is invertible, and the norm of $G_{f_0^\dagger}^{-1}$ is dominated by $\prod_{a \neq b}(a-b)^{-n(a,b)} \cdot \|f_0^\dagger\|^M$ for some large M. We also have the following inequality:

$$\|f_0^\dagger\| \leq A_{20}\left(1 + \frac{e^{A_{21}\|F(h)\|_{L^2}}}{(-\log|z|)}\right).$$

Hence we obtain the following:

$$\begin{aligned}
|q|_h^2 = |\rho - \rho'|_h^2 &= \bigl|G_{f_0^\dagger}^{-1} G_{f_0^\dagger}(\rho - \rho')\bigr|_h^2 \leq |G_{f_0^\dagger}^{-1}|_h^2 \cdot \bigl|G_{f_0^\dagger}(\rho - \rho')\bigr|_h^2 \\
&\leq A_{22}\left(1 + \frac{e^{A_{23}\|F(h)\|_{L^2}}}{-\log|z|}\right)^M \cdot \bigl|G_{f_0^\dagger}(\rho - \rho')\bigr|_h^2 \\
&\leq A_{24}\left(1 + \frac{e^{A_{25}\|F(h)\|_{L^2}}}{-\log|z|}\right)^2 \cdot \bigl|G_{f_0^\dagger}(\rho - \rho')\bigr|_h^2.
\end{aligned} \tag{62}$$

Thus we are done. □

We have the following inequality:

$$\Delta \log|\rho|_h^2 \leq -\frac{\bigl|[\rho, f_0^\dagger]\bigr|_h^2}{|\rho|_h^2} + |F|_h \cdot |z|^{-2}.$$

Due to Lemma 7.15, we obtain the following:

$$\Delta\bigl(\log|\rho|_h^2 - v\bigr) \leq -A_{19} \cdot \frac{(-\log|z|)^2}{\bigl(-\log|z| + e^{A_{20}\|F(h)\|_{L^2}}\bigr)^2} \frac{|q|_h^2}{|z|^2}.$$

We have $k := \log \xi^{-1}|\rho|_h^2 = \log\bigl(1+\xi^{-1}|q|_h^2\bigr) \leq \xi^{-1}|q|_h^2$. Hence we obtain the following:

$$\Delta(k-v) \leq -A_{19}\frac{(-\log|z|)^2\cdot\xi}{(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}})^2}\cdot\frac{k}{r^2}.$$

Since we have $k(Q)>0$, either one of the following holds:

$$k(Q) < 10\cdot v(Q)$$

or

$$\Delta(k-v)(Q) \leq -\frac{9}{10}A_{19}\cdot\frac{(-\log|z|)^2\cdot\xi}{(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}})^2}\cdot\frac{k-v}{r^2}(Q).$$

Lemma 7.16. — *We have a good constant T_4 such that either one of the following holds, for any $Q\in\Delta^*(T_4)$:*

$$k(Q)\leq 10\cdot v(Q)$$

or

$$\Delta(k-v)(Q) \leq -\frac{20\cdot(k-v)}{r^2\cdot(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}})^2}(Q).$$

□

We put $\phi(z) := \bigl(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}}\bigr)^{-4}+\eta\cdot(-\log|z|)$ for any $\eta>0$. Then we have the following:

$$\begin{aligned}(63)\quad \Delta\phi(z) &= -\frac{1}{r^2}\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right)\phi(z)\\ &= -\frac{20}{r^2\cdot\bigl(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}}\bigr)^6} \geq -\frac{20\cdot\phi(z)}{r^2\cdot(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}})^2}.\end{aligned}$$

We have already known $(k-v)(Q)\leq A_{21}\cdot e^{A_{22}\|F(h)\|_{L^2}}$ for some good constants A_{21} and A_{22}. Hence there exist good constants A_{24} and A_{25} such that the inequality $(k-v)(Q)\leq A_{24}\cdot\exp(A_{25}\|F(h)\|)\cdot\phi(Q)$ holds for $Q\in\partial X(T_4)$. We put $S := \bigl\{Q\in X(T_4)\,\big|\,(k-v)(Q) > A_{24}\cdot\exp(A_{25}\|F(h)\|)\cdot\phi(Q)\bigr\}$. We remark that the closure of S does not intersect with D and $\partial X(T_4)$. Then we obtain the following inequality on $X(T_4)-D$, by a standard argument:

$$(k-v)\leq A_{24}\cdot e^{A_{25}\|F(h)\|}\cdot\phi = \frac{A_{24}\cdot e^{A_{25}\|F(h)\|_{L^2}}}{\bigl(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}}\bigr)^4}.$$

Recall the inequality (54). Hence we obtain the following inequality, for some good constant A_{26} and A_{27}:

$$k\leq A_{26}\left(r^{\epsilon}\cdot\epsilon^{(N-1)/2}\|F(h)\|_{L^2}+\frac{e^{A_{25}\|F(h)\|_{L^2}}}{(-\log|z|+e^{A_{20}\|F(h)\|_{L^2}})^4}\right)\leq A_{27}\cdot\frac{e^{A_{28}\|F(h)\|_{L^2}}}{\bigl(-\log|z|\bigr)^4}.$$

Then we arrive at the following inequality, for some good constants A_{28} and A_{29}:

$$|q|_h^2\leq A_{28}\cdot\frac{e^{A_{29}\|F(h)\|_{L^2}}}{(-\log|z|)^4}.$$

Thus the proof of Proposition 7.8 is finished. □

7.3. An estimate on a multiple disc

Let (Y, ω) be a Kahler manifold. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle with a Hermitian metric h. We assume $||F(h)||_{L^2,\omega} < \infty$, where $\|F(h)\|_{L^2\omega}$ denote the L^2-norm with respect to ω and h. For simplicity, we restrict ourselves to the case $\Lambda_\omega F(h) = 0$.

Remark 7.17. — The induced Higgs field and the metric of $\mathrm{End}(E)$ are denoted by $\widetilde{\theta}$ and $\widetilde{h}$. Then the metric $\widetilde{h}$ is a Hermitian-Einstein metric of $\big(\mathrm{End}(E), \widetilde{\theta}\big)$ such that $\Lambda_\omega F(\widetilde{h}) = 0$. □

We would like to estimate the sup norm of θ with respect to ω and h, by the L^2-norm $\|F(h)\|_{L^2,\omega}$ and the eigenvalues of θ, locally on Y. For that purpose, we may assume $Y = \Delta(T_1)^n$ for some $T_1 > 0$. Let g denote the metric $\sum dz_i \cdot d\bar{z}_i$ of $\Delta(T_1)^n$. There are constants $C > 0$ such that $C^{-1} \cdot \omega \leq g \leq C \cdot \omega$. We will use the standard volume forms $\mathrm{dvol}_{z_i} = \sqrt{-1} dz_i \cdot d\bar{z}_i$ and $\mathrm{dvol}_g = \prod \mathrm{dvol}_{z_i}$.

We have the expression $\theta = \sum f_i \cdot dz_i$ for holomorphic sections $f_i \in \mathrm{End}(E)$ on Y. We will obtain the estimate of the norms of f_i on $\Delta(T_2)^n$ for any $T_2 < T_1$ as in the next lemma.

Lemma 7.18. — *Let $(E, \overline{\partial}_E, \theta, h)$ and Y be as above. There are some constants C_1 and C_2 such that the following inequality holds for any $P \in \Delta(T_2)^n$:*

$$\log |f_i|^2(P) \leq C_1 \cdot \big\|F(h)\big\|_{L^2} + C_2.$$

The constants C_1 and C_2 are good in the sense that they depend only on T_j $(j = 1, 2)$, the rank of E, the eigenvalues of f_i $(i = 1, 2, \ldots, n)$ and the constant C.

Proof We take a positive number T_3 such that $T_2 < T_3 < T_1$. Since we have $\widetilde{\theta}(f_i) = 0$, we have the subharmonicity $\Delta_\omega \log |f_i|_h^2 \leq 0$ due to Lemma 2.10. For $P \in \Delta(T_2)^n$, we have the following inequality (see Theorem 9.20 in [**15**], for example):

$$\log |f_i|^2(P) \leq C_3 \cdot \int_{\Delta(T_3)^2} \log^+ |f_i|^2 \cdot \mathrm{dvol}\,.$$

Here we put $\log^+(y) := \max\{0, \log y\}$, and C_3 denotes a good constant in our case.

We have the expression $F = \sum F_{i,j} \cdot dz_i \cdot d\bar{z}_j$. Due to Proposition 7.1, there exist good constants C_j $(j = 4, 5)$ such that the following inequality holds for any point $(z_1, z_2, \ldots, z_n) \in \Delta(T_3)^n$:

$$\log |f_1|^2(z_1, z_2, \ldots, z_n) \leq C_4 \cdot \left(\int_{|w_1| \leq T_1} |F_{1,1}(w_1, z_2, \ldots, z_n)|^2 \cdot \mathrm{dvol}_w \right)^{1/2} + C_5.$$

Then the claim of Lemma 7.18 immediately follows. □

CHAPTER 8

PRELIMINARY FOR A CONVERGENCE RESULT

This chapter is also a preparation for some convergence (Theorem 9.2), which is crucial in our proof of Kobayashi-Hitchin correspondence in the surface case (Theorem 10.1). The results will be used in the section 9.3. Let us briefly mention what we would like to see. (See also the proof of Theorem 9.2.)

Let X be a smooth projective surface with a polarization L, and D be a simple normal crossing divisor. Let $\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m) \,|\, m = 1, 2, \ldots\}$ be a sequence of $\boldsymbol{c}$-parabolic Higgs bundles on (X, D), which converges to a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$. We put $E_m := {}_{\boldsymbol{c}}E_{m|X-D}$.

Assume that the metrics h_m of E_m are given such that $\Lambda_{\omega_m} F(h_m) = 0$, where ω_m denotes Kahler metrics of $X - D$. We also assume that $F(h_m)$ converges to 0 in L^2. In this situation, it is rather easy to see the convergence of $(E_m, \overline{\partial}_{E_m}, \theta_m, h_m)$ locally on $X - D$, once we take an appropriate subsequence. The limit is denoted by $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$, which is a tame harmonic bundle, and then we obtain the $\boldsymbol{c}$-parabolic Higgs bundle $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)$ on (X, D).

Our real problem is to show that $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ and $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)$ are isomorphic. It is rather easy to see that we have only to construct a non-trivial map ${}_{\boldsymbol{c}}E_{|C} \longrightarrow {}_{\boldsymbol{c}}E_{\infty|C}$ compatible with the parabolic structures and the Higgs fields, on a generic curve $C \subset X$.

For that purpose, we would like to show that $\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m)_{|C}\}$ converges to $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta)_{|C}$ in an appropriate sense. Hence we will discuss the following issues:

- We would like to show that a subsequence of $\{(E_m, \overline{\partial}_m, \theta_m, h_m)_{|C\setminus D}\}$ converges to $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)_{|C\setminus D}$ on a general curve C, in a good way. The issue will be discussed in the section 8.1. We have only to care it around the smooth part of D. Hence we will discuss it in the setting of the subsection 8.1.1.
- Let C be as above. Let $\Phi_m : E_m \longrightarrow E_\infty$ be the isomorphisms as in Definition 2.1, given on $X-D$. We would like to replace morphisms $\Phi_{m\,|\,C\setminus D} : E_{m\,|\,C\setminus D} \longrightarrow E_{\infty\,|\,C\setminus D}$ with $\Psi_m : {}_{\boldsymbol{c}}E_{m|C} \longrightarrow {}_{\boldsymbol{c}}E_{\infty|C}$. For the construction of Ψ_m, we will

construct good local holomorphic frames of ${}_{\boldsymbol{c}}E_m$ around $C \setminus D$. The issue will be discussed in the sections 8.2–8.3. Since the problem is local around $C \cap D$, we will discuss a bundle on a punctured disc.

In the section 8.2, we construct an orthonormal frame for which the connection form is small. By modifying it, we will obtain a good holomorphic frame in the section 8.3.

8.1. Selection of a curve and orthogonal frames

8.1.1. Setting. — We put $I := [0,1]$. We have the embedding $I^2 \subset \boldsymbol{C}$ given by $(u,v) \longmapsto u + \sqrt{-1}v$, which gives the complex structure of I^2. We will use the standard measure of I^2 as in the subsection 2.5.2, which will be omitted to denote. We put $K_n := \bigl\{(n-1)\pi < y < (n+1)\pi \bigm| -\pi < x < 2\pi\bigr\}$, which is the subset of the upper half plane $\mathbb{H} = \bigl\{(x,y) \in \boldsymbol{R}^2 \bigm| y > 0\bigr\}$.

Let ω'_m be Kahler forms on $\Delta \times I^2$. We assume that the sequence $\{\omega'_m\}$ converges to a Kahler form ω' on I^2 in the C^∞-sense. Let κ be a positive function on $\Delta \times I^2$. Then we put as follows, for a sufficiently large fixed N, say $N > 10$:

$$\omega_m := \sqrt{-1}\partial\overline{\partial}\Bigl(\epsilon_m^N \cdot \bigl(|z|^2 \cdot \kappa\bigr)^{\epsilon_m}\Bigr) + \omega'_m.$$

We have the following:

$$\omega_m = \sqrt{-1}\epsilon_m^{N+2} \cdot |z|^{2\epsilon_m} \cdot \kappa^{\epsilon_m}\left(\frac{dz}{z} + \frac{\partial\kappa}{\kappa}\right)\left(\frac{d\bar{z}}{\bar{z}} + \frac{\overline{\partial}\kappa}{\kappa}\right) + \sqrt{-1}\epsilon_m^{N+1}|z|^{2\epsilon_m} \cdot \kappa^{\epsilon_m}\bigl(\partial\overline{\partial}\log\kappa\bigr) + \omega'_m. \tag{64}$$

Then ω_m is quasi isometric to the following metric, independently of m:

$$\widetilde{\omega}_m := \epsilon_m^{N+2} \cdot |z|^{2\epsilon_m} \cdot \frac{dz \cdot d\bar{z}}{|z|^2} + dw \cdot d\bar{w}$$

Namely, there exists a positive constant C such that $C^{-1}\widetilde{\omega}_m \leq \omega_m \leq C \cdot \widetilde{\omega}_m$.

8.1.2. Statement. — Let $(E_m, \overline{\partial}_{E_m}, \theta_m)$ be Higgs bundles on $\Delta^* \times I^2$, and let h_m be a Hermitian-Einstein metric of $(E_m, \overline{\partial}_{E_m}, \theta_m)$ with respect to ω_m. We assume the following:

- $\Lambda_{\omega_m}F(h_m) = 0$ and $\int |F(h_m)|^2_{h_m,\omega_m} \cdot \mathrm{dvol}_{\omega_m} \leq \delta_m$.
- We have the expression $\theta_m = f_{m,1} \cdot dz/z + f_{m,2} \cdot dw$. Assumption 7.5 holds for $P_w(z,t) := \det(t - f_1(z,w))$ and for any w. The sets S_0 of the solutions $P_w(0,t)$ are independent of w.

We will prove the following proposition in the rest of this section.

Proposition 8.1. — *There exist subset $Z \subset I^2$ with the positive measure, a large number C and a large integer n_0, such that the following holds for any point $P \in Z$:*

- *There exist C^1-orthonormal frames $\boldsymbol{v}_{m,n}$ of $E_{m\,|\,K_n \times \{P\}}$ for any $n \geq n_0$.*

– *Let* $A_{m,n} \cdot dz/z$ *be the connection form with respect to* $\boldsymbol{v}_{m,n}$*, i.e.,* $(\overline{\partial}_{E_m} + \partial_{E_m})\boldsymbol{v}_{m,n} = \boldsymbol{v}_{m,n} \cdot A_{m,n} \cdot dz/z$*. Then* $\sup_{K_n}\big|A_{m,n}\big| \leq C \cdot n^{-2}$*.*

8.1.3. The choice of Z. — For the proof, we will use the notation in the subsection 2.5.2. We denote the projection $\Delta^* \times I^2 \longrightarrow I^2$ by π. The holomorphic coordinates of Δ and I^2 are given by z and w.

Since we have only to take large n_0, we may assume that Assumption 7.6 holds in family. Namely, we may assume that we have the decomposition $E_m = \bigoplus_{\alpha \in S_0} E_{m,\alpha}$ such that $f_{m,i}(E_{m,\alpha}) \subset E_{m,\alpha}$, and hence we have the decomposition $f_{m,i} = \bigoplus f_{m,i,\alpha}$. We may also assume that any eigenvalues β of $f_{m,1,\alpha}$ satisfy $|\alpha - \beta| \leq C \cdot |z|^{\epsilon_0}$ for some constants C and ϵ_0.

We take $E'_{m,i}$, ρ_m and ρ'_m in the section 7.2. For any point $P \in I^2$, we put as follows:

$$G_m(P) := \int \big|F(h_m)\big|^2_{h_m,\omega_m}(z,P) \cdot \epsilon_m^{N+2} \cdot |z|^{2\epsilon_m} \cdot \frac{\sqrt{-1}dz \cdot d\bar{z}}{|z|^2}. \tag{65}$$

Then G_m gives the measurable function on I^2. By applying Lemma 2.15, we obtain the following.

Lemma 8.2. — *Let M be a sufficiently large number. Then there exists a subset $Z \subset I^2$ with the positive measure such that the following holds for any point $P \in Z$:*

– *Take k and l such that $P \in D_{m,n,l}$ and $P \in \widehat{D}_{m,n,k}$. Then we have the following:*

$$\int_{D^1_{m,n,l}} G_m \leq M \cdot \epsilon_m^{N+2} \cdot e^{-2n\epsilon_m}, \qquad \int_{\widehat{D}^1_{m,n,k}} G_m \leq n^2 \cdot M \cdot \epsilon_m^{N+2} \cdot e^{-2n\epsilon_m}.$$

The integrals are taken with respect to the standard Lesbegue measure. Moreover $G_m(P) < M$ holds. □

We will show that Z is the desired set in Proposition 8.1.

8.1.4. Coordinate change. — Let $\varphi : \mathbb{H} \longrightarrow \Delta^*$ be the universal covering given by $\zeta \longmapsto z = e^{\sqrt{-1}\zeta}$. The metric $\varphi^*\omega_m$ is quasi isometric to $\varphi^*\widetilde{\omega}_m = \epsilon_m^{N+2} \cdot e^{-2y\epsilon_m} \cdot d\zeta \cdot d\bar{\zeta} + dw \cdot d\bar{w}$ on $\mathbb{H}$. The subset $K_n^1 \subset \mathbb{H}$ is given by $K_n^1 := \big\{(n-2)\pi < y < (n+2)\pi \,\big|\, -2\pi < x < 3\pi\big\}$. We put as follows:

$$\mathcal{D}_{m,n,l} := \big\{w^\circ = u + \sqrt{-1}v \,\big|\, -1 \leq u \leq 1,\ -1 \leq v \leq 1\big\},$$

$$\mathcal{D}^1_{m,n,l} := \big\{w^\circ = u + \sqrt{-1}v \,\big|\, -3 \leq u \leq 3,\ -3 \leq v \leq 3\big\}.$$

We have the affine map $\psi_{m,n} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ for each (m,n) with $dw^\circ = L_{m,n} \cdot dw$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{D}_{m,n,l} & \xrightarrow{\psi_{m,n}} & D_{m,n,l} \\ \bigcap & & \bigcap \\ \mathcal{D}^1_{m,n,l} & \longrightarrow & D^1_{m,n,l} \end{array}$$

We put $\omega_{m,n} := \psi_{m,n}^{-1}\varphi^*\omega_m$. Because we have $\psi_{m,n}^{-1}\varphi^*\widetilde{\omega}_m = \epsilon_m^{N+2} \cdot e^{-2y\epsilon_m} \cdot d\zeta \cdot d\overline{\zeta} + L_{m,n}^{-2} dw^\circ \cdot d\overline{w}^\circ$ on $K_n^1 \times \mathcal{D}_{m,n,l}^1$, and because $\epsilon_m^{N+2} \cdot e^{-2y\epsilon_m}$ and $L_{m,n}^{-2}$ are very close by our choice, the metrics $\omega_{m,n}$ and $\omega_{m,n}^\circ := L_{m,n}^{-2}\big(d\zeta \cdot d\overline{\zeta} + dw^\circ \cdot d\overline{w}^\circ\big)$ are mutually bounded. Moreover the Laplacians $\Delta_{\omega_{m,n}}$ and $\Delta_{\omega_{m,n}^\circ}$ are sufficiently close.

8.1.5. The estimate of $q_m = \rho_m - \rho_m'$. — A constant is called good in the following, if it is independent of m and n. We put $q_m := \rho_m - \rho_m'$ on $\Delta^* \times I^2$, and the pull back $\psi_{m,n}^{-1}\varphi^{-1}q_m$ is also denoted by q_m.

Lemma 8.3. — *There exists a good constant C_{500} such that the following holds for any $Q \in K_n \times \mathcal{D}_{m,n,l}$:*

$$|q_m|_{h_m}(Q) \leq \frac{C_{500}}{n^2}. \tag{66}$$

Proof We denote $\psi_{m,n}^{-1}\varphi^*\rho_m$ also by $\rho_{m,n}$. We have the subharmonicity, i.e., $\Delta_{\omega_{m,n}} \log|\rho_{m,n}|_{h_m}^2 \leq 0$. Since $\Delta_{\omega_{m,n}}$ is sufficiently close to the Laplacian for $\omega_{m,n}^\circ = L_{m,n}^{-2}\big(d\zeta \cdot d\overline{\zeta} + dw^\circ \cdot d\overline{w}^\circ\big)$, there exists a good constant $C_{501} > 0$ such that the following inequality holds, for any $Q = (Q_1, Q_2) \in K_n \times \mathcal{D}_{m,n,l}$: (see [**15**], for example):

$$\log\big(\xi^{-1}|\rho_{m,n}|_{h_m}^2\big)(Q) \leq C_{501} \cdot \int_{K_n^1 \times \mathcal{D}_{m,n,l}^1} \log^+\big(\xi^{-1}\big|\rho_{m,n}\big|_{h_m}^2\big) \cdot \mathrm{dvol}_{st}\,.$$

On the other hand, there exists a good constant C_{502} and C_{503} such that the following inequality holds for any $(\zeta, w^\circ) \in K_n^1 \times \mathcal{D}_{m,n,l}^1$, due to the result in the section 7.2:

$$\log\big(\xi^{-1}|\rho_{m,n}|_{h_m}^2\big)(\zeta, w^\circ) \leq C_{502} \cdot \frac{\exp\big(C_{503} \cdot G_m(w^\circ)^{1/2}\big)}{n^4}.$$

Therefore, there exist good constants C_{504} and C_{505} such that the following holds for any point $Q \in K_n \times \mathcal{D}_{m,n,l}$:

$$\begin{aligned}
\log\big(\xi^{-1}|\rho_{m,n}|_{h_m}^2\big)(Q) &\leq \frac{C_{504}}{n^4} \int_{\mathcal{D}_{m,n,l}^1} \exp\big(C_{503} \cdot G_m^{1/2}\big) \cdot \sqrt{-1}dw^\circ d\overline{w}^\circ \\
&\leq \frac{C_{504}}{n^4} \cdot \exp\Big(\int_{\mathcal{D}_{m,n,l}^1} C_{503} \cdot G_m^{1/2} \cdot \sqrt{-1}dw^\circ \cdot d\overline{w}^\circ\Big) \\
&\leq \frac{C_{504}}{n^4} \cdot \exp\left(C_{505} \cdot \Big(\int_{\mathcal{D}_{m,n,l}^1} G_m \cdot \sqrt{-1}dw^\circ d\overline{w}^\circ\Big)^{1/2}\right).
\end{aligned} \tag{67}$$

Due to our choice of $D_{m,n,l}^1$, we have the following, for a good constant C_{506}:

$$\int_{\mathcal{D}_{m,n,l}^1} G_m \cdot \sqrt{-1}dw^\circ \cdot d\overline{w}^\circ = \int_{D_{m,n,l}^1} G_m \cdot \sqrt{-1}dw \cdot d\overline{w} \cdot L_{n,m}^2 \leq C_{506} \cdot M.$$

Hence we obtain the following inequality for some good constant C_{507}:

$$\log\big(\xi^{-1}\big|\rho_{m,n}\big|_{h_m}^2\big)(Q) \leq \frac{C_{507}}{n^4}. \tag{68}$$

For any point $Q\in\Delta^*\times I^2$, we take $\widetilde{\rho}_{m,Q}$ as in (55) for $(E_m,\overline{\partial}_{E_m},\theta_m,h_m)$. Then there exist positive constants C_{508} and ϵ_0 such that the following holds:

$$\left|\rho'_Q-\widetilde{\rho}_{m,Q}\right|^2_{h_m}\leq C_{508}\cdot|z(Q)|^{\epsilon_0}.$$

Then it is easy to show the existence of a good constant C_{509} such that the following inequality holds, for any $Q\in\Delta^*\times I^2$:

$$\left|\xi^{-1}|\rho_m(Q)|^2_{h_m}-\bigl(1+|q_m(Q)|^2_{h_m}\cdot\xi^{-1}\bigr)\right|\leq C_{509}\cdot|z(Q)|^{\epsilon_0}\tag{69}$$

Hence we obtain the following inequality for some good constants C_{510} and C_{511}:

$$\left|\xi^{-1}|\rho_m(Q)|^2_{h_m}-\bigl(1+|q_m(Q)|^2_{h_m}\cdot\xi^{-1}\bigr)\right|\leq C_{510}\cdot e^{-C_{511}\cdot n}.$$

Then (66) follows from (68) and (69). □

8.1.6. The estimate of $f_{m,n,1}$. —

Lemma 8.4. — *There exist good positive constants C_{601} and C_{602} such that the following holds, for any $Q\in K_n\times\mathcal{D}_{m,n,l}$:*

$$|f_{m,n,1}(Q)-\rho'_{m,n}(Q)|_h\leq\frac{C_{601}}{n},\qquad|f_{m,n,1}(Q)|_h\leq C_{602}\tag{70}$$

Proof Since we have the subharmonicity $\Delta_{\omega_{m,n}}\log\bigl(|f_{m,n,1}|^2_{h_m}/\xi\bigr)\leq 0$, there exists a good constant C_{603} such that the following inequality holds for any $Q\in K_n\times\mathcal{D}_{m,n,l}$:

$$\log\bigl(|f_{m,n,1}|^2_{h_m}/\xi\bigr)(Q)\leq C_{603}\int_{K_n^1\times\mathcal{D}^1_{m,n,l}}\log^+\bigl(|f_{m,n,1}|^2_{h_m}/\xi\bigr)\cdot\mathrm{dvol}_{st}\,.$$

Due to the result in the section 7.2, there exist good constants C_{604} and C_{605} such that the following inequality holds for any $Q'\in K_n^1\times\mathcal{D}^1_{m,n,l}$:

$$\log\bigl(|f_{m,n,1}|^2_{h_m}/\xi\bigr)(Q')\leq\frac{C_{604}}{n^2}\cdot\exp\bigl(C_{605}\cdot G_m(\pi(Q'))\bigr).$$

Therefore we obtain the following, for any $Q\in K_n\times\mathcal{D}_{m,n,l}$:

$$\log\bigl(|f_{m,n,1}|^2_{h_m}/\xi\bigr)(Q)\leq\frac{C_{603}\cdot C_{604}}{n^2}\int_{\mathcal{D}^1_{m,n,l}}\exp\Bigl(C_{605}\cdot G_m^{1/2}\Bigr)\cdot dw^\circ\cdot d\overline{w}^\circ\leq\frac{C_{606}}{n^2}.\tag{71}$$

Here C_{606} denotes a good constant. Then (70) follows from (71) by an argument similar to Lemma 8.3. □

8.1.7. The estimate of $f_{m,n,2}$. —

Lemma 8.5. — *There exists a good positive constant C_{700} such that the following inequality holds, for any $Q\in K_n\times\mathcal{D}_{m,n,l}$:*

$$\bigl|f_{m,n,2}(Q)\bigr|_{h_m}\leq\frac{C_{700}}{n}.$$

Proof We put as follows:

$$\widehat{\mathcal{D}}_{m,n,k} := \bigl\{ w^{\wedge} = u + \sqrt{-1}v \bigm| -1 \leq u \leq 1, -1 \leq v \leq 1 \bigr\},$$
$$\widehat{\mathcal{D}}^1_{m,n,k} := \bigl\{ w^{\wedge} = u + \sqrt{-1}v \bigm| -3 \leq u \leq 3, -3 \leq v \leq 3 \bigr\}.$$

Similarly, we can take the affine map $\widehat{\psi}_{m,n}$ with $\widehat{L}_{m,n} dw = dw^{\wedge}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathcal{D}}_{m,n,k} & \xrightarrow{\widehat{\psi}_{m,n}} & \widehat{D}_{m,n,k} \\ \bigcap & & \bigcap \\ \widehat{\mathcal{D}}^1_{m,n,k} & \xrightarrow{\widehat{\psi}_{m,n}} & \widehat{D}^1_{m,n,k} \end{array}$$

We have the expression $\theta_m = f_{m,n,1} \cdot d\zeta + \widehat{f}_{m,n,2} \cdot dw^{\wedge}$ in the coordinate $(\zeta, w^{\wedge})$. For a point $P \in \Delta \times I^2$, we take k and l such that $P \in D^1_{m,n,l} \subset \widehat{D}^1_{m,n,k}$. Correspondingly we obtain $\widetilde{\mathcal{D}}^1_{m,n,l} \subset \widehat{\mathcal{D}}^1_{m,n,k}$.

Let $Q' \in K^1_n \times \widetilde{\mathcal{D}}^1_{m,n,l}$ be the point corresponding to $Q \in K^1_n \times \mathcal{D}^1_{m,n,l}$. Since $L_{n,m}/\widehat{L}_{n,m}$ are almost same as n by our construction, there exist good constants C_{701} and C_{702} such that the following holds:

$$C_{701} \cdot |f_{m,n,2}(Q)|_{h_m} \leq n^{-1} \cdot |\widehat{f}_{m,n,2}|_{h_m}(Q') \leq C_{702} \cdot |f_{m,n,2}(Q)|_{h_m}. \tag{72}$$

For any $\zeta \in K^1_n$, we have the following expression:

$$\widehat{\psi}^*_{m,n} \varphi^* F(h_m)_{|\{\zeta\} \times \widehat{\mathcal{D}}^1_{m,n,l}}(w^{\wedge}) = B_{m,n}(\zeta, w^{\wedge}) \cdot dw \wedge d\overline{w}^{\wedge}.$$

We put as follows:

$$J_{m,n}(\zeta) := \int_{\widehat{\mathcal{D}}^1_{m,n,l}} \bigl| B_{m,n}(\zeta, w^{\wedge}) \bigr|^2 (\zeta, w^{\wedge}) \cdot \sqrt{-1} \cdot dw^{\wedge} \cdot d\overline{w}^{\wedge}.$$

Due to the result in the section 7.1, there exist good positive constants C_{703} and C_{704} such that the following estimates holds for any $(\zeta, w^{\wedge}) \in K^1_n \times \widetilde{\mathcal{D}}^1_{m,n,l}$:

$$\log \bigl| \widehat{f}_{m,n,2} \bigr|_{h_m} (\zeta, w^{\wedge}) \leq C_{703} \cdot J_{m,n}(\zeta)^{1/2} + C_{704}.$$

On the other hand, we have the subharmonicity $\Delta_{\omega_{m,n}} \log \bigl(\bigl| n \cdot f_{m,n,2} \bigr|^2_{h_m} \bigr) \leq 0$ on $K^1_n \times \mathcal{D}^1_{m,n,l}$. Hence we obtain the following inequality for any $Q \in K_n \times \mathcal{D}_{m,n,l}$:

$$\log \bigl| n \cdot f_{m,n,2} \bigr|^2_{h_m}(Q) \leq C_{705} \cdot \int_{K^1_n \times \mathcal{D}^1_{m,n,l}} \log^+ \bigl| n \cdot f_{m,n,2} \bigr|^2_{h_m} \cdot \mathrm{dvol}_{st} \tag{73}$$

$$\leq C_{706} \cdot \int_{K^1_n} J^{1/2}_{m,n} \cdot \sqrt{-1} d\zeta \cdot d\overline{\zeta} + C_{707} \leq C_{708} \cdot \left(\int_{K^1_n} J_{m,n} \cdot \sqrt{-1} d\zeta \cdot d\overline{\zeta} \right)^{1/2} + C_{709}.$$

Here C_i denote the good constants. We have the following:

$$\int_{K^1_n} J_{m,n} \cdot \sqrt{-1} d\zeta \cdot d\overline{\zeta} = \int_{K^1_n \times \widehat{\mathcal{D}}^1_{m,n,l}} |B_{m,n}|^2_{h_m} \sqrt{-1} d\zeta \cdot d\overline{\zeta} \cdot \sqrt{-1} dw^{\wedge} \cdot d\overline{w}^{\wedge}.$$

We have $B_{m,n} \cdot dw^{\wedge} \cdot d\overline{w}^{\wedge} = B_{m,n} \cdot L_{m,n}^2 \cdot dw \cdot d\overline{w}$. Hence we obtain the following:

$$\big|B_{m,n}\big|_{h_m}^2 \leq \widehat{L}_{m,n}^{-4} \cdot \big|F(h_m)\big|_{h_m,\omega_m}^2.$$

Here the right hand side is the norm with respect to h_m and ω_m. Since we have $dw^{\wedge} \wedge d\overline{w}^{\wedge} = dw \wedge d\overline{w} \cdot \widehat{L}_{m,n}^2$, we have good constants C_{710} such that the following holds:

$$C_{710} \cdot d\,\mathrm{dvol}_{\omega_m} \cdot L_{n,n}^2 \cdot \widehat{L}_{m,n}^2 \leq \sqrt{-1} d\zeta \wedge d\overline{\zeta} \wedge \sqrt{-1} dw^{\wedge} \wedge d\overline{w}^{\wedge} \leq C_{710}^{-1} \cdot d\,\mathrm{dvol}_{\omega_m} \cdot L_{n,n}^2 \cdot \widehat{L}_{m,n}^2.$$

Therefore we obtain the following, for some good constants:

$$(74) \quad \int_{K_n^1} J_{m,n} \cdot \sqrt{-1} d\zeta \cdot d\overline{\zeta} \leq C_{711} \cdot \frac{L_{m,n}^2}{\widehat{L}_{m,n}^2} \int_{K_n^1 \times \widehat{D}_{m,n,l}^1} \big|F(h_m)\big|_{h_m,\omega_m}^2 \mathrm{dvol}_{\omega_m}$$
$$\leq C_{712} \cdot n^2 \int_{K_n^1 \times \widehat{D}_{m,n,l}^1} \big|F(h_m)\big|_{h_m,\omega_m}^2 \cdot \mathrm{dvol}_{\omega_m} \leq C_{713} \cdot n^4 \cdot \epsilon_m^{N+2} \cdot e^{-2n\epsilon_m}.$$

Lemma 8.6. — *There exists a good constant C_{714} such that the following inequality holds:*

$$n^4 \cdot \epsilon_m^{N+2} \cdot e^{-2n\epsilon_m} \leq C_{714} \cdot \epsilon_m^{N-2}$$

Proof We put $f(t) = t^4 \epsilon^{N+2} e^{-2t\epsilon}$. We have

$$f'(t) = 4t^3 \epsilon^{N+2} e^{-2t\epsilon} + t^4 \epsilon^{N+2}(-2\epsilon) e^{-2t\epsilon} = t^3 \epsilon^{N+2} e^{-2t\epsilon}(4 - 2t\epsilon) = 0.$$

Hence $f'(t) = 0$ if and only if $t = 2\epsilon^{-1}$. Hence we obtain $f(t) \leq 16\epsilon^{N-2} e^{-4}$. □

Now (72) is immediately obtained. □

8.1.8. The end of the proof of Proposition 8.1. —

Lemma 8.7. — *In all, we obtain the following inequalities, $(\alpha \neq \beta)$:*

$$\big|f_{m,n,1,\alpha,\alpha} - \alpha\big|_{h_m} \leq \frac{C}{n}, \quad \big|f_{m,n,1,\alpha,\alpha}^{\dagger} - \overline{\alpha}\big|_{h_m} \leq \frac{C}{n},$$
$$\big|f_{m,n,1,\alpha,\beta}\big|_{h_m} \leq \frac{C}{n^2}, \quad \big|f_{m,n,1,\alpha,\beta}^{\dagger}\big|_{h_m} \leq \frac{C}{n^2},$$
$$\big|f_{m,n,2,\alpha,\alpha}\big|_{h_m} \leq \frac{C}{n}, \quad \big|f_{m,n,2,\alpha,\alpha}^{\dagger}\big|_{h_m} \leq \frac{C}{n},$$
$$\big|f_{m,n,2,\alpha,\beta}\big|_{h_m} \leq \frac{C}{n^3}, \quad \big|f_{m,n,2,\alpha,\beta}^{\dagger}\big| \leq \frac{C}{n^3}.$$

Proof Let us see the estimate for $f_{m,n,1,\alpha,\beta}$. The first inequality immediately follows from Lemma 8.4. The third inequality follows from Lemma 8.4, Lemma 8.3 and $[f_{m,1}, \rho'_m] = [f_{m,1}, \rho'_m - \rho_m]$. The estimates for $f_{m,n,1,\alpha,\beta}^{\dagger}$ immediately follow from those for $f_{m,n,1,\alpha,\beta}$. The estimates for $f_{m,n,2,\alpha,\beta}$ and their adjoints can be shown similarly by using Lemma 8.3, Lemma 8.5 and $[f_{m,2}, \rho'_m] = [f_{m,2}, \rho'_m - \rho_m]$. □

Let us return to the proof of Proposition 8.1. From Lemma 8.7, we obtain the following estimate on $K_n \times \mathcal{D}_{m,n,l}$:

$$\big[\theta_m, \theta_m^{\dagger}\big] = O\left(\frac{d\zeta \cdot d\overline{\zeta} + dw \cdot d\overline{w}}{n^2}\right).$$

We have $R(h_m) = -\big[\theta_m, \theta_m^{\dagger}\big] + F(h_m)$ and $\Lambda_{\omega_m} R(h_m) = -\Lambda_{\omega_m}\big[\theta_m, \theta_m^{\dagger}\big]$. We have the following on $K_n \times \mathcal{D}_{m,n,l}$, with respect to the standard metric of $A \times \mathcal{D}_{m,n,l}$:

$$\big\|R(h_m)\big\|_{L^2,\omega_m,h_m} \leq \frac{C}{n^2}, \qquad \sup\Big|\Lambda_{\omega_m} R(h_m)\Big|_{h_m} \leq \frac{C}{n^2}.$$

If m is sufficiently large, then the metric $L_{n,m}^{-2} \cdot \omega_{m,n}$ is sufficiently close to the standard metric of $K_n \times \mathcal{D}_{m,n,l}$. By using the argument in [**12**], we obtain the following lemma.

Lemma 8.8. — *Let p be any sufficiently large number. Let P be any point of Z. Take (n,m,l) such that $P \in D_{n,m,l}$. There exist constants $M > 0$ and L_2^p-orthonormal frames $\boldsymbol{v}_{m,n}$ of $E_{m\,|\,K_n \times \mathcal{D}_{m,n,l}}$ for which the following holds:*

- *Let $A_{m,n} \cdot dz/z + A'_{m,n} \cdot dw^{\circ}$ be the connection form with respect to $\boldsymbol{v}_{m,n}$, i.e., $(\overline{\partial}_{E_m} + \partial_{E_m})\boldsymbol{v}_{m,n} = \boldsymbol{v}_{m,n} \cdot \big(A_{m,n} \cdot dz/z + A'_{m,n} \cdot dw^{\circ}\big)$. Then $\big|A_{m,n}\big|_{L_1^p} \leq M \cdot n^{-2}$ holds.* □

Recall that L_i^p implies C^{i-1} if p is sufficiently large. By specializing the frames to the curves $K_n \times \{P\}$, we obtain Proposition 8.1. □

8.2. Uhlenbeck type theorem on a punctured disc

8.2.1. Some notation. — We use the notation in the section 8.1. We put $\overline{K}_n = \varphi(K_n)$. We put as follows:

$$K_{n,1} := \big\{x + \sqrt{-1}y \in K_n \,\big|\, -\pi < x < 0\big\}, \quad K_{n,2} := \big\{x + \sqrt{-1}y \in K_n \,\big|\, \pi < x < 2\pi\big\}.$$

We put $\overline{L}_n := \varphi\big(K_{n,1}\big) = \varphi\big(K_{n,2}\big)$. We also put as follows:

$$M_n := \Big\{(x,y) \in K_n \,\Big|\, -\frac{2}{3}\pi < x < \frac{5}{3}\pi\Big\}, \quad \widehat{M}_n := \big\{(x,y) \in M_n \,\big|\, y < (n + 2^{-1})\pi\big\},$$

$$M_{n,1} := \Big\{(x,y) \in K_n \,\Big|\, -\frac{2}{3}\pi < x < -\frac{1}{3}\pi\Big\}, \quad M_{n,2} := \Big\{(x,y) \in K_n \,\Big|\, \frac{4}{3}\pi < x < \frac{5}{3}\pi\Big\}.$$

In this section, we use the Euclidean metric $g = dx \cdot dx + dy \cdot dy$ on the upper half plane $\mathbb{H}$.

8.2.2. Statement. — Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle on Δ^* with a hermitian metric h. We put $\nabla := \partial_E + \overline{\partial}_E$.

Assumption 8.9. — Assume that we are given C^1-orthonormal frames $\boldsymbol{v}_n$ of $\varphi^* E_{|K_n}$ satisfying the following:

– Let A_n denote the connection one form of ∇ with respect to $\boldsymbol{v}_n$ on K_n. Then the norm of A_n is dominated as follows:

$$|A_n|_{h,g} \leq \frac{C}{n^2}. \tag{75}$$

Here the norm of A_n is taken with respect to h and the standard metric $g = dx \cdot dx + dy \cdot dy$ of $\mathbb{H}$. □

It is the purpose in this section to show the following proposition.

Proposition 8.10. — *There exists a constant γ_0 and a C^1-orthonormal frame $\boldsymbol{w}$ of E on $\Delta^*(\gamma_0)$ for which $\overline{\partial}_E$ is expressed as follows:*

$$\overline{\partial}_E \boldsymbol{w} = \boldsymbol{w} \cdot \Big(A - \frac{1}{2}\Gamma\Big) \cdot \frac{d\bar{z}}{\bar{z}}$$

– Γ *is a constant diagonal matrix whose (i,i)-th components α_i satisfy $0 \leq \alpha_r \leq \alpha_{r-1} \leq \cdots \leq \alpha_1 < 1$.*
– $|A| \leq C_1 \cdot \big(-\log|z|\big)^{-1}$.
– $\sup_K |A| \leq C_2(K)$ *for any compact subset $K \subset \Delta^*(\gamma_0)$.*

The constants γ_0, C_1 and $C_2(K)$ depends only on C.

8.2.3. Proof. — In the following argument in this section, "a constant C' is good" means that a constant C' depends only on the constant C. Similarly, 'a constant $C'(K)$ is good' means that a constant $C'(K)$ depends only on the constant C and a given compact subset $K \subset \Delta^*$.

Let $s_n : K_n \cap K_{n-1} \longrightarrow U(r)$ be the function determined by $\boldsymbol{v}_{n-1} = \boldsymbol{v}_n \cdot s_n$.

Lemma 8.11. — *s_n is C^1, and $|ds_n|_{C^0} \leq C_3 \cdot n^{-2}$ for some good constant C_3.*

Proof It follows from (75) and the relation $ds_n = s_n \cdot A_n - A_{n-1} \cdot s_n$. □

The following lemma can be shown easily.

Lemma 8.12. — *There exist good constants N_1 and C_4, such that we have the expression $s_n = S_n \cdot \exp(\widetilde{s}_n)$ for each $n \geq N_1$. Here $\widetilde{s}_n : K_n \cap K_{n-1} \longrightarrow \mathfrak{u}(r)$ satisfies $|\widetilde{s}_n|_{C^1} \leq C_4 \cdot n^{-2}$, and S_n denotes an element of $U(r)$.* □

Let us consider $\boldsymbol{v}'_n = \boldsymbol{v}_n \cdot S_n \cdot S_{n-1} \cdots \cdot S_{N_1}$ instead of $\boldsymbol{v}_n$. Then we have the function $s'_n : K_n \cap K_{n-1} \longrightarrow U(r)$ satisfying $\boldsymbol{v}'_n = \boldsymbol{v}'_{n-1} \cdot \exp(s'_n)$ and $|s'_n|_{C^1} \leq C_4 \cdot n^{-2}$. The connection one form A'_n of ∇ with respect to the frame $\boldsymbol{v}'_n$ satisfies the estimate (75). Hence we may and will assume $S_n = 1$.

We put $\boldsymbol{v}_{n,a} := \boldsymbol{v}_{n\,|\,K_{n,a}}$. Via the identification $K_{n,1} \simeq \overline{L}_n \simeq K_{n,2}$, both of $\boldsymbol{v}_{n,a}$ give the frames of $p^*E_{|K_{n,1}}$, which are denoted by the same notation. Let $t_n : K_{n,1} \longrightarrow U(r)$ be the function determined by $\boldsymbol{v}_{n,1} = \boldsymbol{v}_{n,2} \cdot t_n$. As in Lemma 8.11, it can be shown that t_n is C^1 and that $|dt_n|_{C^0} \leq C_5 \cdot n^{-2}$ for some good constant C_5. The following lemma is easy.

Lemma 8.13. — *There exist good constants N_2 and C_6 such that we have the expression $t_n = T_n \cdot \exp\bigl(\widetilde{t}_n\bigr)$ for any $n \geq N_2$. Here $\widetilde{t}_n : K_{n,1} \longrightarrow \mathfrak{u}(r)$ satisfies $\bigl|\widetilde{t}_n\bigr|_{C^1} \leq C_6 \cdot n^{-2}$, and T_n denotes an element of $U(r)$.* □

Let $d_{U(r)}$ denote the canonical distance on the unitary group $U(r)$.

Lemma 8.14. — *There exists a good constant C_7 such that the following holds for any $n, m \geq N_2$:*

$$d_{U(r)}(T_n, T_m) \leq C_7 \cdot \max\left(\frac{1}{n}, \frac{1}{m}\right).$$

Proof The distance of the monodromies with respect to the two loops contained in $\overline{K}_n$ can be dominated by $C_7' \cdot n^{-2}$ for some good constant C_7'. Hence the difference of the monodromies with respect to the two loops contained in $\overline{K}_n$ and $\overline{K}_m$ are dominated by $C_7'' \cdot \max\bigl\{n^{-1}, m^{-1}\bigr\}$. Then the claim immediately follows. □

We put $T := \lim_{n\to\infty} T_n$. We take $B \in U(r)$ such that $B^{-1}TB = \exp\bigl(2\pi\sqrt{-1}\cdot\Gamma\bigr)$, where Γ is the diagonal matrix whose (i,i)-th components α_i satisfies $0 \leq \alpha_r \leq \cdots \leq \alpha_1 < 1$. Since we can consider $\boldsymbol{v}_n' = \boldsymbol{v}_n \cdot B$ instead of $\boldsymbol{v}_n$, we may and will assume $T = \exp\bigl(2\pi\sqrt{-1}\Gamma\bigr)$.

Lemma 8.15. — *There are good constants N_3 and C_8 such that we have the function $\check{t}_n : K_{n,1} \longrightarrow \mathfrak{u}(r)$ satisfying $t_n = T \cdot \exp\bigl(\check{t}_n\bigr)$ and $\bigl|\check{t}_n\bigr|_{C^1} \leq C_8 \cdot n^{-1}$ for any $n \geq N_3$.* □

We put $\beta(x,y) := \exp\bigl(x\sqrt{-1}\Gamma\bigr)$ and $\widehat{\boldsymbol{v}}_n := \boldsymbol{v}_n \cdot \beta$. We put $\widehat{\boldsymbol{v}}_{n,a} := \widehat{\boldsymbol{v}}_{n\,|\,K_{n,a}}$, and we regard them as the frames of $p^*E_{|K_{n,1}}$. We put $\widehat{t}_n := \beta^{-1}_{|K_{n,1}} \cdot \check{t}_n \cdot \beta_{|K_{n,1}}$. Then it is easy to check that $\widehat{\boldsymbol{v}}_{n,1} = \widehat{\boldsymbol{v}}_{n,2} \cdot \exp\bigl(\widehat{t}_n\bigr)$, and thus we have $\bigl|\widehat{t}_n\bigr|_{C^1} \leq C_9 \cdot n^{-1}$ for some good constant C_9.

We put $\widehat{s}_n = \beta^{-1} \cdot \widetilde{s}_n \cdot \beta$, and then we have $\widehat{\boldsymbol{v}}_{n-1} = \widehat{\boldsymbol{v}}_n \cdot \exp\bigl(\widehat{s}_n\bigr)$, and the estimates $\bigl|\widehat{s}_n\bigr|_{C^1} \leq C_{10} \cdot n^{-2}$ for some good constant C_{10}. We put $\widehat{A}_n := \beta^{-1} \cdot A_n \cdot \beta + \sqrt{-1}\Gamma \cdot dx$. Then we have the relation $\nabla\widehat{\boldsymbol{v}}_n = \widehat{\boldsymbol{v}}_n \cdot \widehat{A}_n$ and the estimates $\bigl|\widehat{A}_n - \sqrt{-1}\Gamma \cdot dx\bigr|_{C^0} \leq C_{11} \cdot n^{-2}$.

Take a C^∞-function $\chi : \mathbb{H} \longrightarrow [0,1]$ satisfying the following:

$$\chi(x,y) = \begin{cases} 1 & (x \leq -3^{-1}\cdot\pi), \\ 0 & (x \geq 0). \end{cases}$$

We put $\Psi_n := \exp\bigl(-x \cdot \widehat{t}_n\bigr)$, and $\boldsymbol{u}_n := \widehat{\boldsymbol{v}}_n \cdot \Psi_{n\,|\,M_n}$. Under the natural identification $M_{n,1} \simeq M_{n,2}$, we have the following:

$$\boldsymbol{u}_{n\,|\,M_{n,2}} = \widehat{\boldsymbol{v}}_{n,2} = \widehat{\boldsymbol{v}}_{n,1} \cdot \exp\bigl(-\widehat{t}_n\bigr) = \boldsymbol{u}_{n\,|\,M_{n,1}}.$$

Hence $\boldsymbol{u}_n$ gives the orthogonal frame of $E_{|\overline{K}_n}$. The following lemma is easy to see.

Lemma 8.16. — *There exist good constants N_4 and C_{12} such that the following holds:*

- *We have the functions $s'_n : K_n \cap K_{n-1} \longrightarrow \mathfrak{u}(r)$ satisfying $\boldsymbol{u}_{n-1} = \boldsymbol{u}_n \cdot \exp\bigl(s'_n\bigr)$ and $\bigl|s'_n\bigr|_{C^1} \leq C_{12} \cdot n^{-1}$ for any $n \geq N_4$.*
- *The connection forms B_n of ∇ with respect to the frames $\boldsymbol{u}_n$ satisfy $\bigl|B_n - \sqrt{-1} \cdot \Gamma \cdot dx\bigr|_{C^0} \leq C_{12} \cdot n^{-1}$.* □

Let us take a C^∞-function $\rho_n : \mathbb{H} \longrightarrow [0,1]$ satisfying the following:

$$\rho_n(x,y) := \begin{cases} 0 & \bigl(y \leq (n-2^{-1})\pi\bigr), \\ \\ 1 & (y \geq n\pi). \end{cases}$$

We put $\boldsymbol{w}_n := \boldsymbol{u}_n \cdot \exp\bigl(\rho_n \cdot s'_n\bigr)$ on $\widehat{M}_n$. Let $\widehat{B}_n$ denote the connection one form of ∇ with respect to $\boldsymbol{w}_n$. Then we have the estimate $\bigl|\widehat{B}_n - \sqrt{-1}\Gamma dx\bigr|_{C^0} \leq C_{14} \cdot n^{-1}$ for some good constant C_{14}.

On the intersection $\widehat{M}_n \cap \widehat{M}_{n-1} = \bigl\{(x,y) \in \widehat{M}_n \bigm| (n-1)\pi < y < (n-2^{-1})\pi\bigr\}$, we have the following relation:

$$\boldsymbol{w}_{n-1} = \boldsymbol{u}_{n-1} = \boldsymbol{u}_n \cdot \exp\bigl(s'_n\bigr) = \boldsymbol{w}_n.$$

Hence $\bigl\{\boldsymbol{w}_n\bigr\}$ gives the orthogonal frame $\boldsymbol{w}$ of E on $\Delta^*(\gamma_1)$ for some good constant γ_1. Let B be a connection one form of ∇ with respect to the frame $\boldsymbol{w}$. Let us denote p^*B as $B^x \cdot dx + B^y \cdot dy$. Then there exist good constants C_{15} and N_5 such that the following holds on $\bigl\{y > N_5\bigr\}$:

$$\bigl|B^x - \sqrt{-1}\Gamma\bigr| \leq C_{15} \cdot y^{-1}, \quad \bigl|B^y\bigr| \leq C_{15} \cdot y^{-1}.$$

We have the following formula on Δ^*:

$$\overline{\partial}\boldsymbol{w} = \boldsymbol{w} \cdot \left(-\frac{1}{2}\Gamma + \frac{\sqrt{-1}}{2}\bigl(B^x - \sqrt{-1}\Gamma\bigr) - \frac{1}{2}B^y\right) \cdot \frac{d\bar{z}}{\bar{z}}.$$

Then $\boldsymbol{w}$ gives the frame desired. Therefore the proof of Proposition 8.10 is accomplished. □

8.3. Construction of local holomorphic frames

8.3.1. Setting. — Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on Δ^*. Let h be a hermitian metric of E. Let $R(h)$ and $F(h)$ denote the curvatures of the connections $\nabla = \partial_E + \overline{\partial}_E$ and $\nabla + \theta + \theta^\dagger$. We have the relation $R(h) = F(h) - \bigl[\theta, \theta^\dagger\bigr]$. The metrics g, $\widetilde{g}$ and g_ϵ are given as follows:

$$g := dz \cdot d\bar{z}, \quad \widetilde{g} := \frac{dz \cdot d\bar{z}}{|z|^2 \cdot \bigl(-\log|z|\bigr)^2}, \quad g_\epsilon = \epsilon^{M+2} \cdot |z|^{2\epsilon} \cdot \frac{dz \cdot d\bar{z}}{|z|^2}.$$

Here ϵ is a small positive number, and M is a positive number such that $M > 10$.

Assumption 8.17. — Assume the following.

– $F(h)$ is L^2 with respect to the metrics h and g_ϵ:

$$\|F(h)\|_{L^2,g_\epsilon} := \left(\int |F(h)|_{h,g_\epsilon} \cdot \mathrm{dvol}_{g_\epsilon}\right)^{1/2} \leq C_1.$$

– The norm of $[\theta,\theta^\dagger]$ with respect to h and g is dominated as follows:

$$\big|[\theta,\theta^\dagger]\big|_{h,g} \leq C_2 \cdot \frac{1}{|z|^{-2}\cdot\big(-\log|z|\big)^2}.$$

– There exists C^1-orthonormal frame $\boldsymbol{v}$ of E, for which $\overline{\partial}_E$ is represented as follows:

$$\overline{\partial}_E \boldsymbol{v} = \boldsymbol{v}\cdot\left(-\frac{\Gamma}{2}+A\right)\cdot\frac{d\bar{z}}{\bar{z}}$$

Here Γ is a constant diagonal matrix whose (i,i)-th components α_i satisfy $0 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq 1$, and A is a matrix-valued continuous function such that $|A| \leq C_3 \cdot (-\log|z|)^{-1}$. □

Remark 8.18. — The conjugacy class of $\exp\big(\sqrt{-1}\Gamma\big)$ is determined for $(E,\overline{\partial}_E,\theta,h)$, because it is characterized as the limit of the monodromy around the loops $\big\{z \,\big|\, |z| = r\big\}$ for $r \longrightarrow 0$. In particular, the numbers $\alpha_1,\ldots,\alpha_r$ are uniquely determined. □

In the following of this section, we say a constant C is good if it depends only on C_i $(i=1,2,3)$. When a compact set K is given, we say a constant $C(K)$ is good if it depends only on C_i $(i=1,2,3)$ and K.

8.3.2. Rank 1 case. — First, let us consider the case where $\operatorname{rank} E = 1$. We have $R(h) = F(h) \in \Omega^{1,1}_{\Delta^*}$ in this case, and it is L^2 with respect to the metric g_ϵ of Δ^*. Due to the result in the subsection 2.5.1, we can take the function τ on Δ^*, which satisfies the following:

$$\partial\overline{\partial}\tau = R(h), \qquad |\tau| \leq C\cdot\epsilon^{(N-1)/2}\cdot|z|^\epsilon\cdot\|F(h)\|_{L^2,g_\epsilon},$$

$$\big|\partial\tau\big|_g \leq C\cdot\epsilon^{(N+1)/2}\cdot|z|^{\epsilon-1}\cdot\big\|F(h)\big\|_{L^2,g_\epsilon}.$$

Here C is a constant which is independent of $(E,\overline{\partial}_E,h,\theta)$. We put $\widetilde{h} := h\cdot e^\tau$, then $\widetilde{h}$ gives a flat metric. Hence we have a holomorphic section s of E on Δ^* and the real number α, such that the following holds:

$$|s|_{\widetilde{h}} = |z|^{-\alpha}, \quad \partial_{\widetilde{h}} s = -\alpha\cdot s\cdot\frac{dz}{z}.$$

The number α is uniquely determined up to integers. On the other hand, let v be a frame of E as in Assumption 8.17, then we have $\overline{\partial}v = v\cdot(-2^{-1}\alpha' + A)\cdot d\bar{z}/\bar{z}$ for some $\alpha' \in \boldsymbol{R}$ and a function A such that $|A| = O\big((-\log|z|)^{-1}\big)$. It is easy to see that $\alpha \equiv \alpha'$ modulo $\mathbb{Z}$, due to Remark 8.18.

Once we fix α, then s is unique up to multiplication of complex numbers whose absolute values are 1. We have good positive constants B_i $(i=1,2,3)$ such that the following holds:

$$B_1\cdot|z|^{-\alpha}\leq|s|_h\leq B_2\cdot|z|^{-\alpha},\qquad \partial_h s=-\bigl(\alpha+A'\bigr)\cdot s\cdot\frac{dz}{z},\qquad |A'|\leq\frac{B_3}{|z|\cdot(-\log|z|)}.$$

8.3.3. Statement. — Let us consider the case where $\operatorname{rank}E$ is arbitrary. First we recall some results due to Simpson's theory ([**45**] and [**46**]). From $(E,\overline{\partial}_E,h)$, we obtain the prolongmemt ${}_cE={}_cE(h)$ as is explained in the section 3.3, and it was shown that ${}_cE$ is locally free for any $c\in\boldsymbol{R}$. Thus we have the parabolic filtration F of ${}_cE_{|O}$, the sets $\mathcal{P}ar({}_cE):=\bigl\{b\bigm|\operatorname{Gr}^F_b({}_cE)\neq0\bigr\}$ and $\mathcal{P}ar(E):=\bigcup_c\mathcal{P}ar({}_cE)$. For any a, we put $\mathfrak{m}(b):=\dim\operatorname{Gr}^F_b({}_aE_{|O})$ for $b<a\leq b+1$, and we put as follows:

$$\widetilde{a}:=\sum_{b\in\mathcal{P}ar({}_aE)}\mathfrak{m}(b)\cdot b.$$

It is easy to see that the determinant bundle $\bigl(\det(E),\overline{\partial},\det(h),\operatorname{tr}(\theta)\bigr)$ also satisfies Assumption 8.17. It can be shown that the parabolic structure is compatible with the operation taking the determinant, in the following sense $\det\bigl({}_aE\bigr)={}_{\widetilde{a}}\det(E)$ and $\mathcal{P}ar\bigl(\det(E)\bigr)=\bigl\{\widetilde{a}+n\bigm|n\in\mathbb{Z}\bigr\}$. Similarly, it can be shown that the operation taking the dual is compatible with the parabolic structure. (See [**45**] and [**46**] for more detail.)

By applying the result in the subsection 8.3.2 to $\bigl(\det(E),\overline{\partial},\det(h),\operatorname{tr}\theta\bigr)$ which is of rank one, there exist good constants B_i $(i=1,2,3)$, $\widehat{B}_i$ $(i=1,2)$ and the holomorphic section s of $\det(E)$ such that the following holds:

$$B_1\cdot|z|^{-\widetilde{a}}\leq|s|_h\leq B_2\cdot|z|^{-\widetilde{a}},\qquad \partial_{\det(E)}s=\bigl(-\widetilde{a}+A'\bigr)\cdot s\cdot\frac{dz}{z},\quad |A'|_h\leq\frac{B_3}{(-\log|z|)}.$$

Let a be any real number such that $a\notin\mathcal{P}ar(E)$. We impose an additional condition.

Assumption 8.19. — $\epsilon<\operatorname{gap}\bigl({}_aE_*\bigr)/10$. □

Proposition 8.20. — *Fix a sufficiently small positive number η such that $\eta<\epsilon/2$. Fix some number $p>2$.*

- *There exist good constants γ_0, C_{10} and N, and there exist holomorphic sections $F_1,\ldots,F_r$ of ${}_aE$ on $\Delta(\gamma_0)$ and the numbers $a_1,\ldots,a_r$, such that the following holds:*

$$\int_{\Delta(\gamma_0)}|F_i|_h^2\cdot|z|^{2a_i+\eta}\cdot\bigl(-\log|z|\bigr)^N\cdot\mathrm{dvol}_{\widetilde{g}}\leq C_{10}.\tag{76}$$

- *Let s be the holomorphic section of ${}_{\widetilde{a}}\det(E)$ as in the subsection* 8.3.2. *The holomorphic function F on $\Delta(\gamma_0)$ given by $F_1\wedge\cdots\wedge F_r=F\cdot s$ satisfies $L_1\leq|F|\leq L_2$ for some good constants L_i $(i=1,2)$.*

– *Let K be any compact subset of Δ^*. There exists a good constant $C_{20}(K)$ such that $||F||_{L_1^p(K)} \leq C_{20}(K)$.*

Remark 8.21. — The second condition implies that $F_i \in {}_{a_i}E$, and $F_1, \ldots, F_r$ give holomorphic frame of ${}_aE$. □

8.3.4. Preliminary for a proof. — We recall some result on the solvability of the $\overline{\partial}$-equation, based on an idea contained in [**5**]. See [**37**] and [**38**] for more detail. Let $\langle \cdot, \cdot \rangle_{h,\widetilde{g}}$ and $\| \cdot \|_{h,\widetilde{g}}$ denote the naturally defined hermitian inner product and the L^2-norm for the sections of $E \otimes \Omega^{p,q}$ with respect to h and the Poincaré metric $\widetilde{g}$. On the other hand, $(\cdot, \cdot)_{h,\widetilde{g}}$ and $| \cdot |_{h,\widetilde{g}}$ denote the inner product and the norm on the fibers.

We have the unitary connection $\widetilde{\nabla}$ of $E \otimes \Omega^{0,1}$ induced by h and $\widetilde{g}$. Then we have the following formula for any section ρ of $E \otimes \Omega^{0,1}$:

$$\big\langle \overline{\partial}^*_E \rho, \overline{\partial}^*_E \rho \big\rangle_{h,\widetilde{g}} = \big\langle \widetilde{\nabla}'' \rho, \widetilde{\nabla}'' \rho \big\rangle_{h,\widetilde{g}} + \sqrt{-1} \cdot \big\langle \Lambda_{\widetilde{g}} \big(R(\Omega^{0,1}) + R(h) \big) \rho, \rho \big\rangle_{h,\widetilde{g}}.$$

If we put $\widetilde{h} = h \cdot e^{-\chi}$ for some function χ, then we obtain the following:

$$\begin{aligned}(77) \quad \big\langle \overline{\partial}^*_E \rho, \overline{\partial}^*_E \rho \big\rangle_{\widetilde{h},\widetilde{g}} = \big\langle \widetilde{\nabla}'' \rho, \widetilde{\nabla}'' \rho \big\rangle_{\widetilde{h},\widetilde{g}} + \sqrt{-1} \big\langle \Lambda_{\widetilde{g}} \big(R(\Omega^{0,1}_Y) + R(h) \big) \rho, \rho \big\rangle_{\widetilde{h},\widetilde{g}} \\ - \sqrt{-1} \big\langle \Lambda_{\widetilde{g}} (\overline{\partial} \partial \chi) \rho, \rho \big\rangle_{\widetilde{h},\widetilde{g}}.\end{aligned}$$

Due to Assumption 8.17, there exist good constants B_i ($i = 4, 5$) such that the following *pointwize* inequality holds:

$$\Big| \big(\Lambda_{\widetilde{g}} R(h) \rho, \rho \big)_{\widetilde{h},\widetilde{g}} \Big| \leq B_4 \cdot \big|\rho\big|^2_{\widetilde{h}} + B_5 \cdot \big| \Lambda_{\widetilde{g}} F(h) \big|_h \cdot \big|\rho\big|^2_{\widetilde{h}}.$$

Due to the finiteness of $\|F(h)\|_{h,g_\epsilon}$ and the result in the subsection 2.5.1, we have the function τ satisfying the following for some good constants B_6 and B_7 (the signature does not matter):

$$\frac{\partial^2 \tau}{\partial z \partial \bar{z}} = \pm B_5 \cdot |\Lambda_g F(h)|_h \quad |\tau| \leq B_6 \cdot \epsilon^{(M-1)/2} \cdot |z|^\epsilon, \quad \big|\partial \tau\big|_{g_\epsilon} \leq B_7 \cdot \epsilon^{(M+1)/2}.$$

We note $\sqrt{-1} \Lambda_{\widetilde{g}} \partial \overline{\partial} \tau = \pm B_5 \cdot \big| \Lambda_{\widetilde{g}} F(h) \big|_h$. Let b be any real number. Let N be a good constant such that $-N$ is sufficiently larger than B_4. We put $\chi(b, N) := b \cdot \log|z| - N \cdot \log\big(-\log|z|^2\big)$ and $h(b, N, \tau) := h \cdot e^{\chi(b,N)+\tau}$ and $\widetilde{g}$. Due to (77) and our choices of τ and N, there exists a good constant B_8 such that the following inequality holds for any L^2-section ρ of $E \otimes \Omega^{0,1}$:

$$\big\| \overline{\partial}^*_E \rho \big\|_{h(b,N,\tau),\widetilde{g}} \geq B_8 \cdot \big\| \rho \big\|_{h(b,N,\tau),\widetilde{g}}.$$

By a standard argument using the representation theorem by Riesz, we obtain the following lemma.

Lemma 8.22. — *Let f_1 be any L^2-section of $E \otimes \Omega^{0,1}$ with respect to $h(b,N,\tau)$ and $\widetilde{g}$. Then there exists an L^2-section f_2 of E with respect to $h(b,N,\tau)$ satisfying $\overline{\partial} f_2 = f_1$ and $\|f_2\|_{h(b,N,\tau),\widetilde{g}} \leq B_9 \cdot \|f_1\|_{h(b,N,\tau),\widetilde{g}}$, where B_9 denotes a good constant.* □

Corollary 8.23. — *Let b and N be as above. Let f_1 be any L^2-section of $E \otimes \Omega^{0,1}$ with respect to $h(b,N) = h \cdot e^{\chi(b,N)}$ and $\widetilde{g}$. Then there exists an L^2-section f_2 of E with respect to $h(b,N)$ satisfying $\overline{\partial} f_2 = f_1$ and $\|f\|_{h(b,N),\widetilde{g}} \leq B_9 \cdot \|g\|_{h(b,N),\widetilde{g}}$, where B_9 is a good constant.*

Proof Since the metrics $h(b,N)$ and $h(b,N,\tau)$ are mutually bounded, the corollary immediately follows from the previous lemma. □

Let $A^{0,1}_{b,N}(E)$ denote the space of sections of $E \otimes \Omega^{0,1}$, which are L^2 with respect to $h(b,N)$ and $\widetilde{g}$. Let $A^{0,0}_{b,N}(E)$ denote the space of sections f of E such that f and $\overline{\partial} f$ are L^2 with respect to $h(b,N)$ and $\widetilde{g}$. The hermitian product and the norm of $A^{p,q}_{b,N}(E)$ are denoted by $\langle\cdot,\cdot\rangle_{b,N}$ and $\|\cdot\|_{b,N}$.

8.3.5. Proof. — We have only to consider the case where a is positive and sufficiently close to 0 in the sense $\mathcal{P}ar({}_aE) \cap \{b > 0\} = \emptyset$, which we impose in the following argument.

Let Γ and $\boldsymbol{v} = (v_1, \ldots, v_r)$ be as in Assumption 8.17. We put $S(\Gamma) := \{\alpha_1, \ldots, \alpha_r\}$. We have $F(\det(h)) = \operatorname{tr} F(h) = \operatorname{tr} R(h)$. We put $\widetilde{v} = v_1 \wedge \cdots \wedge v_r$, and then we have $\overline{\partial}\widetilde{v} = \widetilde{v} \cdot \big(-2^{-1} \operatorname{tr}\Gamma + \operatorname{tr} A\big) \cdot d\bar{z}/\bar{z}$. Hence we have $-\widetilde{\alpha} := -\operatorname{tr}\Gamma \equiv \widetilde{a} \mod \mathbb{Z}$, due to the remarks given in the subsection 8.3.2. We take the holomorphic section $\widetilde{s} \in {}_{-\widetilde{\alpha}}\det(E)$ as in the subsection 8.3.2 such that $\widetilde{s} = z^n \cdot s$ for some integer n.

Remark 8.24. — We will show that $-\widetilde{\alpha} = \widetilde{a}$, i.e. $s = \widetilde{s}$ later (Lemma 8.28). □

Let T_A denote the section of $\underline{End}(E) \otimes \Omega^{0,1}$ determined by $\boldsymbol{v}$ and $A \cdot d\bar{z}/\bar{z}$, i.e., $T_A(\boldsymbol{v}) = \boldsymbol{v} \cdot A \cdot d\bar{z}/\bar{z}$. We put $\overline{\partial}_0 := \overline{\partial} - T_A$. We put $f_i := |z|^{\alpha_i} \cdot v_i$. Then we have $\overline{\partial}_0 f_0 = 0$ and $|f_i|_h = |z|^{\alpha_i}$. In particular, we have $f_i \in A^{0,0}_{\alpha_i+\eta,N}(E)$, and $\overline{\partial}(f_i) = T_A(f_i)$.

For a moment, we assume that $\eta > 0$ satisfies the following:

$$0 < \eta < \frac{1}{2} \min\big\{|b + \alpha_j| \neq 0 \,\big|\, b \in \mathcal{P}ar'\big({}_aE\big),\ \alpha_j \in S(\Gamma)\big\}. \tag{78}$$

Remark 8.25. — It will be shown the set $\mathcal{P}ar\big({}_aE\big)$ coincides with $\big\{-\alpha_1, \ldots, -\alpha_r\big\}$ (Lemma 8.29). Hence we may assume that (78) holds. □

Let us take $g_i \in (\operatorname{Ker}\overline{\partial})^{\perp}_{\alpha_i+\eta,N}$ satisfying $\overline{\partial} g_i = T_A(f_i)$ and $\big\|g_i\big\|_{\alpha_i+\eta,N} \leq C_3 \cdot \big\|T_A(f_i)\big\|_{\alpha_i+\eta,N}$ as in Lemma 8.22. We put $F_i := f_i - g_i$. Then we have $\overline{\partial} F_i = 0$, $F_i \in A^{0,0}_{\alpha_i+\eta,N}(E)$, and the following estimate:

$$\big\|F_i\big\|_{\alpha_i+\eta,N} \leq \big\|f_i\big\|_{\alpha_i+\eta,N} + C_3 \cdot \big\|T_A(f_i)\big\|_{\alpha_i+\eta,N}.$$

We put $a_i := \max\bigl\{b \in \mathcal{P}ar\bigl({}_aE\bigr) \bigm| b \leq -\alpha_i\bigr\}$. Due to our choice of η, we have $F_i \in {}_{a_i}E$. We have the following:

$$\overline{\partial}_0 g_i = -T_A(g_i) + T_A(f_i). \tag{79}$$

Hence we obtain $g_i \in L_1^2(K)$ for any compact subset $K \subset \Delta^*$, and the L_1^2-norm is dominated by $||T_A(f_i)||_{\alpha_i+\eta,N}$ multiplied by some constant depending only on K. Hence for some number $p > 2$ and some good constant $C'(K)$, we have the following:

$$\bigl\|g_i\bigr\|_{L^p(K)} \leq C'(K) \cdot \bigl\|T_A(f_i)\bigr\|_{\alpha_i+\eta,N}$$

Due to (79), we have the following, for some good constant $C''(K)$:

$$\bigl\|g_i\bigr\|_{L_1^p(K)} \leq C''(K) \cdot \Bigl(\bigl\|T_A(f_i)\bigr\|_{\alpha_i+\eta,N} + \sup_K \bigl|T_A(f_i)\bigr|_{h,\widetilde{g}}\Bigr). \tag{80}$$

By a standard boot strapping argument, p can be sufficiently large.

Let us consider the function $\widetilde{F}$ determined by $\widetilde{F} \cdot \widetilde{s} = F_1 \wedge \cdots \wedge F_r$. We put $K_0 := \bigl\{z \bigm| 3^{-1} < |z| < 2 \cdot 3^{-1}\bigr\}$.

Lemma 8.26. — *There exists a small good constant C_5 with the following property:*

- *Assume the following inequalities hold:*

$$\sup_{K_0} \bigl|A\bigr|_{\widetilde{g}} < C_5, \quad \bigl\|T_A \cdot f_i\bigr\|_{\alpha_i+\eta,N} < C_5, \quad (i = 1, \ldots, r). \tag{81}$$

Then we have a good positive constant B_{100} such that $B_{100} < |\widetilde{F}| < B_{100}^{-1}$.

Proof From (80) and (81), we obtain $\bigl|F_1 \wedge \cdots \wedge F_r - f_1 \wedge \cdots \wedge f_r\bigr| < 4^{-1}$ holds on K_0, if C_5 is sufficiently small. Since $v_1, \ldots, v_r$ are orthonormal and f_i are given as $|z|^{\alpha_i} \cdot v_i$, we have $f_1 \wedge \cdots \wedge f_r = \exp\bigl(\sqrt{-1}\kappa + \nu\bigr) \cdot \widetilde{s}$ for some real valued functions κ and ν, where we have $\sup_{K_0} |\nu| \leq B_{200}$ for some good constant B_{200}. If C_5 is sufficiently small, κ is a sum of a constant κ_0 and a function κ_1 satisfying $\sup_{K_0} |\kappa_1(z)| < 100^{-1}$, due to $\sup_{K_0} |A| < C_5$.

Then we obtain $\widetilde{F}(K_0) \subset W := \bigl\{\exp(s + \sqrt{-1}t) \bigm| |t - \kappa_0| \leq 100^{-1},\ |s| \leq B_{200}\bigr\}$. Due to the maximum principle, we obtain $\widetilde{F}(\Delta(2/3)) \subset W$. Thus we are done. $\square$

For any number $0 < \gamma < 1$, let us consider the map $\phi_\gamma : \Delta^* \longrightarrow \Delta^*$ given by $z \longmapsto \gamma \cdot z$. We have the orthonormal frame $\phi_\gamma^* \boldsymbol{v}$ of $\phi_\gamma^*(E, \overline{\partial}_E, \theta, h)$, for which we have the following:

$$\overline{\partial}\bigl(\phi_\gamma^* \boldsymbol{v}\bigr) = \phi_\gamma^* \boldsymbol{v} \cdot \left(-\frac{1}{2}\Gamma + \phi_\gamma^* A\right) \cdot \frac{d\bar{z}}{\bar{z}}.$$

Note we have the following:

$$\bigl|\phi_\gamma^* A\bigr|_{h,\widetilde{g}} \leq C_3 \cdot \frac{-\log|z|}{-\log|z| - \log|\gamma|} \cdot \bigl(-\log|z|\bigr)^{-1}. \tag{82}$$

Hence it is easy to check that the assumptions hold also for $\phi_\gamma^*(E, \overline{\partial}_E, \theta, h)$. We also have $\|\phi_\gamma^* F(h)\|_{L^2} \leq \gamma^\epsilon \|F(h)\|_{L^2}$. We put $f_i^{(\gamma)} := |z|^{\alpha_i} \cdot \phi_\gamma^* v_i$. We take $g_i^{(\gamma)}$ and $F_i^{(\gamma)}$ as above.

Lemma 8.27. — *There exists a good constant γ_1 such that the assumptions of Lemma 8.26 are satisfied for $\phi_{\gamma_1}^*(E,\overline{\partial}_E,\theta,h)$ and $F_i^{(\gamma_1)}$ $(i=1,\dots,r)$*

Proof If γ is sufficiently small, then we may assume $\sup_{K_0}\big|\phi_\gamma^* A\big|_{\widetilde{g}} \le C_5$ due to (82). We also have the following:

$$\int \big|T_{\phi_\gamma^* A}\cdot f_i^{(\gamma)}\big|_{h,\widetilde{g}}^2 \cdot |z|^{2\alpha_i+2\eta}\cdot\big(-\log|z|\big)^N\cdot \mathrm{dvol}_{\widetilde{g}} \\ \le C_6\cdot\int\left|\frac{-\log|z|}{-\log|z|-\log\gamma}\right|^2\cdot|z|^{2\eta}\cdot\big(-\log|z|\big)^N\cdot\mathrm{dvol}_{\widetilde{g}}\,. \tag{83}$$

Since the right hand side converges to 0 in $\gamma\longrightarrow 0$, we can take a good constant γ_1 such that the inequality $\big\|T_{\phi_\gamma^* A} f_i^{(\gamma)}\big\|_{\alpha_i+\eta,N} < C_5$ holds. □

Hence we obtain the holomorphic sections $F_1^{(\gamma_1)},\dots,F_r^{(\gamma_1)}$ of ${}_a\big(\phi_{\gamma_1}^* E\big)$, satisfying the following:

- There is a good constant C_7 such that $\big\|F_i^{(\gamma_1)}\big\|_{\alpha_i+\eta,N}\le C_7$.
- For the holomorphic function $\widetilde{F}^{(\gamma_1)}$ determined by $\widetilde{F}^{(\gamma_1)}\cdot\widetilde{s} = F_1^{(\gamma_1)}\wedge\cdots\wedge F_r^{(\gamma_1)}$, we have $B_{100}\le|\widetilde{F}^{(\gamma_1)}|\le B_{100}^{-1}$ on $\Delta(2/3)$ for some good constant B_{100}.
- For each compact subset $K\subset\Delta^*$, there exists a good positive constant $C_8(K)$ such that $\big\|\widetilde{F}_i^{(\gamma_1)}\big\|_{L_2^p(K)}\le C_8(K)$.

Lemma 8.28. — *We have $-\widetilde{\alpha}=\widetilde{a}$. In particular, $s=\widetilde{s}$.*

Proof $F_1^{(\gamma_1)}\wedge\cdots\wedge F_r^{(\gamma_1)}$ gives the section of ${}_{\widetilde{a}}\phi_{\gamma_1}^*\det(E)$. Since $B_{100}<|\widetilde{F}^{(\gamma_1)}|$ on Δ, we obtain $-\widetilde{\alpha}\ge\widetilde{a}$.

By applying the above result to the *dual* of $(E,\overline{\partial}_E,\theta,h)$, we easily obtain the inequality $-\widetilde{\alpha}\le\widetilde{a}$. □

Lemma 8.29. — *$\mathcal{P}ar({}_aE)=S(\Gamma)$.*

Proof We have $a_i\le-\alpha_i$ and $\sum a_i=\widetilde{a}=-\widetilde{\alpha}=-\sum\widetilde{\alpha}_i$. Hence we obtain $a_i=-\alpha_i$. □

The holomorphic sections $F_i^{(\gamma_1)}\in\Gamma\big(\Delta,{}_a\phi_\gamma^* E\big)$ naturally give the holomorphic sections $\widetilde{F}_i^{(\gamma_1)}\in\Gamma\big(\Delta(\gamma_1),{}_aE\big)$. We take $\gamma_0<\gamma_1$ appropriately, and we put $F_i:=\widetilde{F}^{(\gamma_1)}_{i\,|\,\Delta(\gamma_0)}$. Now it is clear that they gives the desired holomorphic sections in Proposition 8.20. □

CHAPTER 9

CONVERGENCE

In this chapter, we give two convergence results (Theorem 9.2 and Theorem 9.10). Such convergence problems are rather trivial in the non-parabolic case, and so it can be said that the point is one of the main issues for parabolic Higgs bundles. We will use the results prepared in the chapters 7 and 8.

9.1. The convergence of a sequence of Hermitian-Einstein metrics

Let X be a smooth projective surface over $\boldsymbol{C}$, and $D = \bigcup_{i\in S} D_i$ be a simple normal crossing divisor. Let $({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m)$ $(m = 1, 2, \ldots)$ be $\boldsymbol{c}$-parabolic Higgs bundles on (X, D). We put $E_m := {}_{\boldsymbol{c}}E_{m|X-D}$. Let ϵ_m be numbers such that $0 < \epsilon_m < \mathrm{gap}({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m)$.

Let ω_{ϵ_m} be Kahler metrics of $X - D$ as in the subsection 4.3.1. For simplicity of the notation, we denote it by ω_m. Assume that we are given Hermitian-Einstein metrics h_m of $(E_m, \overline{\partial}_{E_m}, \theta_m)$ on $X - D$ with respect to ω_m, which is adapted to the parabolic structure $\boldsymbol{F}_m$. We assume the following:

Assumption 9.1. —

- $\Lambda_{\omega_m} F(h_m) = 0$ for any m, and $\big\|F(h_m)\big\|_{L^2,\,\omega_m} \to 0$.
- The sequence $\big\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m)\big\}$ converges to $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$ in the sense of Definition 3.21. □

The following theorem will be proved in the sections 9.2–9.3.

Theorem 9.2. —

- *There exists a subsequence $\big\{(E_m, \overline{\partial}_m, \theta_m, h_m)\,\big|\, m \in I\big\}$ which converges to a tame harmonic bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ on $X - D$, weakly in L^p locally on $X - D$, in the sense of Definition* 2.1. *Here p denotes any sufficiently large number.*

– *We have a non-trivial holomorphic map ${}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}}E_\infty$ which is compatible with the parabolic structures and the Higgs fields.*

Remark 9.3. — We can remove the assumption $\dim X = 2$ with an additional argument. In the section 9.4, we give a similar result in the case where $\dim X$ is arbitrary but $F(h_m) = 0$. □

9.2. Local convergence

9.2.1. Statement of local convergence. — Let X and D be as above. Let ω_i $(i = 1, 2, \ldots)$ and ω be Kahler forms of $X - D$. We assume that $\omega_i \to \omega$ in the C^∞-sense locally on $X - D$. Let $(E_i, \overline{\partial}_{E_i}, \theta_i)$ be Higgs bundles on $X - D$. For simplicity, we assume that the topological type of E_i are same. Let h_i be Hermitian-Einstein metrics of $(E_i, \overline{\partial}_{E_i}, \theta_i)$ with respect to ω_i.

Assumption 9.4. — We assume the following:

– The sequence of sections $\{\det(t - \theta_i) \,|\, i = 1, 2, \ldots\}$ of $\mathrm{Sym}^{\cdot}\, \Omega^{1,0}_{X-D}[t]$ is convergent, locally on $X - D$.
– $\Lambda_{\omega_i} F(h_i) = 0$ for any i, and $\big\|F(h_i)\big\|_{L^2,\omega_i} \to 0$. □

Proposition 9.5. — *There exists a subsequence $\big\{(E_i, \overline{\partial}_i, \theta_i, h_i) \,\big|\, i \in I\big\}$ which converges to a harmonic bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ in weakly L^p_2 locally on $X - D$ in the sense of Definition* 2.1.

The first claim of Theorem 9.2 immediately follows from the proposition. For that purpose, we have only to see that the harmonic bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ is tame. But the characteristic polynomials $\{\det(t - \theta_i)\}$ converges to $\det(t - \theta)$ for a parabolic Higgs bundle $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)$, and hence the tameness is obvious.

9.2.2. Preliminary for local convergence. — In this subsection, we give a preliminary for the proof of Proposition 9.5. The setting is as follows: Let (Y, ω) be a Kahler surface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on Y, and h be a Hermitian Einstein metric for $(E, \overline{\partial}_E, \theta)$ with respect to ω. Moreover, we assume $\big\|F(h)\big\|_{L^2,\omega}$ is sufficiently small. From θ, we obtain the section $\det(t - \theta)$ of $\mathrm{Sym}^{\cdot}\, \Omega^{1,0}_Y[t]$.

Let P be any point of Y. Let us take a holomorphic coordinate (U''_P, z''_1, z''_2) such that $z''_i(P) = 0$. We put $g''_P := \sum dz''_i \cdot d\bar{z}''_i$. We have a constant C_0 such that $C_0 \cdot g''_P \leq \omega \leq C_0^{-1} \cdot g''_P$.

Lemma 9.6. — *There exists a constant C such that the following holds:*

– *We put $z_i = C^{-1} \cdot z''_i$ and $U_P := \big\{(z_1, z_2) \,\big|\, |z_i| < 1\big\}$. We put $g_P := \sum dz_i \cdot d\bar{z}_i$ and $\omega_P := C^{-2}\omega$. Then the L^2-norm of $R(h)_{|U_P}$ is sufficiently small with respect to g_P and h, and the sup norm of $\Lambda_{\omega_P} R(h)_{|U_P}$ is also sufficiently small with respect to h.*

The constant C is good in the sense that it depends only on $\det(t-\theta)$, $\|F(h)\|_{L^2,\omega}$ *and* C_0.

Proof We have the expression $\theta = \sum f_i'' \cdot dz_i''$. There exists a good small constant C_1 such that the eigenvalues of $C_1 \cdot f_i''$ are sufficiently small. We put $z_i' = C_1^{-1} \cdot z_i''$, and we have the expression $\theta = \sum f_i' \cdot dz_i'$.

We put $\omega' = C_1^{-2} \cdot \omega$ and $g_P' := \sum dz_i' \cdot d\bar{z}_i'$. Then we have $C_0 \cdot g_P' \leq \omega' \leq C_0^{-1} \cdot g_P'$. We put $U_P' := \{(z_1', z_2') \mid |z_i'| < 1\}$. Let $\|F(h)\|_{L^2,g_P'}$ denote the L^2-norm of $F(h)_{|U_P'}$ with respect to h and g_P. Since $\dim Y = 2$, there exists a good constant B_1 such that $\|F(h)\|_{L^2,\omega'} \leq B_1 \cdot \|F(h)\|_{L^2,\omega}$. Since the eigenvalues of f_i' is sufficiently small, the sup norms $|f_i'|_h$ are dominated by a good constant B_2 due to Lemma 7.18.

Take a sufficiently small good constant C_3 such that $C_3 \cdot |f_i|_h'$ are sufficiently small. We put $z_i = C_3^{-1} \cdot z_i'$ $(i = 1, 2)$ and we have the expression $\theta = \sum f_i \cdot dz_i$. We put $U_P := \{(z_1, z_2) \mid |z_i| < 1\}$. Thus we obtain a holomorphic coordinate neighbourhood (U_P, z_1, z_2) of P. We put $\omega_P := C_3^{-2} \cdot \omega'$ and $g_P := \sum dz_i \cdot d\bar{z}_i$. We have $C_0 \cdot g_P \leq \omega_P \leq C_0^{-1} \cdot g_P$.

Recall we have $R(h) = -[\theta, \theta^\dagger] + F(h)$ and $\Lambda_{\omega_P} R(h) = -\Lambda_{\omega_P}[\theta, \theta^\dagger]$. Since the norms $|f_i|_h$ are sufficiently small, the sup norm of $\Lambda_{\omega_P} R(h)_{|U_P}$ is also sufficiently small with respect to h and ω_P. The L^2-norm of $\|R(h)_{|U_P}\|_{L^2,\omega_P}$ is dominated by $\|[\theta, \theta^\dagger]_{|U_P}\|_{L^2,\omega_P}$ and $\|F(h)\|_{L^2,\omega}$, which is also sufficiently small. □

Recall the argument in [**10**]. Let $\|\cdot\|_{L^2,g_P}$ denote the L^2-norms with respect to g_P. Due to Uhlenbeck's theorem, we can take an orthonormal frame $\boldsymbol{v}_P$ on U_P such that the inequality $\big\|A\big\|_{L^2,g_P} \leq C'' \cdot \big\|R(h)_{|U_P}\big\|_{L^2,g_P}$ holds for the connection form A with respect to $\boldsymbol{v}_P$, namely, $(\partial_E + \overline{\partial}_E)\boldsymbol{v}_P = \boldsymbol{v}_P \cdot A$. Then there exists a constant $C(p) > 0$ such that the L_1^p-norm of A is dominated as follows (See the argument in the page 20–21 of [**10**].):

$$\|A\|_{L_1^p} \leq C(p)\Big(\|R(h)\|_{L^2,\omega} + \sup_{U_P}\big|\Lambda_{\omega_P} R(h)\big|\Big).$$

Let Θ_P be the matrix valued $(1,0)$-form given by $\theta \cdot \boldsymbol{v}_P = \boldsymbol{v}_P \cdot \Theta_P$. Since θ is holomorphic, it is easy to check that L_1^p-norm of Θ_P with respect to g_P is also dominated by the norm of θ and $\|F(h)\|_{L^2,\omega}$.

9.2.3. Local convergence. — Let us return to the proof of Proposition 9.5. Due to the results in the subsection 9.2.2, we immediately obtain the following lemma.

Lemma 9.7. — *Let P be any point of Y. There exist the number $i(P)$ and a holomorphic coordinate (U_P, z_1, z_2) around P satisfying the following:*

- *U_P is isomorphic to $\Delta(1)^2$ by the coordinate (z_1, z_2). The metric $g_P = \sum dz_i \cdot d\bar{z}_i$ is given.*
- *For any $i \geq i(P)$, we have an orthogonal frame $\boldsymbol{v}_{P,i}$ of (E_i, h_i) on U_P. The connection one form $A_{P,i}$ is given by $(\partial_{E_i} + \overline{\partial}_{E_i})\boldsymbol{v}_{P,i} = \boldsymbol{v}_{P,i} \cdot A_{P,i}$. Then the*

L_1^p-norm of the connection form $A_{P,i}$ with respect to g_P are sufficiently small, independently of i.

– *The form $\Theta_{P,i}$ is given by $\theta_i \cdot \boldsymbol{v}_{P,i} = \boldsymbol{v}_{P,i} \cdot \Theta_{P,i}$. The L_1^p-norms of $\Theta_{P,i}$ with respect to g_P are sufficiently small, independently of i.* □

Hence we can take a locally finite covering $\bigl\{(U_\alpha, z_1^{(\alpha)}, z_2^{(\alpha)}) \bigm| \alpha \in \Gamma\bigr\}$ of Y as follows:

– Each U_α is relatively compact in Y.
– We have orthonormal frames $\boldsymbol{v}_{\alpha,i}$ of (E_i, h_i) on U_α such that the L_1^p-norms of $A_{\alpha,i}$ are sufficiently small with respect to the metrics $\sum dz_j^{(\alpha)} \cdot d\bar{z}_j^{(\alpha)}$ independently of i. Here $A_{\alpha,i}$ denote the connection forms of $(\partial_{E_i} + \overline{\partial}_{E_i})$ with respect to $\boldsymbol{v}_{\alpha,i}$.
– Let $\Theta_{\alpha,i}$ be the matrix valued $(1,0)$-forms given by $\theta_i \cdot \boldsymbol{v}_{\alpha,i} = \boldsymbol{v}_{\alpha,i} \cdot \Theta_{\alpha,i}$. Then the L_1^p-norms of $\Theta_{\alpha,i}$ are sufficiently small with respect to $\sum dz_i^{(\alpha)} \cdot d\bar{z}_i^{(\alpha)}$, independently of i.

Let $g_{\beta,\alpha,i}$ be the unitary transformation on $U_\alpha \cap U_\beta$ determined by $\boldsymbol{v}_{\alpha,i} = \boldsymbol{v}_{\beta,i} \cdot g_{\beta,\alpha,i}$. Once α and β are fixed, the L_2^p-norms of $g_{\beta,\alpha,i}$ are bounded independently of i. By a standard argument, we can take a subsequence $I \subset \{i\}$ satisfying the following:

– The sequences $\bigl\{A_{\alpha,i} \bigm| i \in I\bigr\}$, $\bigl\{\Theta_{\alpha,i} \bigm| i \in I\bigr\}$ are weakly L_1^p-convergent for each α.
– The sequence $\bigl\{g_{\alpha,\beta,i} \bigm| i \in I\bigr\}$ is weakly L_2^p-convergent for each (α, β).

Then we obtain the limit Higgs bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty)$ with the metric h_∞ on Y. Due to the assumption $\bigl\|F(h_i)\bigr\|_{L^2,\omega_i} \to 0$, we obtain $\bigl|F(h_\infty)\bigr|_{L^2,\omega} = 0$, and hence $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ is a harmonic bundle. By using the argument of Uhlenbeck, we obtain locally L_2^p-isometries $\Phi_i : (E_i, h_i) \longrightarrow (E_\infty, h_\infty)$ such that $\{\Phi_i(\overline{\partial}_i)\}$ and $\bigl\{\Phi(\theta_i)\bigr\}$ are weakly L_1^p-convergent to $\overline{\partial}_\infty$ and θ_∞ respectively, on any compact subset of Y. Thus the proof of Proposition 9.5 is accomplished. □

9.3. The existence of non-trivial map

Let us show the second claim in Theorem 9.2.

9.3.1. On a punctured disc. — Let us explain the setting in this subsection. Let $(E_m, \overline{\partial}_m, \theta_m)$ $(m = 1, 2, \ldots, \infty)$ be a sequence of Higgs bundles with hermitian metrics h_m on Δ^*. Let ϵ_m $(m = 1, 2, \ldots)$ be a sequence of small positive numbers. Assume the following:

– We have C^1-isomorphisms $\Phi_m : E_m \longrightarrow E_\infty$, such that $\bigl\{\Phi_m(\overline{\partial}_m) - \overline{\partial}_\infty\bigr\}$ and $\bigl\{\Phi_m(\theta_m) - \theta_\infty\bigr\}$ weakly converge to 0 in C^0 locally on $X - D$.
– The assumptions 8.17 are satisfied, where the L^2-norm of $F(h_m)$ is taken with respect to h_m and g_{ϵ_m}. The constants are independent of m.
– $a \notin \mathcal{P}ar(E_\infty) \cup \bigcup_m \mathcal{P}ar(E_m)$.

– The assumption 8.19 for ϵ_m and $\mathcal{P}ar\big({}_aE_m\big)$ is satisfied. The constants are independent of m.

We put $\widetilde{a}(m) := \sum_{b\in\mathcal{P}ar({}_aE_m)} \mathfrak{m}(b)\cdot b$, and we take the holomorphic sections $s^{(m)}$ of ${}_{\widetilde{a}(m)}\det(E_m)$ as in the subsection 8.3.2. We take $F_1^{(m)},\ldots,F_r^{(m)}$ and $a_1^{(m)},\ldots,a_r^{(m)}$ for $(E_m,\delta_m,\theta_m,h_m)$ as in Proposition 8.20. The following claim can be easily shown.

Lemma 9.8. — *There exists a subsequence $\{m'\}$ for which $\big\{\Phi_m(s^{(m')})\big\}$, $\big\{a_i^{(m')}\big\}$ and $\big\{\Phi_m(F_i^{(m')})\big\}$ are convergent in weakly L_1^p locally on Δ^*. The limits are denoted by $s^{(\infty)}$, $a_i^{(\infty)}$ and $F_i^{(\infty)}$ respectively.* □

In this case, we have $F_i^{(\infty)} \in {}_{a_i^{(\infty)}}E_\infty$. We put $\widetilde{a}^{(\infty)} = \sum a_i^{(\infty)}$.

Lemma 9.9. — *The sections $F_1^{(\infty)},\ldots,F_r^{(\infty)}$ give a holomorphic frame of ${}_aE_\infty$, which is compatible with the parabolic structure, and $a_i^{(\infty)}$ is the degree of $F_i^{(\infty)}$ with respect to the parabolic structure.*

Proof We put $S_a := \{F_i^{(\infty)}\,|\,a_i^{(\infty)} = a\}$. We take $F^{(\infty)}$ by $F_1^{(\infty)}\wedge\cdots\wedge F_r^{(\infty)} = F^{(\infty)}\cdot s^{(\infty)}$, and then we have $L_1 \leq \big|F^{(\infty)}\big| \leq L_2$. for some positive constants L_i. We also have $0 < C_{500} < |s^{(\infty)}|\cdot|z|^{\widetilde{a}} < C_{501}$ for some constants. It is easy to see that S_a induces the frame of $\mathrm{Gr}_a^{F^{(\infty)}}$ for each a. Then the claim of the lemma immediately follows. □

We construct the holomorphic map $\Psi_m : {}_aE_m \longrightarrow {}_aE_\infty$ by the correspondence $\Psi_m(F_i^{(m)}) = F_i^{(\infty)}$. By our construction, the following claims hold:

– $\Psi_m - \Phi_m \longrightarrow 0$ weakly in L_1^p locally on Δ^*, and $\Psi_m(\theta_m) - \theta_\infty \longrightarrow 0$ weakly in L^p locally on Δ^*.
– Let $F^{(m)}({}_aE_m)$ denote the parabolic filtrations of ${}_aE_m$ induced by h_m. Then the sequence of the filtrations $\big\{\Psi_m\big(F^{(m)}({}_aE_m)\big)\big\}$ converges to $F^{(\infty)}({}_aE_\infty)$ in the sense of Definition 3.21.

9.3.2. On a curve. — Let us explain the setting in this subsection. Let C be a smooth projective curve, and $D_C = \{P_1,\ldots,P_l\}$. Let $\{\epsilon_m\}$ be a sequence of positive numbers. Let $(E_m,\overline{\partial}_m,h_m,\theta_m)$ $(m = 1,2,\ldots,\infty)$ be a sequence on $C - D_C$, satisfying the following:

– We have C^1-isomorphisms $\Phi_m : E_m \longrightarrow E_\infty$, such that $\big\{\Phi_m(\overline{\partial}_m) - \overline{\partial}_\infty\big\}$ and $\big\{\Phi_m(\theta_m) - \theta_\infty\big\}$ weakly converge to 0 in C^0 locally on $C - D_C$.
– Around each point $P \in D_C$, the assumptions 8.17 are satisfied for any m, where the L^2-norm of $F(h_m)$ is taken with respect to h_m and g_{ϵ_m}. The constants are independent of m.
– Take $\boldsymbol{c} = (c(P)\,|\,P \in D)$ such that $c(P) \notin \mathcal{P}ar(E_\infty,P) \cup \bigcup_m \mathcal{P}ar(E_m,P)$.
– The assumption 8.19 for ϵ_m and $\mathcal{P}ar\big({}_{\boldsymbol{c}}E_m,P\big)$ is satisfied for each point $P \in D_C$.

For each point $P \in D_C$, we take the holomorphic maps ${}^P\Psi_m : {}_{c(P)}E_m \longrightarrow {}_{c(P)}E_\infty$ on U_P as in the previous subsection, which are defined on a neighbourhood $U_P \ni P$.

Let $\chi_P : C \longrightarrow [0,1]$ denote a C^∞-function which is constantly 1 around P, and constantly 0 on $C - U_P$. The L_1^p-maps $\Psi_m : E_m \longrightarrow E_\infty$ are given as follows:

$$\Psi_m := \sum_P \chi_P \cdot {}^P\Psi_m + \Big(1 - \sum_P \chi_P\Big) \cdot \Phi_m.$$

If m is sufficiently large, then Ψ_m are isomorphisms. We have the following:

$$(84)\quad \Psi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Psi_m = \sum \overline{\partial}\chi_P \cdot {}^P\Psi_m - \Big(\sum \overline{\partial}\chi_P\Big) \cdot \Phi_m + \Big(1 - \sum \chi_P\Big) \cdot \Big(\Phi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Phi_m\Big).$$

Hence the sequence $\{\Psi_m \circ \overline{\partial}_m - \overline{\partial}_\infty \circ \Phi_m\}$ converges to 0 weakly in L^p on C. By the remark given in the last of the previous subsection, the sequence of the parabolic filtrations $\{F({}_{\boldsymbol{c}}E_m)\}$ converges to $F({}_{\boldsymbol{c}}E_\infty)$. We also have the convergence of $\Psi_m(\theta_m) - \theta_\infty$ to 0 weakly in L^p locally on $C - D_C$. Hence we obtain the convergence of $\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}^{(\infty)}, \theta_m)\}$ to $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}^{(\infty)}, \theta_\infty)$ weakly in L_1^p on C, in the sense of Definition 3.21.

9.3.3. The end of Proof of Theorem 9.2. — Let us return to the setting for Theorem 9.2. Let $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ be a harmonic bundle obtained as a limit, in the section 9.2. We obtain the parabolic Higgs bundle $({}_{\boldsymbol{c}}E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$. We would like to show the existence of a non-trivial holomorphic homomorphism ${}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}}E_\infty$. By a standard argument, we have only to show the existence of a non-trivial map $f_C : ({}_{\boldsymbol{c}}E_*, \theta)_{|C} \longrightarrow ({}_{\boldsymbol{c}}E_{\infty\,*}, \theta_\infty)_{|C}$ for some sufficiently ample curve $C \subset X$.

By taking an appropriate subsequence, we may assume that $\sum_m \|F(h_m)\|_{L^2,\omega_m} < \infty$. Then we may assume the following for such a curve C, due to the result in the sections 8.1 and 8.2.

– For each $P \in C \cap D$, let U_P denote a neighbourhood of P in C. Then the assumption 8.17 is satisfied for each $P \in C \cap D$ and for each $(E, \overline{\partial}_E, \theta, h_m)_{|U_P}$. The constants are independent of m.

Then by applying the argument in the subsection 9.3.2, we obtain that the parabolic Higgs bundles $({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m)_{|C}$ converges to $({}_{\boldsymbol{c}}E_\infty, \boldsymbol{F}_\infty, \theta_\infty)_{|C}$ weakly in L_1^p on C. On the other hand, we also have the convergence of the sequence $\{({}_{\boldsymbol{c}}E_m, \boldsymbol{F}_m, \theta_m)_{|C}\}$ to $({}_{\boldsymbol{c}}E, \boldsymbol{F}, \theta)_{|C}$ on C. Thus we obtain the existence of desired non-trivial map f_C due to Corollary 3.23. Thus the proof of Theorem 9.2 is finished. $\square$

9.4. The convergence of a sequence of tame harmonic bundles

Let X be a smooth projective variety of arbitrary dimension, and D be a normal crossing divisor of X. Let $(E_m, \overline{\partial}_m, \theta_m, h_m)$ $(m = 1, 2, \dots,)$ be tame harmonic bundles on $X - D$.

Theorem 9.10. — *Assume the following:*

– *The sequence of the sections* $\{\det(t-\theta_m)\}$ *of* $\mathrm{Sym}^{\cdot}\,\Omega_X^{1,0}(\log D)[t]$ *are convergent.*

Then the following claims hold:

– *There exists a subsequence* $\bigl\{(E_m, \overline{\partial}_m, \theta_m, h_m)\bigm| m \in I\bigr\}$ *which converges to a tame harmonic bundle* $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$ *on* $X - D$*, weakly in* L^p *locally on* $X - D$*, in the sense of Definition* 2.1*. Here* p *denotes any sufficiently large number.*
– *Moreover assume that we have a parabolic Higgs sheaf* $({}_{\boldsymbol{c}}E_*, \theta)$ *such that* $\bigl\{({}_{\boldsymbol{c}}E_{m*}, \theta)_{|C}\bigr\}$ *converges to* $({}_{\boldsymbol{c}}E_*, \theta)_{|C}$ *for any generic curve* C*. Then we have a non-trivial holomorphic map* ${}_{\boldsymbol{c}}E \longrightarrow {}_{\boldsymbol{c}}E_\infty$ *which is compatible with the parabolic structures and the Higgs fields.*

Proof It can be shown similarly to Theorem 9.2. In fact, the argument is much easier due to $F(h_m) = 0$. We give only an indication of the argument.

The sequence of sections $\{\det(t-\theta_m)\}$ of $\mathrm{Sym}^{\cdot}\,\Omega_X^{1,0}[t]$ converges to $\det(t-\theta)$. Hence we obtain the estimate of the norms of θ_m locally on $X - D$, due to Proposition 7.1. Hence we also obtain the estimate of the curvatures $R(h_m)$ because of $R(h_m) + [\theta_m, \theta_m^\dagger] = 0$. Therefore we obtain the local convergence result like the first claim by a standard argument. (See [**47**] the page 26–28, for example.) Thus we obtain a harmonic bundle $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)$.

Then we would like to show the existence of a non-trivial map $f : {}_{\boldsymbol{c}}E_\infty \longrightarrow {}_{\boldsymbol{c}}E$ which is compatible with the parabolic structure and the Higgs fields. As in the proof of Theorem 9.2, we have only to show the existence of a non-trivial map $f_C : {}_{\boldsymbol{c}}E_{\infty|C} \longrightarrow {}_{\boldsymbol{c}}E_{|C}$, where C is a sufficiently ample curve in X. We may assume that $\{(E_m, \overline{\partial}_m, \theta_m, h_m)_{|C}\}$ converges to $(E_\infty, \overline{\partial}_\infty, \theta_\infty, h_\infty)_{|C}$.

Let P be any point of $C \cap D$, and (U_P, z) be an appropriate coordinate neighbourhood around P in C, such that $z(P) = 0$. By using the estimate in Proposition 7.8 (or the section 7.1 of [**38**]), we obtain the estimate of the curvatures:

$$\bigl|R(h_m)_{|C}\bigr| \leq B_1 \cdot \frac{dz \cdot d\bar{z}}{|z|^2\bigl(-\log|z|^2\bigr)^2}.$$

Here B_1 denotes a constant which is independent of m. Then it is easy to see that Assumption 8.9 is satisfied for any m, and that the constants are independent of m. Therefore Assumption 8.17 is satisfied for any m, and the constants are independent of m. Hence we obtain the existence of f_C by the same argument as that in the section 9.3. Thus we are done. □

CHAPTER 10

EXISTENCE OF THE ADAPTED PLURI-HARMONIC METRIC

Recall that the half of Kobayashi-Hitchin correspondence for tame harmonic bundles (Theorem 1.4) is given in the chapter 5. Now, we give the other half, after a rather long preliminary. Together with the perturbation of the parabolic structure (the section 3.4), the preliminary correspondence (Proposition 6.1) and the convergence result (Theorem 9.2), we immediately obtain the surface case (Theorem 10.1). The higher dimensional case (Theorem 10.2) can easily be reduced to the surface case.

10.1. The surface case

Let X be a smooth projective surface, and D be a normal crossing divisor of X. We assume that each irreducible component D_i of D is smooth. Let L be an ample line bundle, and ω be a Kahler form representing $c_1(L)$.

Theorem 10.1. — *Let $\big({}_{\boldsymbol{c}}E_*,\theta\big)$ be a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle on (X,D). Assume that the characteristic numbers vanish:*

$$\text{par-deg}_L({}_{\boldsymbol{c}}E_*)=\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E_*)=0.$$

Then there exists a pluri-harmonic metric h of $(E,\theta)=({}_{\boldsymbol{c}}E,\theta)_{|X-D}$ which is adapted to the parabolic structure.

Proof We may assume that $c_i\notin \mathcal{P}ar({}_{\boldsymbol{c}}E_*,i)$. We take a sequence $\{\epsilon_m\}$ converging to 0, such that $\epsilon_m=N_m^{-1}$ for some integers N_m. We take perturbation of parabolic structures $\big({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\epsilon_m)},\theta\big)$, as in the section 3.4. We also take the Kahler metrics ω_{ϵ_m} of $X-D$ as in the subsection 4.3.1. We may assume the following:

- $\big({}_{\boldsymbol{c}}E,\boldsymbol{F}^{(\epsilon_m)}\big)$ are μ_ω-stable.
- $\text{par-deg}_\omega({}_{\boldsymbol{c}}E_*)=0$.

– We have the finiteness:

$$\sum \left(\int_X \text{par-ch}_2({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon_m)}) \right)^{1/2} < \infty.$$

Due to Proposition 6.1, we have the Hermitian-Einstein metric h_m of $(E, \overline{\partial}_E, \theta)$ with respect to ω_{ϵ_m}, which is adapted to the parabolic structure $({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon_m)})$. We remark that $\sum \|F(h_m)\|_{L^2,\omega_{\epsilon_m}} < \infty$, for we have $\|F(h_m)\|^2_{L^2,\omega_{\epsilon_m}}$=par-ch$_{2,\omega}({}_{\boldsymbol{c}}E, \boldsymbol{F}^{(\epsilon_m)})$.

We have a subsequence $I \subset \{m\}$ such that $\left\{ \left(E, \overline{\partial}_E, h_m, \theta_m\right) \,\middle|\, m \in I \right\}$ converges to a tame harmonic bundle $(E_\infty, \overline{\partial}_\infty, h_\infty, \theta_\infty)$ weakly in L^p_2 locally on $X - D$, due to Theorem 9.2. Moreover, we also have the non-trivial map $f : {}_{\boldsymbol{c}}E_\infty \longrightarrow {}_{\boldsymbol{c}}E$ which is compatible with the parabolic structure and the Higgs fields. Due to the μ_L-stability of $\left({}_{\boldsymbol{c}}E_*, \theta\right)$ and the μ_L-polystability of $\left({}_{\boldsymbol{c}}E_\infty, \theta_\infty\right)$ (Proposition 5.1), the map f is isomorphic. Thus we are done. □

10.2. The higher dimensional case

Now the main existence theorem is given.

Theorem 10.2. — *Let X be a smooth projective variety over the complex number field of dimension n. Let $D = \bigcup_i D_i$ be a simple normal crossing divisor of X. Let L be an ample line bundle on X. Let $\left(\boldsymbol{E}_*, \theta\right)$ be a μ_L-stable regular filtered Higgs bundle with the trivial characteristic numbers, i.e.,* par-deg$_L(\boldsymbol{E}_*) = \int_X$ par-ch$_{2,L}(\boldsymbol{E}_*) = 0$. *We put $E := \boldsymbol{E}_{|X-D}$.*

- $P(n)$ **:** *Then there exists a pluri-harmonic metric h of $(E, \overline{\partial}_E, \theta)$, which is adapted to the parabolic structure.*
- $Q(n)$ **:** *Such a metric is unique up to constant multiplication.*

Proof We can assume $\dim X \geq 3$. We can also assume that L is sufficiently ample as in Lemma 6.11. The claim $Q(n)$ follows from the more general result (Proposition 5.2). We use an induction on n. We have already known the existence for $n = 2$ (Theorem 10.1). Hence we have only to show that $P(n-1)$ and $Q(n-2)$ imply $P(n+1)$.

Let $(\boldsymbol{E}_*, \theta)$ be a regular filtered Higgs bundle on (X, D). Assume that it is stable with par-deg$_L(\boldsymbol{E}_*) = \int_X$ par-ch$_{2,L}(\boldsymbol{E}_*) = 0$. For any element $s \in \mathbb{P} := \mathbb{P}\big(H^0(X, L)^\vee\big)$ determines the hypersurface $Y_s = \big\{x \in X \,\big|\, s(x) = 0\big\}$. The subset $\mathcal{X}_L \subset X \times \mathbb{P}$ is given by $\mathcal{X}_L := \{(x, s) \,|\, x \in Y_s\}$. Let U be a Zariski open subset of $\mathbb{P}$ which consists of $s \in \mathbb{P}$ such that $(\boldsymbol{E}_*, \theta)_{|Y_s}$ is μ_L-stable. Since L is assumed to be sufficiently ample, U is not empty. The image W of the naturally defined map $\mathcal{X}_L \times_{\mathbb{P}} U \longrightarrow X$ is Zariski open in X. In fact, $X - W$ consists of, at most, finite points of X due to the ampleness of L.

Let s be any element of U. We have a pluri-harmonic metric h_s of $(E,\theta)_{|Y_s}$, which is adapted to the induced parabolic structure, due to the hypothesis $P(n-1)$. It is unique up to constant multiplication.

Let s_i $(i=1,2)$ be elements of U such that Y_{s_1} and Y_{s_2} are transversal and that $Y_{s_1,s_2} := Y_{s_1}\cap Y_{s_2}$ is transversal to D. We remark that $\dim Y_{s_1}\cap Y_{s_2}\geq 1$. We may also assume that $({}_{\boldsymbol{c}}E,\theta)_{|Y_{s_1,s_2}}$ is μ_L-stable (Lemma 6.11). Hence $h_{s_1\,|\,Y_{s_1,s_2}}$ and $h_{s_2\,|\,Y_{s_1,s_2}}$ are same up to constant multiplication, due to the hypothesis $Q(n-2)$.

Then we obtain the pluri-harmonic metric h of $(E,\overline{\partial}_E,\theta)_{|X-(D\cup W)}$, by using an argument given in the section 6.5 in [**39**]. It can be shown that h gives the C^∞-metric of E on $X-D$, due to the elliptic regularity. Hence, it is pluri-harmonic metric of E on $X-D$, in fact. Thus we obtain the tame harmonic bundle $(E,\overline{\partial}_E,\theta,h)$.

We take the prolongment $({}_{\boldsymbol{c}}E(h)_*,\theta)$ which is a parabolic Higgs bundle on (X,D). (See the section 3.3 for the prolongment.)

Lemma 10.3. — *There exists a closed subset $W'\subset D$ with the following properties:*

- *The codimension of W' in X is larger than 2.*
- *The identity of E is extended to the holomorphic isomorphism ${}_{\boldsymbol{c}}E_{|X-W'}\longrightarrow {}_{\boldsymbol{c}}E(h)_{|X-W'}$.*

Proof The restriction of ${}_{\boldsymbol{c}}E$ and ${}_{\boldsymbol{c}}E(h)$ to any generic hypersurface Y of L is isomorphic by the construction. Then the claim of the lemma easily follows from Corollary 2.53 in [**38**]. □

Since both of ${}_{\boldsymbol{c}}E$ and ${}_{\boldsymbol{c}}E(h)$ are locally free, they are isomorphic. In particular, we can conclude that h is adapted to the parabolic structure. □

CHAPTER 11

THE TORUS ACTION AND THE DEFORMATION OF REPRESENTATIONS

We see that any flat bundle on a quasi projective variety can be deformed to variation of polarized Hodge structure. We can derive some results on the fundamental group of quasi projective varieties.

We owe the essential ideas in this chapter to Simpson [**47**]. In fact, our purpose is a natural generalization of his results on smooth projective varieties. We will use his idea without mentioning his name. The section is included for a rather expository purpose.

11.1. Torus action on the moduli space of representations

11.1.1. Notation. — We begin with a general remark. Let V and V' be a vector space over $\boldsymbol{C}$, and $\Phi : V \longrightarrow V'$ be a linear isomorphism. Let Γ be any group, and $\rho : \Gamma \longrightarrow \mathrm{GL}(V)$ be a homomorphism. Then Φ and ρ induce the homomorphism $\Gamma \longrightarrow \mathrm{GL}(V')$, which is denoted by $\Phi_*(\rho)$. We also remark that the notation in the subsection 2.6 will be used.

11.1.2. Continuity. — Let X be a smooth projective variety with a polarization L, and D be a normal crossing divisor. Let x be a point of $X - D$. We put $\Gamma := \pi_1(X - D, x)$. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-polystable regular filtered Higgs bundle on (X, D) with $\text{par-deg}_L(\boldsymbol{E}_*) = \int_X \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. We put $E := \boldsymbol{E}_{|X-D}$. Since $(\boldsymbol{E}_*, t \cdot \theta)$ are also μ_L-polystable, we have a pluri-harmonic metric h_t for $(E, \overline{\partial}_E, t \cdot \theta)$ on $X - D$ adapted to the parabolic structure, due to Theorem 10.2. Therefore, we obtain the family of the representations $\rho'_t : \Gamma \longrightarrow \mathrm{GL}(E_{|x})$ $(t \in \boldsymbol{C}^*)$. We remark that the ρ'_t is independent of a choice of pluri-harmonic metrics h_t.

Let V be a $\boldsymbol{C}$-vector space whose rank is same as $\operatorname{rank} E$. Let h_V be a hermitian vector space of V. For any $t \in \boldsymbol{C}^*$, we take isometries $\Phi_t : (E_{|x}, h_{t\,|\,x}) \longrightarrow (V, h_V)$, and then we obtain the family of representations $\rho_t := \Phi_{t\,*}(\rho'_t) \in R(\Gamma, \mathrm{GL}(V))$. We remark that $\pi_{\mathrm{GL}(V)}(\rho_t)$ are independent of choices of Φ_t. Thus we obtain the map $\mathcal{P} : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$ by $\mathcal{P}(t) = \pi_{\mathrm{GL}(V)}(\rho_t)$.

Theorem 11.1. — *The induced map $\mathcal{P}$ is continuous.*

Proof We may and will assume that $(\boldsymbol{E}_*, \theta)$ is μ_L-stable for the proof. Let $\{t_i \in \boldsymbol{C}^* \,|\, i \in \mathbb{Z}_{>0}\}$ be a sequence converging to t_0. The theorem can easily be reduced to the following lemma.

Lemma 11.2. — *There exist a subsequence $\{t_i \,|\, i \in S\}$ and a sequence of isometries $\bigl\{\Psi_i : \bigl(E_{|x}, h_{t_i\,|\,x}\bigr) \longrightarrow \bigl(E_{|x}, h_{t_0\,|\,x}\bigr) \bigm| i \in S\bigr\}$ such that $\bigl\{\Psi_{i\,*}\bigl(\rho_{t_i}\bigr) \bigm| i \in S\bigr\}$ converges to ρ_{t_0}.*

Proof Since the sections $\det(T - t_i \cdot \theta)$ of $\mathrm{Sym}^{\cdot}\,\Omega^{1,0}[T]$ converges to $\det(T - t_0 \cdot \theta)$, we may apply Theorem 9.10. Hence there exists a subsequence $\bigl\{t_i \bigm| i \in S'\bigr\}$ such that $\bigl\{\bigl(E, \overline{\partial}_E, h_{t_i}, t_i \cdot \theta_i\bigr) \bigm| i \in S'\bigr\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, h', \theta')$ in L_2^p locally on $X - D$ via some isometries $\Phi_i : (E, h_{t_i}) \longrightarrow (E', h')$ $(i \in S')$. It is easy to see that the representations $\Phi_{i\,|\,x\,*}(\rho_{t_i})$ converges to ρ' in $R(\Gamma, E'_{|x}, h'_{|x})$, where ρ' is associated to the flat connection $\overline{\partial}_{E'} + \partial_{E'} + \theta' + \theta'^{\,\dagger}$.

We also have the non-trivial holomorphic map $f : {}_{\boldsymbol{c}}E' \longrightarrow {}_{\boldsymbol{c}}E$ which is compatible with the parabolic structure and the Higgs fields due to Theorem 9.10. Since $({}_{\boldsymbol{c}}E'_*, \theta')$ is μ_L-polystable and $({}_{\boldsymbol{c}}E_*, t_0 \cdot \theta)$ is μ_L-stable, the map f is isomorphic. Then we have $f_{|x\,*}(\rho') = \rho_{t_0}$. By replacing f appropriately, we may assume $f : E' \longrightarrow E$ is isometric with respect to h' and h_{t_0}. Hence $\Psi_i := \bigl(f \circ \Phi_i\bigr)_{|x}$ gives the desired isometries. Thus Proposition 11.2 and Theorem 11.1 are proved. □

11.1.3. Limit. —

Lemma 11.3. — *$\mathcal{P}\bigl(\{t \in \boldsymbol{C}^* \,|\, |t| < 1\}\bigr)$ is relatively compact in $M(\Gamma, V, h_V)$.*

Proof The sequence of sections $\det(T - t \cdot \theta)$ of $\mathrm{Sym}^{\cdot}\,\Omega^{1,0}[T]$ clearly converges to $T^{\operatorname{rank} E}$ when $t \to 0$. Hence we may apply the first claim of Theorem 9.10, and we obtain a subsequence $\{t_i\}$ converging to 0 such that $\bigl\{(E, \overline{\partial}_E, t_i \cdot \theta, h_{t_i})\bigr\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, \theta', h')$ weakly in L_2^p locally on $X - D$. Then we easily obtain the convergence of the sequence $\bigl\{\pi_{\mathrm{GL}(V)}(\rho_{t_i})\bigr\}$ in $M(\Gamma, V, h_V)$. □

Ideally, the sequence $\{\mathcal{P}(t)\}$ should converge in $t \to 0$, and the limit should come from a variation of polarized Hodge structure. We discuss only a partial but useful result about it.

Let us recall relative Higgs sheaves. In the following, we put $\boldsymbol{C}_t := \operatorname{Spec} \boldsymbol{C}[t]$ and $\boldsymbol{C}_t^* := \operatorname{Spec} \boldsymbol{C}[t, t^{-1}]$. For a smooth morphism $Y_1 \longrightarrow Y_2$, the sheaf of relative holomorphic $(1,0)$-forms are denoted by $\Omega^{1,0}_{Y_1/Y_2}$. We put $\mathfrak{X} := X \times \boldsymbol{C}_t$ and $\mathfrak{X}^* := X \times \boldsymbol{C}_t^*$. Similarly, $\mathfrak{D}$ and $\mathfrak{D}^*$ are given. We put ${}_{\boldsymbol{c}}\widetilde{E}_* := {}_{\boldsymbol{c}}E_* \otimes \mathcal{O}_{\boldsymbol{C}_t^*}$ which is $\boldsymbol{c}$-parabolic bundle on $(\mathfrak{X}^*, \mathfrak{D}^*)$. Then, $t \cdot \theta$ gives the relative Higgs field $\widetilde{\theta}$, which is a homomorphism ${}_{\boldsymbol{c}}\widetilde{E}_* \longrightarrow {}_{\boldsymbol{c}}\widetilde{E}_* \otimes \Omega^{1,0}_{\mathfrak{X}^*/\boldsymbol{C}^*_t}(\log \mathfrak{D}^*)$ such that $\widetilde{\theta}^2 = 0$. Using the standard argument of Langton [**27**], we obtain the $\boldsymbol{c}$-parabolic sheaf ${}_{\boldsymbol{c}}\widetilde{E}'_*$ and relative Higgs field $\widetilde{\theta}' : {}_{\boldsymbol{c}}\widetilde{E}'_* \longrightarrow {}_{\boldsymbol{c}}\widetilde{E}'_* \otimes \Omega^{1,0}_{\mathfrak{X}/\boldsymbol{C}_t}$ satisfying the following (see [**58**]):

- ${}_{\boldsymbol{c}}\widetilde{E}'_*$ is flat over $\boldsymbol{C}_t$, and the restriction to $\mathfrak{X}^*$ is ${}_{\boldsymbol{c}}\widetilde{E}_*$.
- The restriction of $\widetilde{\theta}'$ to $\mathfrak{X}^*$ is $\widetilde{\theta}$.
- $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta}) := \big({}_{\boldsymbol{c}}\widetilde{E}'_*, \widetilde{\theta}'\big)_{|X\times\{0\}}$ is μ_L-semistable.

Proposition 11.4. — *Assume that $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ is a μ_L-stable $\boldsymbol{c}$-parabolic Higgs sheaf. We put $\widehat{E} := {}_{\boldsymbol{c}}\widehat{E}_{|X-D}$. Then the following holds:*

- *$({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ is a Hodge bundle, i.e., $({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta}) \simeq ({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ for any $\alpha \in \boldsymbol{C}^*$.*
- *We have a pluri-harmonic metric $\widehat{h}$ of a Hodge bundle $(\widehat{E}, \widehat{\theta})$, which is adapted to the parabolic structure. It induces the variation of polarized Hodge structure. Thus we obtain the corresponding representation $\widehat{\rho} : \pi_1(X-D, x) \longrightarrow \mathrm{GL}(\widehat{E}_{|x})$ which underlies a variation of polarized Hodge structure.*
- *Take any isometry $G : (\widehat{E}_x, \widehat{h}_{|x}) \simeq (V, h_V)$. Then the sequence $\big\{\pi_{\mathrm{GL}(V)}(\rho_t)\big\}$ converges to $\pi_{\mathrm{GL}(V)}\big(G_*(\widehat{\rho})\big)$ in $M(\Gamma, V, h_V)$ for $t \to 0$.*
- *In particular, the map $\pi_{\mathrm{GL}(V)}(\rho_t) : \boldsymbol{C}^* \longrightarrow M(\Gamma, V, h_V)$ is continuously extended to the map of $\boldsymbol{C}$ to $M(\Gamma, V, h_V)$.*

Proof The argument is essentially due to Simpson [**47**]. The fourth claim follows from the third one. Let $\{t_i \,|\, i \in \mathbb{Z}_{>0}\}$ be a sequence converging to 0. Due to Theorem 9.10, there exists a subsequence $\{t_i \,|\, i \in S\}$ such that the sequence $\big\{\big(E, \overline{\partial}_E, h_{t_i}, t_i\cdot\theta\big) \,\big|\, i \in S\big\}$ converges to a tame harmonic bundle $(E', \overline{\partial}_{E'}, h', \theta')$ in L^p_2 locally on $X-D$, via isometries $\Phi_i : (E, h_{t_i}) \longrightarrow (E', h')$. Let $\rho' : \pi_1(X-D, x) \longrightarrow \mathrm{GL}(E'_{|x})$ denote the representation associated to the flat connection $\overline{\partial}_{E'} + \partial_{E'} + \theta' + \theta'^{\dagger}$. Then we have the convergence of $\big\{\Phi_{i|x\,*}(\rho_{t_i}) \,\big|\, i \in S''\big\}$ to ρ' in $M(\Gamma, \widehat{E}_{|x}, \widehat{h}_{|x})$. Due to Theorem 9.10, we also have a non-trivial morphism $f : {}_{\boldsymbol{c}}E' \longrightarrow {}_{\boldsymbol{c}}\widehat{E}$ which is compatible with the parabolic structures and the Higgs fields. Then it must be isomorphic due to μ_L-polystability of $({}_{\boldsymbol{c}}E'_*, \theta')$ and μ_L-stability of $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$. In particular, $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ is a μ_L-stable $\boldsymbol{c}$-parabolic Higgs bundle. The metric $\widehat{h}$ of $\widehat{E}$ is given by h' and f.

Let us consider the morphism $\phi_\alpha : \boldsymbol{C}_t \longrightarrow \boldsymbol{C}_t$ given by $t \longmapsto \alpha\cdot t$. We have the natural isomorphism $\phi^*_\alpha\big({}_{\boldsymbol{c}}\widetilde{E}_*, \widetilde{\theta}\big) \simeq \big({}_{\boldsymbol{c}}\widetilde{E}_*, \alpha\cdot\widetilde{\theta}\big)$ which can be extended to the morphism $\phi^*_\alpha({}_{\boldsymbol{c}}\widetilde{E}'_*, \widetilde{\theta}') \longrightarrow ({}_{\boldsymbol{c}}\widetilde{E}'_*, \alpha\cdot\widetilde{\theta}')$ such that the specialization $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta}) \longrightarrow ({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta})$ at $t = 0$ is not trivial. Since $({}_{\boldsymbol{c}}\widehat{E}_*, \widehat{\theta})$ and $({}_{\boldsymbol{c}}\widehat{E}_*, \alpha\cdot\widehat{\theta})$ are μ_L-stable, the map is isomorphic. Hence $({}_{\boldsymbol{c}}\widehat{E}, \widehat{\theta})$ is a Hodge bundle. Thus the first and the third claims are proved.

Since $(\widehat{E}, \overline{\partial}_{\widehat{E}}, \widehat{\theta})$ is a Hodge bundle, we have the S^1-action κ on $\widehat{E}$ such that $\kappa(t) : \big(\widehat{E}, \overline{\partial}_{\widehat{E}}, \widehat{\theta}\big) \simeq \big(\widehat{E}, \overline{\partial}_{\widehat{E}}, t\cdot\widehat{\theta}\big)$ for any $t \in S^1$. The metric $\kappa(t)_*\widehat{h}$ is determined by $\kappa(t)_*\widehat{h}(u, v) = \widehat{h}\big(\kappa(t)(u), \kappa(t)(v)\big)$, which is also the pluri-harmonic metric of $(\widehat{E}, \overline{\partial}_{\widehat{E}}, t\cdot\theta)$. Since $(\widehat{\boldsymbol{E}}_*, t\cdot\widehat{\theta})$ is μ_L-stable, the pluri-harmonic metric is unique up to a positive constant multiplication. Hence we obtain the map $\nu : S^1 \longrightarrow \boldsymbol{R}_{>0}$ such that $\kappa(t)_*\widehat{h} =$

$\nu(t)\cdot\widehat{h}$. Since ν is a homomorphism of groups, we obtain $\nu(t)=1$ for any $t\in S^1$. Namely, $\widehat{h}$ is S^1-invariant, which means $\big(\widehat{E},\overline{\partial}_{\widehat{E}},\widehat{\theta},\widehat{h}\big)$ gives a variation of polarized Hodge structure. Thus the second claim is proved. □

Lemma 11.5. — *Assume $({}_{\boldsymbol{c}}\widehat{E}_*,\widehat{\theta})$ is not μ_L-stable. Let ρ_0 be an element of $R(\Gamma,V)$ such that $\pi_{\mathrm{GL}(V)}(\rho_0)$ is the limit of a subsequence $\big\{\pi_{\mathrm{GL}(V)}(\rho_{t_i})\big\}$ for $t_i\to 0$. Then ρ_0 is not simple.*

Proof Let $\{t_i\}$ be a sequence converging to 0 such that $\big\{(E,\overline{\partial}_E,t_i\cdot\theta,h_{t_i})\big\}$ converges to a tame harmonic bundle $(E',\overline{\partial}_{E'},\theta',h')$ in L_2^p locally on $X-D$. We may assume that ρ_0 is the associated representation to $(E',\overline{\partial}_{E'},\theta',h')$. We have a nontrivial map $f:{}_{\boldsymbol{c}}E'\longrightarrow{}_{\boldsymbol{c}}\widehat{E}$ compatible with the parabolic structures and the Higgs fields. If ρ_0 is simple, then $({}_{\boldsymbol{c}}E'_*,\theta')$ is μ_L-stable, and it can be shown that the map f has to be isomorphic by the same argument in the proof of Proposition 11.4. But it contradicts the assumption that $({}_{\boldsymbol{c}}\widehat{E},\widehat{\theta})$ is not μ_L-stable. □

11.1.4. Deformation to a variation of polarized Hodge structure. — Let Y be a smooth quasi projective variety over $\boldsymbol{C}$ with a base point x. We may assume that $Y=X-D$, where X and D denote a smooth projective variety and its simple normal crossing divisor respectively. A representation $\rho:\pi_1(Y,x)\longrightarrow\mathrm{GL}(V)$ induces a flat bundle (E,∇). We say that ρ comes from a variation of polarized Hodge structure, if (E,∇) underlies a variation of polarized Hodge structure. For simplicity of the notation, we put $\Gamma:=\pi_1(Y,x)$.

Theorem 11.6. — *Let $\rho\in R(\Gamma,V)$ be a representation. Then it can be deformed to a representation $\rho'\in R(\Gamma,V)$ which comes from a variation of polarized Hodge structure on Y.*

Proof We essentially follow the argument of Theorem 3 in [**47**]. Any representation $\rho\in R(\Gamma,V)$ can be deformed to a semisimple representation $\rho'\in R(\Gamma,V)$. Therefore we may assume that ρ is semisimple from the beginning. Let (E,∇) be the corresponding semisimple flat bundle on $X-D$. We can take a Jost-Zuo metric h of (E,∇), and hence we obtain the tame pure imaginary harmonic bundle $(E,\overline{\partial}_E,\theta,h)$. Let $(\boldsymbol{E}_*,\theta)$ denote the associated regular filtered Higgs bundle on (X,D). We have the canonical decomposition (Corollary 3.4):

$$(\boldsymbol{E}_*,\theta)=\bigoplus_{j\in\Lambda}(\boldsymbol{E}_{i*},\theta_i)\otimes\boldsymbol{C}^{m(j)}.$$

We put $r(\rho):=\sum_{j\in\Lambda}m(j)$. Note that $r(\rho)\leq\operatorname{rank}E$, and we have $r(\rho)=\operatorname{rank}E$ if and only if $(\boldsymbol{E}_*,\theta)$ is a direct sum of Higgs bundles of rank one. We use a descending induction on $r(\rho)$.

We obtain the family of regular filtered Higgs bundles $\big\{(\boldsymbol{E}_*,t\cdot\theta)\,\big|\,t\in\boldsymbol{C}^*\big\}$ ($t\in\boldsymbol{C}^*$). In particular, we have the associated deformation of representations $\{\rho_t\in R(\Gamma,V)\,|\,t\in\boldsymbol{R}_{>0}\}$ as in the subsection 11.1.2. We may assume that $\rho_1=\rho$. We have the induced map $\mathcal{P} :]0,1] \longrightarrow M(\Gamma, V, h_V)$ given by $\mathcal{P}(t) := \pi_{\mathrm{GL}(V)}(\rho_t)$, which is continuous due to Theorem 11.1. The image is relatively compact due to Lemma 11.3. We take a representation $\rho_0 \in R(\Gamma, V)$ such that $\pi_{\mathrm{GL}(V)}(\rho_0)$ is the limit of a subsequence of $\{\pi_{\mathrm{GL}(V)}(\rho_t) \,|\, t \in]0,1]\}$. We may assume that it comes from a tame harmonic bundle as in the proof of Lemma 11.3.

The case 1. Assume that each family $\{(\boldsymbol{E}_{i\,*}, t\cdot\theta_i) \,|\, t \in \boldsymbol{C}^*\}$ converges to the stable regular filtered Higgs bundle. Then ρ_0 comes from a variation of polarized Hodge structure due to Proposition 11.4.

We remark that the rank one Higgs bundle is always stable. Hence the case $r(\rho) = \operatorname{rank} E$ is done, in particular.

The case 2. Assume that one of the families $\{(E_i, \boldsymbol{F}_i, t\cdot\theta_i) \,|\, t \in \boldsymbol{C}^*\}$ converges to the semistable parabolic Higgs bundle, which is not μ_L-stable. Then we have $r(\rho) < r(\rho')$ due to Lemma 11.5. Hence the induction can proceed. □

11.2. Monodromy group

We discuss the monodromy group for the Higgs bundles or flat bundles, by following the ideas in [**47**].

11.2.1. The Higgs monodromy group. — Let (X, D) be a pair of smooth projective variety and a simple normal crossing divisor, as before. Let L be a polarization of X. Let $(\boldsymbol{E}_*, \theta)$ be a μ_L-polystable regular filtered Higgs bundle on (X, D) with $\text{par-deg}_L(\boldsymbol{E}_*) = \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$. For any non-negative integers a and b, we have the regular filtered Higgs bundles $(T^{a,b}\boldsymbol{E}_*, \theta)$. (See the subsection 3.2.1 for the explanation.) Since we have a pluri-harmonic metric h of $(E, \overline{\partial}_E, \theta)$ adapted to the parabolic structure, the regular filtered Higgs bundles $T^{a,b}(\boldsymbol{E}_*, \theta)$ are also μ_L-polystable. In particular, we have the canonical decompositions of them. We recall the definition of the Higgs monodromy group given in [**47**].

Definition 11.7. — Let x be a point of $X - D$. The Higgs monodromy group $M(\boldsymbol{E}_*, \theta, x)$ of μ_L-polystable Higgs bundle $(\boldsymbol{E}_*, \theta)$ is the subgroup of $\mathrm{GL}(E_{|x})$ defined as follows: An element $g \in \mathrm{GL}(E_{|x})$ is contained in $M(\boldsymbol{E}_*, \theta, x)$ if and only if the induced endomorphisms $T^{a,b}g$ preserve the decompositions of $T^{a,b}E_{|x}$ induced by the canonical decompositions of $T^{a,b}(\boldsymbol{E}_*, \theta)$ for any $(a, b) \in \mathbb{Z}^2_{\geq 0}$. □

Remark 11.8. — Although such a Higgs monodromy group should be defined for semistable parabolic Higgs bundles as in [**47**], we do no need it in this paper. □

We have an obvious lemma.

Lemma 11.9. — *We have $M(\boldsymbol{E}_*, \theta, x) = M(\boldsymbol{E}_*, t\cdot\theta, x)$ for any $t \in \boldsymbol{C}^*$, i.e., the Higgs monodromy group is invariant under the torus action.* □

Let us take a pluri-harmonic metric h of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ on $X - D$, which is adapted to the parabolic structure. Then we obtain the flat connection $\mathbb{D}^1 = \overline{\partial}_E + \partial_E + \theta + \theta^\dagger$. Then we obtain the monodromy group $M(E, \mathbb{D}^1, x) \subset \mathrm{GL}(E_{|x})$ of the flat connection. (See the subsection 12.1.4.)

Lemma 11.10. — *In general, we have $M(E, \mathbb{D}^1, x) \subset M(\boldsymbol{E}_*, \theta, x)$. For a tame pure imaginary harmonic bundle, we have $M(E, \mathbb{D}^1, x) = M(\boldsymbol{E}_*, \theta, x)$.*

Proof The canonical decomposition $T^{a,b}(\boldsymbol{E}_*, \theta) = \bigoplus_{j \in S(a,b)} (\boldsymbol{E}_{j*}, \theta_j)$ induces the decomposition of the flat bundles: $T^{a,b}(E, \mathbb{D}^1) = \bigoplus_{j \in S(a,b)} (E_j, \nabla_j)$. Since any element $g \in M(E, \mathbb{D}^1, x)$ preserves $E_{j\,|\,x} \subset T^{a,b} E_{|x}$, we have $M(E, \mathbb{D}^1, x) \subset M(\boldsymbol{E}_*, \theta, x)$. In the pure imaginary case, the decomposition $T^{a,b}(E, \mathbb{D}^1)$ is same as the canonical decomposition of the flat bundle. Hence we have $M(E, \mathbb{D}^1, x) = M(\boldsymbol{E}_*, \theta, x)$. □

11.2.2. The deformation and the monodromy group. — Let X be a smooth projective variety with a polarization L, D be a simple normal crossing divisor, and x be a point of $X - D$. For simplicity of the description, we put $\Gamma := \pi_1(X - D, x)$.

Let (E, ∇) be a semisimple flat bundle over $X - D$. We have a Jost-Zuo metric h of (E, ∇), and thus we obtain a tame pure imaginary harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$. The associated regular filtered Higgs bundle is denoted by $(\boldsymbol{E}_*, \theta)$, which is μ_L-polystable with $\text{par-deg}_L(\boldsymbol{E}_*) = \text{par-ch}_{2,L}(\boldsymbol{E}_*) = 0$.

As in the subsection 11.1.2, we have pluri-harmonic metrics h_t for any $(E, \overline{\partial}_E, t \cdot \theta)$ $(t \in \boldsymbol{C}^*)$. Hence we obtain the flat connections $\mathbb{D}^1_t$ of E, and the representations $\rho_t : \pi_1(X - D, x) \longrightarrow \mathrm{GL}(E_{|x})$.

Lemma 11.11. — *We have $M(E, \mathbb{D}^1_t) \subset M(E, \mathbb{D}^1_1)$ for $t \in \boldsymbol{C} - \{0\}$, and $M(E, \mathbb{D}^1_t) = M(E, \mathbb{D}^1_1)$ for $t \in \boldsymbol{R} - \{0\}$.*

Proof It follows from Lemma 11.9 and Lemma 11.10. □

We put $G_0 := M(E, \mathbb{D}^1_t, x)$ for $(t \in \boldsymbol{R}_{>0})$ which is independent of choice of t. Let $U(E, h_t, x)$ denote the unitary group for the metrized space $(E_{|x}, h_{t\,|\,x})$. Due to Lemma 12.17, the intersection of $K_{0,t} := G_0 \cap U(E, h_t, x)$ are maximal compact in G_0.

We put $V := E_{|x}$ and $h_V := h_{1\,|\,x}$. We denote G_0 and $K_{0,1}$ by G and K respectively, when we regard it as the subgroup of $\mathrm{GL}(V)$. Then we can take an isometry $\nu_t : (E_{|x}, h_{t\,|\,x}) \simeq (V, h_V)$ such that $\nu_t(G_0) = G$ and $\nu_t(K_{0\,t}) = K$ for each t. Such a map is unique up to the adjoint of $N_G(h_V)$. Thus we obtain the family of representations $\widetilde{\rho}_t := \nu_{t\,*}(\rho'_t) \in R(\Gamma, G)$ $(t \in \boldsymbol{R}_{>0})$.

Lemma 11.12. — *The induced map $\pi_G(\widetilde{\rho}_t) : \boldsymbol{R}_{>0} \longrightarrow M(\Gamma, G, h_V)$ is continuous.*

Proof It follows from Lemma 2.18 and Theorem 11.1. □

Lemma 11.13. — *The image $\pi_G(\rho_t)\big(]0,1]\big)$ is relatively compact in $M(\Gamma, G, h_V)$.*

Proof It follows from Lemma 11.3 and the properness of the map $M(\Gamma, G, h_V) \longrightarrow M(\Gamma, V, h_V)$. $\square$

11.2.3. Deformation to variation of polarized Hodge structure. — Let Y be a quasi projective variety. We put $\Gamma := \pi_1(Y, x)$. Let V be a finite dimensional $\boldsymbol{C}$-vector space. Let G be a reductive subgroup of $\mathrm{GL}(V)$.

Lemma 11.14. — *Let ρ be an element of $R(\Gamma, G)$. We assume that there exists a subgroup Γ_0 such that $\rho_{|\Gamma_0} : \Gamma_0 \longrightarrow G$ is Zariski dense and rigid. Then we can take a deformation $\rho' \in R(\Gamma, G)$ of ρ which comes from a variation of polarized Hodge structure on Y.*

Proof We remark that $\rho : \Gamma \longrightarrow \mathrm{GL}(V)$ is semisimple. Let us take a tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ whose associated representation gives ρ. Then let us consider the associated deformation $\rho_t : \Gamma \longrightarrow G$ $(t \in \boldsymbol{R}_{>0})$ as above. It induces the continuous map $\pi_G(\rho_{t\,|\,\Gamma_0}) : \boldsymbol{R}_{>0} \longrightarrow M(\Gamma_0, G, h_V)$.

Let B be the connected component of $M(\Gamma, G, h_V)$. Since $\rho_{|\Gamma_0} = \rho_{1\,|\,\Gamma_0}$ is rigid and Zariski dense, the natural map $B \longrightarrow M(\Gamma, V, h_V)$ is injective.

Let us take $\rho_0 \in R(\Gamma, G)$ and a subsequence $\{t_i\}$ converging to 0 such that $\{\pi_G(\rho_{t_i})\}$ converges to ρ_0. We remark that $\rho_{0\,|\,\Gamma_0} : \Gamma_0 \longrightarrow G$ is also Zariski dense and rigid. If it comes from a variation of polarized Hodge structure, we are done. If it does not come from a variation of polarized Hodge structure, we deform ρ_0 as above, again. The process will stop in the finite steps by Theorem 11.6. $\square$

11.2.4. Non-existence result about fundamental groups. — The following lemma is a straightforward generalization of Lemma 4.4 in [**47**].

Lemma 11.15. — *Let $\rho : \pi_1(Y, x) \longrightarrow G$ be a Zariski dense homomorphism. If ρ comes from a variation of polarized Hodge structure, then the real Zariski closure W of ρ is a real form of G, and W is a group of Hodge type in the sense of Simpson. (See the page* 46 *in* [**47**].*)* $\square$

The following lemma is essentially same as Corollary 4.6 in [**47**].

Proposition 11.16. — *Let G be a complex reductive algebraic group, and W be a real form of G. Let $\rho : \pi_1(Y, x) \longrightarrow G$ be a representation such that $\mathrm{Im}\,\rho \subset W$. Assume that there exists a subgroup $\Gamma_0 \subset \pi_1(Y, x)$ such that $\rho_{|\Gamma_0}$ is rigid and Zariski dense in G. Then W is a group of Hodge type, in the sense of Simpson.*

Proof We reproduce the argument of Simpson. We take a deformation ρ' of ρ, which comes from a variation of polarized Hodge structure as in Lemma 11.14. Then there exists an element $u \in N(G, U)$ such that $\mathrm{ad}(u) \circ \rho_{|\Gamma_0} \simeq \rho'_{|\Gamma_0}$. Let W'' denote the real Zariski closure of $\rho'(\Gamma_0)$, and W' denote the real Zariski closure of ρ'. Then we have $W'' \subset W'$. Since ρ' comes from a variation of Hodge structure, W' is a real form in G (Lemma 11.15). Since W is a real form of G, $W'' = \mathrm{ad}(u)(W)$ is also a real form of G. Therefore we have $W' = W''$. Since W' is a group of Hodge type, we are done. □

Corollary 11.17. — *Let* Γ_0 *be a rigid discrete subgroup of a real algebraic group, which is not of Hodge type. Then* Γ_0 *cannot be a split quotient of the fundamental groups of any smooth quasi projective variety.*

Proof It follows from Lemma 11.15 and Proposition 11.16. (See the pages 52–54 of [**47**]). □

CHAPTER 12

G-HARMONIC BUNDLE (APPENDIX)

12.1. G-principal bundles with flat structure or holomorphic structure

We recall the Tannakian consideration about harmonic bundles given in [**47**] by Simpson.

12.1.1. A characterization of algebraic subgroup of GL. — We recall some facts on algebraic groups. (See also I. Proposition 3.1 in [**8**], for example.) Let V be a vector space over a field k of characteristic 0. We put $T^{a,b}V := Hom(V^{\otimes a}, V^{\otimes b})$. Let G be an algebraic subgroup of $\mathrm{GL}(V)$, defined over k. We have the induced G-action on $T^{a,b}V$. Let $\mathcal{S}(V,a,b)$ denote the set of G-subspaces of $T^{a,b}V$, and we put $\mathcal{S}(V) = \coprod_{a,b} \mathcal{S}(V,a,b)$.

Let g be an element of $\mathrm{GL}(V)$. We have the induced element $T^{a,b}(g) \in \mathrm{GL}\big(T^{a,b}V\big)$. Then it is known that $g \in \mathrm{GL}(V)$ is contained in G if and only if $T^{a,b}(g)W \subset W$ holds for any $(W,a,b) \in \mathcal{S}(V)$. Suppose G is reductive. Then there is an element v of $T^{a,b}(V)$ for some (a,b) such that g is contained in G if and only if $g \cdot v = v$ holds.

We easily obtain the corresponding characterization of Lie subalgebras of $\mathfrak{gl}(V)$ corresponding to algebraic subgroups of $\mathrm{GL}(V)$.

12.1.2. A characterization of connections of principal G-bundle. — Let k denote the complex number field $\boldsymbol{C}$ or the real number field $\boldsymbol{R}$. Let G be an algebraic group over k. Let P_G be a G-principal bundle on a manifold X in the C^∞-category. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be a representation defined over k, such that the induced morphism $d\kappa : \mathfrak{g} \longrightarrow End(V)$ is injective. We put $E := P_G \times_G V$. We have $T^{a,b}E := Hom(E^{\otimes a}, E^{\otimes b}) \simeq P_G \times_G T^{a,b}V$. We have the subbundle $E_U = P_G \times_G U$ of $T^{a,b}E$ for each $U \in \mathcal{S}(V,a,b)$. A connection ∇ on E induces the connection $T^{a,b}\nabla$ on $T^{a,b}E$. Let $\mathcal{A}_G(E)$ be the set of the connections ∇ of E such that the induced connections $T^{a,b}\nabla$ preserve the subbundle E_U for any $(U,a,b) \in \mathcal{S}(V)$.

Let $\mathcal{A}(P_G)$ denote the set of the connections of P_G. If we are given a connection of P_G, the connection ∇ of E is naturally induced. It is clear that the connection

$T^{a,b}\nabla$ preserves $E_U \subset T^{a,b}E$ for any $(U,a,b) \in \mathcal{S}(V)$. Hence we have the map $\varphi : \mathcal{A}(P_G) \longrightarrow \mathcal{A}_G(E)$.

Lemma 12.1. — *The map φ is bijective.*

Proof Since $d\kappa$ is injective, the map φ is injective. Let us take a connection $\nabla \in \mathcal{A}_G(E)$ and a connection ∇_0 which comes from a connection of P_G. Then $f = \nabla - \nabla_0$ is a section of $\mathrm{End}(E) \otimes \Omega^1$. Since $T^{a,b}f$ preserves E_U for any (a,b) and $U \subset \mathcal{S}(V,a,b)$, f comes from a section of $\mathrm{ad}(P_G) \otimes \Omega^1 \subset \mathrm{End}(E) \otimes \Omega^1$. $\square$

12.1.3. K-reduction of holomorphic G-principal bundle and the induced connection. — Let G be a linear reductive group defined over $\boldsymbol{C}$. Let P_G be a *holomorphic* G-principal bundle on X. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be a representation defined over $\boldsymbol{C}$, such that $d\kappa : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is injective. We put $E = P_G \times_G V$. Let K be a maximal compact group of G, or equivalently, a compact real form. Let $P_K \subset P_G$ be a K-reduction in the C^∞-category, i.e., $P_K \times_K G \simeq P_G$. Then the connection of P_K is automatically induced. Namely, we have the canonical G-decomposition for each (a,b):

$$T^{a,b}V = \bigoplus_{\rho \in \mathrm{Irrep}(G)} V_\rho^{(a,b)}. \tag{85}$$

Here $\mathrm{Irrep}(G)$ denotes the set of the equivalence classes of irreducible representations of G. Each $V_\rho^{(a,b)}$ is isomorphic to the tensor product of the irreducible representation ρ and the trivial representation $\boldsymbol{C}^{m(a,b,\rho)}$. The decomposition (85) is same as the canonical K-decomposition. Let us take a K-invariant hermitian metric h of V. It induces the hermitian metric $T^{a,b}h$ of $T^{a,b}V$, for which the decomposition (85) is orthogonal. The restriction of $T^{a,b}h$ to $V_\rho^{(a,b)}$ is isomorphic to a tensor product of a K-invariant hermitian metric on ρ and a hermitian metric on $\boldsymbol{C}^{m(a,b,\rho)}$. The metric h induces the hermitian metric of E, which is also denoted by h. From the holomorphic structure $\overline{\partial}_E$ and the metric h, we obtain the unitary connection $\nabla = \partial_E + \overline{\partial}_E$. The induced connection $T^{a,b}\nabla$ on $T^{a,b}E$ is the unitary connection determined by $T^{a,b}h$ and the holomorphic structure of $T^{a,b}E$. Then it is easy to see that $T^{a,b}\nabla$ preserves E_U for any $U \in \mathcal{S}(a,b,V)$. Hence the connection ∇ comes from P_G. Since ∇ also preserves the unitary structure, we can conclude that ∇ comes from the connection of P_K.

12.1.4. The monodromy group. — We recall the monodromy group of flat bundles ([**47**]). Let X be a complex manifold with a base point x. The monodromy group of a flat bundle (E,∇) at x is defined to be the Zariski closure of the induced representation $\pi_1(X,x) \longrightarrow \mathrm{GL}(E_{|x})$. It is denoted by $M(E,\nabla,x)$. Let us recall the case of principal bundles. Let G be a linear algebraic group over $\boldsymbol{R}$ or $\boldsymbol{C}$, and P_G be a G-principal bundle on X with a flat connection in the C^∞-category. Let us take a point $\widetilde{x} \in P_{G|x}$. Then we obtain the representation $\rho : \pi_1(X,x) \longrightarrow G$. Then the monodromy group $M(P_G, \widetilde{x}) \subset G$ is defined to be the Zariski closure of the image of ρ. We obtain the canonical reduction of principal bundles $P_{M(P_G,\widetilde{x})} \subset P_G$. The monodromy groups of the flat vector bundles and flat principal bundles are related as follows. Let $\kappa : G \longrightarrow \mathrm{GL}(V)$ be an injective representation. Then we have the flat bundle $E = P_G \times_G V = P_{M(P_G,\widetilde{x})} \times_{M(P_G,\widetilde{x})} V$. Via the identification $V = E_{|x}$ given by $\widetilde{x}$, we are given the inclusion $M(P_G, \widetilde{x}) \subset \mathrm{GL}(E_{|x})$. Clearly $M(P_G, \widetilde{x})$ is same as $M(E, \nabla, x)$ and it is independent of a choice of $\widetilde{x}$. Hence we can reduce the problems of the monodromy groups of flat principal G-bundles to those for flat vector bundles.

For a flat bundle (E, ∇), let $T^{a,b}E$ denote the flat bundle $Hom(E^{\otimes a}, E^{\otimes b})$ provided the canonically induced flat connection. Let $\mathcal{S}(E, a, b)$ denote the set of flat subbundles U of $T^{a,b}E$, and we put $\mathcal{S}(E) := \coprod_{(a,b)} \mathcal{S}(E, a, b)$. Let g be an element of $\mathrm{GL}(E_{|x})$. Then g is contained in $M(E, \nabla, x)$ if and only if $T^{a,b}g$ preserves U_x for any $(U, a, b) \in \mathcal{S}(E)$. If $M(E, \nabla, x)$ is reductive, we can find some (a, b) and $v \in T^{a,b}E_{|x}$ such that $g \in M(E, \nabla, x)$ if and only if $g \cdot v = v$. Hence there exists a flat subbundle $W \subset T^{a,b}E$ such that $g \in M(E, \nabla, x)$ if and only if $T^{a,b}g_{|W} = \mathrm{id}_W$.

12.2. Definitions

12.2.1. A G-principal Higgs bundle and a pluri-harmonic reduction. — Let G be a linear reductive group defined over $\boldsymbol{C}$, and K be a maximal compact group. Let X be a complex manifold and P_G be a holomorphic G-principal bundle on X. Let $\mathrm{ad}(P_G)$ be the adjoint bundle of P_G, i.e., $\mathrm{ad}(P_G) = P_G \times_G \mathfrak{g}$. Recall that a Higgs field of P_G is defined to be a holomorphic section θ of $\mathrm{ad}(P_G) \otimes \Omega^{1,0}$ such that $\theta^2 = 0$.

Let $P_K \subset P_G$ be a K-reduction of P_G in C^∞-category, then we have the natural connection ∇ of P_K, as is seen in the subsection 12.1.3. We also have the adjoint $\theta^\dagger$ of θ, which is a C^∞-section of $\mathrm{ad}(P_G) \otimes \Omega^{0,1}$. Then we obtain the connection $\mathbb{D}^1 := \nabla + \theta + \theta^\dagger$ of the principal bundle P_G.

Definition 12.2. — If $\mathbb{D}^1$ is flat, then the reduction $P_K \subset P_G$ is called pluri-harmonic, and the tuple $(P_K \subset P_G, \theta)$ is called a G-harmonic bundle. □

Let V be a $\boldsymbol{C}$-vector space. A representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ is called immersive if $d\kappa$ is injective, in this paper. Let us take an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric h_V. From a G-principal Higgs bundle (P_G, θ) with a K-reduction $P_K \subset P_G$, we obtain the Higgs bundle $(E, \overline{\partial}_E, \theta)$ with the hermitian metric h.

Lemma 12.3. — *Let (P_G, θ) be a G-principal Higgs bundle, and $P_K \subset P_G$ be a K-reduction. The following conditions are equivalent.*

1. *The reduction $P_K \subset P_G$ is pluri-harmonic.*

2. *Let us take any immersive representation $G \longrightarrow \mathrm{GL}(V)$, and any K-invariant hermitian metric of $\boldsymbol{C}$-vector space V. Then the induced Higgs bundle with the hermitian metric is a harmonic bundle.*
3. *There exist an immersive representation $G \longrightarrow \mathrm{GL}(V)$ and a K-invariant hermitian metric of $\boldsymbol{C}$-vector space V, such that the induced Higgs bundle with the hermitian metric is a harmonic bundle.*

Proof If $G \longrightarrow \mathrm{GL}(V)$ is immersive, then a connection of P_G is flat if and only if the induced connection on $P_G \times_G V$ is flat. Therefore the desired equivalence is clear. □

12.2.2. A flat G-bundle and a pluri-harmonic reduction. — Let G be a linear reductive group over $\boldsymbol{C}$ or $\boldsymbol{R}$, and K be a maximal compact group of G. Let $\pi : \widetilde{X} \longrightarrow X$ denote a universal covering. Let us take base points $x \in X$ and $x_1 \in \widetilde{X}$ such that $\pi(x_1) = x$. Let (P_G, ∇) be a flat G-principal bundle over a complex manifold X. Once we pick a point $\widetilde{x} \in P_{G|x}$, the homomorphism $\pi_1(X, x) \longrightarrow G$ is given. If a K-reduction $P_K \subset P_G$ is given, we obtain a $\pi_1(X, x)$-equivariant map $F : \widetilde{X} \longrightarrow G/K$, where the $\pi_1(X, x)$-action on G/K is given by the homomorphism $\pi_1(X, x) \longrightarrow G$.

Definition 12.4. — If the map F is pluri-harmonic, then the reduction $P_K \subset P_G$ is called pluri-harmonic. The property is independent of a choice of the points x, x_1 and $\widetilde{x}$. □

Lemma 12.5. — *The following conditions are equivalent.*

1. *The reduction $P_K \subset P_G$ is pluri-harmonic, in the sense of Definition* 12.4.
2. *Let us take any immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of a vector space V over $\boldsymbol{C}$. Then the induced flat bundle with the hermitian metric is a harmonic bundle.*
3. *There exist an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric of a vector space V over $\boldsymbol{C}$, such that the induced flat bundle with the hermitian metric is a harmonic bundle.*

Proof Let $G \longrightarrow \mathrm{GL}(V)$ be an immersive representation. Let us take a K-invariant hermitian metric of V, and let U denote the unitary group of V with respect to h. Then we have the inclusion $\iota : G/K \subset \mathrm{GL}(V)/U$, which is totally geodesic. Hence F in Definition 12.4 is pluri-harmonic if and only if $\iota \circ F$ is pluri-harmonic. The desired equivalence immediately follows. □

Corollary 12.6. — *If G is a linear reductive group over $\boldsymbol{C}$, the definitions* 12.2 *and* 12.4 *are equivalent.* □

12.2.3. A tame pure imaginary G-harmonic bundle. — Let G be a linear reductive group over $\boldsymbol{C}$. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$, and let W denote the Weyl group. We have the natural real structure $\mathfrak{h}_{\boldsymbol{R}} \subset \mathfrak{h}$. Hence we have the subspace $\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}} \subset \mathfrak{h}$. We have the W-invariant metric of $\mathfrak{h}$, which induces the distance d of $\mathfrak{h}/W$. Let $B\big(\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}, \epsilon\big)$ denote the set of the points x of $\mathfrak{h}/W$ such that there exists a point $y \in \sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}$ satisfying $d(x,y) < \epsilon$.

Let $(P_K \subset P_G, \theta)$ be a G-harmonic bundle on Δ^*. We have the expression $\theta = f \cdot dz/z$, where f is a holomorphic section of $\mathrm{ad}(P_G)$ on Δ^*. It induces the continuous map $[f] : \Delta^* \longrightarrow \mathfrak{h}/W$.

Definition 12.7. —

- A G-harmonic bundle $(P_K \subset P_G, \theta)$ is called tame, if $[f]$ is bounded.
- A tame G-harmonic bundle $(P_K \subset P_G, \theta)$ is called pure imaginary, if for any $\epsilon > 0$ there exists a positive number r such that $\big[f(z)\big] \in B\big(\sqrt{-1}\mathfrak{h}_{\boldsymbol{R}}, \epsilon\big)$ for any $|z| < r$. □

Lemma 12.8. — *Let $(P_K \subset P_G, \theta)$ be a harmonic bundle on Δ^*. The following conditions are equivalent.*

1. *It is tame (pure imaginary).*
2. *For any $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*
3. *For some immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and some K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*

Proof The implications $1 \Longrightarrow 2 \Longrightarrow 3$ are clear. The implication $3 \Longrightarrow 1$ follows from the injectivity of $d\kappa : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. □

Let X be a smooth projective variety, and D be a normal crossing divisor.

Definition 12.9. — A harmonic G-bundle $(P_K \subset P_G, \theta)$ on $X - D$ is called tame (pure imaginary), if the restriction $(P_K \subset P_G, \theta)_{|C \setminus D}$ is tame (pure imaginary) for any curve $C \subset X$ which is transversal with D. □

Remark 12.10. — Tameness and pure imaginary property are defined for principal G-Higgs bundles. □

Remark 12.11. — Tameness and pure imaginary property are preserved by pull back. We also remark the curve test for usual tame harmonic bundles. □

Let us consider the case where G is a linear reductive group defined over $\boldsymbol{R}$, with a maximal compact group K. We have the complexification $G_{\boldsymbol{C}}$ with a maximal compact group $K_{\boldsymbol{C}}$ such that $K = K_{\boldsymbol{C}} \cap G$.

Definition 12.12. — Let (P_G, ∇) be a flat bundle. A pluri-harmonic K-reduction $(P_K \subset P_G, \nabla)$ is called a tame pure imaginary, if the induced reduction $(P_{K_{\boldsymbol{C}}} \subset P_{G_{\boldsymbol{C}}}, \nabla)$ is a tame pure imaginary. □

Lemma 12.13. — *Let $(P_K \subset P_G, \theta)$ be a harmonic bundle on $X - D$. The following conditions are equivalent.*

1. *It is tame (pure imaginary).*
2. *For any $\kappa : G \longrightarrow \mathrm{GL}(V)$ and any K-invariant metric of V, the induced harmonic bundle is tame (pure imaginary).*
3. *There exists an immersive representation $\kappa : G \longrightarrow \mathrm{GL}(V)$ and a K-invariant metric of V such that the induced harmonic bundle is tame (pure imaginary).*

□

12.3. Semisimplicity and tame pure imaginary pluri-harmonic K-reduction

12.3.1. Preliminary. — Recall the existence and the uniqueness of tame pure imaginary pluri-harmonic metric ([**39**], [**23**]), which is called the Jost-Zuo metric. Let (E, ∇) be a semisimple flat bundle, and let $\rho : \pi_1(X, x) \longrightarrow \mathrm{GL}(E_{|x})$ denote the corresponding representation. We have the canonical decomposition of $E_{|x}$:

$$E_{|x} = \bigoplus_{\chi \in \mathrm{Irrep}(\pi_1(X,x))} E_{|x,\chi}.$$

Here $\mathrm{Irrep}(\pi_1(X, x))$ denotes the set of irreducible representations, and $E_{|x,\chi}$ denotes a $\pi_1(X, x)$-subspace of $E_{|x}$ isomorphic to $\chi^{\oplus m(\chi)}$. Correspondingly, we have the canonical decomposition of the flat bundle (E, ∇):

$$(E, \nabla) = \bigoplus_{\chi \in \mathrm{Irrep}(\pi_1(X,x))} E_\chi.$$

The flat bundle E_χ is isomorphic to a tensor product of trivial bundle $\boldsymbol{C}^{m(\chi)}$ and a flat bundle L_χ whose monodromy is given by χ.

Lemma 12.14. —

– *There exists a Jost -Zuo metric h_χ of L_χ, which is unique up to positive constant multiplication.*
– *Under the isomorphism $(E, \nabla) \simeq \bigoplus_\chi L_\chi \otimes \boldsymbol{C}^{m(\chi)}$, any Jost-Zuo metric of (V, ∇) is of the following form:*

$$\bigoplus_\chi h_\chi \otimes g_\chi.$$

Here g_χ denote any hermitian metrics of $\boldsymbol{C}^{m(\chi)}$. In other words, the ambiguity of the Jost-Zuo metrics is a choice of hermitian metrics g_χ of $\boldsymbol{C}^{m(\chi)}$, once we fix h_χ.

– *The decomposition of flat connection $\nabla = \partial + \overline{\partial} + \theta + \theta^\dagger$ is independent of a choice of g_χ.*

Proof The first claim is proved in [**39**]. The second claim easily follows from the proof of the uniqueness result in [**39**]. The third claim follows from the second claim. □

We also have the following lemma (see [**39**] or [**44**]).

Lemma 12.15. — *If there is a Jost-Zuo metric on a flat bundle (E, ∇), then the flat bundle is semisimple.* □

12.3.2. Compatibility with real structure. — We have the involution $\chi \longmapsto \overline{\chi}$ on $\mathrm{Irrep}\big(\pi_1(X, x)\big)$ such that $\chi \otimes_{\boldsymbol{R}} \boldsymbol{C} = \chi \oplus \overline{\chi}$. If $\overline{\chi} = \chi$, we have the real structure of L_χ. If $\overline{\chi} \neq \chi$, we have the canonical real structure of $L_\chi \otimes \boldsymbol{C} = L_\chi \oplus L_{\overline{\chi}}$.

Let us consider the case where (E, ∇) has the flat real structure $E_{\boldsymbol{R}}$ such that $E = E_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Let $\iota : E \longrightarrow E$ denote the conjugate with respect to $E_{\boldsymbol{R}}$. Then (E, ∇) is isomorphic to the following:

$$\bigoplus_{\overline{\chi}=\chi} L_\chi \otimes \boldsymbol{C}^{m(\chi)} \oplus \bigoplus_{\overline{\chi}\neq\chi} \big(L_\chi \oplus L_{\overline{\chi}}\big) \otimes \boldsymbol{C}^{m(\chi)}.$$

The real structure of (E, ∇) is induced from the real structures of L_χ ($\overline{\chi} = \chi$) and $L_\chi \otimes \boldsymbol{C}$ ($\overline{\chi} \neq \chi$). Then the following lemma is clear.

Lemma 12.16. — *When (E, ∇) has a real structure, there exists a Jost-Zuo metric of (E, ∇) which is invariant under the conjugation. The ambiguity of the metric is a choice of the metrics of the vector spaces $\boldsymbol{C}^{m(\chi)}$.* □

12.3.3. The case of the principal bundle associated with the monodromy group. — Let X be a quasi projective variety with a base point x, and (E, ∇) be a flat bundle. Let $G_0 \subset \mathrm{GL}(E_{|x})$ denote the monodromy group $M(E, \nabla, x)$. We obtain the principal G_0-bundle P_{G_0} with the flat connection.

If the flat bundle (E, ∇) is semisimple, we have a Jost-Zuo metric h of (E, ∇). Let $U = U(E_{|x}, h_{|x})$ denote the unitary group of the metrized vector space $(E_{|x}, h_{|x})$, and we put $K_0 := G_0 \cap U$.

Lemma 12.17. — *K_0 is a maximal compact subgroup of G_0.*

Proof The argument was given by Simpson (Lemma 4.4 in [**47**]) for a different purpose. We reproduce it here. Let $\tau : \mathrm{GL}(E_{|x}) \longrightarrow \mathrm{GL}(E_{|x})$ be the anti-holomorphic involution such that $\tau(g) = (g^\dagger)^{-1}$, where $g^\dagger$ denotes the adjoint of g with respect to the metric $h_{|x}$ of $E_{|x}$. Let us take a flat subbundle $S \subset T^{a,b}(E)$ with the following property: Let g be an element of $\mathrm{GL}(E_{|x})$. Then g is contained in G_0 if and only if $T^{a,b}(g)_{|S_{|x}} = \mathrm{id}_{S_{|x}}$. We have the G_0-decomposition $T^{a,b}V = S_{|x} \oplus F_{|x}$. We may assume that $F_{|x}$ does not contain the trivial representation. Let $T^{a,b}E = S \oplus F$ is the corresponding decomposition.

Lemma 12.18. — *The decomposition $T^{a,b}V = S \oplus F$ is orthogonal with respect to the induced metric $T^{a,b}h$.*

Proof The canonical decomposition is orthogonal with respect to the Jost-Zuo metric. Then the claim immediately follows. □

Let us return to the proof of Lemma 12.17. For any $g \in \mathrm{GL}(E_{|x})$, we have the expression $g = u \cdot \exp(y)$, where $u \in U$ and $y \in End(E_{|x})$ such that $\tau(y) = -y$. The decomposition is compatible with tensor products and orthogonal decompositions. It follows that $T^{a,b}u$ and $T^{a,b}y$ preserves $S_{|x}$. Since $T^{a,b}g = \mathrm{id}_{S_{|x}}$ is trivial on S, we have $T^{a,b}u = \mathrm{id}_{S_{|x}}$ and $T^{a,b}y = 0$ on $S_{|x}$. Therefore we obtain $u \in G_0 \cap U = K_0$ and $y \in \mathfrak{g}_0 \subset End(E_{|x})$, where $\mathfrak{g}_0$ denote the Lie subalgebra of $\mathrm{End}(E_{|x})$ corresponding to G_0. Hence $\tau(g) = u \cdot \exp\bigl(-y\bigr)$ is contained in G_0. Namely, τ preserves G_0.

Since we have the decomposition $g = u \cdot \exp(y)$ for any $g \in G_0$, K_0 intersects with any connected components of G_0. Let G_0^0 denote the connected component of G_0 containing the unit element. Since K_0^0 is the fixed point set of $\tau_{|G_0^0}$, it is easy to see that K_0^0 is a maximal compact subgroup of G_0^0. Then we can conclude that K_0 is a maximal compact subgroup of G_0. Namely the proof of Lemma 12.17 is accomplished. □

Let us consider the case where (E, ∇) has the real structure. We use the notation in the subsection 12.3.2. We have the real parts $E_{\boldsymbol{R}\,|\,x} \subset E_{|x}$ and $G_{0\boldsymbol{R}} := G_0 \cap GL(E_{\boldsymbol{R}\,|\,x})$. Let us take a Jost-Zuo metric of h which is invariant under the conjugation ι. We put $K_{0\boldsymbol{R}} = G_{0\boldsymbol{R}} \cap K_0 = G_{0\boldsymbol{R}} \cap U$.

Lemma 12.19. — *$K_{0\,\boldsymbol{R}}$ is maximal compact in $G_{0\,\boldsymbol{R}}$.*

Proof We use the notation in the proof of Lemma 12.17. Since $h_{|x}$ is invariant under the conjugation ι, U is stable under ι, and τ and ι are commutative. Let g be an element of $G_{0\,\boldsymbol{R}}$. We have the decomposition $g = u \cdot \exp(y)$ as in the proof of Lemma 12.17, where u denotes an element of K_0 and y denotes an element of $\mathfrak{g}_0$ such that $\tau(y) = -y$. Since $\iota(g) = g$, we have $\iota(u) \cdot \exp\bigl(\iota(y)\bigr) = u \cdot \exp(y)$. Since we have $\iota(u) \in \iota(U) = U$ and $\tau(\iota(y)) = \iota(\tau(y)) = -\iota(y)$, we obtain $\iota(u) = u$ and $\iota(y) = y$. Namely $u \in K_{0\boldsymbol{R}}$ and $y \in \mathfrak{g}_{0\boldsymbol{R}}$. Then we can show $K_{0\boldsymbol{R}}$ is maximal compact in $G_{0\boldsymbol{R}}$, by an argument similar to the proof of Lemma 12.17. □

Proposition 12.20. — *Assume that (E, ∇) is semisimple. Then there exists the unique tame pure imaginary pluri-harmonic K-reduction $P_{K_0} \subset P_{G_0}$. Assume (E, ∇) has the flat real structure, moreover. Then it is induced from the pluri-harmonic reduction of $P_{G_{0\boldsymbol{R}}}$.*

Proof Let h be a Jost-Zuo metric of (E, ∇). For any point $z \in X$, let $M(E, \nabla, z)$ denote the monodromy group at z, and $U(E_{|z}, h_{|z})$ denote the unitary group of $E_{|z}$ with the metric $h_{|z}$. Then the intersection $M(E, \nabla, z) \cap U(E_{|z}, h_{|z})$ is a maximal compact subgroup of $M(E, \nabla, z)$, due to Lemma 12.17. Hence they give the reduction $P_{K_0} \subset P_{G_0}$, which is pluri-harmonic. By using a similar argument and Lemma 12.19, we obtain the compatibility with the real structure, if (E, ∇) has the flat real structure. The uniqueness of the pluri-harmonic reduction follows from the uniqueness result in Lemma 12.14. Hence we are done. □

Corollary 12.21. — *(E, ∇) is semisimple if and only if the monodromy group G_0 is reductive.*

Proof It is clear that the reductivity of G_0 implies the semisimplicity of (E, ∇). If (E, ∇) is semisimple, a maximal compact subgroup of G_0 is a real form of G_0. Hence it is reductive. □

12.3.4. Zariski dense case. — Let G be a linear reductive algebraic group over $\boldsymbol{C}$ or $\boldsymbol{R}$. Let X be a quasi projective variety with a base point x. The following corollary immediately follows from Proposition 12.20.

Corollary 12.22. — *Let P_G be a flat G-principal bundle over X. Assume that the image of the induced representation $\pi_1(X, x) \longrightarrow G$ is Zariski dense in G. Then there exists the unique tame pure imaginary pluri-harmonic reduction of P_G.* □

We can reword the corollary as follows. Let $\widetilde{X}$ be a universal covering of X. Let $\pi_1(X, x) \longrightarrow G$ be a homomorphism whose image is Zariski dense in G, which gives a $\pi_1(X, x)$-action on G/K.

Corollary 12.23. — *There is a $\pi_1(X, x)$-equivariant pluri-harmonic map $\widetilde{X} \longrightarrow G/K$, which induces a tame pure imaginary harmonic G-bundle on X. It is unique up to the G-action.* □

12.3.5. General case. —

Theorem 12.24. — *Let G be a linear reductive algebraic group over $\boldsymbol{C}$ or $\boldsymbol{R}$, and X be a quasi projective variety. Let (P_G, ∇) be a flat G-bundle on a quasi projective variety. The monodromy group G_0 is reductive if and only if there exists a tame pure imaginary pluri-harmonic reduction $P_K \subset P_G$. If such a reduction exists, the decomposition $\nabla = \nabla_K + (\theta + \theta^\dagger)$ does not depend on a choice of a pluri-harmonic reduction $P_K \subset P_G$.*

Proof If a pluri-harmonic reduction exists, the monodromy group is reductive due to Lemma 12.8 and Corollary 12.21. If G_0 is reductive, let K_0 be a maximal compact group of G_0. Then we have the unique tame pure imaginary pluri-harmonic reduction $P_{K_0} \subset P_{G_0}$. We take K such as $K \cap G_0 = K_0$. Then the pluri-harmonic reduction $P_K \subset P_G$ is induced, and thus the first claim is proved. The second claim is clear. □

Corollary 12.25. — *Let G be a linear reductive group with a maximal compact subgroup K. Let X be a quasi projective variety with a base point x, and $\widetilde{X}$ denote the universal covering of X. Let $\pi_1(X,x) \longrightarrow G$ be a homomorphism such that the Zariski closure is reductive. Then there is a $\pi_1(X,x)$-equivariant pluri-harmonic map $\widetilde{X} \longrightarrow G/K$, which induces a tame pure imaginary harmonic G-bundle on X.* □

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