Lefschetz property, Schur-Weyl duality and a q-deformation of Specht polynomial

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Abstract

We describe the Schur-Weyl duality for a polynomial representation of the quantum group and the Hecke algebra of type A from a viewpoint of a q-analogue of the strong Lefschetz property. A q-deformation of the Specht polynomial appears as a constituent of bases for irreducible components.

Introduction

We investigate the Schur-Weyl duality for a representation of the quantum group $U_q(\mathfrak{sl}_d)$ and the Hecke algebra $\mathcal{H}_{S_n}(q)$ realized on the algebra $A = \mathbf{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$ by means of a q-analogue of the strong Lefschetz property, which is a useful tool for combinatorial studies of finite-dimensional Gorenstein graded rings. We describe explicitly the irreducible decomposition of A as a $(U_q(\mathfrak{sl}_d), \mathcal{H}_{S_n}(q))$-module. A q-analogue of the Specht polynomial gives a generator of each irreducible component. In this paper, all the vector spaces and algebras are over the field of complex numbers $\mathbf{C}$.

Let $V = \bigoplus_{i \geq 0} V_i$ be a finite-dimensional graded vector space. Consider linear endomorphisms E and F on V such that $\deg E = 1$, $\deg F = -1$ and the set $\{E, F, H := [E, F]\}$ forms an $\mathfrak{sl}_2$-triple. Let us call $V_i^0 := V_i \cap \mathrm{Ker} F$ the primitive component of degree i. Then the part of degree i decomposes as the direct sum of the images of the primitive components:

$$V_i = \bigoplus_{j \geq 0} E^j V_{i-j}^0.$$

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This follows from the complete reducibility and the fact that every finite dimensional representation of $\mathfrak{sl}_2$ is isomorphic to a highest weight representation with nonnegative integral highest weight. For example, if V is the cohomology group $H^*(X, \mathbf{C})$ of a compact Kähler manifold X and if E is the multiplication operator by the Kähler class, then the decomposition above is called the Lefschetz decomposition, see e.g. Griffiths and Harris [4]. The idea of the Lefschetz decomposition is useful also for analyzing the structure of Artinian Gorenstein rings, see e.g. Stanley [11], Watanabe [14], Harima, Migliore, Nagel and Watanabe [6]. The notion of (strong or weak) Lefschetz property for Artinian Gorenstein graded rings is an abstraction of the Hard Lefschetz Theorem.

The algebra $A = \mathbf{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d) = (\mathbf{C}[x]/(x^d))^{\otimes n}$ is an example of Gorenstein rings with the strong Lefschetz property. In fact, it is isomorphic to the cohomology ring of a product of the projective spaces. The symmetric group S_n acts on A via permutation of indices of $x_1, \ldots, x_n$. At the same time, the linear group $GL_d(\mathbf{C})$ acts on A by regarding $\mathbf{C}[x]/(x^d)$ as the vector representation of $GL_d(\mathbf{C})$. Hence A admits a decomposition as a (GL_d, S_n)-module, which is implied by the Schur-Weyl duality. Morita, Wachi and Watanabe [10] proposed a method to give a description of the irreducible decomposition of A from the view point of the strong Lefschetz property of A. They explicitly constructed a basis for each irreducible component in terms of Specht polynomials for $d = 2$. We will give a generalization of their result in Section 1.

The main purpose of this paper is to understand a q-deformed framework of their approach in order to study the Schur-Weyl duality proved by Jimbo [8] for the quantum group $U_q(\mathfrak{sl}_d)$ and the Hecke algebra $\mathcal{H}_{S_n}(q)$. The polynomial representation of the Hecke algebra appearing here is the one studied by Martin [9] and by Duchamp, Krob, Lascoux, Leclerc, Sharf and Thibon (DKLLST for short) [1]. We define q-deformed Specht polynomials by using the q-analogue of the skew-symmetrizer introduced by Gyoja [5]. These polynomials are the same as those studied in Martin [9] and DKLLST [1]. We show as the main result that q-deformed Specht polynomials are part of the basis of the primitive irreducible component of A as a $(U_q(\mathfrak{sl}_d), \mathcal{H}_{S_n}(q))$-module (Theorem 4.1). In particular, they form a basis of the primitive component for $d = 2$.

On the other hand, the coinvariant algebra R_n of the symmetric group S_n is another important example of the Gorenstein ring with the strong Lefschetz property on which the symmetric group acts. In this case, Terasoma and Yamada [13] have constructed a standard polynomial basis for each irreducible component as an S_n-module. Note that in contrast to the case of the algebra A, the coinvariant algebra R_n does not have any S_n-invariant Lefschetz elements.

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1 Lefschetz decomposition and Specht polynomial

Definition 1.1 *A commutative Artinian graded algebra $R = \oplus_{i=0}^{D} R_i$ is said to have the strong Lefschetz property if there exists an element ℓ of degree one such that $(\ell^{D-2i}\times) : R_i \to R_{D-i}$ is bijective. The element ℓ satisfying the condition above is called a Lefschetz element.*

It is easy to see that the algebra $A = \mathbf{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$ has the strong Lefschetz property with a Lefschetz element $L = x_1 + \cdots + x_n$. Let us define a linear endomorphism $\Lambda : A \to A$ by

$$\Lambda(x_1^{m_1} \cdots x_n^{m_n}) = \sum_{j=1}^{n} m_j(d-m_j) x_1^{m_1} \cdots x_j^{m_j-1} \cdots x_n^{m_n}, \ 0 \le m_1, \ldots, m_n \le d-1.$$

Then the set $\{L, \Lambda, H := [L, \Lambda]\}$ forms an $\mathfrak{sl}_2$-triple. Note that the operator Λ is induced by the differential operator $\sum_{j=1}^{n}(-x_j(\partial/\partial x_j)^2 + (d-1)(\partial/\partial x_j))$ acting on $\mathbf{C}[x_1, \ldots, x_n]$. Since the Lefschetz element L is invariant under the action of the symmetric group S_n, the Lefschetz decomposition

$$A_i = \bigoplus_{j \ge 0} L^j A_{i-j}^0$$

is a decomposition as a representation of S_n. Hence, the irreducible decomposition of A as an S_n-module is reduced to that of each primitive component A_i^0.

Let $\mathcal{T}$ be a semi-standard tableau. See e.g. Fulton [3], Stanley [12] for the definition of the (semi-)standard tableau.

Definition 1.2 *The Specht polynomial for a semi-standard tableau $\mathcal{T}$ is defined by*

$$\Delta_{\mathcal{T}}(x) := \prod_i \Delta(x_{\mathcal{T}_i}),$$

where $\mathcal{T}_i$ is the set of indices contained in the i-th column of $\mathcal{T}$ and $\Delta(x_{\mathcal{S}}) = \prod_{i,j\in\mathcal{S},\, i<j}(x_i - x_j)$.

Let $m_{\mathcal{T}}$ be the initial monomial of $\Delta_{\mathcal{T}}(x)$ with respect to the lexicographic ordering. Define the column skew-symmetrizer $b_{\mathcal{T}}$ by

$$b_{\mathcal{T}} := \sum_{w\in C(\mathcal{T})} \varepsilon(w)w, \;\; \varepsilon(w) := (-1)^{l(w)},$$

where $C(\mathcal{T}) \subset S_n$ is the column group of $\mathcal{T}$, which preserves the set of indices on each column of $\mathcal{T}$. If $\mathcal{T}$ is a standard tableau, then it is well-known that $\Delta_{\mathcal{T}}(x) = b_{\mathcal{T}}(m_{\mathcal{T}})$.

Proposition 1.1 *For any standard tableau $\mathcal{T}$, one has*

$$\Lambda(\Delta_{\mathcal{T}}(x)) = 0.$$

Proof. Note that if (I_1, I_2) is a partition of $\{1,\dots,n\}$, i.e. $I_1 \sqcup I_2 = \{1,\dots,n\}$, then

$$\Lambda(f(x_{I_1})g(x_{I_2})) = \Lambda(f(x_{I_1}))\cdot g(x_{I_2}) + f(x_{I_1})\cdot \Lambda(g(x_{I_2})).$$

Since the action of the symmetric group commutes with Λ,

$$\Lambda(\Delta_{\mathcal{T}}) = b_{\mathcal{T}}(\Lambda(m_{\mathcal{T}})).$$

It is enough to show $b_{\mathcal{T}}(\Lambda(m_{\mathcal{T}})) = 0$ when the shape of $\mathcal{T}$ is $(1,\dots,1)$, the number of boxes of $\mathcal{T}$ is l, $1 \le l \le n$, and $m_{\mathcal{T}}(x) = x_1^{l-1}x_2^{l-2}\cdots x_{l-1}$. One has

$$\Lambda(x_1^{l-1}x_2^{l-2}\cdots x_{l-1}) = \sum_{j=1}^{l-1} j(d-j)x_1^l\cdots x_j^{l-j}x_{j+1}^{l-j}\cdots x_{l-1}.$$

So we can conclude that $b_{\mathcal{T}}(\Lambda(m_{\mathcal{T}})) = 0$.

Corollary 1.1 (1) *The Specht polynomial $\Delta_{\mathcal{T}}(x)$ is a generator of an irreducible (GL_d, S_n)-component. The algebra A decomposes as a direct sum of irreducible GL_d-modules:*

$$A = \bigoplus_{\mathcal{T}} GL_d\cdot(\mathbf{C}\Delta_{\mathcal{T}}(x)),$$

where $\mathcal{T}$ runs over all the standard tableaux of shape $\lambda \vdash n$ with the length $l(\lambda) \leq d$.
(2) *For each standard tableaux satisfying the condition above, the polynomials*

$$\Delta_{\mathcal{T}}(x), L \cdot \Delta_{\mathcal{T}}(x), \ldots, L^{(d-1)n-2\deg \Delta_{\mathcal{T}}} \cdot \Delta_{\mathcal{T}}(x)$$

form a linear basis of an irreducible $\mathfrak{sl}_2$-submodule of A.

Remark 1.1 For $d = 2$, Morita, Wachi and Watanabe [10] have shown that the Specht polynomials for standard tableaux consisting of two rows form a basis of $\mathrm{Ker}\Lambda$. For $d > 2$, the Specht polynomials are only part of a basis of $\mathrm{Ker}\Lambda$.

Lemma 1.1 *Let B_m be the set of the nonzero polynomials in A of form $L^i(\Delta_{\mathcal{T}}(x))$ with $\deg(L^i(\Delta_{\mathcal{T}}(x))) = m$. Then B_m is linearly independent for $m \leq [(d-1)n/2]$.*

Proof. Since L is the Lefschetz element, this follows from the fact that each element of B_m belongs to different irreducible $(\mathfrak{sl}_2, S_n)$-component from others. ■

Here we show a procedure to produce polynomials generating irreducible $(\mathfrak{sl}_2, S_n)$-components other than the Specht polynomials. We identify the Lie algebra $\mathfrak{gl}_d$ with the matrix algebra $\mathrm{Mat}_d(\mathbf{C})$ consisting of $(d \times d)$-matrices $Z = (Z_{ab})_{a,b=0}^{d-1}$. Under this identification, the matrix elements $E_{a,b}$ $(a, b = 0, \ldots, d-1)$ acts on the algebra $\mathbf{C}[x]/(x^d)$ by $E_{a,b}x^i = \delta_{b,i}x^a$. The coproduct of the universal enveloping algebra $U(\mathfrak{gl}_d)$ determines the natural action of $E_{a,b}$ on A. The action of $E_{a,b}$ on A is given by the formula

$$E_{a,b}(x_1^{m_1} \cdots x_n^{m_n}) = \sum_{j=1}^{n} \delta_{b,m_j} \cdot x_1^{m_1} \cdots x_j^{a} \cdots x_n^{m_n}.$$

The following proposition shows an inductive way to find polynomials that generate irreducible S_n-submodules and belong to the primitive part $\mathrm{Ker}\Lambda$.

Proposition 1.2 (1) *Let φ be an element of $U(\mathfrak{gl}_d)$ and $\mathcal{T}$ a standard tableau. If the polynomial $\Lambda^j(\varphi(\Delta_{\mathcal{T}}(x)))$ is not equal to zero in the algebra A, it generates an irreducible S_n-submodule isomorphic to the irreducible representation corresponding to λ.*
(2) *Let $\varphi \in U(\mathfrak{gl}_d)$ be a monomial in the matrix elements of $\mathrm{Mat}_d(\mathbf{C})$. Assume*

that φ increases the degree of $\Delta_{\mathcal{T}}$ by i. If $\Lambda^j(\varphi(\Delta_{\mathcal{T}}(x))) = \sum_{\nu=0}^{\deg \Delta_{\mathcal{T}}+i-j} L^\nu y_\nu$, $y_\nu \in (H+\nu)(\mathrm{Ker}\Lambda)$, for some $j > 0$, then

$$\Lambda^{j-1}(\varphi\Delta_{\mathcal{T}}) + \sum_{\nu=0}^{\deg \Delta_{\mathcal{T}}+i-j} \frac{L^{\nu+1}\cdot y'_\nu}{\nu+1} \in \mathrm{Ker}\Lambda,$$

where y'_ν is an element in $\mathrm{Ker}\Lambda$ such that $(H+\nu)y'_\nu = y_\nu$.

Proof. The first statement (1) follows from the commutativity of the action of $U(\mathfrak{gl}_d)$ with that of S_n.

In order to prove (2), we use the commutation relation

$$[L^{\nu+1}, \Lambda] = (\nu+1)L^\nu(H+\nu).$$

Since $y'_\nu \in \mathrm{Ker}\Lambda$, we get

$$\Lambda\left(\Lambda^{j-1}(\varphi\Delta_{\mathcal{T}}) + \sum_{\nu=0}^{\deg \Delta_{\mathcal{T}}+i-j} \frac{L^{\nu+1}\cdot y'_\nu}{\nu+1}\right) = \Lambda^j(\varphi\Delta_{\mathcal{T}}) - \sum_{\nu=0}^{\deg \Delta_{\mathcal{T}}+i-j} L^\nu y_\nu = 0.$$

This completes the proof. ■

Let f^λ be the number of the standard tableaux with shape λ, which can be computed by means of the Frame-Robinson-Thrall formula (Frame, Robinson and Thrall [2]):

$$f^\lambda = \frac{|\lambda|!}{h(\lambda)},$$

where $|\lambda|$ is the weight of λ and $h(\lambda)$ is the product of all the hook lengths. On the other hand, since the Hilbert polynomial of A is

$$\mathrm{Hilb}(A;t) = \left(\frac{1-t^d}{1-t}\right)^n,$$

it is easy to see the following from the Schur-Weyl duality and the Weyl character formula.

Proposition 1.3 *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we set $n(\lambda) := \sum_i (i-1)\lambda_i$. Then we have*

$$\sum_{\lambda\vdash n,\, l(\lambda)\le d} f^\lambda t^{n(\lambda)} \prod_{1\le i<j\le n} \left(\frac{1-t^{\lambda_i-\lambda_j+j-i}}{1-t^{j-i}}\right) = \left(\frac{1-t^d}{1-t}\right)^n.$$

Remark 1.2 As for the sum of the numbers f^λ, the following interesting formula is known, see e.g. Stanley [12, 7.13.9]:

$$\sum_{\lambda \vdash n} f^\lambda = \text{Coefficient of } x^n \text{ in } \exp(x + \frac{x^2}{2}).$$

Example 1.1 Let us consider the case $d = 3$ and $n = 4$. The Hilbert polynomial of A is

$$\mathrm{Hilb}(A; t) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.$$

The set of the partitions $\lambda \vdash 4$ with $l(\lambda) \leq 3$ is

$$\{(4, 0, 0, 0),\ (3, 1, 0, 0),\ (2, 2, 0, 0),\ (2, 1, 1, 0)\}.$$

The following is the list of the Specht polynomials in this case.

$\deg = 0$	1
$\deg = 1$	$x_1 - x_2,\ x_1 - x_3,\ x_1 - x_4$
$\deg = 2$	$(x_1 - x_2)(x_3 - x_4),\ (x_1 - x_3)(x_2 - x_4)$
$\deg = 3$	$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),\ (x_1 - x_3)(x_1 - x_4)(x_3 - x_4),$ $(x_1 - x_2)(x_1 - x_4)(x_2 - x_4)$

Let us find the basis of $\mathrm{Ker}\Lambda$ by using Proposition 1.2. Take the standard tableau $\mathcal{T} = 1\ 2\ 3\ 4$ which corresponds to the trivial representation and $\Delta_{\mathcal{T}}(x) = 1$. We can obtain the polynomials of higher degree that generate the trival representations of S_n by applying the operators $(E_{1,0})^k \in U(\mathfrak{gl}_d)$, $k = 1, 2, 3, 4$. Here, we take the product of the operator $E_{1,0}$ *not in* $\mathrm{Mat}_d(\mathbf{C})$, but in $U(\mathfrak{gl}_d)$. In fact, we obtain the elementary symmetric polynomials in this case:

$$E_{1,0}(\Delta_{\mathcal{T}}) = e_1(x_1, x_2, x_3, x_4),\ (E_{1,0})^2(\Delta_{\mathcal{T}}) = 2e_2(x_1, x_2, x_3, x_4),$$

$$(E_{1,0})^3(\Delta_{\mathcal{T}}) = 6e_3(x_1, x_2, x_3, x_4),\ (E_{1,0})^4(\Delta_{\mathcal{T}}) = 24e_4(x_1, x_2, x_3, x_4),$$

where e_i is the i-th elementary symmetric polynomial. The polynomial $(E_{1,0})^4\Delta_{\mathcal{T}}(x) = 24x_1x_2x_3x_4$ is a generator of a copy of the trivial representation in A_4. However, it does not belong to $\mathrm{Ker}\Lambda$. We have

$$\Lambda(e_4) = 2e_3(x_1, x_2, x_3, x_4),\ \Lambda^2(e_4) = 8e_2(x_1, x_2, x_3, x_4),$$

$$\Lambda^3(e_4) = 48e_1(x_1,x_2,x_3,x_4) = 48L\Delta_{\mathcal{T}},$$

Hence, we have the following polynomials generating the trivial S_n-submodules in A_2^0 and A_4^0 :

$$\begin{array}{cc} A_2^0 & 7e_2 - 3L^2, \\ A_4^0 & 15e_4 - 5L^2e_2 + 2L^4. \end{array}$$

Now let us take the standard tableau

$$\mathcal{T} = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array}.$$

Then the corresponding Specht polynomial is $\Delta_{\mathcal{T}}(x) = x_1 - x_2$. The polynomial $E_{2,1}(\Delta_{\mathcal{T}}) = x_1^2 - x_2^2$ also generates the irreducible S_n-module of type $\lambda = (3,1,0,0)$. Since

$$\Lambda(L(x_1-x_2)) = 6x_1 - 6x_2,\ \Lambda(x_1^2 - x_2^2) = 2x_1 - 2x_2,$$

the polynomial $3(x_1^2 - x_2^2) - L(x_1 - x_2)$ belongs to $\mathrm{Ker}\Lambda$. We can find the rest of the basis of $\mathrm{Ker}\Lambda$ in a similar way. The list of the basis of each primitive component is as follows.

A_0^0	$\lambda = (4,0,0,0)$	1
A_1^0	$\lambda = (3,1,0,0)$	$x_1 - x_2,\ x_1 - x_3,\ x_1 - x_4$
A_2^0	$\lambda = (4,0,0,0)$	$7e_2(x) - 3L^2$
	$\lambda = (3,1,0,0)$	$3(x_1^2 - x_2^2) - L(x_1 - x_2),\ 3(x_1^2 - x_3^2) - L(x_1 - x_3),$ $3(x_1^2 - x_4^2) - L(x_1 - x_4)$
	$\lambda = (2,2,0,0)$	$(x_1 - x_2)(x_3 - x_4),\ (x_1 - x_3)(x_2 - x_4)$
A_3^0	$\lambda = (3,1,0,0)$	$(10e_2(x) - 3L^2)(x_1 - x_2) - 5L(x_1^2 - x_2^2),$ $(10e_2(x) - 3L^2)(x_1 - x_3) - 5L(x_1^2 - x_3^2),$ $(10e_2(x) - 3L^2)(x_1 - x_4) - 5L(x_1^2 - x_4^2)$
	$\lambda = (2,1,1,0)$	$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),\ (x_1 - x_2)(x_1 - x_4)(x_2 - x_4),$ $(x_1 - x_3)(x_1 - x_4)(x_3 - x_4)$
A_4^0	$\lambda = (4,0,0,0)$	$15e_4(x) - 5L^2e_2(x) + 2L^4$
	$\lambda = (2,2,0,0)$	$3(x_1^2 - x_2^2)(x_3^2 - x_4^2) - L^2(x_1 - x_2)(x_3 - x_4),$ $3(x_1^2 - x_3^2)(x_2^2 - x_4^2) - L^2(x_1 - x_3)(x_2 - x_4)$

2 Spin $(d-1)/2$ representation of $U_q(\mathfrak{sl}_2)$ and q-analogue of Lefschetz element

In this section, we construct a q-analogue of the Lefschetz element $L \in A$. Let us realize the spin $(d-1)/2$ representation on the algebra $\mathbf{C}[x]/(x^d)$.

Take the standard generators $X^{\pm}, K^{\pm 1}$ of $U_q(\mathfrak{sl}_2)$. They satisfy the relations:

$$KK^{-1} = 1,\ KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm},\ [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Define their action on $\mathbf{C}[x]/(x^d)$ by

$$X^+(x^i) = \begin{cases} x^{i+1}, & i = 0, \ldots, d-2, \\ 0, & i = d-1, \end{cases}$$
$$X^-(x^i) = [i][d-i]x^{i-1},$$
$$K(x^i) = q^{2i-d+1}x^i.$$

Here, the q-integer $[i]$ is defined by the formula $[i] := (q^i - q^{-i})/(q - q^{-1})$. The action of X^+ on $\mathbf{C}[x]/(x^d)$ is the multiplication operator by x. One has the action of X^+ on A via successive application of the coproduct of $U_q(\mathfrak{sl}_2)$. The action of X^+ on A obtained in this way can be regarded as a q-analogue of the Lefschetz element L, which we denote by L_q. More explicitly, L_q is the operator that acts on monomials as

$$L_q(x_1^{m_1} \cdots x_n^{m_n}) = \sum_{i=1}^{n} q^{N_i(m_1,\ldots,m_n)} x_1^{m_1} \cdots x_i^{m_i+1} \cdots x_n^{m_n},$$

where

$$N_i(m_1, \ldots, m_n) := \sharp\{j < i | m_j = m_i\} - \sharp\{j < i | m_j = m_i + 1\}.$$

Similarly, one has a q-analogue of the operator Λ given by

$$\Lambda_q(x_1^{m_1} \cdots x_n^{m_n}) = \sum_{i=1}^{n} q^{N_i'(m_1,\ldots,m_n)} [m_i][d-m_i] x_1^{m_1} \cdots x_i^{m_i-1} \cdots x_n^{m_n},$$

where

$$N_i'(m_1, \ldots, m_n) := \sharp\{j > i | m_j = m_i\} - \sharp\{j > i | m_j = m_i - 1\}.$$

Now we restate the Lefschetz decomposition of A under this situation.

Proposition 2.1 *Suppose that q is neither zero nor root of the unity. Then the component of degree i decomposes as*

$$A_i = \bigoplus_{j \geq 0} L_q^j A_{i-j}^0,$$

where $A_i^0 := \mathrm{Ker}(\Lambda_q) \cap A_i$.

Remark 2.1 (1) Let $\xi_1, \ldots, \xi_n$ be operators on A defined by

$$\xi_i(x_1^{m_1} \cdots x_n^{m_n}) = q^{N_i(m_1,\ldots,m_n)} x_1^{m_1} \cdots x_i^{m_i+1} \cdots x_n^{m_n}.$$

If we define the operators κ_{ij} for $i < j$ by

$$\kappa_{ij}(x_1^{m_1} \cdots x_n^{m_n}) = \begin{cases} q^2 x_1^{m_1} \cdots x_n^{m_n}, & \text{if } m_i = m_j, \\ q^{-1} x_1^{m_1} \cdots x_n^{m_n}, & \text{if } |m_i - m_j| = 1, \\ x_1^{m_1} \cdots x_n^{m_n}, & \text{otherwise,} \end{cases}$$

then the operators κ_{ij} commute each other. In the algebra A_q generated by the operators $\xi_1, \ldots, \xi_n$ and $\kappa_{ij}^{\pm 1}$, $1 \leq i < j \leq n$, we have relations

$$\xi_1^d = \cdots = \xi_n^d = 0, \ \xi_i \xi_j = \kappa_{ij} \xi_j \xi_i \ (i < j).$$

We can consider the operator $L_q = \xi_1 + \cdots + \xi_n$ as a "Lefschetz element" in the noncommutative algebra A_q.
(2) More generally, the algebra

$$A_{(d_1,\ldots,d_n)} = \mathbf{C}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$$

has the Lefschetz property with the Lefschetz element $L = x_1 + \cdots + x_n$. Since $A_{(d_1,\ldots,d_n)}$ is considered as the tensor product of spin $(d_i - 1)/2$ representations of $U_q(\mathfrak{sl}_2)$, we can also construct the q-analogue of L in the same manner as the construction of L_q for A. However, the symmetric group S_n does not act on $A_{(d_1,\ldots,d_n)}$ any longer unless $d_1 = \cdots = d_n$.

3 Schur-Weyl duality

The algebra $\mathbf{C}[x]/(x^d)$ can be regarded as a vector representation of $U_q(\mathfrak{sl}_d)$. The action of the standard generators $X_i^{\pm}$, $K_i^{\pm 1}$, $i = 1, \ldots, d$, of $U_q(\mathfrak{sl}_d)$ on the monomial x^j is given by

$$X_i^+(x^j) = \delta_{i,j+1} x^{j+1}, \ X_i^-(x^j) = \delta_{i,j} x^{j-1}, \ K_i(x^j) = q^{\delta_{i,j} - \delta_{i,j+1}} x^j.$$

Let us remind of the action of the Hecke algebra $\mathcal{H} = \mathcal{H}_{S_n}(q)$ on $(\mathbf{C}[x]/(x^d))^{\otimes n}$ to state the Schur-Weyl duality between $U_q(\mathfrak{sl}_d)$ and $\mathcal{H}$. The Hecke algebra $\mathcal{H}$ is a $\mathbf{C}$-algebra defined by the following data:

- Generators $\quad T_1, \ldots, T_n$

- Relations

$$\begin{array}{cc} (T_i - q)(T_i + q^{-1}) = 0, & i = 1, \ldots, n, \\ T_i T_j = T_j T_i, & |i-j| > 1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & i = 1, \ldots, n-1. \end{array}$$

Define the action of $\mathcal{H}$ on A by

$$T_i(x_1^{m_1} \cdots x_n^{m_n})$$

$$= \begin{cases} q x_1^{m_1} \cdots x_n^{m_n}, & \text{if } m_i = m_{i+1}, \\ x_1^{m_1} \cdots x_i^{m_{i+1}} x_{i+1}^{m_i} \cdots x_n^{m_n} + (q - q^{-1}) x_1^{m_1} \cdots x_n^{m_n}, & \text{if } m_i < m_{i+1}, \\ x_1^{m_1} \cdots x_i^{m_{i+1}} x_{i+1}^{m_i} \cdots x_n^{m_n}, & \text{if } m_i > m_{i+1}. \end{cases}$$

Denote by $\mathcal{S}_q$ the image of the algebra homomorphism $\mathcal{H} \to \mathrm{End}_{\mathbf{C}}(A)$ induced by the action defined above. We also denote by $\mathcal{G}_q$ the image of the homomorphism $U_q(\mathfrak{sl}_d) \to \mathrm{End}_{\mathbf{C}}(A)$ obtained by regarding A as the tensor product of the vector representation of $U_q(\mathfrak{sl}_d)$.

Proposition 3.1 (Schur-Weyl duality)(Jimbo [8]) *Let q be generic. The subalgebras $\mathcal{S}_q$ and $\mathcal{G}_q$ are mutually commutants in* $\mathrm{End}_{\mathbf{C}}(A)$.

Since the q-analogue of the Lefschetz element L_q belongs to $\mathcal{G}_q$, it commutes with the action of $\mathcal{H}$. Hence, the Lefschetz decomposition in Proposition 2.1 is a decomposition as a representation of $\mathcal{H}$.

4 q-deformation of Specht polynomial

Now we define a q-deformation of the Specht polynomial by using the q-skew symmetrizer introduced by Gyoja [5].

Let Y be a Young diagram. Assume that its j-th column has length l_j. Define the standard tableau Y^0 on Y so that the (i,j)-entry of Y^0 is $i + (l_1 + \cdots + l_{j-1})$. For example,

$$\begin{array}{cccc} 1 & 6 & 10 & 12 \\ 2 & 7 & 11 & \\ 3 & 8 & & \\ 4 & 9 & & \\ 5 & & & \end{array}$$

is the tableau Y^0 corresponding to $Y = (4,3,2,2,1)$. When a standard tableau $\mathcal{T}$ has shape Y, the corresponding tableau Y^0 is also denoted by $\mathcal{T}^0$.

For an element $w \in S_n$, one can define the element T_w in $\mathcal{H}$ by $T_w := T_{i_1} \cdots T_{i_l}$ if w has a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$. Then the q-analogue of the skew-column symmetrizer corresponding to a tableau of form $\mathcal{T}^0$ is defined as an element in $\mathcal{H}$ by

$$b^q_{\mathcal{T}^0} := \sum_{w \in C(\mathcal{T}^0)} \varepsilon(w) q^{-l(w)} T_w.$$

The element $b^q_{\mathcal{T}^0} \in \mathcal{H}$ is denoted by e_- in Gyoja [5]. For general standard tableaux $\mathcal{T}$, the q-skew symmetrizer $b^q_{\mathcal{T}}$ is defined as $b^q_{\mathcal{T}} := T_{w(\mathcal{T})} \cdot b^q_{\mathcal{T}^0} \cdot T^{-1}_{w(\mathcal{T})}$, where $w(\mathcal{T})$ is the permutation which transforms $\mathcal{T}^0$ to $\mathcal{T}$.

Definition 4.1 *We define the q-Specht polynomial for a standard tableau $\mathcal{T}$ by*

$$\Delta^q_{\mathcal{T}}(x) := b^q_{\mathcal{T}}(m_{\mathcal{T}}),$$

where $m_{\mathcal{T}}$ is the initial monomial of $\Delta_{\mathcal{T}}$ with respect to the lexicographic ordering.

Remark 4.1 The polynomials $\Delta^q_{\mathcal{T}}$ are essentially same as those introduced by Martin [9] and by DKLLST [1] (in different notation). See the following Proposition 4.1.

Lemma 4.1 *Let $\mathcal{T}$ be a standard tableau. Then, there exists a sequence of simple transpositions $s_{i_1}, \ldots, s_{i_m}$ such that all the images $\mathcal{T}(a) := s_{i_a} \cdots s_{i_1}(\mathcal{T})$ $a = 1, \ldots, m$, are standard tableaux, $\mathcal{T}(m) = \mathcal{T}^0$ and $l(w(\mathcal{T}(i+1))) = l(w(\mathcal{T}(i))) - 1$.*

Proof. Here we use a variant of a term "northwest" from Fulton [3], Section 4.2. Let us say a box B' on a Young tableau is *northeast* of B, if the row of B' is strictly above that of B, and the column of B' is strictly right to that of B.

For the given standard tableau $\mathcal{T}$, repeatedly apply the following operation starting with the initial condition $i = 0$ and $\mathcal{T}(0) := \mathcal{T}$.
At each i-th step, apply the procedure $P_i(j)$ from $j = 1$ to $j = |\mathcal{T}|$.
Procedure $P_i(j)$

• If the box j is northeast of the box $j+1$, then apply the transposition s_j to the tableau $\mathcal{T}(i,j)$ and put $\mathcal{T}(i,j+1) := s_j\mathcal{T}(i,j)$.
• Otherwise, just put $\mathcal{T}(i,j+1) := \mathcal{T}(i,j)$.

After finishing $P_i(1), \ldots, P_i(|\mathcal{T}|)$, we get a standard tableau $\mathcal{T}(i, |\mathcal{T}|+1)$. Then, put $\mathcal{T}(i+1) = \mathcal{T}(i+1,1) := \mathcal{T}(i, |\mathcal{T}|+1)$ and go to the $(i+1)$-st step.

One can reach the standard tableau $\mathcal{T}^0$ from an arbitrary standard tableau $\mathcal{T}$ within finite steps.

Moreover, if the box j is northeast of the box $j+1$ in a standard tableau $\mathcal{T}$, then $w(\mathcal{T})^{-1}(i) > w(\mathcal{T})^{-1}(i+1)$. Hence $l(w(s_i\mathcal{T})) = l(s_iw(\mathcal{T})) = l(w(\mathcal{T}))-1$.

Proposition 4.1 *One has*

$$\Delta^q_{\mathcal{T}}(x) = T_{w(\mathcal{T})}b^q_{\mathcal{T}^0}(m_{\mathcal{T}^0}) = T_{w(\mathcal{T})}\left(\prod_k \prod_{i,j\in\mathcal{T}^0_k,\, i<j}(x_i - q^{-1}x_j)\right),$$

where $\mathcal{T}^0_k$ is the k-th column of $\mathcal{T}^0$.

Proof. For a standard tableau $\mathcal{T}$, choose the sequence of simple reflections $s_{i_1}, \ldots, s_{i_m}$ as in Lemma 4.1. Then $T^{-1}_{w(\mathcal{T})} = T_{i_1}\cdots T_{i_m}$ and $T_{i_a}\cdots T_{i_1}(m_{\mathcal{T}}) = m_{\mathcal{T}(a)}$. This shows the first equality.

The second equality is a consequence of the identity

$$(*)\quad b^q_{\mathcal{T}^0} = \prod_k\Big(\sum_{w\in S(\mathcal{T}^0_k)}\varepsilon(w)q^{-l(w)}T_w\Big),$$

where $S(\mathcal{T}^0_k)$ is the permutation group on the set of indices in the k-th column of $\mathcal{T}^0$.

Theorem 4.1 *For any standard tableau $\mathcal{T}$, one has*

$$\Lambda_q(\Delta^q_{\mathcal{T}}(x)) = 0.$$

Proof. The proof can be done in a similar manner to that of Proposition 1.1 after replacing Λ and $b_{\mathcal{T}}$ by Λ_q and $b^q_{\mathcal{T}}$. However, a more detailed analysis for cancellation is needed.

Since Λ_q commutes with the action of $\mathcal{H}$, one has

$$\Lambda_q(\Delta^q_{\mathcal{T}}(x)) = T_{w(\mathcal{T})}b^q_{\mathcal{T}^0}(\Lambda_q m_{\mathcal{T}^0}).$$

For a partition (I_1, I_2) of $\{1, \ldots, n\}$ such that $i < j$ for all $i \in I_1$ and $j \in I_2$, the operator Λ_q satisfies

$$(**) \quad \Lambda_q(f(x_{I_1})g(x_{I_2})) = q^N \Lambda_q(f(x_{I_1}))g(x_{I_2}) + f(x_{I_1})\Lambda_q(g(x_{I_2}))$$

for some integer N, if f and g are monomials. If we denote by l_i the length of the i-th column of $\mathcal{T}^0$, the initial monomial $m_{\mathcal{T}^0}$ can be expressed as

$$m_{\mathcal{T}_0} = (x_1^{l_1-1}x_2^{l_1-1}\cdots x_{l_1-1})(x_{l_1+1}^{l_2-1}x_{l_1+2}^{l_2-2}\cdots x_{l_1+l_2-1})\cdots.$$

From $(*)$ and $(**)$, we can see that it is enough to show $b^q_{\mathcal{T}^0}(\Lambda_q m_{\mathcal{T}^0}) = 0$ when $\mathcal{T}^0$ consists of only one column. So we consider the case $m_{\mathcal{T}^0} = x_1^{a-1}x_2^{a-2}\cdots x_{a-1}$. We will show that $\Lambda_q(m_{\mathcal{T}^0}) = 0$. In this case, one has

$$\Lambda_q(m_{\mathcal{T}^0}) = \sum_{i=1}^{a-1} q^{-1}[a-i][d-a+i]x_1^{a-1}\cdots x_i^{a-i-1}x_{i+1}^{a-i-1}\cdots x_{a-1},$$

In the following we compute the image of the monomials $M_{(i)}$:

$$M_{(i)} := x_1^{a-1}\cdots x_i^{a-i-1}x_{i+1}^{a-i-1}\cdots x_{a-1},$$

by the skew-symmetrizer $b^q_{\mathcal{T}^0}$ for the permutation group S_a on the set $\{1, \ldots, a\}$. Denote by C_i the set of the minimal (right) coset representatives (cf. Humphreys [7]) for the parabolic subgroup $S^{(i)} := S_{\{1,\ldots,i\}} \times S_{\{i+1,\ldots,a\}}$. Then, $b^q_{\mathcal{T}^0}$ can be factorized as follows:

$$b^q_{\mathcal{T}^0} = (\sum_{u \in S^{(i)}} (-q)^{-l(u)}T_u)(\sum_{v \in C_i} (-q)^{-l(v)}T_v).$$

Let us decompose C_i into the disjoint of the two subsets $D_i^{\pm} := \{v \in C_i \mid l(vs_i) = l(v) \pm 1\}$. From the Exchange Condition in Humphreys [7, Chapter 1, 1.7], each element in D_i^- has a reduced decomposition ending in s_i. For $v \in D_i^-$, there exists a unique $v' \in C_i$ such that $vs_i \in S^{(i)}v'$. If we take $t \in S^{(i)}$ such that $vs_i = tv'$, then $l(v') \geq l(t^{-1}u) - 1$. Since v and v' are the unique elements of minimal length in their right cosets respectively, $l(v') \geq l(u) - 1$, and so $v' = us$. Therefore, we have $D_i^- s_i = D_i^+$. For $i = 1, \ldots, l-1$, we obtain

$$\sum_{v \in C_i} (-q)^{-l(v)}T_v(M_{(i)})$$

$$= \sum_{v\in D_i^+} (-q)^{-l(v)} T_v(M_{(i)}) + \sum_{v\in D_i^-} (-q)^{-l(v)} T_v(M_{(i)})$$

$$= \sum_{v\in D_i^+} (-q)^{-l(v)} T_v(M_{(i)}) + \sum_{v'\in D_i^- s_i} (-q)^{-l(v')-1} T_{v'} T_i(M_{(i)})$$

$$= \sum_{v\in D_i^+} (-q)^{-l(v)} T_v(M_{(i)}) + \sum_{v'\in D_i^+} -(-q)^{-l(v')} T_{v'}(M_{(i)}) = 0.$$

This completes the proof.

Corollary 4.1 *Let q be generic.*
(1) *The q-Specht polynomial $\Delta^q_{\mathcal{T}}(x)$ is a generator of an irreducible $(U_q(\mathfrak{sl}_d), \mathcal{H})$-component. The algebra A decomposes as a direct sum of irreducible $U_q(\mathfrak{sl}_d)$-modules:*

$$A = \bigoplus_{\mathcal{T}} U_q(\mathfrak{sl}_d) \cdot \Delta^q_{\mathcal{T}}(x),$$

where $\mathcal{T}$ runs over all the standard tableaux of shape $\lambda \vdash n$ with the length $l(\lambda) \leq d$.
(2) *For each standard tableaux satisfying the condition above, the elements*

$$\Delta^q_{\mathcal{T}}(x), L_q \cdot \Delta^q_{\mathcal{T}}(x), \ldots, L_q^{(d-1)n-2\deg \Delta^q_{\mathcal{T}}} \cdot \Delta^q_{\mathcal{T}}(x)$$

form a linear basis of an irreducible $U_q(\mathfrak{sl}_2)$-submodule of A. In particular, the q-Specht polynomials form a linear basis of the primitive part A_i^0 for $d = 2$. Hence, $\mathbf{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ has a decomposition

$$\mathbf{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2) = \bigoplus_j \bigoplus_{l(\mathcal{T})\leq 2} \mathbf{C} \cdot (L_q^j \Delta^q_{\mathcal{T}}(x)),$$

where $l(\mathcal{T})$ is the number of rows of $\mathcal{T}$.

Example 4.1 We consider the case $d = 3$ and $n = 4$ again. Take the standard tableau $\mathcal{T} = 1\ 2\ 3\ 4$. The corresponding q-Specht polynomial is $\Delta^q_{\mathcal{T}}(x) = 1$. The polynomials

$$X_1^+(\Delta^q_{\mathcal{T}}) = x_1 + qx_2 + q^2x_3 + q^3x_4 = L_q(1),$$

$$(X_1^+)^2(\Delta^q_{\mathcal{T}}) = [2](x_1x_2 + qx_1x_3 + q^2x_1x_4 + q^2x_2x_3 + q^3x_2x_4 + q^4x_3x_4),$$

$$(X_1^+)^3(\Delta^q_{\mathcal{T}}) = [2][3](x_1x_2x_3 + qx_1x_2x_4 + q^2x_1x_3x_4 + q^3x_2x_3x_4),$$

$$(X_1^+)^4(\Delta_{\mathcal{T}}^q) = [2][3][4]x_1x_2x_3x_4$$

give the q-deformation of the elementary symmetric polynomials. So, we define the polynomials $e_i^q(x_1, x_2, x_3, x_4)$, $i = 1, 2, 3, 4$, as follows:

$$e_1^q(x_1, x_2, x_3, x_4) = x_1 + qx_2 + q^2x_3 + q^3x_4,$$

$$e_2^q(x_1, x_2, x_3, x_4) = x_1x_2 + qx_1x_3 + q^2x_1x_4 + q^2x_2x_3 + q^3x_2x_4 + q^4x_3x_4,$$
$$e_3^q(x_1, x_2, x_3, x_4) = x_1x_2x_3 + qx_1x_2x_4 + q^2x_1x_3x_4 + q^3x_2x_3x_4,$$
$$e_4^q(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4.$$

Then we have $\Lambda_q(e_4^q) = [2]e_3^q$, $\Lambda_q(e_3^q) = [2]^2e_2^q$, $\Lambda_q(e_2^q) = [2][3]e_1^q$. We can find the polynomials that generate the irreducibe $\mathcal{H}$-module corresponding to $\lambda = (4, 0, 0, 0)$ in A_2^0 and A_4^0 as follows:

$$\begin{array}{lc} A_2^0 & (1 + [2][3])e_2^q - [3]L_q^2(1), \\ A_4^0 & [2][3]([2][3] - 1)e_4^q - [2]([3] + 2)L_q^2e_2^q + ([3] + 1)L_q^4(1). \end{array}$$

For the standard tableau

$$\mathcal{T} = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array},$$

we have $\Delta_{\mathcal{T}}^q = x_1 - q^{-1}x_2$ and $X_2^+(\Delta_{\mathcal{T}}^q) = x_1^2 - q^{-1}x_2^2$. Then we can see that

$$\Lambda_q((1 + [2])(x_1^2 - q^{-1}x_2^2) - L_q(x_1 - q^{-1}x_2)) = 0$$

by direct computation. Similarly, the q-deformed version of the basis of the primitive part KerΛ listed in Example 1.1 can be computed as follows.

A_0^0	$\lambda=(4,0,0,0)$	1
A_1^0	$\lambda=(3,1,0,0)$	$x_1-q^{-1}x_2, qx_1-q^{-1}x_3, q^2x_1-q^{-1}x_4$
A_2^0	$\lambda=(4,0,0,0)$	$(1+[2][3])e_2^q-[3]L_q^2(1)$
	$\lambda=(3,1,0,0)$	$(1+[2])(x_1^2-q^{-1}x_2^2)-L_q(x_1-q^{-1}x_2)$, $(1+[2])(qx_1^2-q^{-1}x_3^2)-L_q(qx_1-q^{-1}x_3)$, $(1+[2])(q^2x_1^2-q^{-1}x_4^2)-L_q(q^2x_1-q^{-1}x_4)$
	$\lambda=(2,2,0,0)$	$(x_1-q^{-1}x_2)(x_3-q^{-1}x_4)$, $T_2(x_1-q^{-1}x_2)(x_3-q^{-1}x_4)$
A_3^0	$\lambda=(3,1,0,0)$	$((2[3]-1)(X_1^+)^2-[3]L_q^2)(x_1-q^{-1}x_2)$ $+([2]+[3])L_q(x_1^2-q^{-1}x_2^2)$, $T_2\{((2[3]-1)(X_1^+)^2-[3]L_q^2)(x_1-q^{-1}x_2)$ $+([2]+[3])L_q(x_1^2-q^{-1}x_2^2)\}$, $T_3T_2\{((2[3]-1)(X_1^+)^2-[3]L_q^2)(x_1-q^{-1}x_2)$ $+([2]+[3])L_q(x_1^2-q^{-1}x_2^2)\}$,
	$\lambda=(2,1,1,0)$	$(x_1-q^{-1}x_2)(x_1-q^{-1}x_3)(x_2-q^{-1}x_3)$, $T_3(x_1-q^{-1}x_2)(x_1-q^{-1}x_3)(x_2-q^{-1}x_3)$, $T_2T_3(x_1-q^{-1}x_2)(x_1-q^{-1}x_3)(x_2-q^{-1}x_3)$
A_4^0	$\lambda=(4,0,0,0)$	$[2][3]([2][3]-1)e_4^q-[2]([3]+2)L_q^2e_2^q+([3]+1)L_q^4(1)$,
	$\lambda=(2,2,0,0)$	$([2]+1)(x_1^2-q^{-1}x_3^2)(x_2^2-q^{-1}x_4^2)$ $-L_q^2(x_1-q^{-1}x_3)(x_2-q^{-1}x_4)$, $T_2\{([2]+1)(x_1^2-q^{-1}x_3^2)(x_2^2-q^{-1}x_4^2)$ $-L_q^2(x_1-q^{-1}x_3)(x_2-q^{-1}x_4)\}$

Problem 4.1 (1) The algebra A is isomorphic to the cohomology ring of the product of n copies of the projective space $\mathbf{P}^{d-1}$. The Lefschetz element L corresponds to the multiplication by the class of a hyperplane section. Is it possible to construct the q-analogue of the Lefschetz decomposition geometrically?
(2) What happens if q is a root of the unity?

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