SELF HOMOTOPY GROUPS WITH LARGE NILPOTENCY CLASSES

HIROAKI HAMANAKA, DAISUKE KISHIMOTO, AND AKIRA KONO

Abstract. We demonstrate that, for any $n > 0$, there exists a compact connected Lie group G such that the self homtopy group $[G, G]$ has the nilpotency class greater than n, where $[G, G]$ is a nilpotent group for a compact connected Lie group G.

1. Introduction

Let G be a path connected topological group and X a finite complex. The homotopy set $[X, G]$ becomes a group by the point-wise multiplication and, in fact, it is a nilpotent group with nilpotency of class $\leq \operatorname{cat} X$, where $\operatorname{cat} X$ is the normalized Lusternik-Schnirelmann category of X ([11]). For a compact connected Lie group G, we call a group of the self homotopy set $[G, G]$ the self homotopy group. The nilpotency class of the self homotopy groups is studied by various authors ([1], [5], [6], [8], [10]), but they show the nilpotency class of the self homotopy groups is greater than 4 at most.

The purpose of this paper is to demonstrate that there exists a compact connected Lie group such that the self homotopy group has large nilpotency class and we obtain:

Theorem 1.1. *Let p be an odd prime, then*

$$\operatorname{nil}\,[PU(p), PU(p)]_{(\frac{1}{2})} > p - 3,$$

where, for a nilpotent group H, $\operatorname{nil} H$ and $H_{(\frac{1}{2})}$ denote the nilpotency class of H and the localization of H at all primes but 2 respectively.

Corollary 1.1. *For any $n > 0$, there exists a compact connected Lie group such that the self homotopy group has nilpotency class greater than n.*

In §2 we decompose $PU(n)_{(\frac{1}{2})}$ into two factors and the proof of Theorem 1.1 is given in §3 by using the decomposition in §2.

2. Decomposition of $PU(n)$

Let $\mathbf{c} : SU(n) \to SU(n)$ be the complex conjugation, then we have

$$\mathbf{c}_*(e_{2i-1}) = (-1)^i e_{2i-1}, \tag{2.1}$$

where $H_*(SU(n); \mathbf{Z}) = \bigwedge(e_3, e_5, \ldots, e_{2n-1})$ and e_{2i-1} is primitive. We denote by X_+ and X_- the infinite telescopes

$$SU(n) \xrightarrow{\mathbf{c}+1} SU(n) \xrightarrow{\mathbf{c}+1} SU(n) \xrightarrow{\mathbf{c}+1} \cdots$$

and

$$SU(n) \xrightarrow{\mathbf{c}-1} SU(n) \xrightarrow{\mathbf{c}-1} SU(n) \xrightarrow{\mathbf{c}-1} \cdots .$$

Then, by (2.1), it is easily seen that the natural map $SU(n) \to X_+ \times X_-$ induces an isomorphism

$$H_*(SU(n); \mathbf{Z}[\tfrac{1}{2}]) \cong H_*(X_+ \times X_-; \mathbf{Z}[\tfrac{1}{2}]).$$

By the J.H.C. Whitehead theorem, we obtain

$$SU(n)_{(\frac{1}{2})} \simeq X_{+(\frac{1}{2})} \times X_{-(\frac{1}{2})}.$$

Remark 2.1. Similar decompositions of $SU(n)$ at an odd prime are obtained by several authors ([7], [9], [12]).

The complex conjugation $\mathbf{c} : SU(n) \to SU(n)$ induces the map $\mathbf{c}' : PU(n) \to PU(n)$ and we have the infinite telescopes

$$PU(n) \xrightarrow{\mathbf{c}'+1} PU(n) \xrightarrow{\mathbf{c}'+1} PU(n) \xrightarrow{\mathbf{c}'+1} \cdots$$

and

$$PU(n) \xrightarrow{\mathbf{c}'-1} PU(n) \xrightarrow{\mathbf{c}'-1} PU(n) \xrightarrow{\mathbf{c}'-1} \cdots$$

denoted by Y_+ and Y_- respectively. The commutative diagram

$$\begin{array}{ccc} SU(n) & \longrightarrow & X_+ \times X_- \\ \downarrow & & \downarrow \\ PU(n) & \longrightarrow & Y_+ \times Y_- \end{array}$$

yields that

$$\pi_i(PU(n)_{(\frac{1}{2})}) \cong \pi_i(Y_{+(\frac{1}{2})} \times Y_{-(\frac{1}{2})}) \text{ for } i \geq 2.$$

The direct calculation shows that

$$\mathbf{c}' = -1 : \pi_1(PU(n)) \to \pi_1(PU(n)).$$

Then we have

$$\pi_1(PU(n)_{(\frac{1}{2})}) \cong \pi_1(Y_{+(\frac{1}{2})} \times Y_{-(\frac{1}{2})})$$

and, in particular, $\pi_1(Y_+) = 0$, hence

$$X_+ \simeq Y_+. \tag{2.2}$$

By the J.H.C. Whitehead theorem, we obtain

$$PU(n)_{(\frac{1}{2})} \simeq Y_{+(\frac{1}{2})} \times Y_{-(\frac{1}{2})}.$$

Let $n = p^r m$, where p is an odd prime and $(p, m) = 1$. In Baum-Browder [2], it is shown that

$$H^*(PU(n); \mathbf{Z}/p) = \mathbf{Z}/p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_2, \ldots, \widehat{x_{p^r}}, \ldots, x_n),$$

where $|y| = 2, |x_i| = 2i - 1$. Then, by (2.2), we have

$$H^*(Y_{+(\frac{1}{2})}; \mathbf{Z}/p) = \bigwedge(x_2, x_4, \ldots)$$

and

$$H^*(Y_{-(\frac{1}{2})}; \mathbf{Z}/p) = \mathbf{Z}/p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_3, \ldots, \widehat{x_{p^r}}, \ldots).$$

We summarize the results above as:

Proposition 2.1. *Let $n = p^r m$, where p is an odd prime and $(p, m) = 1$. Then there exist spaces Y_+ and Y_- with*

$$H^*(Y_+; \mathbf{Z}/p) = \bigwedge(x_2, x_4, \ldots)$$

and

$$H^*(Y_-; \mathbf{Z}/p) = \mathbf{Z}/p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_3, \ldots, \widehat{x_{p^r}}, \ldots)$$

such that

$$PU(n)_{(\frac{1}{2})} \simeq Y_+ \times Y_-,$$

where $|y| = 2, |x_i| = 2i - 1$.

Remark 2.2. A similar decomposition of $PU(n)$ at an odd prime is obtained in Broto-Møller [4] by using the theory of the homotopy fixed point and the complex conjugation which is considered as the unstable Adams operation of degree -1.

3. Proof of Theorem 1.1

Let p be an odd prime. We consider the commutator map

$$\gamma : PU(p) \times PU(p) \to PU(p)$$

on the mod p cohomology.

As is seen above, in Baum-Browder [2], it is shown that

$$H^*(PU(n); \mathbf{Z}/p) = \mathbf{Z}/p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_2, \ldots, \widehat{x_{p^r}}, \ldots, x_n)$$

and

$$\bar{\phi}(y) = 0, \bar{\phi}(x_i) = x_1 \otimes y^{i-1} + \sum_{j=2}^{i-1} \binom{i-1}{j-1} x_j \otimes y^{i-j},$$

where $|y| = 2, |x_i| = 2i-1$ and $\bar{\phi}$ is the reduced co-multiplication. We denote the multiplication of $PU(p)$, the inverse map of $PU(p)$, the diagonal map and the alternating map by μ, ι, Δ and T respectively. Put

$$H = H^*(PU(p); \mathbf{Z}/p) \text{ and } I_k = \widetilde{H}^*(PU(p)^k; \mathbf{Z}/p).$$

Then we have:

$$\begin{aligned}
x_i \xmapsto{\mu^*}\ & x_i \otimes 1 + 1 \otimes x_i + (i-1)x_{i-1} \otimes y \quad \bmod (I_2)^3 \\
\xmapsto{(\mu\times\mu)^*}\ & (i-1)(x_{i-1} \otimes y \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x_{i-1} \otimes y \\
& \qquad + x_{i-1} \otimes 1 \otimes 1 \otimes y + 1 \otimes x_{i-1} \otimes y \otimes 1) \\
& \qquad \bmod (H \otimes 1 \otimes H \otimes 1) + (1 \otimes H \otimes 1 \otimes H) + (I_4)^3 \\
\xmapsto{(1\times1\times\iota\times\iota)^*}\ & (i-1)(x_{i-1} \otimes y \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x_{i-1} \otimes y \\
& \qquad - x_{i-1} \otimes 1 \otimes 1 \otimes y - 1 \otimes x_{i-1} \otimes y \otimes 1) \\
& \qquad \bmod (H \otimes 1 \otimes H \otimes 1) + (1 \otimes H \otimes 1 \otimes H) + (I_4)^3 \\
\xmapsto{(1\times T\times1)^*}\ & (i-1)(x_{i-1} \otimes 1 \otimes y \otimes 1 + 1 \otimes x_{i-1} \otimes 1 \otimes y \\
& \qquad - x_{i-1} \otimes 1 \otimes 1 \otimes y - 1 \otimes y \otimes x_{i-1} \otimes 1) \\
& \qquad \bmod (H \otimes H \otimes 1 \otimes 1) + (1 \otimes 1 \otimes H \otimes H) + (I_4)^3 \\
\xmapsto{(\Delta\times\Delta)^*}\ & (i-1)(x_{i-1} \otimes y - y \otimes x_{i-1}) \quad \bmod (H \otimes 1) + (1 \otimes H) + (I_2)^3.
\end{aligned}$$

Since γ passes through $PU(p) \wedge PU(p)$, we obtain:

Proposition 3.1. *For $x_i \in H^{2i-1}(PU(p); \mathbf{Z}/p)$,*

$$\gamma^*(x_i) \equiv (i-1)(x_{i-1} \otimes y - y \otimes x_{i-1}) \quad \bmod (I_2)^3.$$

Since $\gamma^*(y) = 0$, we have:

Corollary 3.1. *Let γ_n be the n-fold iterated commutator map*

$$\gamma(\gamma \times 1) \cdots (\gamma \times 1 \times \cdots \times 1) : PU(p)^{n+1} \to PU(p).$$

Then

$$\gamma_{i-2}^*(x_i) = (i-1)!(x_2 \otimes y \otimes \cdots \otimes y - y \otimes x_2 \otimes y \otimes \cdots \otimes y) \quad \bmod (I_{i-1})^i.$$

Let $f : PU(p)_{(\frac{1}{2})} \to PU(p)_{(\frac{1}{2})}$ be the composition

$$PU(p)_{(\frac{1}{2})} \simeq Y_+ \times Y_- \xrightarrow{\pi} Y_+ \subset Y_+ \times Y_- \simeq PU(p)_{(\frac{1}{2})},$$

where Y_+, Y_- are as in Proposition 2.1 and π denotes the first projection. By Proposition 2.1 and Corollary 3.1, we have

$$l\Delta^*(f \times 1 \times \cdots \times 1)^* \gamma_{p-3}{}^*_{(\frac{1}{2})}(x_{p-1}) \equiv (p-2)! x_2 y^{p-3} \not\equiv 0 \mod (I_1)^{p-1}.$$

Since, by [3],

$$[PU(p), PU(p)]_{(\frac{1}{2})} = [PU(p)_{(\frac{1}{2})}, PU(p)_{(\frac{1}{2})}],$$

this proves Theorem 1.1.

References

[1] M.A. Arkowitz, H. Ōshima and J. Strom, *Noncommutativity of the group of self homotopy classes of Lie groups*, Topology Appl. **125** (2002), no. 1, 87-96.

[2] P.F. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305-336.

[3] A.K. Bousfield and D.M. Kan, *Homotopy limits completions and localization*, Lecture Notes in Math. **304**, Springer (1972).

[4] C. Broto and J.M. Møller, *Finite Chevalley versions of p-compact groups*, preprint.

[5] H. Hamanaka, *Nilpotency of unstable K-theory*, to appear in Topology Appl.

[6] H. Hamanaka and A. Kono, *A note on the Samelson products in $\pi_*(SO(2n))$ and the group* $[SO(2n), SO(2n)]$, preprint.

[7] B. Harris, *Suspensions and characteristic maps for symmetric spaces*, Ann. of Math. **76** (1962), 295-305.

[8] A. Kono and H. Ōshima, *Commutativity of the group of self-homotopy classes of Lie groups*, Bull. London Math. Soc. **36** (2004), 37-52.

[9] M. Mimura, G. Nishida and H. Toda, *Mod p decomposition of compact Lie groups*, Publ. Res. Inst. Math Sci **13** (1977/78), 627-680.

[10] H. Ōshima and N. Yagita, *Non commutativity of self homotopy groups*, Kodai Math. J. **24** (2001), 15-25.

[11] G.W. Whitehead, *Elements of homotopy theory*, GTM **61**, Springer (1978).

[12] C.W. Wilkerson, *Self-maps of classifying spaces*, Lecture Notes in Math. **418**, Springer (1974), 150-157.

(H. Hamanaka) Department of Natural Science, Hyogo University of Teacher Education

E-mail address: hammer@sci.hyogo-u.ac.jp

(D. Kishimoto) Department of Mathematics, Kyoto University

E-mail address: kishi@math.kyoto-u.ac.jp

(A. Kono) Department of Mathematics, Kyoto University

E-mail address: kono@math.kyoto-u.ac.jp