Geometric representations of interacting maps

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Introduction

Motivated by phenomena in molecular biology, in [K3] we have formulated an interaction system by use of families of maps on intervals, which consistes of infinte families of compositions between them. By wasting detailed and extracting more rough information by use of projections, one can produce some integrable systems of automata. This enables one to see some patterns which correspond to more macro features.

Let $f : [0,1] \to [0,1]$ be a map, and consider its iteration $\{f^n(x)\}_{n=0}^{\infty}$. This will behave very randomly and will touch sensitively with respect to points $x \in [0,1]$. Let $X_2 = \{(a_0, a_1, \dots) : a_i \in \{0,1\}\}$ be the one sided full shift and take two maps $\{f_0, f_1\}$. By generalizing from iteration of a single map to compositions of two maps, one can consider families of compositions $\{h^n(x) \equiv f_{i_n} \circ \dots f_{i_0}(x)\}_{n=0}^{\infty}$, where the set of indices $\{i_k\}_k \subset \{0,1\}$ is determined for each an element $\bar{k} \in X_2$.

Using them, one can construct continuous maps $\Phi(x) = \Phi(f_1, f_2)(x)$ between X_2 to itself for each point $x \in [0,1]$, which are called the *interaction maps*. These are determined by use of projections on the interval and so by which subinterval $h^n(x)$ lies on for all n. In order to determine a value $\Phi(x)(\bar{k})$, even though one is required to have information on $h^n(x)$ for all n, in fact it is enough to know rough values of them, rather than rigorous ones on $[0,1]$. So $\Phi(x)$ is a map in a macroscopic scale compared with the micro interaction $\{h^n(x)\}$. This might be a very simple mechanism to create patterns in macro scale from random micro dynamics. This method is immediately generalized to use families of maps on intervals $\{f_i\}_i$, and one also obtains interaction maps $\Phi(f_1, \ldots, f_k)$ by the same way.

So far we have known that many important cell automata can be expressible by interaction systems as above ([K3], 1.A). In the first part of this paper, we study geometric properties of such cell automata which include the box and ball system (BBS), Lotka Volterra cell automaton and lamplighter automaton. For example we will construct assignments from BBS flows to braid groups and extensions of BBS actions on compactified spaces passing through group actions on trees. We will see some relation between LV cell automaton and the lamplighter automaton, and generalize the latter to obtain more group actions on trees using such relation.

In the middle part, we study connections of general automata with complex geometry. In real algebraic geometry, some geometric mappings from complex planes to the real ones were discovered, by taking coordinatewisely absolute values and their logarithms ([V]). This connects algebraic varieties with piecewise linear maps and is called the *tropical geometry*. On the other hand in [K3] several relations of pl maps with automata are studied. For example the LV cell automaton are given by a family of pl maps. Passing through the above operations, in this paper we will associate a family of polynomials from an automaton. Using the projective duality on projective varieties and the above assignment, we will find a *duality between cell automata*. In fact for an automaton A, we will associate a dual one $A^{\vee}$. We will calculate some examples in the case of curves.

In general automata are represented by families of interaction maps. Thus direct assignments from families of maps to these geometric objects will show geometric representations of interaction systems.

In the last part, we construct infinite families of graphs from finite families of maps, points and symbolic sequences, and we call them the *interaction graphs*. We will construct these graphs so that they represent some dynamics of interacting states in micro scale (see below). The central dogma in molecular biology tells us that proteins are products of various interaction systems in micro scale, beginning from DNA. From geometric point of view, one would like to obtain a space and an automorphism on it from such interacting data, which will represent more macro features. In analogy to the central dogma as above, the space will correspond to a protein or polymers, and orbits of an automorphism correspond to states on it, since functions of proteins are determined by the shape of themselves.

A *space form problem* is to construct a space V and an automorphism A on it so that the orbits $\{x, A(x), A^2(x), \dots\} \subset V$ are induced from the dynamics of the interaction graphs. In this paper we will formulate and address a space form problem from toric variety view point, passing through several hierarchies of dynamics.

Thoughout this paper, our basic direction is to study geometric properties of them as macro objects in light of dynamical properties of compositions and iterations of families of maps corresponding to the micro one.

In some particular cases of families of maps which include some piecewise linear maps, the interaction maps are reduced to some automata, which allow to connect them to several rigid objects like integrable systems. On the other hand in general, maps on intervals are very flexible objects and their dynamics are very random.

One of our main aim in furthur development is to study macro properties of these dynamics for families of maps which are near such special types of maps. Namely let $\{f_i\}_i$ be a family as above so that the corresponding interaction system is reduced to some automaton A. Let us take any geometric object arising from an interaction system $G = G(\{f_i\}_i)$. We have several examples of G below. Since G passes through A, its structure will have some rigid properties.

Let us take another family of maps $\{g_i\}_i$ which is sufficiently near the original $\{f_i\}_i$, where the corresponding interaction system for $\{g_i\}_i$ will not be reduced to any automaton in general. We would like to study geometric properties of $G(\{g_i\}_i)$ by comparing with $G(\{f_i\}_i)$. This might be one direction to proceed some understanding of mechanisms of pattern formation. In particular several stability of G under small deformation of these maps will be particularly of interest for us.

Now let us describe the contents of the paper more concretely.

The box and ball system is a dynamics on the set of finite subsets in $\mathbf{Z}$. For $\{i_1, \dots, i_l\} \subset \mathbf{Z}$, $i_1 < i_2 < \cdots < i_l$, one can regard that a ball occupies in each position i_m, $m = 1, \dots, l$, and these balls will be moved by the following rule. The ball i_1 in the most left hand side is moved to some $i_1 < j_1$ which is the most left hand side in $\{i_1+1, i_1+2, \dots\} \backslash \{i_2, \dots, i_l\}$. We repeat a similar procedure. i_2 is moved to aonther $i_2 < j_2$ which is most left hand side in $\{i_2+1, i_2+2, \dots\} \backslash \{j_1, i_2, \dots, i_l\}$. By the same way i_k moved to aonther $i_k < j_k$ which is most left hand side in $\{i_k + 1, i_k+2, \dots\} \backslash \{j_1, \dots, j_{k-1}, i_{k+1}, i_{k+2}, \dots, i_l\}$. When this procedure is finished for i_l, then we are done.

Let Σ_2^0 be all the finite subsets in $\mathbf{Z}$. Then the above procedure is expressed as:

$$T : \Sigma_2^0 \cong \Sigma_2^0.$$

Let $\Sigma_2^0(N) \subset \Sigma_2^0$ be the sets of N-subsets in $\mathbf{Z}$. Then in fact T is a map as $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$.

Following our expression of interaction, in section 1, we will describe the BBS system by an interaction of maps for some family of continuous maps.

It is known that the BBS flows by the dynamics of T contain solitons. Among dynamical properties of solitons, the relative positions of the individual waves and how these waves pass through the others will be in the most important structures. In section 2 we will represent such information geometrically by using the braid groups. Elements in braid groups are certainly representing such situation. In order to eliminate infinitely many ambiguities, in this paper we will use quotient groups $\bar{B}_n$ of the braid groups. There are many of them, and in a special case it is a subgroup of the mapping class group of finite index. Each σ admits an index among subsets in $\{1, \dots, N-1\}$, which is determined by $T^t(\sigma)$ near $t = \pm\infty$. We will associate a quotient braid group with respect to individual index. Then we obtain canonical maps:

$$B : \Sigma_2^0(N) \to \bar{B}_N$$

which we call the *braiding map.* $B(\sigma)$ is constructed from the dynamics of iterations

$$\{\dots, T^{-1}, \sigma, T(\sigma), T^2(\sigma), \dots\}$$

Scattering process of BBS is described in [FOY] by using the combinatorial R-matrix. Using such direction, one may obtain some invariants of the dynamics T.

Let Σ_2 be the both sided full shift with the alphabets $\{0, 1\}$. Then Σ_2^0 can be regarded as a subset Σ_2. Then in section 4 we study geometric properties of the BBS map T. For example we see that it can be extended as $T : \Sigma_2 \to \Sigma_2$. BBS is isomorphic to the Lotka Volterra cell automaton. In fact there is an explicit procedure to construct such isomorphism. Thus one obtains an injection $\Sigma_2^0 \hookrightarrow \Sigma_\infty^0$ which assigns a solution of the BBS to the coresponding one of the LV. We will see that it can be extended on some partially compactified space.

The Lotka Volterra cell automaton is given by the equation ([ST]):

$$V_n^{t+1} - V_n^t = \max(0, V_{n+1}^t - L) - \max(0, V_{n-1}^{t+1} - L).$$

This is obtained from the original Lotka Voleterra equation by making differentizations twice. It is known that this possess solitons which are induced from the ones of the difference LV solitons (see [K3]). It is known that BBS is isomorphic the LV cell automaton. In section 3, we formulate several structures of *path spaces* of the set of cell automata.

The lamplighter group is well known as an automata group in geometric group theory ([GZ]), which admits an action on the rooted binary tree. In section 4 we find that in a special case the LV cell automaton is in fact a transition function for the lamplighter automaton:

Lemma 0.1 *Suppose the initial sequence* $(a_0, a_1, \dots)$ *consistes of only* $\{0, 1\}$ *entries. Then the degeneration of the LV cell automaton is the* *same as the transition function ϕ of an automaton whose group is isomorphic to the lamplighter group.*

Thus using the LV cell automaton as transition functions, one can construct a family of automata groups which we call *LV cell automata groups*. They can act on the boundary $\tilde{\partial} T^*_\infty \equiv \cup_i \partial T^*_i$ of the rooted infinite tree.

In section 5, we study automata from *complex geometry* point of view. Tropical geometry connects algebraic geometry with piecewise linear maps. There is a procedure Φ_t, $t \in (1, \infty)$ to obtain a family of real maps from polynomials. It is done by taking absolute values and taking their logarithms ([LM],[V]). Combining with the method in [K3], one obtains an assignment from automata A to a pair of parametrized polynomials (f^1_t, f^2_t), $t \in (1, \infty)$. Namely both $\lim_{t\to\infty} \Phi_t(f^1_t)$ and $\lim_{t\to\infty} \Phi_t(f^2_t)$ become pl maps, and the equation $\Phi_\infty(f^1_\infty) = \Phi_\infty(f^2_\infty)$ represents the automaton A.

Here one will find importance to study *stability* of dynamics of iteration for families of maps on the interval. In fact given a family of infinte sequences of complex numbers $\{\mathbf{z}_t = (z^0_t, z^1_t, \dots)\}_t$, and suppose they give a flow of the polynomial pair (f^1_t, f^2_t) which is equivalent that the sequences $\{(\log_t |z^0_t|, \log_t |z^1_t|, \dots)\}$ is the one of $(\Phi_t(f^1_t), \Phi_t(f^2_t))$. So:

Proposition 0.1 *Let $\mathbf{z}^l_t \subset \mathbf{R}^\infty_+$ be families of positive and real numbers. Suppose $\mathbf{v}^l \equiv \lim_{t\to\infty} \mathbf{z}^l_t$ exist. Then $\mathbf{v}^l$ is a flow of the original automaton A.*

This leads to the following notion of stability:

Definition 0.1 *Let A be an admissible and deterministic automaton, and choose a parametrized families of sequences $\{\mathbf{z}_t\}_{t\geq 1} \subset \mathbf{C}^\infty$. We say that the family $\{\mathbf{z}_t\}_{t\geq 1}$ is a stable sequence, if for the induced family $\{\mathbf{z}^l_t\} \subset \mathbf{C}^N$, there is a large t_0 so that for all $t \geq t_0$, the family:*

$$\{[Log_t(\mathbf{z}^l_t)]\}_{l=1,2,\dots} \subset \mathbf{Z}^{\mathbf{N}}$$

gives a flow of the solutions of the original automaton A.

Now using these polynomials, one can obtain spaces:

Definition 0.2 *The accociated affine hypersurfaces is a parametrized family of hypersurfaces given by the equations:*

$$V(A)_t = \{\mathbf{z} \in \mathbf{C}^N : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z})\}.$$

One important reason to consider such varieties come from *projective duality* on algebraic varieties. Let us denote the projective dual of an algebraic variety X by $X^\vee$. Suppose the projective dual of $V(A)_t$ are hypersurfaces. Then there is a parametrized family of polynomials $\Delta(A)_t$ which define these spaces. They are called $\{A_i\}_i$-*discriminant.*

Again by using Φ_t above and letting $t \to \infty$, one obtains another automaton $A^\vee$. Thus one has obtained an assignment:

$$A \to A^\vee$$

by passing through duality in complex geometry. We call $A^\vee$ the *dual automaton.* For example we have calculated in the case of some curves. Let $a \geq 2$, α and c be integers. Then we have the following:

Proposition 0.2

$$[\max\{au_n, \alpha + au_{n+1}\} = c]^\vee =$$
$$\max\{\frac{a}{a-1}(c - \frac{\alpha}{a}) + \frac{a}{a-1}u_{n+1}, \frac{ac}{a-1} + \frac{a}{a-1}u_n\} = c.$$

In micro level, molecular interactions occur by covalent or hydrogen bonds where electrons of molecules share their orbitals. From this point, in section 6, we have formulated interaction systems of families of maps by constructing some graphs.

Let us take two interval maps $f_0, f_1 : [0,1] \to [0,1]$ and $\Phi(x, f_0, f_1) : X_2 \to X_2$ be the interaction map. Let us choose another map $d : [0,1] \to [0,1]$. In the light of orbitals stated above, if the projection to subintervals of iteration $\{d^n(z)\}_n$ coincides with $\Phi(x, f, g)(\bar{k})$ in X_2, then we construct a marked oriented edge as:

$$(f, x) \xrightarrow{(g,\bar{k})} (d, z).$$

Let us give families of maps $\{f_1, \ldots, f_k\}$, points $\{x_1, \ldots, x_l\} \subset [0,1]$ and $\{\bar{a}(i,j,h)\}_{i,j,h=1}^{i,j=k,h=l} \in X_2$. By the above way, we will construct an oriented marked finite graph:

$$G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)\})$$

which we call the *interaction graph*. We will interpret this graph to represent a state of the system consisted by the triple $(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)\})$.

Let us choose a triple (f_i, f_j, x_h). Then we have the corresponding interaction map $\Phi(f_i, f_j, x_h) : X_2 \to X_2$. In particular one can obtain another elements:

$$\bar{a}(i,j,h)_2 \equiv \Phi(f_i, f_j, x_h)(\bar{a}(i,j,h)) \in X_2.$$

By this way one can obtain another interaction graph:

$$G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)_2\}).$$

Let us denote by $\mathfrak{G}(\{f_i\}, \{x_j\})$ the set of interaction graphs with fixed families of maps and points. The numbers of vertices are all the same in any element in this. Notice that this is a finite set.

Then by the above procedure one can obtain the following map:

$$\Phi_* : \mathfrak{G}(\{f_i\}, \{x_j\}) \to \mathfrak{G}(\{f_i\}, \{x_j\}).$$

By iterating this procedure, one obtains an infinite family of interaction graphs $G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)\})$, $G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)_2\})$, $G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)_3\}), \ldots$ One can regard that this family of interaction graphs might represent a dynamics of states of a micro interaction system, and according our principle at the first of the introduction, one may induce some macro patterns from them. In this paper we will induce some *hierarchies* of combinatoric objects arising from such family of graphs.

Let $\mathfrak{G}$ be the set of finite graphs, and $F : \mathfrak{G}(\{f_i\}, \{x_j\}) \to \mathfrak{G}$ be the forgetful map. Thus passing through F, one obtains a family of fintite graphs:

$$G_1 \equiv F(\bar{G}_1), G_2 \cdots \subset \mathfrak{G}, \quad \bar{G}_t = G(\{f_i\}, \{x_j\}, \{\bar{a}(i,j,h)_t\})$$

which we call just the *associated graphs.* Any G_i have the same number of vertices N. Thus there is a finite number of finite graphs $\{H_1, \dots, H_m\}$ so that each G_i coincides with one of $\{H_j\}$.

We say that a family of finite graphs is *strongly regular*, if they have the same number of edges as others.

Let G be a finite graph. Then the associated *configuration* $\bar{a} \in \mathbf{Z}^m$ in combinatroics is determined. Thus one obtains another family of configurations:

$$\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots \subset \mathbf{Z}^m$$

which we call the *transcripted configurations.*

For each configuration $\bar{a} \in \mathbf{Z}$, one obtains an ideal $I_{\bar{a}} \subset \mathbf{C}[y_1, \dots, y_m]$ which is called the *toric ideal.*

Thus associated to the family of configurations, one obtains the corresponding family of ideals:

$$I_1, I_2, \dots, \quad I_j \subset \mathbf{C}[y_1, \dots, y_{m_j}]$$

which we call the *associated ideals.*

For each ideal $I \subset \mathbf{C}[y_1, \dots, y_m]$, one obtains a complete fan over $\mathbf{R}^m$ and the corresponding toric viariety $X_I \subset \mathbf{CP}^{m-1}$. The fan is called the *Gröbner fan.*

Thus corresponding to the associate ideals, one obtains a family of toric varieties:

$$X_1, X_2, \dots \subset \mathbf{CP}^{m-1}, \quad m = \sup\{m_1, m_2, \dots\}.$$

We call the sequence as the *translated toric variety.* When the associated graphs are strongly regular, then all X_j have the same dimension.

Let us have more abstract setting. Let V be an algebraic variety with the affine coordinates V_i defined by an ideal J_i. Let us take an automorphism A on V. We say that an affine coordinate $\{(V_i, J_i)\}_{i=1}^m$ is an (stable) *algebraic Markov partition* for A, if for each i, there is some j so that:

$$A(V_i) \subset V_j$$

holds.

We say that the the associated ideals $\{I_i\}_i$ are *regular*, if they have the same dimension as others. Let us put $\cup_i I_i = \{J_1, \dots, J_k\}$. Now we have one formulation of a space form problem:

Definition 0.3 *Let* $(\{f_i\}_i, \{x_j\}_j, \{\bar{a}(i,j,h)\})$ *be an interaction data, and suppose the associated ideals* $I = (I_0, I_1, \dots)$ *are regular. The sequence is called a symbolic flow of an automorphism, if there is an algebraic Markov partition for* (V, A) *with an affine coordinate* $\{(V_i, J_i)\}_{i=1}^k$ *and some* $x \in V$ *so that its orbit* $\{A^n(x)\}_{n=0,1,\dots}$ *corresponds to the sequence.*

We would like to call such pair (V, A) a *prohedron* (which comes from 'proteiform'). We will see by an easy example that combinatorics of the interaction graphs will reflect existence of such pairs.

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1 Interaction by composition of families of maps

Let us take two interval maps:

$$f, g : [0,1] \to [0,1]$$

and consider their iterations:

$$O_1(x) = \{f^k(x)\}_{k=0,1,\dots}, \quad O_2(x) = \{g^k(x)\}_{k=0,1,\dots}.$$

We call them the *oscillatins* ([K2]).

Let us define *interaction* of these orbits below. For this, let X_2 be the one sided full shift with two alphabets $\{0,1\}$:

$$X_2 = \{(k_0, k_1, \dots) : k_i \in \{0,1\}\}.$$

Then for each element $\bar{k} = (k_0, k_1, \dots) \in X_2$, we will associate a family of maps:

$$\{h^m(x)\}_{k=0,1,\dots}, \quad h^m : [0,1] \to [0,1]$$

as follows. Let us put:

$$d_i(x) = \begin{cases} f(x) & i = 0, \\ g(x) & i = 1. \end{cases}$$

Then we define h^k by:

$$h^m(x) \equiv d_{k_m} \circ d_{k_{m-1}} \circ \cdots \circ d_{k_0}(x).$$

Let:

$$\pi : [0,1] \to \{0,1\}$$

be a measurable map given by $\pi([0,\frac{1}{2})) \equiv 0$ and $\pi((\frac{1}{2},1]) \equiv 1$. Then for each $x \in [0,1]$, one can compose $\{h^m(x)\}_m$ with π and obtains another element for a.e. x:

$$\bar{k}' \equiv \pi((h^0(x), h^1(x), \dots)) \equiv (\pi \circ h^0(x), \pi \circ h^1(x), \dots) \in X_2.$$

Thus for each element $\bar{k} \in X_2$, one can assign another element $\bar{k}'$. We denote this assignment:

$$\Phi(x,f,g) : X_2 \to X_2$$

by $\Phi(x,f,g)(\bar{k}) \equiv \pi((h^0(x), h^1(x), \dots))$ and call it the *interaction map*.

In [K3], we have constructed the flow of the Lotka Volterra cell automaton by the iteration of the interaction map for some family of interval maps. LV cell automaton is isomoprhic to the BBS system described below. We will study the isomorphism in section 3.

1.B BBS system: Let Σ_2 be the both sided full shift with two alphabets. Let $\sigma = (\dots, v_{-n}, \dots, v_0, v_1, \dots) \in \Sigma_2$ be an infinite sequence by $\{0,1\}$ such that for all sufficiently large $n >> 0$, $v_n = v_{-n} = 0$. Let us denote the set of such sequences by $\Sigma_2^0 \subset \Sigma_2$. It is shift invariant. Notice that this has a canonical identification with Σ_2^0 in the introduction by $(\dots, v_n, \dots) \to \{n : v_n = 1\}$.

Let us choose an element $\sigma \in \Sigma_2^0$ and $(i_1 < i_2 < \dots < i_m)$ be all the indices with $v_{i_l} = 1$. Let $T(\sigma)_1 = (\dots, v^1_{-m}, \dots, v^1_0, v^1_1, \dots) \in \Sigma_2^0$ be another element defined as follows; let $j_1 \geq i_1$ be the smallest index with the property that it is larger than i_1 and $v_{j_1} = 0$. Then $v^1_l = v_l$ except $l = i_1$ and j_1, and we put $v^1_{i_1} = 0$ and $v^1_{j_1} = 1$.

Next we do the same thing for $v^1_{i_2} = v_{i_2}$ in $T(\sigma)_1$, and find another smallest index $j_2 \geq i_2$ with $v^1_{j_2} = 0$. Then we exchange 0 and 1 in $v^1_{i_2}$ and $v^1_{j_2}$ as above. The result is denoted by $T(\sigma)_2$.

We continue this process for $i_3, i_4, \dots$ until i_m, and finally one obtains the desired $T(\sigma) \equiv T(\sigma)_m \in \Sigma_2^0$.

Thus one has obtained a continuous bijective map:

$$T : \Sigma_2^0 \cong \Sigma_2^0$$

which is calld the *box and ball system* (BBS). Let $\Sigma_2^0(N)$ be the set of N-subsets. Then the BBS system induces a bijection $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$.

1.C BBS as interaction of maps: Let us describe the BBS map T by an interaction of a family of interval maps. The basic method will take two steps. Firstly we will describe it by an automaton, and then write down the automaton by an interaction of a family of maps. Here we will use some modified way in order to express it by a family of continuous maps.

Firstly we will construct an automaton A. It will induce a map $M : \Sigma_2^0(N) \cong \Sigma_2^0(N)$ which conrresponds to $T(\sigma)_1$ in $1.B$. Then the BBS map $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$ is given by $T|\Sigma_2^0(N) = M^N$.

Let $S = \{0, 1, 2, 3, 4\}$ be a finite set. Then A is given by:

$$\phi : S \times \{0, 1\} \to S,$$
$$\psi : S \times \{0, 1\} \to \{0, 1\}$$

where these are defined by the followings:

$$\phi : \quad \begin{array}{ccccccc} & | & 0 & 1 & 2 & 3 & 4 \\ - & - & - & - & - & - & - \\ 0 & | & 0 & 2 & 4 & 2 & 4 \\ 1 & | & 1 & 3 & 4 & 3 & 4 \end{array},$$

$$\psi(a) = \begin{cases} \epsilon, & a = 1, 2, \\ \text{id} , & a = 0, 3, 4 \end{cases}$$

where $\epsilon \in S_2$ is the non trivial element in the permutation group on $\{0, 1\}$.

Let us describe the induced map $M : \Sigma_2^0(N) \cong \Sigma_2^0(N)$ as follows; take $\sigma = (\dots, v_m, v_{m+1}, \dots) \in \Sigma_2^0$, and choose sufficiently small m_0 so that $v_m = 0$ holds for all $m \leq m_0$.

The initial state is $s_0 = 0$ in the above diagram, and let us define inductively as:

$$s_k = \phi(s_{k-1}, v_{m_0+k-1}),$$
$$v'_{m_0+k} = \phi(s_{k+1}, v_{m_0+k}),$$

where we put $v'_m = 0$ for all $m \leq m_0$.

This is independent of choice of m_0 and gives an assignment:

$$(\dots, v_m, v_{m+1}, \dots) \to (\dots, v'_m, v'_{m+1}, \dots)$$

which induces the desired map $M : \Sigma_2^0 \cong \Sigma_2^0$.

Let us denote $\phi = \{\phi_0, \phi_1\}$ and $\psi = \{\psi_0, \psi_1\}$, where $\phi_i : S \to S$ and $\psi_i : S \to \{0, 1\}$ for $i = 0, 1$. Let $\pi : [0, 1] \to \{0, 1\}$ and $\pi_5 : [0, 1] \to \{0, 1, 2, 3, 4\}$ be the canonical projections.

If the automaton above is described by compositions of a family of continuous maps $\{\alpha_i\}_{i=0,1}$ and $\{f_i\}_{i=0,1}$ as $\phi_i = \pi_5 \circ \alpha_i$ and $\psi_i = \pi \circ f_i$, then we say that the automaton is given by an *interaction of maps on intervals* ([K3]).

It is immediate to see that the above A cannot be expressible by a family of continuous maps as above. So we will modify as follows. Let us choose two permutations $\epsilon_0, \epsilon_1 : S \cong S$. Then we say that A is given by a *modified interaction of maps* with respect to (ϵ_0, ϵ_1), if there are families of continuous maps $\{\alpha_i\}_{i=0,1}$ and $\{f_i\}_{i=0,1}$ so that the following hold:

$$\phi_i = \epsilon_i \circ \pi_5 \circ \alpha_i,$$
$$\psi_i = \pi \circ f_i$$

Now let us choose permutations as:

$$\epsilon_0 : (0, 1, 2, 3, 4) \to ((0, 2, 4, 1, 3),$$
$$\epsilon_1 : (0, 1, 2, 3, 4) \to (0, 1, 3, 4, 2).$$

Now we choose α_i and f_i so that they satisfy the following properties:

$$\alpha_0|[\frac{i}{5},\frac{i+1}{5}] \subset [\frac{i}{5},\frac{i+1}{5}], i=0,1,2,$$
$$\alpha_0|[\frac{3}{5},\frac{4}{5}] \subset [\frac{1}{5},\frac{2}{5}], \quad \alpha_0|[\frac{4}{5},1] \subset [\frac{2}{5},\frac{3}{5}],$$
$$\alpha_1|[\frac{i}{5},\frac{i+1}{5}] \subset [\frac{i+1}{5},\frac{i+2}{5}], i=0,1,2,$$
$$\alpha_1|[\frac{i}{5},\frac{i+1}{5}] \subset [\frac{i-1}{5},\frac{i}{5}], i=3,4,$$

$$f_0|[\frac{j}{5},\frac{j+1}{5}] \subset [0,\frac{1}{2}], j=0,3,4, \quad f_0|[\frac{j}{5},\frac{j+1}{5}] \subset [\frac{1}{2},1], j=1,2,$$
$$f_1|[\frac{j}{5},\frac{j+1}{5}] \subset [0,\frac{1}{2}], j=1,2, \quad f_1|[\frac{j}{5},\frac{j+1}{5}] \subset [0,\frac{1}{5}], j=0,3,4.$$

Lemma 1.1 *The above families of continuous maps with the permutations give modified interaction of maps representing an automaton A so that the BBS automaton is given by $A^N = A \circ \cdots \circ A$ on $\Sigma_2^0(N)$ for $N = 0, 1, 2, \ldots$*

Notice that these maps can be obtained by piecewise linear way.

2 Quotient of the braid groups and BBS system

2.A Quotient of the braid groups: Let B_n be the braid group with n-strands. Thus it has a presentation:

$$B_n = \{t_1, \ldots, t_{n-1} : t_i t_j = t_j t_i, |i-j| > 1, t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}\}.$$

It has a canonical element representing the half twist:

$$\Omega_n \equiv (t_1 \ldots t_{n-1})(t_1 \ldots t_{n-2}) \ldots (t_1 t_2) t_1 \in B_n$$

and the quotient by Ω_n^2 is a subgroup of the mapping class group on $(n+1)$-punctured sphere. In fact it is a subgroup which fixes ∞ point and index $n+1$ ([ECHLTP]):

$$B_n / < \Omega_n^2 > \subset MPG_{n+1}.$$

Let us generalize it and have quotient braid groups.

Let $i = 1, \ldots, n-1$ and $k = 0, \ldots, n-1$, and denote subsets by $[i,k] \equiv (i, i+1, \ldots, i+k) \subset \{1, \ldots, n-1\}$. Let $B_{[i,k]} \subset B_n$ be the subgroup generated by $\{t_i, \ldots, t_{i+k}\}$. By the same way, each $B_{[i,k]}$ contains the corresponding canonical elements:

$$\Omega_{[i,k]} \in B_{[i,k]}.$$

We will define two types of quotients of the braid groups using these canonical elements.

Definition 2.1 *The mod 2 braid group $M_2\bar{B}_n$ is given by the following:*

$$\begin{aligned} M_2\bar{B}_n &= \{t_1, \ldots, t_{n-1} : t_i t_j = t_j t_i, |i-j| > 1, t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, t_i^2\} \\ &= B_n / \ gen \ < \cup_{i=1}^{n-1} t_i^2 > . \end{aligned}$$

For example, $M_2\bar{B}_2 = \mathbf{Z}_2$ and $M_2\bar{B}_3$ is an infinite group which has a presentation given by $\mathbf{Z}_2 * \mathbf{Z}_2 / < t_1 t_2 t_1^2 t_2 t_1 >= \mathbf{Z}_2 * \mathbf{Z}_2$.

Let us denote the quotient map by:

$$\pi : B_n \to M_2\bar{B}_n.$$

Lemma 2.1 $\pi(\Omega^2_{[i,k]}) = 1 \in M_2\bar{B}_n$ *for any i, k. In particular we have:*

$$M_2\bar{B}_n = B_n / \ gen \ < \cup_{[i,k] \subset \{1,\ldots,n-1\}} \Omega^2_{[i,k]} > .$$

Proof: We check for $n = 4$, and the general case will follow from this immediately.

In fact $\pi(\Omega_4^2) = (t_1t_2t_3t_1t_2t_1)(t_1t_2t_3t_1t_2t_1) = t_1t_2t_3t_1t_2t_1^2t_2t_3t_1t_2t_1 = t_1t_2t_3t_1t_2^2t_3t_1t_2t_1 = t_1t_2t_3t_1t_3t_1t_2t_1 = t_1t_2t_3t_3t_1t_1t_2t_1 = 1$.

This completes the proof.

Let $\mathbf{I} \subset \{1, \ldots, n-1\}^2$ be a set of subsets. Then we define another type of quotient braid groups by:

$$\bar{B}_n(\mathbf{I}) = B_n / \ \text{gen} \ < \Omega_I : I \in \mathbf{I} > .$$

Notice that we do not include elemetns like $(i, 0)$ which correspond to one string. We will denote $\pi_{\mathbf{I}} : B_n \to \bar{B}_n(\mathbf{I})$ for the projection.

Later we will have geometric meaning to divide by the twists in terms of the BBS system. In particular we will see in the following section that solitary behaviour in BBS system can be represented as an element in the quotient braid groups.

2.B Solitary flow: Any $\sigma \in \Sigma_2^0$ can be uniquely expressed by a finite set:

$$\{I_1, \dots, I_l\} \equiv \{(i_1, N_1), \dots, (i_l, N_l)\}$$

where $N_j \geq 1$ and $l \geq 0$. They satisfy the followings:

(1) $v_j = 1$ for all $j \in \cup_{k=1,\dots,l}\{i_k, i_k + 1, \dots, i_k + N_k - 1\}$, and

(2) $v_j = 0$ holds for some j in $\{i_k + N_k, \dots, i_{k+1} - 1\}$, $k = 1, \dots, l$.

We call $(N_1, \dots, N_l)$ the *index* of σ. Thus one may regard each I_j as one soliton.

Let $\sigma \in \Sigma_2^0$ and conside the BBS map T. The indices of $T^t(\sigma)$, $(S_1, \dots, S_m)$ are all constant for sufficiently small $t << 0$, and also for all suffciently large $t >> 0$, they are also constant $(T_1, \dots, T_{m'})$, where $\Sigma S_i = \Sigma T_j$ holds. Here inequalties hold:

$$S_1 \geq S_2 \geq \dots \geq S_m, \quad T_1 \leq T_2 \leq \dots \leq T_{m'}.$$

We say that σ has its *type*:

$$(\{S_i\}_{i=1}^m, \{T_j\}_{j=1}^{m'}).$$

If σ is a soliton, then the equality:

$$(T_1, \dots, T_{m'}) = (S_m, S_{m-1}, \dots, S_1)$$

holds. More precisely let $\{\sigma^t\}_{t \in \mathbf{Z}} \subset \Sigma_2^0$ be a flow. We say that it is *solitary*, if there are a set $\{M_1, \dots, M_m\} \subset \mathbf{N}$, $1 \leq M_1 < M_2 < \dots < M_m$, and families $\{i_1^t, \dots, i_m^t\} \subset \mathbf{Z}$ such that for all sufficiently large $t >> 0$,

$$\sigma^{-t} = \{(i_1^{-t}, M_1), (i_2^{-t}, M_2), \dots, (i_m^{-t}, M_m)\},$$
$$\sigma^t = \{(i_1^t, M_m), (i_2^t, M_{m-1}), \dots, (i_m^t, M_1)\}$$

where:

(1) $i_1^t < i_2^t < \cdots < i_m^t$ and $i_m^{-t} < i_{m-1}^{-t} < \cdots < i_1^{-t}$ hold, and
(2) $|i_j^{\pm t} - i_{j+1}^{\pm t}| \to \infty$ as $t \to \infty$ for all $j = 1, \ldots, m-1$.

Thus for any $\sigma \in \Sigma_2^0$, the corresponding flow $\{T^t(\sigma)\}_{t \in \mathbf{Z}}$ is solitary.

2.C Assignment of braid elements: Let us take $\sigma \in \Sigma_2^0(N)$, and T be the BBS map. Let us take a large $t_0 >> 0$ so that $T^{-t_0}(\sigma)$ and $T^{t_0}(\sigma)$ have indices $(S_1, \ldots, S_m)$ and $(T_1, \ldots, T_{m'})$ respectively. Notice the equality $N \equiv \Sigma_{i=1}^m S_i = \Sigma_{j=1}^{m'} T_j$.

Let us put $\sigma_0 \equiv T^{-t_0}(\sigma)$, and we consider the step $T(\sigma_0)_i$, $i = 1, \ldots, m$ in the definition of the map T $(1.B)$. For the step from σ_0 to $T(\sigma_0)_1$, let us assign a natural element:

$$b_1 \in B_N$$

below, where each string corresponding to an element 1 in σ_0. Namely when the most left hand side 1 moves into a position passing through another r number of 1's, then $t_{r-1} \ldots t_1$ is assigned. For example if $\sigma_0 = (\ldots, 0, 1, 1, 0, \ldots)$ moves as $(\ldots, 0, 0, 1, 1, \ldots)$, then one generating element t_1 is assigned. Similarly if $\sigma_0 = (\ldots, 1, 1, 1, 0, \ldots)$ which moves as $(\ldots, 0, 0, 1, 1, 1, \ldots)$, then $t_2 t_1$ is assigned.

Next for the second step from $T(\sigma_0)_1$ to $T(\sigma_0)_2$, one assigns another element $b_2 \in B_N$ by the same way. Continueing, one obtains another $b_3, \ldots, b_m$.

By this way one has assigned an element:

$$b = b(\sigma_0, t_0) = b_m b_{m-1} \ldots b_1 \in B_N$$

which we call the *braiding element*.

2.C.2 Braiding maps: Notice that b depends on the choice of t_0 and so σ_0 in the above. In fact there will be infinitely many different elements each other with respect to choice of t_0. The ambiguities arise from choices of the starting point $\sigma_0 = T^{-t_0}(\sigma)$ and the ending point $T^{t_0}(\sigma)$. Let σ has the type $(\{S_i\}_{i=1}^m, \{T_j\}_{j=1}^{m'})$. They are essentially given by the twists of the canonical elements

$$\Omega_{[1,S_1-1]}, \Omega_{[S_1+1,S_2-1]}, \Omega_{[S_1+S_2+1,S_3-1]}, \ldots, \Omega_{[S_1+\cdots+S_{i-1}+1,S_i-1]}$$

for the former, and the latter is by:

$$\Omega_{[T_1+\cdots+T_j+1,T_{j+1}-1]}, \Omega_{[T_1+\cdots+T_j+T_{j+1}+1,T_{j+2}-1]}, \dots, \Omega_{[N-T_{m'},T_{m'}-1]}$$

where the rest indices all correspond to one string:

$$S_k = 1 \quad (k = i+1, \dots, m), \quad T_l = 1 \quad (l = 1, 2, \dots, j).$$

Let us denote the set:

$$\begin{aligned}\mathbf{I}(\sigma) = \{[1{,}S_1 - 1], [S_1 + 1, S_2 - 1] \dots, [S_1 + \cdots + S_{i-1} + 1, S_i - 1],\\ [T_1 + \cdots + T_j + 1, T_{j+1} - 1], [T_1 + \cdots + T_{j+1} + 1, T_{j+2} - 1],\\ \dots, [N - T_{m'}, T_{m'} - 1]\}.\end{aligned}$$

We say that $\mathbf{I}(\sigma)$ is the *index* of the BBS flow for σ.

Let $\mathbf{I} \subset \{1, \dots, n-1\}$ be a set of subsets. Then we put:

$$\Sigma_2^0(\mathbf{I}) \equiv \{\sigma \in \Sigma_2^0 : \mathbf{I}(\sigma) = \mathbf{I}\}.$$

Let $\pi_{\mathbf{I}}$ be the pojection as before. Then we define the *restricted braiding map with respect to* $\mathbf{I}$:

$$B(\mathbf{I}) : \Sigma_2^0(\mathbf{I}) \to \bar{B}_N(\mathbf{I})$$

by assigning $\pi_{\mathbf{I}}(b)$, $b = b(\sigma_0, t_0))$. It is independent of choice of t_0 and gives a single map.

The above map depends on $\mathbf{I}$. Below we will have another braiding map from Σ_2^0 as a multi-valued one. The target is also obtained by the quotient of the braid group.

Let $\pi : B_N \to M_2\bar{B}_N$ be the projection. Thus $\pi(b) \in M_2\bar{B}_N$ have ambiguity at most finitely many elements with respect to t_0. Now we define the *mod* 2 *braiding map*:

$$B^2 : \Sigma_2^0(N) \to M_2\bar{B}_N$$

as the images of all various values of t_0 for sufficiently large $|t_0| >> 0$, given by $\pi(b) \equiv \bar{b} \in M_2\bar{B}_N$, where b is as above. Thus B^2 is a finite multi-valued map.

We call $\bar{b}$ the *mod* 2 *braiding element*.

2.D Connected braiding maps: Let us take two elements:

$$\sigma^k = (\dots, 0, a_{i_1,k}, \dots, a_{i_{m_k},k}, 0, \dots) \in \Sigma_2^0, \quad k = 1, 2.$$

Choose sufficiently large $M >> 0$. Then we define the *connected sum* of σ^1 with σ^2 of length M by:

$$\sigma_1 \sharp_M \sigma_2 = (\dots, 0, a_{i_1,1}, \dots, a_{i_{m_1},1}, 0, \dots, 0, a_{i_2,2}, \dots, a_{i_{m_2},2}, 0, \dots) \in \Sigma_2^0$$

where 0 appears M times in the middle.

The index of the *connected sum* and their union are given by:

$$\mathbf{I}(\sigma^1, \sigma^2; M) = \mathbf{I}(\sigma^1) \cup \mathbf{I}(\sigma^2) \cup \mathbf{I}(\sigma_1 \sharp_M \sigma_2) \subset \{1, \dots, m_1 + m_2 - 1\},$$
$$\mathbf{I}(\sigma^1, \sigma^2) \equiv \cup_{M>>0} \mathbf{I}(\sigma_1 \sharp_M \sigma_2).$$

In fact $\mathbf{I}(\sigma^1, \sigma^2; M)$ is completely determined by the triple $(\mathbf{I}(\sigma^1), \mathbf{I}(\sigma^2), M)$.

Let $\mathbf{I}_N$ be all the set of subsets in $\{1, \dots, N-1\}$ such that each element can be an index for some $\sigma \in \Sigma_2^0$. Then for large $M >> 0$, there are maps:

$$H_M : \mathbf{I}_N \times \mathbf{I}_{N'} \to \mathbf{I}_{N+N'}$$

which give the indices of connected sums.

Thus these induce a family of maps:

$$H(\mathbf{I}, \mathbf{I}'; M) : \bar{B}_N(\mathbf{I}) \times \bar{B}_{N'}(\mathbf{I}') \to \bar{B}_{N+N'}(H_M(\mathbf{I}, \mathbf{I}'))$$

satisfying:

$$H(\mathbf{I}, \mathbf{I}'; M)(B(\mathbf{I})(\sigma^1), B(\mathbf{I}')(\sigma^2)) = B(H_M(\mathbf{I}, \mathbf{I}'))(\sigma^1 \sharp_M \sigma^2)$$

for $\mathbf{I} \in \mathbf{I}_N$ and $\mathbf{I}' \in \mathbf{I}_{N'}$. We will say that $H(\mathbf{I}, \mathbf{I}'; M)$ is a *connected braiding map.*

Proposition 2.1 $H(\mathbf{I}, \mathbf{I}'; M)$ *is eventually period with respect to* M*.*

This follows since the target is divided by twists in braid groups.

3 Transformations on cell automata

3.A From BBS to LV: The Lotka Voleterra cell automaton is given by the equation ([ST]):

$$V_n^{t+1} - V_n^t = \max(0, V_{n+1}^t - L) - \max(0, V_{n-1}^{t+1} - L).$$

This is obtained from the original Lotka Voleterra equation by making differentizations twice. It is known that this possess solitons which are induced from the ones of the difference LV solitons (see [K3]).

There is a procedure to transform BBS equation to LV cell automaton, and vice versa:

$$\begin{aligned} &B_n^{t+1} = \min\{1 - B_n^t, \Sigma_{i=-\infty}^{n-1}(B_i^t - B_i^{t+1})\} \longleftrightarrow \\ &V_n^{t+1} - V_n^t = \max\{L, V_{n+1}^t\} - \max\{L, V_{n-1}^{t+1}\} \end{aligned}$$

by changes of variables.

Let $O_0 =$ LV cell automaton $\to O_1 \to \cdots \to O_k =$ BBS be a procedure of transformations. We say that it is *invertible*, if the procedure has its inverse $O_k \to O_{k-1} \to \cdots \to O_0$. We say that O_0 and O_k can be connected by an invertible procedure.

Lemma 3.1 *LV cell automaton and BBS can be connected by an invertible procedure.*

In fact they can be connected by three steps as follows:

$$\begin{aligned} &S_{n+1}^{t+1} - S_n^t = \min\{0, 1 - S_{n+1}^t + S_n^{t+1}\}, \quad S_n^t = \Sigma_{i=-\infty}^n B_i^t, \\ &U_{n+1}^{t+1} - U_n^t = \max\{0, U_n^{t+1} - 1\} - \max\{0, U_{n+1}^t - 1\}, \quad U_n^t = S_{n+1}^t - S_n^{t+1}, \\ &V_n^{t+1} - V_n^t = \max\{1, V_{n+1}^t\} - \max\{1, V_{n-1}^{t+1}\}, \quad V_{t-n}^n = U_n^t. \end{aligned}$$

For the first transformation, one has the relations $B_n^t = S_n^t - S_{n-1}^t$. For the second, notice that $S_{n-a}^{t+a+1} = 0$ for all sufficiently large a. Then we have the relations:

$$\Sigma_{x=0}^{\infty} U_{n-x}^{t+x} = S_{n+1}^t - \lim_{a \to \infty} S_{n-a}^{t+a+1} = S_{n+1}^t.$$

3.A.2 Deformation by commutators: Let us consider the second step in 3.A. We express the linear transformation $U_n^t = S_{n+1}^t - S_n^{t+1}$ as:

$$U = \alpha(S)$$

where we mean $\alpha(S)_n^t \equiv U_n^t$. We say in short that α is a transformation.

By this way, let us express others by:

$$\begin{aligned}
&f(S), \quad f(S)_n^t = S_n^t - S_{n-1}^{t-1}, \\
&\beta(S), \quad \beta(S)_n^t = S_{n+1}^t - S_n^{t+1}, \\
&\gamma(S), \quad \gamma(S)_n^t = \min\{0, 1 - S_{n-1}^{t-1}\}.
\end{aligned}$$

Now the second step is expressed as:

$$\begin{aligned}
&U = \alpha(S), \\
&f(U) = \beta \circ f(S) \quad (*)
\end{aligned}$$

and the defining equation becomes as:

$$f(S) = \gamma \circ \alpha(S) \quad (**).$$

Thus combining with $(*)$ and $(**)$, after the transformation, the equation changes as below:

$$f(U) = \beta \circ \gamma(U) \quad (***).$$

One can consider abstractly general transformations satisfying the above conditions.

Let us consider to deform the transformations. Let h be a transformation. Then we say that it *commutes* with f, if

$$f \circ h = h \circ f$$

holds.

Example 3.1: Let $U = \{U_n^t\}_{t,n}$ satisfy $\lim_{i\to\infty} U_{n-i}^{t-i} = 0$ for each t and n.

Let h be as:

$$h(U)_t^n = \Sigma_{i=-\infty}^0 U_{n-i}^{t-i}$$

and f be $f(U)^t_n = U^t_n - U^{t-1}_{n-1}$ as above. Then h commutes with f.

Let us consider transformations given by $(*)$ and $(**)$ above. Then a *deformation of the transformation* by a commutator h is another one given below:

$$W = h \circ \alpha(S),$$
$$f(W) = h \circ \beta \circ f(S)$$

which follows from $(*)$. When one considers the equation of the form:

$$f(S) = \gamma \circ h \circ \alpha(S)$$

then it is changed as $f(W) = h \circ \beta \circ \gamma(W)$.

We denote all the set of commutators with respect to f by:

$$C(f) = \{h : [h, f] = 0\}.$$

3.B Spaces of cell automata: So far we have considered one type of cell autotama whose defining equations are given by max-plus equations. These are not closed under change of variables. Thus in this section one will consider to generalize classes of cell automata.

It has shown in [K3] that LV cell automaton is given by a family of PL maps and projections over the interval $[0, 1]$. We have seen that BBS is also given by a family of PL maps, projections and permutations in $1.C$.

Let us denote a set of integer valued maps:

$$PL_n = \{f : \mathbf{Z}^{2n} \to \mathbf{Z}, f(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{Z} \quad (*)\}$$

where $(*)$: they are obtained from PL maps on intervals combining with compositions of projections and permutations as above.

A *generalized cell automaton* is given by a family of elements:

$$\{f_1, \ldots, f_m\} \subset PL_\infty = \lim_{n \to \infty} PL_n$$

and a map $F \in PL_\infty$:

$$F : \mathbf{Z}^2 \to \{1, 2, \ldots, m\}, \quad i \in \mathbf{Z}.$$

The defining equation of the cell automaton is given by:

$$T_k^t = f_{l(k)}(\{\dots, T_{k-n}^t, T_{k-n+1}^t, \dots, T_{k-1}^t\}, \{\dots, T_{k-n}^{t-1}, \dots, T_k^{t-1}, \dots\}),$$
$$F_{k-1}(\{\dots, T_{k-2}^t, T_{k-1}^t, 0, 0, \dots\}, \{\dots, T_{k-1}^{t-1}, T_k^{t-1}, \dots\}) = l(k).$$

Let us introduce classes in the set of all generalized cell automata.

(1) If $\{f_1, \dots, f_m\} \subset PL_N$ for some N, then we say that it is of (N, ∞) type. For simplicity, we will say that it is of finite-∞ type.

(2) If there is some $M \geq 0$ so that F is a map as $F : \mathbf{Z}^{M+1} \times \mathbf{Z}^{2M+1} \to \{0, 1, \dots, m\}$,

$$l(k) = F(T_{k-1-M}^t, \dots, T_{k-1}^t\}, \{T_{k-M}^{t-1}, \dots, T_k^{t-1}, \dots, T_{k+M}^{t-1}\})$$

then we say that it is of (∞, M) type. For simplicity, we will say that it is of ∞-finite type.

(3) If both (1) and (2) hold, then we say that it is of (N, M) type. For simplicity, we will say that it is of finite-finite type.

In particular the Lotka Volterra cell automaton is of finite-finite type and the BBS is of infinite-infinite type.

Let us denote all the set of generalized cell automata and its subset consisted by that of finite-∞ type by:

$$\mathbf{GCA}_{\text{f}-\infty} \subset \mathbf{GCA}.$$

Similarly one denotes $\mathbf{GCA}_{\infty-\text{f}}$ and $\mathbf{GCA}_{\text{f-f}}$.

We will also denote the set of cell automata given by the max-plus equations and its subset by:

$$\mathbf{MP}_{\text{f}-\infty} \subset \mathbf{MP}.$$

Similarly one denotes $\mathbf{MP}_{\infty-\text{f}}$ and $\mathbf{MP}_{\text{f-f}}$.

One has the inclusions:

$$\text{BBS} \in \mathbf{MP} \subset \ \mathbf{GCA} \supset \mathbf{GCA}_{\text{f-f}} \supset \mathbf{MP}_{\text{f-f}} \ni \text{L-V CA}.$$

3.B.2 Cell automata of finite-infinite type: Let us start by an example. We wil use the notations in $3.A$. Let $\{B_n^t\}$ and $\{V_n^t\}$ be the BBS and LV cell automaton respectively. In $3.A$, one has obtained a path $\bar{O} = \{O_0 = \{B_n^t\} \to O_1 \to O_2 \to O_3 = \{V_n^t\}\}$.

Let us put:

$$T_n^t = B_{n+1}^t - B_n^{t+1}.$$

Then $O_1' \equiv \{T_n^t\}$ is linearly invertible with $\{U_n^t\}$ by the relations:

$$\begin{aligned} U_n^t &= \Sigma_{i=-\infty}^n T_i^t, \\ T_n^t &= U_n^t - U_{n-1}^t. \end{aligned}$$

Thus one has obtained another path $\bar{O}' = \{O_0 \to O_1' \to O_2 \to O_3\}$.

Now we have the equalities:

$$\begin{aligned} T_n^t - T_{n-1}^{t-1} &= (U_n^t - U_{n-1}^t) - (U_{n-1}^{t-1} - U_{n-2}^{t-1}) \\ &= (U_n^t - U_{n-1}^{t-1}) - (U_{n-1}^t - U_{n-2}^{t-1}) \\ &= \max(0, U_{n-1}^t - 1) - \max(0, U_n^{t-1} - 1) \\ &\qquad - \max(0, U_{n-2}^t - 1) + \max(0, U_{n-1}^{t-1} - 1) \\ &= [\max(0, U_{n-1}^t - 1) - \max(0, U_{n-2}^t - 1)] \\ &\qquad - [\max(0, U_n^{t-1} - 1) - \max(0, U_{n-1}^{t-1} - 1)]. \end{aligned}$$

Let us consider the first term in the last equality. There are four cases: The first term becomes:

(1) T_{n-1}^t when $U_{n-1}^t - 1 \geq 0$ and $U_{n-2}^t - 1 \geq 0$ hold, (2) $T_{n-1}^t - 1$ when $U_{n-1}^t - 1 \geq 2$ and $U_{n-2}^t = 0$ hold, (3) $T_{n-1}^t + 1$ when $U_{n-1}^t = 0$ and $U_{n-2}^t \geq 2$ hold, and (4) 0 when $U_{n-1}^t \leq 0$ and $U_{n-2}^t \leq 0$.

Thus the values are one of the followings:

$$\{T_{n-1}^t, T_{n-1}^t - 1, T_{n-1}^t + 1, 0\}$$

which depends on $U_n^t = \Sigma_{i=-\infty}^n T_i^t$. Thus O_1' is an element in $\mathbf{GCA}_{\mathrm{f}-\infty}$.

3.C Loop groupoid: Let us choose two generalized cell automata $O = \{S_n^t\}$ and $O' = \{T_n^t\}$ in **GCA**. A *transformation* from O to O' is the one given by a change of variables:

$$T_k^t = f(\{\dots, S_{k-n}^t, S_{k-n+1}^t, \dots, S_{k-1}^t\}, \{\dots, S_{k-n}^{t-1}, \dots, S_k^{t-1}, \dots\})$$

for some linear function f. We denote the transformation by:

$$O \to O'.$$

They are mutually *invertible*, if both $O \to O'$ and $O' \to O$ hold.

Let us consider two invertible paths $O_0 = O \to O_1 \to \cdots \to O_m = O'$, and $O'_0 = O' \to O'_1 \to \cdots \to O'_n = O$ in **GCA**. Then the composition:

$$O = O_0 \to \cdots \to O_m \to O'_1 \to O'_2 \cdots \to O'_n = O$$

gives a loop with the origin O. Clearly two loops with the origin O admits a natural composition, and by this operation, the set of loops:

$$\Omega_O = \{\bar{O} \equiv O = O_0 \to O_1 \cdots \to O_m = O : \text{ invertible paths } \}$$

admits a group structure.

Definition 3.1 *The loop groupoid Ω is given by:*

$$\Omega(O, O') = \{O = O_0 \to O_1 \to \cdots \to O_n : \textit{invertible paths}\}.$$

$\Omega(O) \equiv \Omega(O, O)$ is called the loop group.

Let O_0 be the LV cell automaton, and denote all the set of the solutions of the equation:

$$S_{LV} = \{\{v_n^t\}_{t,n} : \text{ solutions of the LV cell automaton}\}.$$

Let us take an element $\bar{O} \in \Omega_{O_0}$. Then correspondingly, there is a bijective maps between solutions of the equations:

$$\bar{O}_* : S_{LV} \cong S_{LV}.$$

We call it as an *induced map* associated with $\bar{O} \in \Omega_{O_0}$.

In $5.D$, we study some liftings of GCA to some families of polynomials which give affine hypersurfaces on complex planes.

4 Actions on the boundary of trees

4.A Actions on the boundary and compactification: For any graph with marking on each edge, we denote by $m(e)$ as the marking at the edge e. A tree T is said to be *bi-infinite*, if for any vertex $v \in T$, it contains a geodesic real line $v \in \mathbf{R} \subset T$.

Let T_2 be the marked bi-infintie binary tree with marking $\{0,1\}$ so that it contains a base path $l_0 : \mathbf{R} \to T_2$ with $m(l_0(t)) = 0$, $t \in \mathbf{Z}$. Let $\mathbf{P} = \{\mathbf{R} \to T_2\}$ be all the set of geodesics. Then there is a canonical inclusion:

$$\mathbf{P} \subset \Sigma_2$$

where Σ_2 is the both sided full shift with the alphabets $\{0,1\}$. Thus Σ_2 can be regarded as a compactification of the set of all geodesics in T_2.

Let us put:

$$\mathbf{P}^0 = \{l \in \mathbf{P} : m(l(t)) = 0 \text{ for all sufficiently large } |t| >> 0\}.$$

Then we have the proper inclusions:

$$\mathbf{P}^0 \subset \mathbf{P} \subset \Sigma_2.$$

It is easy to see $\mathbf{P}^0 \subset \Sigma_2$ is dense. In fact $\Sigma_2 \backslash \mathbf{P}^0$ consiste of elements $l = (\dots, a_{-1}, a_0, a_1, \dots)$ such that the sets $\{i : m(a_i) = 1\}$ are unbounded at least for one direction.

Now we have a natural identification $(1.B)$:

$$\Sigma_2^0 \cong \mathbf{P}^0 \subset \Sigma_2.$$

Then the BBS system is described by an isomorphism:

$$\Phi : \mathbf{P}^0 \cong \mathbf{P}^0.$$

Notice that the only fixed point is $(\dots, 0, 0, \dots)$.

Proposition 4.1 Φ *can be naturally extended to the continuous map:*

$$\Phi : \Sigma_2 \to \Sigma_2$$

where:

$$\Phi((\ldots,1,1,\ldots)) = (\ldots,0,0,\ldots).$$

Thus the extended map is not an isomorphism. More generally one has non injective points:

$$\Phi((\ldots,0,1,0,\ldots)) = (\ldots,0,0,1,\ldots) = \Phi((\ldots,0,1,1,0,1,\ldots)).$$

The extension also has one fixed point $(\ldots,0,\ldots)$.

In 3.A, we have assigned isomorphic procedures:

$$O_0 = \text{ BBS } \to O_1 \to O_2 \to O_3 = \text{ LV cell automaton.}$$

where we denote each step by $F_i : O_i \to O_{i+1}$, $i = 0,1,2$. Each O_i admits an $\mathbf{N}$ action by the flow. By the construction, we have that F_0 and F_1 are both equivariant with respect to the flow, but F_2 is not.

Let us put $\Sigma_\infty^0 \equiv \cup_{m=1}^\infty \Sigma_m^0$, and Σ_∞ defined similarly. Then by the above procedure $F : O_0 \to O_3$, we have an injection:

$$F : \mathbf{P}^0 \hookrightarrow \Sigma_\infty^0$$

which assigns the solution of the BBS to the one of LV.

Notice that F extends as $F : \mathbf{P} \to \Sigma_\infty$ with respect to the compactification of $\mathbf{P}^0$ defined above, on the other hand it cannot be extended as $F : \Sigma_2 \to \Sigma_\infty$.

Lemma 4.1 $F : \mathbf{P} \to \Sigma_\infty$ *is not a surjection to the set of all solutions of the LV cell automaton.*

Proof: Recall that in the transformations above, F_0 and $F_1 : O_1 \to O_2$ are equivariant, and F_2 is just change of indices. Then the result follows since there are at least two fixed points for O_2 as:

$$\{(\ldots,0,0,\ldots),(\ldots,1,1,\ldots)\}.$$

Question: Let us choose a large $N >> 0$. Describe compactification of $F(\mathbf{P}^0) \cap \Sigma_N \subset \Sigma_N^0$.

4.B Embedding of the lamplighter group: Let S and A be finite sets. An *automata group* G is an infintie group acting on the rooted tree T_m^*, $m = \sharp A$ which is determined by a transition function $\phi : S \times A \to S$ and an exit function $\psi : S \times A \to A$ such that $\psi(a) \in S_m$ is an element of the permutation group for each $a \in A$.

Let T_2 be as in $4.A$. Recall that the lamplighter group is an automata group which acts on the rooted binary tree T_2^*, and it has a presentation ([GZ]):

$$G =< a, \gamma : \gamma^2 = 1, [\gamma^{a^i}, \gamma^{a^j}] = 1, i, j \in \mathbf{Z} > .$$

The action of G preserves each level set of T_2^*, since it is constructed using an automaton. The automaton is given below:

$$\phi : \quad \begin{array}{cccc} & | & a & b \\ - & - & - & - \\ 0 & | & a & a \\ 1 & | & b & b \end{array} \quad \psi(a) = \epsilon, \quad \psi(b) = \text{ id}$$

where $\epsilon \in S_2$ is the nontrivial permutation on $\{0, 1\}$. We call it the *lamplighter automaton.*

Let us construct another group $\tilde{G}$ which contains G and acts on T_2. Recall that T_2 contains a line $(\dots, 0, 0, \dots) \subset T_2$, and choose a base vertex $*$ on the line. By identifying $*$ with the root in T_2^*, one can embed T_2^* into T_2:

$$T_2^* \hookrightarrow T_2.$$

Now passing through this embedding, one can make G act on T_2 by letting the same action as the lamplighter group on $T_2^* \hookrightarrow T_2$, and by putting the identity on $T_2 \backslash T_2^*$.

Now Aut $T_2 \supset \tilde{G} \supset G$ is generated by G and another element τ. Let us describe τ. Notice that each edge of T_2 is assigned by one of $\{0, 1\}$. We say that an automorphism g on T_2 *preserves the marking*, if $g(e)$ has the same marking as the one of e in $\{0, 1\}$. Now τ is an automorphism preserving the marking and uniquely defined by the property that it shifts i-zero in $((\dots, 0, 0, \dots))$ to $i + 1$-one. Thus τ preserves the line. Then $\tilde{G}$ is generated by a, γ and τ.

The action of $\tilde{G}$ on $\tilde{T}_2$ has no fixed point. Moreover it is finitely generated and has quotient isomorphic to $\mathbf{Z}$. Thus the Bass-Serre theory suggests the following:

Question: Is $\tilde{G}$ an amalgam ? If so, write down explicitly G_1, G_2 and A with an isomorphism:

$$\tilde{G} \cong G_1 *_A G_2.$$

4.C Actions on tree by cellular automata: The action of the automata group on T_2^* is determined by the levelwise way. In fact it is an action on each vertex of T_2^*. This is not the case for general cellular autmata, like LV cell automaton. In fact the action is determined for each path in T_2^*, rather than points in T_2^*. More precisely, for many automata including LV case, the image of a vertex v by an element $g \in G$ is determined by a *neighboourhood* of v.

4.C.2 LV cell automaton as a transition function: Let us consider a cell automaton:

$$\Phi : \mathbf{N} \times \mathbf{N}^2 \to \mathbf{N}$$

where each U_n^t is determined inductively by $U_n^{t+1} = \Phi(U_{n-1}^{t+1}, U_n^t, U_{n+1}^t)$.

We say that $\{U_n^t\}$ *degenerates* with respect to Φ, if there is another function G so that it satisfies the relation:

$$U_n^{t+1} = \Phi(G(U_n^t, U_{n+1}^t), U_n^t, U_{n+1}^t).$$

Recall that the lamplighter automaton has two states $\{a, b\}$ and the alphabets $\{0, 1\}$. In order to compare the LV cell automaton with the transition function of it, we regard the flow $\{U_t^0\}_t \to \{U_t^1\}_t$ as an output of a transition function. By this way let us regard that the initial flow $(U_0^0, U_1^0, \dots)$, $U_i^0 \in \{0, 1\}$ is a sequence of the alphabets, and $(U_0^1, U_1^1, \dots)$ is another sequence of the states, where one identifies a and b with 0 and 1 respectively.

Lemma 4.2 *Suppose an initial sequence* $(a_0, a_1, \dots)$ *consistes of only* $\{0, 1\}$ *entries.*

Then the degeneration of the LV cell automaton by $G(x, y) = x$ *is the same as the transition function* ϕ *of the lamplighter automaton.*

When one expresses the flow of Φ by an automaton, one puts an exit function by:

$$\psi(a, i) = 0, \quad \psi(b, i) = 1, \quad i = 0, 1.$$

In particular, the corresponding continuous map on the rooted tree is not an automorphism (it is not one to one).

In general flows of the LV cell automaton takes integer elements $U_n^t \in \mathbf{N}$. In order to treat these cases, let us put a sequence of states:

$$(a_0, a_1, \dots)$$

where $a_0 = a$ and $a_1 = b$. Let S_∞ be the group of compactly supported permutation on $\{0, 1, \dots\}$. Then consider an exit function:

$$\psi : \mathbf{N} \to S_\infty$$

with $\psi(0) = \epsilon \in S_2 \subset S_\infty$ and $\psi(1) = \text{id}$.

In general let $\Phi : \mathbf{N} \times \mathbf{N}^2 \to \mathbf{N}$ be a transition function, and choose an initial sequence $(i_0, i_1, \dots) \subset \mathbf{N}$ and any $U_0^1 \in \mathbf{N}$.

Now we define the *generalized automaton* (Φ, ψ) as follws. As an output, we will exit another sequence $(i_0^1, i_1^1, \dots) \subset \mathbf{N}$ as follows.

Firstly determine the sequence of the states $(U_0^1, U_1^1, \dots)$ inductively by Φ:

$$U_n^1 = \Phi(U_{n-1}^1, i_n^t, i_{n+1}^t).$$

Then we inducively obtain the exits by:

$$i_n^1 = \psi(U_n^1)(i_n^0).$$

By this way one obtains an assignment which is in fact an isomorphism:

$$g_{U_0^1} : X_\infty \cong X_\infty$$

where X_N is the one sided full shift with N alphabets and X_∞ is their union. The group $G(\Phi, \psi)$ generated by g_m, $m = 0, 1, \dots$ is also called the *generalized automata group* given by (Φ, ψ).

In the case when Φ_0 is the LV cell automaton, the equation of the transition function becomes as:

$$U^1_{t+1} = i^t_n + \max(L, i^t_{n+1}) - \max(L, U^{t+1}_{n-1}).$$

Definition 4.1 *The LV cell automata group is a group acting on the boundary* $\tilde{\partial} T^*_\infty \equiv \cup_i \partial T^*_i$ *of the rooted infinite tree defined by the transition function* Φ_0 *and an exit function* ψ *as above.*

A generalized automata group is of *bounded type* by N, if it induces an action between $X_N = \partial T^*_N$.

4.D Quasi actions on trees: Let G be a group acting on the boundary of the rooted tree T^*_2.

We say that an element $\gamma \in G$ is a k_0-*quasi action* on T^*_2, if there is some $k_0 \geq 0$ such that if we write $\gamma(a_0, a_1, \dots) = (a'_0, a'_1, \dots)$, then a'_M is determined by the data $(a_0, a_1, \dots, a_{M+k_0})$ for each $M = 0, 1, \dots$ We denote:

$$\gamma(a_0, a_1, \dots)_M = (a'_0, \dots, a'_M) \in F_2$$

where F_2 is the free group generated by $\{\alpha, \beta\}$. We denote the word length by $|g|$.

Suppose γ is a 1-quasi action. Then we have a map:

$$\begin{aligned} & F_\gamma : F_2 \to F_2, \\ & F_\gamma(g) = (\gamma(g\alpha)_{|g|})^{-1} \gamma(g\beta)_{|g|} \in F_2. \end{aligned}$$

which we call the *differential* of γ. Notice that when γ is an automorphism on the tree, then F is the identity map.

We say that a quasi action by γ is (k_0, l_0)-*semi Markov*, if there is some $l_0 \geq 0$ such that a'_M is determined by $(a_{M-l_0}, \dots, a_{M+k_0})$ for each $M = 0, 1, \dots$

Let γ has a 1-quasi action on T^*_2. We say that it is *bounded*, if there is a bounded function $B : F_2 \to F_2$ so that the differential satisfies the equality:

$$F_\gamma(g) = B(g\alpha)^{-1}B(g\beta).$$

Lemma 4.3 *A bounded quasi action γ is semi Markov.*

Proof: In fact one has the equality:

$$\gamma(g\alpha)_{|g|}B(g\alpha)^{-1} = \gamma(g\beta)_{|g|}B(g\beta)^{-1}.$$

This implies the equality:

$$\gamma(g\alpha)_{|g|-N} = \gamma(g\beta)_{|g|-N}$$

where $N = \sup\{|B(g)| : g \in F_2\}$. This completes the proof.

4.D.2 Solitary operatros: Let T_m be the bi-infinite regular m tree with $(\dots, 0, 0, \dots) \in T_m$. Two elements $\sigma_i = (\dots, a^i_{-1}, a^i_0, a^i_1, \dots) \in T_m$, $i = 1, 2$ and $a^i_j \in \{0, 1, \dots, m-1\}$, are saied to have *disjoint support*, if $a^0_i a^1_i = 0$ hold for all $i \in \mathbf{Z}$. Namely when σ_1 does not have zero at the position i, then σ_2 does have zero at the position i.

For two disjoint elements σ_1 and σ_2, there is a canonical sum:

$$\sigma_1 + \sigma_2 \in T_m.$$

Let $T : T_m \cong T_m$ be the shift, and take a function $f :$ Vert $T_m \to \mathbf{R} \in L^2(T_m)$ which is partially linear in the sense that $f(\sigma_1 + \sigma_2) = f(\sigma_1) + f(\sigma_2)$ whenever σ_1 and σ_2 has disjoint support mutually. We denote the set of such functions by $L^2(T_m)_{pl} \subset L^2(T_m)$.

We say that a bounded operator P on $L^2(T_m)_{pl}$ is *solitary*, if it is equivariant with respect to T action, and denote the C^* algebra of all solitary operators by:

$$Sol(T_m) \subset B(L^2(T_m)_{pl}).$$

We call it the *soliton* C^* algebra.

5 Associated algebraic varieties

5.A Tropical algebra: Maslov inroduced the *dequantization* of the real line $\mathbf{R}$, so that for $t > 1$, there is a family of semirings R_t which are all the real number $\mathbf{R}$ as sets.

The multiplications and the additions are respectively given by:

$$x \oplus_t y = \log_t(t^x + t^y), \quad x \otimes_t y = x + y.$$

When one let $t \to \infty$, then one obtains the equation:

$$x \oplus_\infty y = \max\{x, y\}.$$

Corresponding to polynomials in the usual real numbers, one has R_t-*polynomials* whose limit $t \to \infty$ satisfies a max plus equation:

$$\begin{aligned} &\varphi_t(x) = \oplus_t(\alpha_j + jx), \quad x \in \mathbf{R}^n, \quad j \in \mathbf{Z}^n, \\ &\varphi_\infty(x) = \max(\alpha_1 + j^1 x, \dots, \alpha_k + j^k x). \end{aligned}$$

Let $\mathrm{Log}_t : (\mathbf{C}^*)^n \to \mathbf{R}^n$ be defined as:

$$(x_1, \dots, x_n) = (\log|z_1|, \dots, \log|z_n|).$$

Proposition 5.1 (LM,V) *$f_t \equiv (\log_t)^{-1} \circ \varphi_t \circ Log_t : \mathbf{R}^n_+ \to \mathbf{R}_+$ is in fact a polynomial map $f_t(z) = \Sigma_j t^{\alpha_j} z^j$.*

Thus as far as t takes a finite number, then R_t-polynomials and usual $\mathbf{R}$-polynomials can recover each other.

Conversely let $\varphi : \mathbf{R}^n_+ \to \mathbf{R}_+$ be a piecewise linear map described by a max-plus equation. Then one can associate a parametrized R_t and $\mathbf{R}$ polynomials respectively as above, and denote them by:

$$\begin{aligned} &\varphi_t \equiv DQ_t(\varphi), \\ &f_t \equiv P_t(\varphi). \end{aligned}$$

We say that φ_t and f_t are the *associated R_t and $\mathbf{R}$ polynomials* respectively with respect to φ.

5.B Toropical maps in LV cell automaton: Let us denote by $\mathbf{R}_*$ the semiring over $\mathbf{R}$ such that the addtion and the multiplication are given by:

$$\begin{aligned} &x \oplus_* y \equiv \max\{x, y\}, \\ &x \times_* y \equiv x + y \end{aligned}$$

where $+$ is the usual addtion on $\mathbf{R}$. This is called the *tropical semiring.*

Let $F_t : \mathbf{C}^N \to \mathbf{C}$ be a family of polynomials given by:

$$F(z_1, \dots, z_N) = \Sigma_j t_j^a z^j$$

where $j = (j_1, \dots, j_N) \in \mathbf{Z}^N$ and $z = (z_1, \dots, z_N) \in \mathbf{C}^N$. Then we define the corresponding *tropical polynomial* $f = f_{tr}$ as a piesewise linear map on $\mathbf{R}^N$ by:

$$f_{tr}(x_1, \dots, x_N) = \max_j \{a_j + jx\}$$

where $x = (x_1, \dots, x_N) \in \mathbf{R}^N$.

Let us put a piecewise linear map f_{LV} by:

$$\begin{aligned} f_{LV}(x_1, x_2, x_3) &= x_1 + \max\{L_0, x_2\} - \max\{L_0, x_3\} \\ &= x_1 + \max\{0, x_2 - L_0\} - \max\{0, x_3 - L_0\}. \end{aligned}$$

This is called the *ultra discrete Lotka Volterra map.*

Let us choose any large integer $L >> 0$ and put $f_{i,j} : [0,1] \to [0,1]$ by:

$$f_{i,j}(x) = \frac{1}{L} f_{LV}(i, j, Lx) \in [0,1].$$

In [K3], we have formulated interactions of families of maps, and obtained the interaction map:

$$\Phi(x, f_{i,j}) : X_L \to X_L.$$

Proposition 5.2 (K3) *The flow $\Phi(x.f_{i,j})^t$, $t = 0, 1, \dots$ gives the solutions of the Lotka Volterra cell automaton.*

In particular it gives solitary solutions.

5.C Associated hypersurfaces: In general a cell automaton contains minus signs in their equations. After moving some terms in the equation, if an equation can be expressible by max-plus equations in both sides, then we say that the cell automaton is *admissible.*

Example 5.1: The LV cell automaton is admissible. In fact one can immediately rewrite the LV cell automaton as:

$$V_n^{t+1} + \max\{L, V_{n-1}^{t+1}\} = V_n^t + \max\{L, V_{n+1}^t\}.$$

Let φ_1 and φ_2 be two max-plus equations. Suppose a cell automaton A is given by the equation:

$$\varphi_1(V_{n-k}^{t-l}, \dots, V_{n+k'}^{t+l'}) = \varphi_2(V_{n-k}^{t-l}, \dots, V_{n+k'}^{t+l'}), \quad n = 0, 1, \dots$$

for some numbers k, k' and l, l'. Let f_t^1 and f_t^2 be the associated $\mathbf{R}$ polynomials respectively, and assign $(k''+1)(l''+1)$ complex variables $\mathbf{z} = (z_1, z_2, \dots)$ to $(V_{n-k}^{t-l}, \dots, V_{n+k'}^{t+l'})$, $k'' = \max\{k, k'\}$, $l'' = \max\{l, l'\}$.

Definition 5.1 *The accociated affine hypersurfaces is a parametrized family of hypersurfaces given by the equations:*

$$V(A)_t = \{\mathbf{z} \in \mathbf{C}^{(k''+1)(l''+1)} : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z})\}.$$

Example 5.2: For the LV cell automaton, the above equation is the same as:

$$\max\{L + V_n^{t+1}, V_n^{t+1} + V_{n-1}^{t+1}\} = \max\{L + V_n^t, V_n^t + V_{n+1}^t\}.$$

Then correspondingly, one has a parametrized polynomials of degree 2:

$$f_t(z, w) = t^L z + zw.$$

Then with respect to the equation above, one considers the equation:

$$f_t(z_1, z_4) = f_t(z_2, z_3)$$

where each z_i corresponds as:

$$z_1 \leftrightarrow V_n^{t+1}, \quad z_2 \leftrightarrow V_n^t, \quad z_3 \leftrightarrow V_{n+1}^t, \quad z_4 \leftrightarrow V_{n-1}^{t+1}.$$

Thus the associated hypersurfaces $V(\mathrm{LV})_t \subset \mathbf{C}^4$ is a family defined by:

$$V(\mathrm{LV})_t = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : f_t(z_1, z_4) = f_t(z_2, z_3)\}.$$

5.C.2 Deterministic automata: A cell automaton A given by a polynomial F of the form

$$V_n^t = F(\{V_k^s\} : s = t, t-1, k < n \text{ when } s = t)$$

is called as *deterministic*.

Let A be an admissible and deterministic cell automaton such that the number of variables are finite N:

$$V_n^t = F(\{V_k^s\} : (s,k) = (n-m_1,t), \dots, (n-1,t),$$
$$(n-m_2,t-1), \dots, (n+m_3,t-1); m_1+m_2+m_3 = N-1).$$

Then one obtains the associated hypersurfaces $V(A)_t \subset \mathbf{C}^N$, where we associate z_1 to V_n^t above. Since A is a deterministic automaton, there is a parametrized rational map $Q_t(w_2, \dots, w_N)$ on $\mathbf{C}^{N-1}$ so that any point on $V(A)_t$ satisfies:

$$w_1 = Q_t(w_2, \dots, w_N).$$

Example 5.3: Let A be the LV cell automaton. Then one has:

$$Q_t(z_2, z_3, z_4) = (t^L + z_4)^{-1}(t^L z_2 + z_2 z_3).$$

Now let us give an infinite sequence of complex numbers as:

$$(z_1^0, z_2^0, z_3^0, \dots) \in \mathbf{C}^\infty.$$

Then with respect to this, one determines another $(z_1^1, z_2^1, \dots)$, where each z_i^1 is followed by the equation:

$$z_i^1 = Q_t(z_{i-m_1}^1, \dots, z_{i-1}^1, z_{i-m_2}^0, \dots, z_{i+m_3}^0).$$

Thus for large i,

$$\mathbf{z}_t^1(i) \equiv (z_{i-m_1}^1, \dots, z_i^1, z_{i-m_2}^0, \dots, z_{i+m_3}^0) \in V(A)_t$$

gives a dynamics of points on $V(A)_t$.

We say that the sequences $\mathbf{z}_t = \{\mathbf{z}_t^i\}_i$ is the *induced dynamics* on $V(A)_t$.

By iterating the same procedure, one obtains another sequence $(z_1^2, z_2^2, \dots)$ by complex numbers from $(z_1^1, z_2^1, \dots)$. Thus one gets another induced dynamics $\mathbf{z}_t^2 \subset V(A)_t$.

Succesively, one obtains a family of induced sequences:

$$\{\mathbf{z}_t^l\}_{l=1,2,\dots} \subset V(A)_t.$$

5.C.3 Stability: Let A be a cell automaton given by two max-plus functions φ^1 and φ^2. Let us consider the associated polynomials φ_t^i and f_t^i, $i = 1, 2$.

We recall the equality $f_t = \log_t^{-1} \circ \varphi_t \circ \mathrm{Log}_t$ on the positive real numbers. Thus any real and positive point $x \in \mathbf{R}_+ V(A)_t \subset \mathbf{R}^N$ satisfies the equation:

$$\varphi_t^1 \circ \mathrm{Log}_t(x) = \varphi_t^2 \circ \mathrm{Log}_t(x).$$

Let $\{x_t\}_{t \geq 0}$ be a family of positive real points on $\mathbf{R}_+ V(A)_t$ such that the family $\{\ \mathrm{Log}_t(x_t)\}_t \subset \mathbf{R}^N$ converges to $y \in \mathbf{R}^N$ as $t \to \infty$. Then since the limit $\lim_{t \to \infty} \varphi_t^i = \varphi^i$ converges to the original max-plus functions, the equality holds:

$$\lim_{t \to \infty} \varphi^1(\mathrm{Log}_t(x_t)) = \lim_{t \to \infty} \varphi^2(\mathrm{Log}_t(x_t)).$$

Now let us give a family of sequences by real and positive numbers $\mathbf{x}_t = ((x_1)_t, (x_2)_t, \dots) \in \mathbf{R}_+^\infty$, and let $\{\mathbf{x}_t^l\}_l \subset \mathbf{R}V(A)_t$ be the real induced dynamics.

Proposition 5.3 *Let A be an admissible and deterministic automaton. Suppose the induced family $\{\mathbf{x}_t^l\} \subset \mathbf{R}_+^N$ are all positive and satisfies that:*

$$\mathbf{v}^l \equiv \lim_{t \to \infty} Log_t(\mathbf{x}_t^l)$$

converges.

Then the result $\{\mathbf{v}^l\}_{l=1,2,\dots}$ gives a flow of the original automaton A.

Let $[v] \in \mathbf{Z}$ be the integer part of $v \in \mathbf{R}$. For a sequence $\mathbf{v} = (v_1, v_2, \dots)$, we denote by $[\mathbf{v}] \equiv ([v_1], [v_2], \dots)$.

Definition 5.2 *Let A be an admissible and deterministic automaton, and choose a parametrized families of sequences $\{\mathbf{z}_t\}_{t \geq 1} \subset \mathbf{C}^\infty$. We say that the family $\{\mathbf{z}_t\}_{t \geq 1}$ is a stable sequence, if for the induced family $\{\mathbf{z}_t^l\} \subset \mathbf{C}^N$, there is a large t_0 so that for all $t \geq t_0$, the family:*

$$\{[Log_t(\mathbf{z}_t^l)]\}_{l=1,2,\dots} \subset \mathbf{Z}^{\mathbf{N}}$$

gives a flow of the solutions of the original automaton A.

Let A be the LV cell automaton. Then as in $5.B$, there is a family of interval maps $\{f_{i,j}\}_{i,j}$ so that the flow of the solutions of LV CA can be represented by interactions of maps between $\{f_{i,j}\}$. Thus existence of stable families above will be heavily influenced by stability of dynamical properties of the family $\{f_{i,j}\}$.

5.D Transformations: Recall that we have obtained another cell automata O_1 and O_2 during transforming CA from BBS to LV ($3.A$). Similarly as above we rewrite these as:

$$S_{n+1}^{t+1} + \max\{S_n^{t+1} + 1, S_{n+1}^t\} = S_n^t + S_n^{t+1} + 1,$$
$$U_{n+1}^{t+1} + \max\{1, U_{n+1}^t\} = U_n^t + \max\{1, U_n^{t+1}\}.$$

Let us assign:

$$z_1 \leftrightarrow X_{n+1}^{t+1}, \quad z_2 \leftrightarrow X_{n+1}^t, \quad z_3 \leftrightarrow X_n^{t+1}, \quad z_4 \leftrightarrow X_n^t$$

for both $X = S$ and $X = U$. Then we have the corresponding polynomial pairs as:

$$f_t^1(z_1, z_2, z_3) = z_1(tz_3 + z_2), \quad f_t^2(z_3, z_4) = tz_3 z_4, \quad \text{for } S,$$
$$f_t^1(z_1, z_2) = z_1(t + z_2), \quad f_t^2(z_3, z_4) = z_4(t + z_3), \quad \text{for } U.$$

These U and S are both admissible and deterministic.

Question: What are the relations between associated hypersurfaces:

$$V(\text{LV}), V(U), V(S) \subset \mathbf{C}^4.$$

5.D.2 Representatives by homogeneous maps: Let O be a cell automaton. We say that O admits a *homogeneous representative*, if there is an invertible transformations $O = O_0 \to O_1 \to \cdots \to O_k$ so that the associated $\mathbf{R}$-polynomial to O_k, f_t^1 and f_t^2 are both homogeneous of the same degree.

Example: Let $O_0 = BBS \to O_1 \to O_2 \to O_3 = LV$ cell automaton be the invertible path. From the above, it follows:

Proposition 5.4 *LV cell automaton admits a homogeneous representative at O_1.*

We say that O_1 is a *homogeneous* cell automaton.

Let O be an admissible, deterministic and homogeneous cell automaton. Then one obtains the *associated projective hypersurface*:

$$V(O) \subset \mathbf{CP}^{N-1}.$$

We say that $V(O)$ is the *associated projective hypersurface.*

5.D.3 Liftings: Let us take two admissible automata O and O', and suppose that there is an invertible transformation from O to O'. Let us denote the associated $\mathbf{R}$-polynomials by f_t^i and g_t^i respectively, $i = 1, 2$. Their differences $f_t^1 - f_t^2$ and $g_t^1 - g_t^2$ are mutually obtained by rational change of variables.

Let $\bar{O} = \{O = O_0 \to O_1 \to \dots O_m = O'\} \in \Omega(O, O')$ be an invertible path $(3.C)$ such that all O_i are admissible. Then one has the associated $\mathbf{R}$-polynomials $(f_t^i)_i$, $i = 1, \dots, m$ and the associated hypersurfaces $\{V(O_i)\}_{i=1,\dots,m} \subset \mathbf{C}^N$. We call this a *lifting* of $\bar{O} \in \Omega_O$.

Let **CAH** be all the set of hypersurfaces in $\mathbf{C}^N$ which come from admissible cell automata. Then we say that $V(O)$ and $V(O')$ are connected in **CAH**.

Question: What are structures of connected components of **CAH** ?

5.E Cell automatic varieties: Let A_1 and A_2 be two cell automata. If sequences $\{V_n^t\}_{n,t}$ satisfy both equations for A_1 and A_2, then we say that $\{V_n^t\}_{n,t}$ is a *flow* for A_1 and A_2.

Let us denote the set of such flows by:

$$\mathbf{F}(A_1, A_2) = \{\{\mathbf{V}^l\}_{l=1,2,\dots} \subset \mathbf{Z}^\infty : \text{ flow for both } A_1, A_2\}.$$

We say that $\mathbf{F}(A_1, A_2)$ is a *cell automatic variety* for A_1 and A_2.

Let $A_1, \dots, A_m$ be a family of cell automata. By generalizing the above and considering flows for them, one also obtains a cell automatic variety $\mathbf{F}(A_1, \dots, A_m) \subset \mathbf{Z}^\infty$.

5.E.1.2 Compatible automata: Let A_1 and A_2 be two deterministic cell automata given by polynomials F_1 and F_2 respectively of the forms for $i = 1, 2$:

$$V_n^t = F_i(\{V_{n-k}^{t-1}, \ldots, V_{n+l}^{t-1}\}, \{V_{n-k'}^t, \ldots, V_{n-1}^t\}) \quad (*)_i$$

Let us denote the sets:

$$v_n \equiv (V_{n-k}^{t-1}, \ldots, V_{n+l}^{t-1}, V_{n-k'}^t, \ldots, V_n^t) \in \mathbf{Z}^N$$

where $N = k + l + 1 + k'$.

We say that A_2 is *compatible* with A_1, if the following holds; suppose v_n satisfies both $(*)_1$ and $(*)_2$. Let v_{n+1} be the set determined by $(*)_2$. Then it also satisfies $(*)_1$.

In the case of compatible automata (A_1, A_2), the cell automatic variety $\mathbf{F}(A_1, A_2)$ will be non empty which is not the case in general.

Example 5.4: Let A_1 and A_2 are both automata given by linear maps as:

$$u_{n+1} = \alpha u_n + \beta u_{n-1} + \gamma, \quad (A_1),$$
$$u_{n-1} = a u_n + b, \quad (A_2).$$

If these coefficients satisfy the following relations:

$$(1 - \beta) a = \alpha, \quad (1 - \beta) b = \gamma$$

then A_2 is compatible with A_1.

5.E.2 Associated varieties: Let A_1 and A_2 be two admissible CA. Let us denote by f_t^i, g_t^i, $i = 1, 2$ the associated $\mathbf{R}$-polynomials. By the same way as before, one obtains the *parametrized associated affine algebraic varieties* given by the equations:

$$V(A_1, A_2)_t = \{\mathbf{z} \in \mathbf{C}^N : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z}), g_t^1(\mathbf{z}) = g_t^2(\mathbf{z})\}.$$

Similarly as before, one can consider the *induced dynamics*: $\{\mathbf{z}_t^l\} \subset V(A_1, A_2)_t$.

Let $A_1, \ldots, A_m$ be a family of admissible CA. Then by the same way as above, one obtains the parametrized associated affine algebraic variety:

$$V(A_1, \ldots, A_m)_t = \{\mathbf{z} \in \mathbf{C}^N : (f_t^k)^1(\mathbf{z}) = (f_t^k)^2(\mathbf{z}), k = 1, \ldots, m\}$$

where $(f_t^k)^i$, $i = 1, 2$, are the associated **R**-polynomials with respect to A_k.

5.F Duality: Here we will have a procedure to obtain "dual automaton" from a family of automata. This procedure passes through the *projective duality* between algebraic varieties, and in the formulation, one uses the associated varieties of automata essentially.

Let V be a complex n dimensional vector space, and $P(V)$ be its projective space. Recall that there is a natural isomorphism between $P(V)$ and $P^*(V^*)$, where $P^*(W)$ is the set of all hyperplanes in W, and V^* is the dual space to V.

Let $X \subset P(V)$ be an algebraic variety. Then one can associate another variety $X^\vee \subset P(V^*)$ as follows ([GKZ]). A hyperplane $H \subset P(V)$ is said to be tangent to X, if there exists a smooth point $x \in H \cap X$ and the tangent space of X at x is contained in H. Let $X^* \subset P^*(V)$ be all the set of tangent hyperplanes, and passing through the above isomorphism, one obtains a set $X^\vee \subset P(V^*)$ which is the desired one. It is called the *projective dual variety.*

In the case when $X^\vee \subset P(V^*)$ is a hypersurface, then its defining polynomial Δ_X is called the X-*discriminant.*

Now let $\{A_1, \ldots, A_m\}$ be a family of admissible and homogeneous automata. Thus one obtains a parametrized family of projective varieties $V(\{A_i\})_t \subset \mathbf{CP}^N$ by taking closure of the associated affine varieties.

Let $\tilde{V}(\{A_i\})_t^\vee \subset \mathbf{CP}^N$ be the corresponding parametrized projective dual varieties.

Suppose these are hypersurfaces, and denote the defining functions by $\Sigma_j t^{\alpha_j} a^j z^j$. When one can modify the polynomial as $\Sigma_j t^{\alpha_j} w^j$ by change of variables $a_l z_l = w_l$, then we call $\Sigma_j t^{\alpha_j} w^j$ the $\{A_i\}$-*discriminant*, and denote it by:

$$\Delta(\{A_i\})_t.$$

We will denote the corresponding modified varieties by:

$$V(\{A_i\})_t^{\vee}$$

and call them as the *associated dual varieties* with respect to $\{A_i\}_i$. They are isomorphic with $\tilde{V}(\{A_i\})_t^{\vee}$.

Definition 5.3 *If $\Delta(\{A_i\})_t)$ are corresponding to R_t-polynomials φ_t, then we call the dual automaton for the max-plus function φ_∞ given by its limit, and denote it by $\{A_i\}^{\vee}$*

In general it will not be easy to know whether the cell automatic variety $\mathbf{F}(\{A_i\}_i)$ is non empty or not, on the other hand in the case of a single automaton $\{A_i\}^{\vee}$, it will not be difficult. So duality on cell automata may tell some information about this point, which will depend on stability property (5.C.3).

5.F.2 Curves in $\mathbf{CP}^2$: Let $X \subset \mathbf{CP}^2$ be an irreducible curve. Then it is known that $X^{\vee} \subset \mathbf{CP}^2$ is also another irreducible one.

In the affine coordinate, if X has a parametrization $x = x(s)$ and $y = y(s)$, $s \in \mathbf{C}$, then $X^{\vee}$ has a parametrization given by the following ([GKZ]):

$$p(s) = \frac{-y'(s)}{x'(s)y(s) - x(s)y'(s)}, \quad q(s) = \frac{x'(s)}{x'(s)y(s) - x(s)y'(s)}.$$

Using this, let us consider a very simple case. Let $a \geq 2$, α and c be integers, and consider an automaton A given by:

$$\max\{au_n, \alpha + au_{n+1}\} = c.$$

The associated polynomial and the associated varieties are given by:

$$X = \{(x, y) \in \mathbf{C}^2 \subset \mathbf{CP}^2 : x^a + t^{\alpha} y^a = t^c\}.$$

Choosing a parametrization as:

$$x = s, \quad y = t^{-\frac{\alpha}{a}}(t^c - s^a)^{\frac{1}{a}}$$

one can immediately obtain the parametrization of $X^{\vee}$ as:

$$t^{\frac{a}{a-1}(c-\frac{\alpha}{a})} q^{\frac{a}{a-1}} + t^{\frac{ac}{a-1}} p^{\frac{a}{a-1}} = t^c \quad (*)$$

which give the dual varieties:

$$V(A)_t^\vee = \{(p,q) \in \mathbf{C}^2 \subset \mathbf{CP}^2 : (*)\}.$$

Thus the dual automaton becomes:

Proposition 5.5

$$[\max\{au_n, \alpha + au_{n+1}\} = c]^\vee =$$
$$\max\{\frac{a}{a-1}(c - \frac{\alpha}{a}) + \frac{a}{a-1}u_{n+1}, \frac{ac}{a-1} + \frac{a}{a-1}u_n\} = c.$$

5.F.3 Approximation: Let $\{f_i\}_i$ be a family of maps. In general these does not come from **R**-polynomials, since they are not necessarily PL maps.

Let us approximate these by PL maps $\{g_i(n)\}$:

$$g_i(n) \to f_i \quad n \to \infty$$

in C^0 topology.

An *admissible approximation* is the one $\{g_i(n)\}$ such that the corresponding max-plus functions are admissible. Let $\{V(n)_t\}_n$ be a family of the associated varieties.

Definition 5.4 *The family $\{V(n)_t\}_n$ is convergent, if for each t, there is a limit space $V(\infty)_t \subset \mathbf{CP}^n$ in C^0 topology.*

6 Interaction graphs

Let $f, g : [0,1] \to [0,1]$ be two interval maps and $\Phi(x, f, g) : X_2 \to X_2$ be the interaction map. Recall $\pi : [0,1]\backslash\frac{1}{2} \to \{0,1\}$ be the projection. Let us choose another map $d : [0,1] \to [0,1]$.

Suppose for a point $z \in [0,1]$ and some $\bar{k} \in X_2$, the following equality holds:

$$\Phi(x, f, g)(\bar{k}) = \pi((d(z), d^2(z), \dots)) \equiv (\pi(d(z)), \pi(d^2(z)), \dots).$$

Then we express this by a marked oriented edge as:

$$(f, x) \xrightarrow{(g,\bar{k})} (d, z).$$

Let us choose families of maps $\{f_0, \dots, f_k\}$ and points $\{x_0, \dots, x_l\}$. For each $(i, j, x) \in \{0, \dots, k\}^2 \times \{x_0, \dots, x_l\}$, let us assign an element $\bar{k}(i, j, x) \in X_2$. Thus we obtain another family $\{\bar{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l} \subset X_2$. Then we put two sets:

$$V = \{(f_i, x_j) : 0 \leq i \leq k, 0 \leq j \leq l\} \text{ (the set of vertices)},$$
$$E = \{e_{i,j,k} : (f_i, x_h) \stackrel{(f_j, \bar{k}(i,j,x_h))}{\longrightarrow} (f_k, x_v) :\} \text{ (the set of edges)}.$$

Definition 6.1 *An interaction graph is a marked oriented graph, where the set of vertices V and edges E are given as above. We denote it by:*

$$G(\{f_i\}_i^k; \{x_j\}_j^l; \{\bar{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l})$$

We will denote the set of interaction graphs arising from $\{f_0, \dots, f_k\}$ and $\{x_0, \dots, x_l\}$ by:

$$\mathfrak{G}(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l).$$

Notice that this is a finite set.

Let us put:

$$X_2^{k,l} \equiv X_2^{k^2+l} = X_2 \times X_2 \times \cdots \times X_2.$$

Then any element in $X_2^{k,l}$ can be writeen as $\bar{k}(i, j, x)$ as above. Then the family of the interaction map gives a map:

$$\Phi : X_2^{k,l} \to X_2^{k,l}$$

where:

$$\Phi(\{\bar{k}(i, j, x)\}) = \{\bar{k}'(i, j, x)\},$$
$$\bar{k}'(i, j, x) \equiv \Phi(f_i, f_j, x)(\bar{k}(i, j, x)).$$

This induces a map on the set of the interaction graph as:

$$\Phi_* : \mathfrak{G}(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l) \to \mathfrak{G}(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l)$$

by

$$\Phi_*(G(\{f_i\}_i^k;\{x_j\}_j^l; \{\bar{k}(i, j, x_h)\})) \\ = G(\{f_i\}_i^k; \{x_j\}_j^l; \Phi(\{\bar{k}(i, j, x_h)\})).$$

Thus one obtains a sequence of the interaction graphs as:

$$(G_0, G_1, \dots),$$
$$G_i = G(\{f_i\}_i^k; \{x_j\}_j^l; \Phi^i(\{\bar{k}(i,j,x_h)\}_{i,j,h=0}^{i,j=k,h=l})).$$

This gives a dynamics of the interaction graphs. Below we will formulate several geometric spaces arising from dynamics of the interaction graphs which we call *spaces from the interaction graphs.*

6.A.1.2 Veronese map: Here we have an easy example of spaces from the interaction graphs. Let $(\{f_i\}, \{x_j\}, \{\bar{k}(i,j,x_h)\})$ be an interaction system, and denote the corresponding interaction graphs by $(G_0, G_1, \dots)$. Passing through the forgetful map, one obtains a sequence of finite graphs $(G'_0, G'_1, \dots)$.

For each vertex $v \in G'_i$, let $e(v)$ be the number of the edges with a common vertex v. Let us fix $m \geq 0$ and put:

$$P(m, G) = \{(i_0, \dots, i_k) : \Sigma_{a=0}^k e(i_a) = m, \quad k \leq N\}.$$

Let $N+1$ be the number of the edges in the interaction graphs. The *Veronese map* with respect to $\{G'_i\}_i$ is a family of embeddings:

$$I_i : \mathbf{CP}^N \hookrightarrow \mathbf{CP}^M,$$
$$[x_0, \dots, x_N] \to [\{x_0^{i_0} \dots, x_k^{i_k} : (i_0, \dots, i_k) \in P(m, G_i)\}]$$

where $N_i = \sharp P(m_i, G_i) + 1$, and $M = \frac{(N+1)N}{2}$ is the maximum number of the edges in the graphs.

6.A.2 Reduction to dynamics of toric ideals: Let $\mathfrak{G}$ be the set of finite graphs, and:

$$F : \mathfrak{G}(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l) \to \mathfrak{G}$$

be the forgetful map. Then one obtains a family of finite graphs:

$$G_1, G_2, \dots \subset \mathfrak{G}$$

consisted by the images of F of the interaction graphs. We call them just the *associated graphs.*

Let $\{G_0, G_1, \dots\}$ be a family of finite graphs. We say that the family is *strongly regular*, if the number of edges of G_i are all same. In $6.A$, we will always assume that sequences consisted by the images of F of the interaction graphs are strongly regular.

For each $G \in \mathfrak{G}$, let us associate a *configuration*:

$$\mathfrak{A} = \{\bar{a}_1, \dots, \bar{a}_m\} \subset \mathbf{Z}^N$$

as follows, where $N = (k+1)(l+1)$ is the number of the vertices. Let us make a numbering of the set of vertices, $v_1, \dots, v_N$ and let $e_i \in \mathbf{Z}^N$ be the unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 appears only at i-th. Then $e_i + e_j \in \mathfrak{A}$ if and only if v_i and v_j are mutually connected by an edge.

Thus one obtains a reduction from dynamics of the interaction graghs to the one of the configurations:

$$\{\mathfrak{A}_0, \mathfrak{A}_1, \dots\}$$

which we call the *transcripted configurations.*

For a configuration $\mathfrak{A} = \{\bar{a}_1, \dots, \bar{a}_m\} \subset \mathbf{Z}^N$, we associate a Laurent polynomial:

$$\mathbf{C}[\mathfrak{A}] \equiv \mathbf{C}[t^{\bar{a}_1}, \dots, t^{\bar{a}_m}] \subset \mathbf{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$$

where $t^{\bar{a}_k} = t_1^{a_k^1} \dots t_N^{a_k^N}$, $\bar{a}_k = (a_k^1, \dots, a_k^N)$. We call $\mathbf{C}[\mathfrak{A}]$ as the *associated toric ring.*

By assigning $y_i \to t^{\bar{a}_i}$, one obtains a ring homomorphism:

$$\pi : \mathbf{C}[y_1, \dots, y_m] \to \mathbf{C}[\mathfrak{A}]$$

and its kernel $I_{\mathfrak{A}}$ is called the *toric ideal.*

For each interaction graph $G_i \in \mathfrak{G}(\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l)$, one forgets markings on edges and orientation, and then obtains a finite graph $F(G_i)$. Then one can assign a configuration $\mathfrak{A}_i \subset \mathbf{Z}^N$. Now correspondingly one has the associated toric ideal $I_i \subset \mathbf{C}[y_1, \dots, y_m]$. Thus one has obtained a sequence of toric ideals:

$$I_0, I_1, \dots, I_k, I_{k+1}, \dots \subset \mathbf{C}[y_1, \dots, y_m]$$

which we will call the *associated ideals.* Notice that there are finite number of ideals $\{J_1, \dots, J_d\}$ such that each I_i coincides with one of $\{J_j\}$.

We say that the associated ideals are *regular*, if all I_i has the same dimension.

Let us fix a total ordering $<$ on the set of monomials of the polynomial ring $\mathbf{C}[y_1, \dots, y_m]$, for example lexicographic or its reverse ones.

Let $I \subset \mathbf{C}[y_1, \dots, y_m]$ be an ideal, and let $\bar{z} = \{z_1, \dots, z_k\}$ be a generating set. We denote by $\text{int} z_i$ to imply the leading term of the polynomial with respect to the ordering. Let $\text{int} I \subset \mathbf{C}[y_1, \dots, y_N]$ be another ideal generated by $\text{int} z$ for all $z \in I$. It is called the *initial ideal.* We say that $\bar{z}$ is a *Gröbner basis*, if

$$\text{int} I = \text{ gen } \{\text{int} z_1, \dots, \text{int} z_k\}.$$

Notice that the sequence of the toric ideals $\{I_i\}_i$ is obtained originally from the data $(\{f_i\}_i, \{x_j\}_j, \{\bar{k}(i, j, x_h)\})$. The following problem seems natural, since in some cases, these interactions come from some combinatoric structures, like cell automata ([K3]).

Question: Can one find an algorithm to find out Gröbner basis succesively for $I_0, I_1, \dots$ from the interaction data ?

6.A.3 Gröbner fans and translated varieties: Let $w = (w_1, \dots, w_m) \in \mathbf{R}^m$ be a vector which is called the *weight vector.* Then the weight of a monomial $y_1^{a_1} \dots y_m^{a_m}$ is equal to $< w, \bar{a} >= w_1 a_1 + w_2 a_2 + \dots + w_m a_m$. For each polynomial $f = \Sigma c_{\bar{a}} y^{\bar{a}} \in \mathbf{C}[y_1, \dots, y_m]$, let $\text{in}_w(f) = \Sigma_{\bar{b}} c_{\bar{b}} y^{\bar{b}}$ where any $\bar{b}$ satisfy $< w, \bar{b} >= \max_{c_{\bar{a}} \neq 0} < w, \bar{a} >$.

Let I be an ideal. Then we denote the corresponding initial ideal $\text{in}_w(I)$ generated by all elements of the form $\text{in}_w(f)$. It is known that for every ideal I and ordering $<$, there associated with a weight vector w satisfying $\text{in}_<(I) = \text{in}_w(I)$.

For an ideal I, we say that w and w' are equivalent, if they give the same initial ideal $\text{int}_w I = \text{int}_{w'} I$. The equivalence class:

$$\mathfrak{C}(I, w) = \{w' \in \mathbf{R}^m : \text{ int}_w I = \text{ int}_{w'} I\}$$

is an open polyhedral cone in $\mathbf{R}^m$.

The set of the cones $\{\mathfrak{C}(I, w)\}_w$ is finite and defines a polyhedral fan $\mathfrak{F}(I)$. We say that it is the *Gröbner fan* of I (see [Stu]).

For each fan, there associated with a toric variety. Thus for each interaction graph G, one associates with a toric variety X_G which we call the *translated variety.*

Thus corresponding to a sequence of the associated ideals $I_0, I_1, \ldots,$ one obtains a sequence of fans over $\mathbf{R}^m$ and the associated translated varieties:

$$\bar{X} = (X_0, X_1, \ldots, X_i, \ldots)$$

Remark 6.1: In order to study structure of $\bar{X}$, one may use resultants for the defining polynomials of these ideals.

Now we have started from a finite data:

$$\mathbf{D} = (\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l, \{\bar{a}(i,j,h)\}_{i,j,h=0}^{i,j=k,h=l}\}.$$

We will call such data an *interaction data.*

Then we have obtained a sequence of the interaction graphs:

$$G_0, G_1, \ldots,$$
$$G_i = G(\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l, \Phi_*^i(\{\bar{a}(i,j,h)\})).$$

By forgetting extra data, one obtains a sequence of the transcripted configurations:

$$\mathfrak{A}_0, \mathfrak{A}_1, \ldots$$

where $\mathfrak{A}_i \subset \mathbf{Z}^N$. Then one has obtained a sequence of the toric ideals $\{I_0, I_1, \ldots\}$ and the translated toric varieties:

$$\{X_0, X_1, \ldots\} \subset \mathbf{CP}^{m-1}$$

where each X_i coincides with one in a finite set of toric varieties $\{Y_1, \ldots, Y_l\} \subset \mathbf{CP}^{m-1}$.

6.A.4 Correspondence on polytopes: Let $(\{f_i\}^k, \{x_j\}^l, \{\bar{a}(i,j,h)\})$ be a triple, and consider the corresponding interaction graphs $\{G_i\}_{i=0}^{\infty}$.

Then there are finite configurations $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_l\} \subset \mathbf{Z}^N$ such that each G_i associates one of $\mathfrak{A}_j$ for some $j = j(i)$.

Let $\mathfrak{A} \subset \mathbf{Z}^N$ be a configuration, and m be the number of the elements in $\mathfrak{A}$.

Let Δ be a triangruation of $\mathfrak{A}$. Then every $\psi \in \mathbf{R}^m$ defines the corresponding piecewise linear function $g_{\psi,\Delta}$ on $\mathbf{R}^m$ satisfying $g_{\psi,\Delta}(a_i) = \psi(a_i)$ for each vertex a_i of Δ which is affine on each simplex of Δ.

Then we put:

$$\mathfrak{C}(\mathfrak{A}, \Delta) \equiv \{\psi \in \mathbf{R}^m : g_{\psi,\Delta} \text{ is concave and } g_{\psi,\Delta} \geq \psi(a_i) \text{ whenever } a_i \text{ is not a vertex of } \Delta\}.$$

$\mathfrak{C}(\mathfrak{A}, \Delta)$ is a closed polyhedral cone.

$$\mathfrak{F}(\mathfrak{A}) \equiv \{\mathfrak{C}(\mathfrak{A}, \Delta) : \Delta \text{ is a triangulation}\}$$

forms a complete fan on $\mathbf{R}^m$, and is called the *secondary fan* of $\mathfrak{A}$ ([GKZ] p219). It is known that the Gröbner fan is a refinement of the secondary fan ([Stu]).

Let $Q \subset \mathbf{R}^m$ be a polytope. Then for each $p \in Q$, let us define the *normal cone*:

$$N(Q, p) \equiv \{v \in \mathbf{R}^m :< v, p > \geq < v, y > \text{ for all } y \in Q\}.$$

The set of the normal cones:

$$N(Q) = \cup_{p \in \text{ Vert } Q} N(Q, p)$$

is called the *normal fan*.

Theorem 6.1 (GKZ) *For a configuration $\mathfrak{A}$, there is a polytope $Q \equiv \Sigma(\mathfrak{A})$ and an assignment of a vertex $\varphi_\Delta \in Q$ for each triangruation Δ such that the normal cone $N(\Sigma(\mathfrak{A}), \varphi_\Delta)$ coincides with $\mathfrak{C}(\mathfrak{A}, \Delta)$.*

In particular there is a natural correspondence from the secondary fan $\mathfrak{F}(\mathfrak{A})$ to a the corresponding normal fan $N(Q)$.

$\Sigma(\mathfrak{A})$ is called the *secondary polytope*.

Let us denote $\mathfrak{A} = \{a_1, \dots, a_m\}$. Then one can describe Q explicitly as follows. Let us put:

$$\phi_\Delta = (\phi_\delta^1, \dots, \phi_\Delta^m) \in \mathbf{R}^m,$$
$$\phi_\Delta^i = \Sigma\{ \text{ vol } (\tau) : \tau \in \Delta, a_i \in \tau\}.$$

Then $Q = \Sigma(\mathfrak{A})$ is given by:

$$Q \equiv \text{ Conv } \{\phi_\Delta : \Delta \text{ is a trianguration of } \mathfrak{A}\}$$

where vol (τ) is the volume of τ.

Now let $\{\mathfrak{A}_0, \dots, \mathfrak{A}_l, \dots\} \subset \mathbf{Z}^N$ be the transcripted configurations. Then correspondingly one obtains another sequences of polytopes:

$$\{\Sigma(\mathfrak{A}_0), \Sigma(\mathfrak{A}_1), \dots\}$$

which is called the *secondary transcripted polytopes.*

6.B Dynamics over local charts: Let us fix two sets $\{f_i\}_{i=0}^k$ and $\{x_j\}_{j=0}^l$. Then there associates with finite numbers of toric ideals $\{J_1, \dots, J_m\} \subset \mathbf{C}[y_1, \dots, y_N]$.

If we choose a finite subset $\{\bar{a}(i,j,h)\} \subset X_2$, then one obtains an infinite sequence of ideals:

$$I(\{\bar{a}(i,j,h)\}) = (I_0, I_1, \dots, I_i, \dots)$$

among a finite set $\{J_j\}_j$ above.

Let us put all the set of sequences:

$$\mathbf{I} = \{I(\{\bar{a}(i,j,h)\}) : \{\bar{a}(i,j,h)\} \subset X_2\}.$$

We call it the *sequence of local charts.*

One may consider this as though it might be a symbolic dynamics of some 'Markov partition' over some algebraic variety V, by regarding each I_k as a definig ideal of a local chart of V.

6.B.2 Algebraic Markov partition: Let V be an algebraic variety with the affine coordinates V_i defined by an ideal J_i. Let us take an automorphism A on V. We say that an affine coordinate $\{(V_i, J_i)\}_{i=1}^m$

is an (stable) *algebraic Markov partition* for A, if for each i, there is some j so that:

$$A(V_i) \subset V_j$$

holds.

Example 6.1: Let A be an automorphism on $\mathbf{CP}^N$ by $[z_0, z_1, \dots, z_N] \to [z_1, z_0, z_2, z_3, \dots, z_N]$. Then for $V_i = \{[z_0, z_1, \dots, z_N] : z_i \neq 0\}$, $\mathbf{CP}^N$ admits an affine covering by $V_0 \cup V_1$. Moreover $A(V_i) = V_{i+1}$ mod 2.

Let $A : V \cong V$ be an automorphism, and consider its iteration $A^t : V \cong V$. Let us have an algebraic Markov partition by the set $\{V_i\}_i$. Then one obtains a symbolic dynamics of the algebraic Markov partition:

$$\Sigma(A; \{V_l\}_l) = \{(a_0, a_1, \dots) : A(V_{a_i}) \subset V_{a_{i+1}})\} \subset X_m$$

where m is the number of the local charts. It can be expressed by a set of sequences of the defining ideals $\{(I_0, I_1, \dots)\}$.

One can consider its converse. Let us put the set of all sequences coming from the algebraic Markov partition:

$$\mathbf{A} = \{(I_0, I_1, \dots)\}.$$

When one is given an interaction data $(\{f_i\}_i, \{x_j\}_j, \{\bar{a}(i,j,h)\})$, then one obtains a sequence of ideals. Thus a fundamental question in symbolic dynamics of ideals will be to construct correspondence from interaction data to algebraic Markov partitions:

Question: Let $\mathbf{I}$ be a set of sequences of ideals among a finite set of ideals. Then can one construct an algebraic Markov partition $\{J_1, \dots, J_k\}$ for an algebraic variety V and an automorphism A, so that $\mathbf{I} \subset \mathbf{A}$ might hold ? Namely when a set of sequences of ideals are symbolic dynamics of algebraic Markov partitions ?

Conversely, given $\mathbf{A}$, can one find some $\{f_i\}_{i=0}^k$ and $\{x_j\}_{j=0}^l$ so that the corresponding $\mathbf{I}$ might satisfy $\mathbf{I} \subset \mathbf{A}$?

We say that an associated ideals $I = (I_0, I_1, \dots)$ are *regular*, if they have the same dimension as the others. Let us put $\cup_i I_i = \{J_1, \dots, J_k\}$.

Definition 6.2 *Let* $(\{f_i\}_i, \{x_j\}_j, \{\bar{a}(i,j,h)\})$ *be an interaction data, and suppose the associated ideals by* $I = (I_0, I_1, \dots)$ *are regular. The sequence is called a symbolic flow of an automorphism, if there is an algebraic Markov partition for* (V, A) *with an affine coordinate* $\{(V_i, J_i\}_{i=1}^k$ *and some* $x \in V$ *so that its orbit* $\{A^n(x)\}_{n=0,1,\dots}$ *corresponds to the sequence.*

We call such pair (V, A) corresponding to I a *prohedron* (One may imagine as though it represents some state of a protein).

Let us consider the simplest case. Let us consider the partition of $\mathbf{CP}^N = V_0 \cup V_1$ and the involution A in example 6.1.

Lemma 6.1 *Let* $(\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l, \{\bar{a}(i,j,h)\})$ *be an interaction data such that the corresponding sequence of the interaction graphs* $G_0, G_1, \dots$ *satisfy that (1) the numbers of edges are all constant* m*, (2)* $F(G_{2i})$ *and* $F(G_{2i+1})$ *are mutually the same finite graphs* G *and* G' *respectively for all* i*, and (3) there are no primitive loops of even length for any graph.*

Then the corresponding sequence of ideals gives an algebraic Markov partition for (V_0, V_1, A) *above.*

This follows from the following general facts:

Sublemma 6.1 *The toric ideal is generated by the set of primitive loops of the even length.*

It follows from this that the corresponding ideals are all zero, which defines the affine plane $\mathbf{C}^N$. This completes the proof.

This simple case suggests that in general, possibility of construction of alg. Markov partitions will be reflected by combinatorics of the transcripted configurations.

6.B.3 Automorphism groups: Let us choose a family of interval maps $\{f_i\}^k$, and V be an algebraic variety. We denote:

$$D(\{f_i\}^k, V) \equiv \{(\{x_j\}^l, \{\bar{a}(i,j,h)\}) : (\{f_i\}^k, \{x_j\}^l, \{\bar{k}(i,j,h)\}) \text{ give alg. Markov partitions for some } A \text{ on } V\}$$

in $(\cup_l([0,1]^{l+1} \times X_2^{k,l})$.

Let us put the set of the associated automorphisms E and the automorphism groups G generated by E:

$$\begin{aligned} E(\{f_i\}^k, V) &\equiv \{A = A(\{f_i\}^k, \{x_j\}^k, \{\bar{k}(i,j,h)\}) : \\ &\qquad (\{x_j\}^l, \{\bar{k}(i,j,h)\}) \in D(\{x_j\}^l, V)\}, \\ G(\{f_i\}^k, V) &= \text{ gen } E(\{f_i\}^k, V). \end{aligned}$$

We also put the closure of $G(\{f_i\}^k, V)$ by $\bar{G}(\{f_i\}^k, V) \subset \text{Aut } V$.

Thus for each algebraic variety V, one has obtained a map from a set of interval maps and a Lie subgroup of Aut V:

$$(\{f_i\}^k, V) \to \bar{G}(\{f_i\}^k, V) \subset \text{ Aut } V.$$

It is known that when V is compact and non singular toric variety, then Aut V is linear, and its root system can be written explicitly from the fan of V. Thus in this case $\bar{G} \subset \text{Aut } V$ is a closed subgroup of a linear algebraic group. Thus it will be natural to ask the following:

Question: Whether one might write down the above subgroups from the information of the associated family of Gröbner fans.

6.C Zariski subsets on the moduli of interaction graphs: Let us choose an interaction data, (1) a set of interval maps $\{f_1, \dots, f_k\}$, (2) an index $\{1, \dots, l\}$ and (3) a set of $\{0,1\}$ sequences $\{\bar{a}(i,j,h)\}_{i,j,h} \subset X_2$.

For each assignment from $\{1, \dots, l\}$ to $\{x_1, \dots, x_l\}$, one obtains the associated interaction graph G. Each edge $e = ((i,h),(k,v)) \in G$ is assigned with some $j \in \{1, \dots, k\}$ so that $(f_i, x_h) \stackrel{(f_j, \bar{a}(i,j,h))}{\longrightarrow} (f_k, x_v)$ hold. So an interaction graph G_k is an weighted finite graph such that each edge is assigned with an element in $\{1, \dots, k\}$.

Let us fix k and l as above, and denote the set of interaction graphs:

$$\mathbf{G}(k,l) = \{G(k,l) : \text{ interaction graphs }\}.$$

$\mathbf{G}(k,l)$ can be parametrized as:

$$\begin{aligned}\mathbf{G}_N &\cong [0,1]^l \times \text{ Map } [(\{1,\dots,k\} \times \{1,\dots,l\})^2 \to \{0,1,\dots,k\}] \\ &\cong [0,1]^M\end{aligned}$$

where 0 in the last term implies no edges, and $M = M(k,l)$.

So once one gives an interaction data:

$$(\{f_1,\dots,f_k\}, \{\bar{a}(i,j,h)\})$$

then one obtains the family of the associated interaction graphs:

$$\mathbf{G}(\{f_i\}_i, \{\bar{a}(i,j,h)\}) = \cup_{\{x_j\}_j \in [0,1]^l} G(\{f_i\}_i, \{x_j\}_j, \{\bar{a}(i,j,h)\})$$

in $\mathbf{G}(k,l) \cong [0,1]^M$.

Definition 6.3 *A Zariski subset $X \subset \mathbf{G}(k,l) \cong [0,1]^M$ is a subset of the form:*

$$X = \mathbf{G}(\{f_i\}_i, \{\bar{a}(i,j,h)\}) \subset [0,1]^M.$$

Let $\mathfrak{A}$ be all the set of the interval maps. Then we have obtained a map:

$$\mathbf{J} : \mathfrak{A}^k \times [0,1]^l \times X_2^{k^2+l} \mapsto \mathbf{G}(k,l).$$

Let $\{f_i\}$ and $\{g_i\}$ be two k interval maps. Then:

Question: (1) In order to gurantee that $\mathbf{J}(\{f_i\}, \quad) = \mathbf{J}(\{g_i\}, \quad)$ implies $\{f_i\} = \{g_i\}$, how k and l should be large ?

(2) Can one find some continuous properties for $\mathbf{J}$?

Let X be a Zariski subset, and consider a proper decreasing Zariski subsets:

$$X = X_0 \supset X_1 \supset \cdots \supset X_n.$$

We define the *dimension* of X to be the largest number n with the above property.

6.C.2 Dynamics on Zariski subsets: Let us choose a set $\{x_j\}^l \subset [0,1]^l$. Then one obtains sequences $\{\bar{a}(i,j,h)^n\}_{n=0,1,\dots}$ by using the interaction map. Thus one obatains a sequence of Zariski subsets:

$$X_0, X_1, \dots, X_n, X_{n+1}, \dots \subset [0,1]^M,$$
$$X_n = \mathbf{G}(\{f_i\}, \{\bar{a}(i,j,h)^n\}).$$

We say that $\cap_n X_n$ is an *invariant* subset. If $\lim_n X_n \subset \mathbf{G}(k,l)$ exists, then we say that the associated dynamics of the Zariski subsets converges.

References

[CG] A. Carbone and M. Gromov, *Mathematical slices of molecular biology*, Gaz. Math. **88** suppl. (2001).

[ECHLPT] D.Epstein, J.Cannon, D.Holt, S.Levy, M.Peterson and W.Thurston, *Word processing in groups*, Jones and Bartlett (1992).

[GKZ] I.Gelfand, M.Kapranove and A.Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhauser (1994).

[FOY] K.Fukuda, M.Okado and Y.Yamada, *Energy functions in box ball systems*, arXiv:math. QA/9908116 v2 (1999).

[GZ] R.Grigorchuk, A.Zuk, *The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geometric Dedicata, **87**, pp. 209-244 (2001).

[Gr] M.Gromov, *Asymptotic invariants of infinite groups*, London Math.Soc. LNS **182**, (1993).

[K1] T.Kato, *Operator dynamics in molecular biology*, to appear in the Proceedings of the first international conference on natural computation, L.N. in computer science, Springer.

[K2] T.Kato, *Interaction states by iterating maps in molecular biology*, in RIMS kokyuroku **1420** (2005).

[K3] T.Kato, *Interacting maps, symbolic dynamics and automorphisms in microscopic scale*, to appear in International Journal of Pure and Applied Mathematics.

[Ki] B.Kitchens, *Symbolic dynamics*, Springer (1998).

[LM] G.Litvinov and V.Maslov, *The correspondence principle for idempotent calculus and some computer applications*, Idempotency, Ed. J.Gunawardena, Cambridge Univ. Press, pp420-443 (1998).

[MS] W. de Melo and S. van Strien, *One dimensional dynamics*, Springer (1993).

[Mi] G.Mikhalkin, *Amoebas and tropical geoemtry*, in Different faces of geometry eds, S.Donaldson, Y.Eliashberg and M.Gromov, Kluwer academic plenum publ., (2004).

[O] T.Oda, *Convex bodies and algebraic geometry*, Birkhauser (1988).

[ST] J.Satsuma and D.Takahashi, *A soliton cellular automaton*, Journal of Phys. Soc.Japan, **59** pp. 3514 (1990).

[S] J-P. Serre, *Trees*, Springer (2003).

[St] R.Stanley, *Combinatorics and commutative algebra*, Birkhauser (1996).

[Stu] B.Sturmfels, *Gröbner bases of toric varieties*, Tohoku Math. Journal, **43** pp. 249-261 (1991).

[TM] D.Takahashi and J.Matsukidaira, *On discrete soliton equations related to cellular automata*, Phys. Lett, **A209** pp. 184-188 (1995).

[TTMS] T.Tokihiro, D.Takahashi, J.Matsukidaira and J.Satsuma, *From soliton equations to intagrable cellular automaton through a limiting procedure*, Phys. Rev. Let. **76** pp. 3247-3250 (1996).

[T] A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. B237, pp. 37-72 (1952).

[V] O.Viro, *Dequantization of real algebraic geometry on logarithmic paper*, Proc. of the European Congrress of Math., (2000).

[W] S.Wolfram, *Cellular automata and complexity*, Addison Wesley (1994).

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