GLUING OF INFINITELY MANY INSTANTONS

MASAKI TSUKAMOTO

Abstract. This paper is one step toward infinite energy gauge theory and the geometry of infinite dimensional moduli spaces. We generalize a gluing construction in the usual Yang-Mills gauge theory to an "infinite energy" situation. We show that we can glue infinitely many instantons at once, and that the resulting instantons have infinite L^2-energy in general. Moreover we show that they have infinite degrees of freedom. Our construction is a generalization of Donaldson's "alternating method".

1. Introduction

Since the remarkable paper of S. K. Donaldson [D1] appeared in 1983, Yang-Mills gauge theory has been one of the central themes in geometry and mathematical physics. There are plenty of papers studying this subject, and many interesting theories have been developed.

But most of them study only *finite energy* anti-self-dual instantons and their *finite dimensional* moduli spaces. (Here an energy means a value of the Yang-Mills functional.) We little know about *infinite energy* anti-self-dual instantons and their moduli spaces. In fact we don't know even whether such instantons exist in general situations. This paper is one step in this direction. We shall show that we can construct many, actually tremendously many, infinite energy instantons by using a "gluing construction".

Gluing of anti-self-dual instantons is a well-known technique in gauge theory [D1], [D2], [DFK], [DK], [FU], [T1], [T2], but we usually study gluing of two (or finitely many) instantons. In this paper we shall generalize it to glue infinitely many instantons at once. Of course, gluing infinitely many instantons causes several serious difficulties in analysis. Especially it is hard to construct the usual Fredholm deformation theory. We overcome these difficulties by adapting Donaldson's "alternating method" in [D2] to this situation. This decomposes the problem into infinitely many pieces where we can use the usual Fredholm deformation theory. We establish estimates over each piece and then put them together to get the total estimate. This method is so powerful that the constructed instantons have infinite degrees of freedom.

In Section 2, we present notation and state the main theorems. We establish basic estimates in Section 3. These estimates are essentially the reproduction of those in [D2]. The details of our gluing construction are given in Section 4. We discuss the moduli problem in Section 5.

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Key words and phrases. Yang-Mills gauge theory, gluing of ASD instantons, infinite energy, infinite degrees of freedom, Donaldson's alternating method.

2. Notation and Statement of the Main Results

We first review some basic notions of gauge theory. For the details, see [DK] or [FU].

Suppose that X is a closed oriented Riemannian 4-manifold and that $E \to X$ is a principal $SU(2)$ bundle. Let adE be a Lie algebra bundle associated with E. A connection A on E is called anti-self-dual, or ASD, if the self-dual part of its curvature is zero:

$$F_A^+ = 0.$$

Associating with an ASD connection A, we have the Atiyah-Hitchin-Singer complex:

$$0 \to \Omega^0(adE) \xrightarrow{d_A} \Omega^1(adE) \xrightarrow{d_A^+} \Omega^+(adE) \to 0,$$

where Ω^+ is a space of self-dual 2-forms and d_A^+ denotes the self-dual part of d_A. An ASD connection A is called acyclic if the above complex is exact (acyclic). This is equivalent to that

$$d_A^* + d_A^+ : \Omega^1(adE) \to (\Omega^0 \oplus \Omega^+)(adE),$$

is isomorphism. Here d_A^* is a formal adjoint of $d_A : \Omega^0(adE) \to \Omega^1(adE)$. A is called regular if $d_A^* + d_A^+$ is surjective.

Next we will make some convenient definitions.

Definition 2.1. We define "*GlD*" and "*gluing data*" as follows:

$$GlD \overset{\text{def}}{=} \{(X, x^L, x^R, E, A) | \text{This satisfies the following (i)}\sim\text{(v)}\}$$

(i) X is a closed oriented Riemannian 4-manifold.
(ii) x^L and x^R are two distinct points of X.
(iii) E is a $SU(2)$ bundle over X.
(iv) A is an acyclic ASD connection on E. (This can be replaced with the condition that A is a regular ASD connection. See the remark at the end of Section 5.)
(v) The metric on X is flat in neighborhoods of x^L and x^R. (This is just for simplicity. It can be easily removed; cf. [D2], (4.48) Proposition)

We refer to an element of GlD as a *gluing data*.

Definition 2.2. Let $(X_i, x_i^L, x_i^R, E_i, A_i)$ $(i \in \mathbb{Z})$ be a sequence of gluing data indexed by integers. This sequence is *of finite type* if it satisfies the following:

- As a set, $\{(X_i, x_i^L, x_i^R, E_i, A_i) | i \in \mathbb{Z}\}$ is a finite set.

This condition means that the *varieties* of gluing data in the sequence are finite although the sequence itself is an *infinite* sequence. So the same element appears infinitely many times in the sequence.

We will define a connected sum of gluing data. This construction is a generalization of the "conformal connected sum" construction in [D2] and [DK].

Let (X, x^L, x^R, E, A) be a gluing data. By the condition (v) in Definition 2.1, we have an oriented Euclidean coordinate ξ centered at x^R, and also an oriented Euclidean coordinate η centered at x^L.

We define an annular region Ω^R, and "shells" R^- and R^+ in the neighborhood of x^R (Fig.1) as follows:

$$\Omega^R = \{\xi | kN^{-1}\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\}$$
$$R^- = \{\xi | kN^{-1}\sqrt{\lambda} < |\xi| < N^{-1}\sqrt{\lambda}\}$$
$$R^+ = \{\xi | N\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\}$$

Here k is a constant smaller than 1, say $k = 0.9$, and N and λ are positive parameters. We define Ω^L, L^- and L^+ in a similar way by using the Euclidean coordinate η centered at x^L. So L^- is an inner shell of Ω^L and L^+ is an outer one. In addition we set:

$$U = X \setminus (\bar{B}(x^L, kN^{-1}\sqrt{\lambda}) \cup \bar{B}(x^R, kN^{-1}\sqrt{\lambda}))$$

Here $\bar{B}(x^L, kN^{-1}\sqrt{\lambda})$ and $\bar{B}(x^R, kN^{-1}\sqrt{\lambda})$ are balls $|\eta| \le kN^{-1}\sqrt{\lambda}$ and $|\xi| \le kN^{-1}\sqrt{\lambda}$.

Remark 2.3. We choose N large and λ small. In particular we take λ is so small that

$$\lambda N^{100} \ll 1.$$

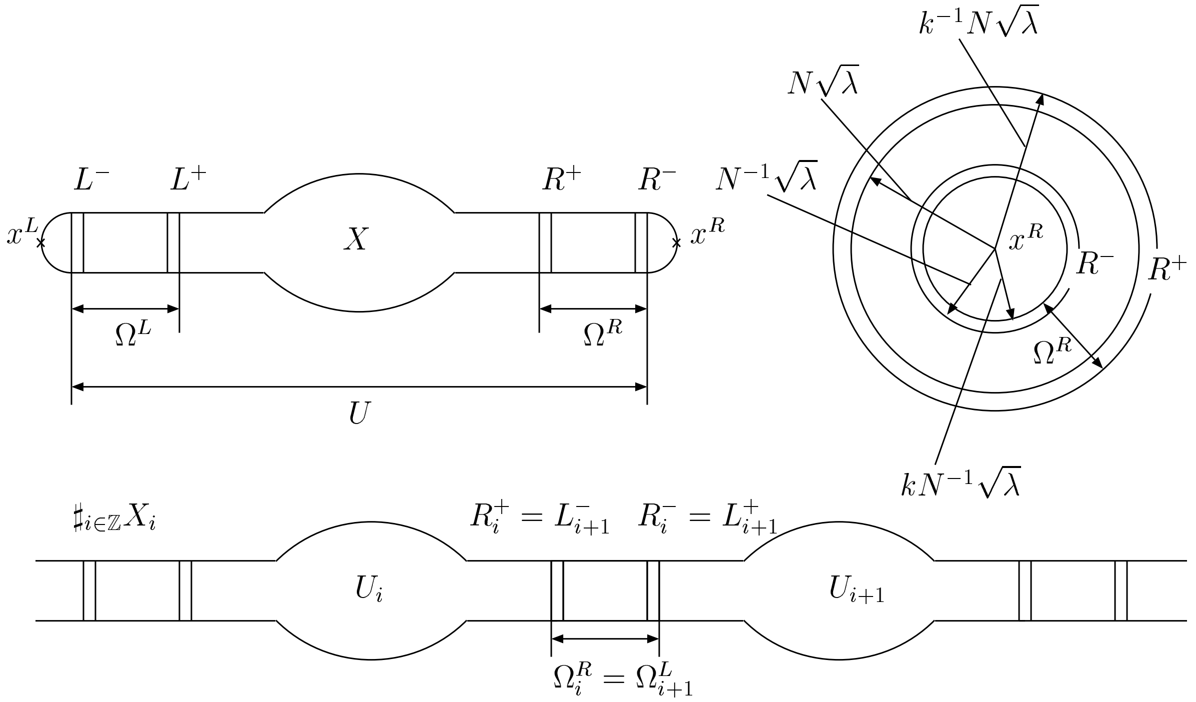

FIGURE 1. A gluing data and a connected sum.

Let $(X_i, x_i^L, x_i^R, E_i, A_i)_{i\in\mathbb{Z}}$ be a sequence of gluing data *of finite type* (cf. Definition 2.2). (The necessity of the finite type condition will be revealed in Section 4.) We have Ω_i^R, R_i^-, R_i^+, and Ω_i^L, L_i^-, L_i^+, and U_i for each $i \in \mathbb{Z}$ by the above construction. We construct the connected sum:

$$\sharp_{i\in\mathbb{Z}}X_i = \cdots \sharp X_{-1}\sharp X_0 \sharp X_1 \sharp \cdots .$$

by identifying Ω_i^R and Ω_{i+1}^L under the following map:

$$\eta = \lambda\bar{\xi}/|\xi|^2. \tag{1}$$

Here ξ is an oriented Euclidean coordinate centered at x_i^R, η is an oriented Euclidean coordinate centered at x_{i+1}^L, and $\xi \mapsto \bar{\xi}$ is any reflection. This identifies R_i^- with L_{i+1}^+, and R_i^+ with L_{i+1}^-. It follows that $\sharp_{i\in\mathbb{Z}}X_i = \bigcup_{i\in\mathbb{Z}} U_i$.

(1) is a conformal map. Hence $\sharp_{i\in\mathbb{Z}}X_i$ has a natural conformal structure which is compatible with the metrics of X_i. (Note that an ASD condition depends only on a conformal structure.)

We have constructed the conformal connected sum $\sharp_{i\in\mathbb{Z}}X_i$. Next we glue the $SU(2)$ bundles E_i.

Definition 2.4. In the above situation, we define:

$$GlP \stackrel{\mathrm{def}}{=} \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{SU(2)}((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L}) \quad (= SU(2)^{\mathbb{Z}}) \ .$$
$$EGlP \stackrel{\mathrm{def}}{=} \prod_{i\in\mathbb{Z}} (\mathrm{Hom}_{SU(2)}((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L})/\{\pm 1\}) \quad (= SO(3)^{\mathbb{Z}}).$$

Here $\mathrm{Hom}_{SU(2)}((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L})$ is a set of $SU(2)$-isomorphisms between the fibres $(E_i)_{x_i^R}$ and $(E_{i+1})_{x_{i+1}^L}$.

We refer to an element of GlP as a "*gluing parameter*" and an element of $EGlP$ as an "*effective gluing parameter*". (Note that gauge transformation (-1) acts trivially on connections.)

The exponential gauges centered at x_i^L and x_i^R give the trivializations of E_i in their neighborhoods. (See [DK] Chapter 2, or [FU] Chapter 9.) So, with these trivializations a gluing parameter $\rho \in GlP$ gives isomorphisms between E_i and E_{i+1} over $\Omega_i^R = \Omega_{i+1}^L$ covering the base identification map (1). We define a bundle $\mathbb{E}(\rho)$ over $\sharp_{i\in\mathbb{Z}}X_i$ by using this identification.

Our main theorem will assert that we can also glue connections. The construction in Section 4 will give an ASD connection $\mathbb{A}(\rho)$ on $\mathbb{E}(\rho)$ which is "close" to A_i over each U_i:

Theorem 2.5. For any positive number $\varepsilon > 0$ and $p > 1$, if we choose λ small, we can construct an ASD connection $\mathbb{A}(\rho)$ which satisfies:

$$\|\mathbb{A}(\rho) - A_i\|_{L^p(U_i)} < \varepsilon \quad (\text{for all } i \in \mathbb{Z}).$$

Note that there is a natural identification between $\mathbb{E}(\rho)$ and E_i over U_i. Therefore $\mathbb{A}(\rho) - A_i$ has a natural meaning over U_i.

Moreover, if there are infinitely many non-flat ASD connections A_i in the given sequence of gluing data, the L^2-energy of $\mathbb{A}(\rho)$ is infinite: $\|F(\mathbb{A}(\rho))\|_{L^2} = +\infty$.

The more precise statement of this theorem will be given in Section 4.

In Section 5, we will discuss the moduli problem. Our main result is as follows:

Theorem 2.6. Let ρ and ρ' be two gluing parameters. $\mathbb{A}(\rho)$ is gauge equivalent to $\mathbb{A}(\rho')$ if and only if $[\rho] = [\rho']$ in $EGlP$.

Of course this theorem is meaningless without knowing how to construct $\mathbb{A}(\rho)$. It will be explained in Section 4. But before explaining the details, I want the readers to note one important implication of this theorem; $EGlP = SO(3)^{\mathbb{Z}}$ has infinite degrees of freedom $(3 \times \infty)$, hence the degrees of freedom of our ASD connections $\mathbb{A}(\rho)$ are infinite.

3. The Main Estimates

In this section we will establish basic estimates for the proofs of Theorem 2.5 and Theorem 2.6. These are essentially the reproduction of the estimates in [D2], (4.22) Lemma and (4.24) Remarks. We will use notations similar to those in [D2].

Fix $p > 6$ and let (X, x^L, x^R, E, A) be a *gluing data*. We have L^-, L^+, R^-, R^+ and U as in Section 2. We consider the following map:

$$\Omega^1(adE)\times\Omega^1(adE)\to(\Omega^0\oplus\Omega^+)(adE).$$
$$(b,a)\to D_A b+[a\wedge b]^+ +(b\wedge b)^+.$$

Here D_A is a map $d_A^*+d_A^+ : \Omega^1(adE)\to(\Omega^0\oplus\Omega^+)(adE)$. We regard the above as a smooth map $L_1^p\times L^{2p}\to L^p$. Since the b-derivative of it at (0,0) is D_A which is invertible by Definition 2.1 (iv), we can apply implicit function theorem. Hence if a is small in L^{2p} and $\sigma\in\Omega^+(adE)$ is small in L^p, we can solve the following equation for b:

$$d_A^*b=0,\quad d_A^+b+[a\wedge b]^+ +(b\wedge b)^+=-\sigma, \tag{2}$$

and we get a small solution b which satisfies:

$$\|b\|_{L_1^p(X)}\le const\cdot\|\sigma\|_{L^p(X)}. \tag{3}$$

This is because $b=0$ is a solution to (2) if $\sigma=0$, for any a.

We use this in the situation that a is a smooth form defined on U such that the self-dual part of the curvature $\sigma=F^+(A+a)$ is supported in $L^+\cup R^+$. (σ is initially defined on U.) We extend a and σ to X by zero and call them a and σ again. Put $\delta=\|\sigma\|_{L^\infty}$. The volume of $(L^+\cup R^+)$ which contains the support of σ is bounded by

$$const\cdot N^4\lambda^2.$$

Hence if a is small in L^{2p} and δ is bounded by some constant, we have the solution b to (2) and (3) for sufficiently small λ.

Let us introduce a cut-off function ψ which is 1 on $U\setminus(L^-\cup R^-)$ and 0 on $X\setminus U$. Hence $supp(d\psi)\subset L^-\cup R^-$ and we can adjust ψ so that

$$|d\psi|\le\frac{20N}{\sqrt{\lambda}}. \tag{4}$$

We will investigate the effect of replacing the connection $A+a$ with $A+a+\psi b$. Put $\tau=F^+(A+a+\psi b)$. τ is initially defined on U, but we extend it over X by zero and call it τ again.

$$\begin{aligned}\tau &= F^+(A+a)+d_A^+(\psi b)+\psi[a\wedge b]^+ +\psi^2(b\wedge b)^+\\ &= \sigma+(d\psi\wedge b)^+ +\psi(d_A^+b+[a\wedge b]^+ +(b\wedge b)^+)+\psi(\psi-1)(b\wedge b)^+\\ &= (1-\psi)\sigma+(d\psi\wedge b)^+ +\psi(\psi-1)(b\wedge b)^+\\ &= (d\psi\wedge b)^+ +\psi(\psi-1)(b\wedge b)^+\quad (supp(\sigma)\cap supp(1-\psi)=\emptyset).\end{aligned}$$

Hence τ is supported in $L^-\cup R^-$, and we have

$$\|\tau\|_{L^\infty}\le\frac{20N}{\sqrt{\lambda}}\,\|b|_{L^-\cup R^-}\|_{L^\infty}+\|b|_{L^-\cup R^-}\|^2_{L^\infty}. \tag{5}$$

Proposition 3.1. For any $\kappa>0$, there is $\eta>0$ such that if $\delta\le\kappa$, $\|a\|_{L^{2p}}\le\eta$ and $\lambda\ll 1$, we have the following:

(i) $\|b\|_{L^{2p}}\le C_1(N)\cdot\lambda^{(p+2)/2p}\cdot\delta$.
(ii) $\|\tau\|_{L^\infty}\le C_2\cdot K_N\cdot N^4\cdot\delta$.
Here we put $K_N=N/(N-N^{-1})^3$.
(iii) $\|\psi\cdot b\|_{L_1^p}\le C_3(N)\cdot\lambda^{2/p}\cdot\delta$.
(iv) $\|F(A+a+\psi\cdot b)-F(A+a)\|_{L^2(U)}\le C_4(N)\cdot\lambda\cdot\delta$.

Here $C_1(N)$, $C_3(N)$, $C_4(N)$ are constants depending on N. On the other hand, C_2 *does not* depend on N; It is determined by (X,x^L,x^R,E,A).

Proof. (This follows [D2], (4.22) Lemma.)
(i) This is a result from (3) and the Sobolev embedding, $L^{4p/(p+2)}_1 \hookrightarrow L^{2p}$. (Note that this Sobolev embedding *does not* hold at p $= \infty$.)

$$\begin{aligned} \|b\|_{L^{2p}} &\le const \cdot \|b\|_{L^{4p/(p+2)}_1} \\ &\le const \cdot \|\sigma\|_{L^{4p/(p+2)}} \\ &\le C_1(N) \cdot \lambda^{(p+2)/2p} \cdot \delta. \end{aligned}$$

(ii) This needs a more delicate estimation. First we will estimate a leading term of b.
Let $\tilde{b}$ be the solution to

$$d_A^* \tilde{b} = 0, \quad d_A^+ \tilde{b} = -\sigma.$$

Since $D_A = d_A^* \oplus d_A^+$ is invertible by our assumption, we have the Green kernel $G(x,y)$ for D_A:

$$\tilde{b}(x) = -\int_X dy\ G(x,y) \cdot \sigma(y).$$

$G(x,y)$ has a singularity along the diagonal, and we have

$$|G(x,y)| \le const \cdot \frac{1}{d(x,y)^3}. \quad (d(x,y) \ll 1.)$$

Here $d(x,y)$ is a distance between x and y. This is essentially because D_A is a *first order* elliptic differential operator. (For the details, see [D2], (4.22) Lemma.) In the complement of a neighborhood of the diagonal, $G(x,y)$ is bounded.

We first consider the case $x \in R^-$. Since σ is supported in $L^+ \cup R^+$,

$$\tilde{b}(x) = -\int_{L^+} dy\ G(x,y)\cdot\sigma(y) - \int_{R^+} dy\ G(x,y)\cdot\sigma(y).$$

Because the distance between R^- and L^+ is sufficiently large, the first term can be easily estimated:

$$\left|\int_{L^+} dy\ G(x,y)\cdot\sigma(y)\right| \le const \cdot vol(L^+) \cdot \delta \le const \cdot N^4 \cdot \lambda^2 \cdot \delta.$$

This implies that the interaction between R^- and L^+ is comparatively small. The second term is more delicate; since $d(R^+, R^-) = N\sqrt{\lambda} - N^{-1}\sqrt{\lambda}$, we have

$$\begin{aligned} \left|\int_{R^+} dy\ G(x,y)\cdot\sigma(y)\right| &\le const \cdot vol(R^+) \cdot \frac{\delta}{(N\sqrt{\lambda} - N^{-1}\sqrt{\lambda})^3} \\ &\le const \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \delta. \quad (K_N = \frac{N}{(N-N^{-1})^3}.) \end{aligned}$$

Using the fact: $N^4 \cdot \lambda^2 \ll \sqrt{\lambda}$ (cf. Remark 2.3), we get

$$|\tilde{b}(x)| \le const \cdot K_N \cdot N^3 \sqrt{\lambda} \cdot \delta.$$

We can deduce the same estimate in the case $x \in L^-$.

Next we will examine the lower term. Let $b = \tilde{b} + \beta$, then β satisfies

$$D_A \beta + [a \wedge b]^+ + (b \wedge b)^+ = 0.$$

From the Sobolev embeddings, we get a estimate of β:

$$\begin{aligned}
\|\beta\|_{L^\infty} &\leq const \cdot \|D_A\beta\|_{L^6} \quad (L^6_1 \hookrightarrow L^\infty)\\
&\leq const \cdot (\|a\|_{L^{12}} \cdot \|b\|_{L^{12}} + \|b\|^2_{L^{12}})\\
&\leq const \cdot (\|a\|_{L^{2p}} \cdot \|b\|_{L^3_1} + \|b\|^2_{L^3_1}) \quad (L^3_1 \hookrightarrow L^{12})\\
&\leq const \cdot (\eta \cdot \|b\|_{L^3_1} + \|b\|^2_{L^3_1}).
\end{aligned}$$

From (3) we have

$$\|b\|_{L^3_1} \leq const \cdot \|\sigma\|_{L^3} \leq const \cdot N^{4/3} \cdot \lambda^{2/3} \cdot \delta.$$

Hence,

$$\|\beta\|_{L^\infty} \leq C(N) \cdot \lambda^{2/3} \cdot \delta \quad (C(N) \text{ is a constant depending on N}).$$

Since $C(N) \cdot \lambda^{2/3} \lll \sqrt{\lambda}$, $\tilde{b}$ is a leading term of b on $R^- \cup L^-$:

$$\|b|_{R^-\cup L^-}\|_{L^\infty} \leq const \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \delta. \tag{6}$$

Substituting this into (5), we conclude that

$$\|\tau\|_{L^\infty} \leq \frac{20N}{\sqrt{\lambda}} \cdot \|b|_{R^-\cup L^-}\|_{L^\infty} + \|b|_{R^-\cup L^-}\|^2_{L^\infty} \leq C_2 \cdot K_N \cdot N^4 \cdot \delta.$$

(iii) $\nabla_A(\psi \cdot b) = d\psi \otimes b + \psi \cdot \nabla_A b$, and $|d\psi| \leq 20N/\sqrt{\lambda}$, hence

$$\begin{aligned}
\|\psi \cdot b\|_{L^p_1} &\leq \frac{20N}{\sqrt{\lambda}} \cdot vol(L^- \cup R^-)^{1/p} \|b|_{L^-\cup R^-}\|_{L^\infty} + \|b\|_{L^p_1}\\
&\leq const \cdot N^{1+4/p} \cdot \lambda^{-1/2+2/p} \cdot \|b|_{L^-\cup R^-}\|_{L^\infty} + \|b\|_{L^p_1}.
\end{aligned}$$

Using (3) and (6)

$$\|\psi \cdot b\|_{L^p_1} \leq C_3(N) \cdot \lambda^{2/p} \cdot \delta.$$

(iv)

$$F(A + a + \psi b) = F(A + a) + d_A(\psi b) + \psi[a \wedge b] + \psi^2(b \wedge b).$$

Hence

$$\|F(A + a + \psi b) - F(A + a)\|_{L^2(U)} \leq \|\psi b\|_{L^2_1} + \|a\|_{L^4} \|b\|_{L^4} + \|b\|^2_{L^4}.$$

The same proofs in (i) and (iii) show

$$\|\psi b\|_{L^2_1} \leq C(N) \cdot \lambda \cdot \delta, \quad \|b\|_{L^4} \leq C'(N) \cdot \lambda \cdot \delta.$$

It follows that

$$\|F(A + a + \psi b) - F(A + a)\|_{L^2(U)} \leq C_4(N) \cdot \lambda \cdot \delta.$$

□

This Proposition 3.1 is sufficient to prove Theorem 2.5, but the moduli problem (Theorem 2.6) needs more. Therefore we will next establish the parametrized version of Proposition 3.1.

Suppose that a and σ smoothly depend on a parameter $t \in [0, 1]$: $a = a(t)$, $\sigma = \sigma(t)$. (To be precise, it is enough to assume that $a : [0, 1] \to L^{2p}$ and $\sigma : [0, 1] \to L^\infty$ are smooth maps.) Put $\delta = \delta(t) = \|\sigma\|_{L^\infty(X)}$.

Provided that $\|a\|_{L^{2p}(X)} \leq \eta$ and $\delta \leq \kappa$ for each t (η and κ are constants in Proposition 3.1.), we have the following proposition ([D2], (4.24) Remarks):

Proposition 3.2. In the above situation, if we take η and λ sufficiently small, b also smoothly depends on t and we get the estimates as follows:

$$\text{(i)} \quad \left\|\frac{\partial b}{\partial t}\right\|_{L^{2p}(X)} \leq C_5(N)\cdot\lambda^{(p+2)/2p}\left(\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty(X)} + \delta\cdot\left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}(X)}\right).$$

$$\text{(ii)} \quad \left\|\frac{\partial \tau}{\partial t}\right\|_{L^\infty(X)} \leq C_6\cdot K_N\cdot N^4\cdot\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty(X)} + C_7(N)\cdot\lambda^{1/p}\cdot\delta\cdot\left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}(X)}.$$

C_6 is a constant which *does not* depend on N, although $C_5(N)$ and $C_7(N)$ depend on N.

Proof. (i) It is clear that $b:[0,1]\to L^p_1$ is smooth. Differentiating $D_Ab+[a\wedge b]^+ + (b\wedge b)^+ = -\sigma$ with respect to t

$$D_A\left(\frac{\partial b}{\partial t}\right) + \left[\frac{\partial a}{\partial t}\wedge b\right]^+ + \left[a\wedge\frac{\partial b}{\partial t}\right]^+ + \left(\frac{\partial b}{\partial t}\wedge b\right)^+ + \left(b\wedge\frac{\partial b}{\partial t}\right)^+ = -\frac{\partial\sigma}{\partial t.}$$

Put $D'_A(\cdot) = D_A(\cdot) + [a\wedge\cdot]^+ + (\cdot\wedge b)^+ + (b\wedge\cdot)^+$. Then

$$D'_A\left(\frac{\partial b}{\partial t}\right) = -\left[\frac{\partial a}{\partial t}\wedge b\right]^+ - \frac{\partial\sigma}{\partial t}.$$

Since D_A is invertible and $\|a\|_{L^{2p}} < \eta$, $\|b\|_{L^{2p}} < C_1(N)\cdot\lambda^{(2+p)/2p}\delta$: sufficiently small,

$$\begin{aligned}
\left\|\frac{\partial b}{\partial t}\right\|_{L^{2p}} &\leq const\cdot\left\|D'_A\left(\frac{\partial b}{\partial t}\right)\right\|_{L^{4p/(p+2)}} \\
&\leq const\left(\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^{4p/(p+2)}} + \left\|\left[\frac{\partial a}{\partial t}\wedge b\right]^+\right\|_{L^{4p/(p+2)}}\right) \\
&\leq const\left(\lambda^{(2+p)/2p}\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty} + \left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}\|b\|_{L^{2p}}\right) \\
&\leq C_5(N)\cdot\lambda^{(2+p)/2p}\left(\left\|\frac{\partial\sigma}{\partial t}\right\|_{L^\infty} + \delta\cdot\left\|\frac{\partial a}{\partial t}\right\|_{L^{2p}}\right).
\end{aligned}$$

Here we used the fact: $supp(\partial\sigma/\partial t)\subset L^+\cup R^+$, whose volume is $O(\lambda^2)$.
(ii) Differentiating $\tau = (d\psi\wedge b)^+ + \psi(\psi-1)(b\wedge b)^+$ with respect to t

$$\frac{\partial\tau}{\partial t} = \left(d\psi\wedge\frac{\partial b}{\partial t}\right)^+ + \psi(\psi-1)\{\left(\frac{\partial b}{\partial t}\wedge b\right)^+ + \left(b\wedge\frac{\partial b}{\partial t}\right)^+\}.$$

Using (4) and (6)

$$\begin{aligned}
\left\|\frac{\partial\tau}{\partial t}\right\|_{L^\infty} &\leq \frac{20N}{\sqrt{\lambda}}\left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} + 2\cdot\|b|_{L^-\cup R^-}\|_{L^\infty}\left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} \\
&\leq \frac{20N}{\sqrt{\lambda}}\left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty} + C(N)\cdot\sqrt{\lambda}\cdot\delta\cdot\left\|\left.\frac{\partial b}{\partial t}\right|_{L^-\cup R^-}\right\|_{L^\infty}.
\end{aligned}$$

As in the proof of Proposition 3.1 (ii), let $b = \tilde{b} + \beta$ such that $D_A \tilde{b} = -\sigma$. Then

$$\begin{aligned}\frac{\partial \tilde{b}}{\partial t}(x) &= -\int_X dy\ G(x,y) \cdot \frac{\partial \sigma}{\partial t}(y), \quad (G(x,y) \text{ is the Green kernel.})\\ &= -\int_{L^+} dy\ G(x,y) \cdot \frac{\partial \sigma}{\partial t}(y) - \int_{R^+} dy\ G(x,y) \cdot \frac{\partial \sigma}{\partial t}(y).\end{aligned}$$

This can be estimated in the same way as before and we get

$$\left\| \left. \frac{\partial \tilde{b}}{\partial t} \right|_{L^- \cup R^-} \right\|_{L^\infty} \leq const \cdot K_N \cdot N^3 \cdot \sqrt{\lambda} \cdot \left\| \frac{\partial \sigma}{\partial t} \right\|_{L^\infty}.$$

$\partial b/\partial t = \partial \tilde{b}/\partial t + \partial \beta/\partial t$, and $\partial \beta/\partial t$ satisfies

$$D_A \left(\frac{\partial \beta}{\partial t} \right) + \left[\frac{\partial a}{\partial t} \wedge b \right]^+ + \left[a \wedge \frac{\partial b}{\partial t} \right]^+ + \left(\frac{\partial b}{\partial t} \wedge b \right)^+ + \left(b \wedge \frac{\partial b}{\partial t} \right)^+ = 0.$$

Then the Sobolev embedding and the above (i) give

$$\begin{aligned}\left\| \frac{\partial \beta}{\partial t} \right\|_{L^\infty} &\leq const \cdot \left\| D_A \left(\frac{\partial \beta}{\partial t} \right) \right\|_{L^6}, \quad (L^6_1 \hookrightarrow L^\infty.)\\ &\leq C(N) \cdot \lambda^{(2+p)/2p} \left(\left\| \frac{\partial \sigma}{\partial t} \right\|_{L^\infty} + \delta \cdot \left\| \frac{\partial a}{\partial t} \right\|_{L^{2p}} \right).\end{aligned}$$

Hence

$$\left\| \left. \frac{\partial b}{\partial t} \right|_{L^- \cup R^-} \right\|_{L^\infty} \leq C \cdot K_N N^3 \sqrt{\lambda} \left\| \frac{\partial \sigma}{\partial t} \right\|_{L^\infty} + C(N) \cdot \lambda^{(2+p)/2p} \cdot \delta \left\| \frac{\partial a}{\partial t} \right\|_{L^{2p}}.$$

Substituting this into the formula of $\|\partial \tau/\partial t\|_{L^\infty}$, we conclude that

$$\left\| \frac{\partial \tau}{\partial t} \right\|_{L^\infty} \leq C_6 \cdot K_N N^4 \left\| \frac{\partial \sigma}{\partial t} \right\|_{L^\infty} + C_7(N) \cdot \lambda^{1/p} \cdot \delta \left\| \frac{\partial a}{\partial t} \right\|_{L^{2p}}.$$

□

Here we fix the value of N. Any value of N with

$$C_2 \cdot K_N \leq \frac{1}{2}, \quad C_6 \cdot K_N \leq \frac{1}{2}.$$

will do. (Note that $K_N = N/(N - N^{-1})^3$ converges to zero as $N \to \infty$.) So let N be a large positive number which satisfies the above. (Of course this choice depends on a gluing data (X, x^L, x^R, E, A).)

Hence (ii) of Proposition 3.1 and (ii) of Proposition 3.2 become:

(ii)' $\|\tau\|_{L^\infty} \leq \dfrac{1}{2} \cdot N^4 \cdot \delta.$

(ii)' $\left\| \dfrac{\partial \tau}{\partial t} \right\|_{L^\infty} \leq \dfrac{1}{2} \cdot N^4 \cdot \left\| \dfrac{\partial \sigma}{\partial t} \right\|_{L^\infty} + C_7(N) \cdot \lambda^{1/p} \cdot \delta \left\| \dfrac{\partial a}{\partial t} \right\|_{L^{2p}}.$

4. The Gluing Construction

In this section we give a proof of Theorem 2.5. Let $(X_i, x_i^L, x_i^R, E_i, A_i)_{i\in\mathbb{Z}}$ be a sequence of gluing data of finite type as in Section 2.

The finite type condition implies that we can apply the several estimates in Section 3 to each $(X_i, x_i^L, x_i^R, E_i, A_i)$ with the *same* constants. (This is the reason for introducing this notion.)

Before starting the construction, we make one remark below about the conformal transformation (1) which identifies Ω_i^R and Ω_{i+1}^L. (cf. Section 2.)

Remark 4.1. The distortion factor of (1) over $U_i \cap U_{i+1} = \{kN^{-1}\sqrt{\lambda} < |\xi| < k^{-1}N\sqrt{\lambda}\}$ is bounded independently of λ:

$$\left\| \frac{\partial \eta}{\partial \xi} \right\| = \frac{\lambda}{|\xi|^2} \leq \frac{N^2}{k^2}, \quad \left\| \frac{\partial \xi}{\partial \eta} \right\| = \frac{\lambda}{|\eta|^2} \leq \frac{N^2}{k^2}.$$

In particular, if $\xi \in R_i^- = L_{i+1}^+$,

$$\left\| \frac{\partial \xi}{\partial \eta} \right\| \leq \frac{1}{N^2}.$$

It follows that

$$\|\omega\|_{L^\infty(L_{i+1}^+)} \leq \frac{1}{N^4} \|\omega\|_{L^\infty(R_i^-)}$$

for a 2-form ω under the identification $R_i^- = L_{i+1}^+$.

Let $\rho = (\rho_i)_{i\in\mathbb{Z}}$ be a *gluing parameter* (each $\rho_i \in \mathrm{Hom}_{SU(2)}((E_i)_{x_i^R}, (E_{i+1})_{x_{i+1}^L})$) and $\mathbb{E}(\rho)$ be a $SU(2)$ bundle over $\sharp_{i\in\mathbb{Z}} X_i$ constructed in Section 2. We will inductively construct connections $\mathbb{A}^{(0)}(\rho)$, $\mathbb{A}^{(1)}(\rho)$, $\mathbb{A}^{(2)}(\rho)$, $\cdots$, on $\mathbb{E}(\rho)$ which converge to an ASD connection $\mathbb{A}(\rho)$. This is a generalization of the alternating method in [D2].

At the n-th stage of the iteration we will have a connection $\mathbb{A}^{(n)}(\rho)$ which is represented as

$$\mathbb{A}^{(n)}(\rho) = (A_i + a_i^{(n)}(\rho))_{i\in\mathbb{Z}}.$$

This expression means that each $a_i^{(n)}(\rho)$ is an element of $\Omega^1(adE_i)$ over U_i, and that $A_i + a_i^{(n)}(\rho) = A_{i+1} + a_{i+1}^{(n)}(\rho)$ on $U_i \cap U_{i+1}$ under the identification ρ_i, and then they compatibly define a connection $\mathbb{A}^{(n)}(\rho)$ over $\sharp_{i\in\mathbb{Z}} X_i$. We assume the following inductive hypotheses:

Hypotheses 4.2. (i) Let $\sigma^{(n)}$ be a self-dual curvature of $\mathbb{A}^n(\rho)$; $\sigma^{(n)} := F^+(\mathbb{A}^{(n)}(\rho))$. This is supported as follows:

$$supp(\sigma^{(n)}) \subset \begin{cases} \coprod_{i\in\mathbb{Z}} (L_{2i}^+ \cup R_{2i}^+) & \text{if } n \text{ is even.} \\ \coprod_{i\in\mathbb{Z}} (L_{2i+1}^+ \cup R_{2i+1}^+) & \text{if } n \text{ is odd.} \end{cases}$$

And let

$$\delta_n := \begin{cases} sup_{i\in\mathbb{Z}} \|\sigma_{2i}^{(n)}\|_{L^\infty(X_{2i})} & \text{if } n \text{ is even.} \\ sup_{i\in\mathbb{Z}} \|\sigma_{2i+1}^{(n)}\|_{L^\infty(X_{2i+1})} & \text{if } n \text{ is odd.} \end{cases}$$

Here $\sigma_i^{(n)}$ is the restriction of $\sigma^{(n)}$ to U_i. (We extend $\sigma_i^{(n)}$ to X_i by zero and call it also $\sigma_i^{(n)}$.) This δ_n satisfies:

$$\delta_n \leq \kappa. \quad (\kappa \text{ is a constant in Proposition 3.1 and will be given later.})$$

(ii) $\|a_i^{(n)}\|_{L^{2p}(U_i)} \leq \eta$. ($\eta$ is also a constant in Proposition 3.1.)

If these condition hold at the n-th stage of the iteration and $\lambda \ll 1$, we can pass to the $(n+1)$-th stage as follows.

In the case that n is even, we use Proposition 3.1 on each X_{2i} and get $b_{2i}^{(n)}$. Then replace $a_{2i}^{(n)}(\rho)$ and $a_{2i+1}^{(n)}(\rho)$ by

$$a_{2i}^{(n+1)}(\rho) := a_{2i}^{(n)}(\rho) + \psi_{2i} \cdot b_{2i}^{(n)}, \tag{7}$$

$$a_{2i+1}^{(n+1)}(\rho) := a_{2i+1}^{(n)}(\rho) + \psi_{2i} \cdot \rho_{2i} b_{2i}^{(n)} \rho_{2i}^{-1} + \psi_{2i+2} \cdot \rho_{2i+1}^{-1} b_{2i+2}^{(n)} \rho_{2i+1}. \tag{8}$$

Here ψ_i is a cut-off over X_i as in Section 3. Note that $\psi_{2i} \cdot \rho_{2i} b_{2i}^{(n)} \rho_{2i}^{-1}$ and $\psi_{2i+2} \cdot \rho_{2i+1}^{-1} b_{2i+2}^{(n)} \rho_{2i+1}$ are well-defined over U_{2i+1}. Then the construction in Section 3 and Proposition 3.1 (ii)' at the end of Section 3 give

$$supp(\sigma^{(n+1)}) \subset \coprod_{i \in \mathbb{Z}} (R_{2i}^- \cup L_{2i}^-) = \coprod_{i \in \mathbb{Z}} (L_{2i+1}^+ \cup R_{2i+1}^+),$$

$$\|\sigma_{2i}^{(n+1)}\|_{L^\infty(X_{2i})} \leq \frac{1}{2} \cdot N^4 \delta_n.$$

Taking account of Remark 4.1, from this inequality we have

$$\|\sigma_{2i+1}^{(n+1)}\|_{L^\infty(X_{2i+1})} \leq \frac{1}{2} \delta_n.$$

So

$$\delta_{n+1} \leq \frac{1}{2} \delta_n. \tag{9}$$

If n is odd, we use Proposition 3.1 over each X_{2i+1}, and do a similar procedure to pass to the $(n+1)$-th stage.

The estimate (9) holds as long as this procedure can continue. Hence

$$\delta_n \leq \frac{1}{2^n} \delta_0. \tag{10}$$

We begin the iteration by defining a connection $\mathbb{A}^{(0)}(\rho) = (A_i + a_i^{(0)})_{i \in \mathbb{Z}}$ as follows:

$$a_{2i+1}^{(0)} := (\psi_{2i+1} - 1)(A_{2i+1} - \rho_{2i} A_{2i} \rho_{2i}^{-1} - \rho_{2i+1}^{-1} A_{2i+2} \rho_{2i+1}), \tag{11}$$

$$a_{2i}^{(0)} := \psi_{2i+1}(-A_{2i} + \rho_{2i}^{-1} A_{2i+1} \rho_{2i}) + \psi_{2i-1}(-A_{2i} + \rho_{2i-1} A_{2i-1} \rho_{2i-1}^{-1}). \tag{12}$$

Here each A_i represents the connection matrix in the exponential gauges centered at x_i^L and x_i^R. The above expressions are well-defined on U_{2i+1} or U_{2i}, and compatible over the overlap regions.

The self-dual curvature of $\mathbb{A}^{(0)}(\rho)$ is supported in $\coprod(L_{2i+1}^- \cup R_{2i+1}^-) = \coprod(L_{2i}^+ \cup R_{2i}^+)$. Since each $|A_i|$ is $O(\sqrt{\lambda})$ and $|d\psi_i|$ is $O(1/\sqrt{\lambda})$ (cf. (4)), we have a constant κ independent of λ such that

$$\|\sigma_i^{(0)}\|_{L^\infty(X_i)} \leq \kappa. \tag{13}$$

We take this κ as the constant in Proposition 3.1; The condition (i) in Hypotheses 4.2 is satisfied at $n = 0$.

Each $a_i^{(0)}$ is supported in the overlaps whose volume is $O(\lambda^2)$. Hence

$$\|a_i^{(0)}\|_{L^{2p}(U_i)} \leq const \cdot \lambda^{(2+p)/2p}. \tag{14}$$

Hence we also satisfy the condition (ii) if we choose $\lambda \ll 1$. Then we can start the iteration.

The geometric decay (10) implies that the condition (i) in Hypotheses 4.2 is always satisfied as long as the condition (ii) is valid.

At the n-th stage of the iteration, if n is even, Proposition 3.1 (i) shows

$$\|a_{2i}^{(n+1)} - a_{2i}^{(n)}\|_{L^{2p}(U_{2i})} \leq const \cdot \lambda^{(2+p)/2p}\delta_n \leq const \cdot \lambda^{(2+p)/2p}\frac{\delta_0}{2^n}.$$

And we also get the estimate of $a_{2i+1}^{(n+1)}$ from this and Remark 4.1:

$$\|a_{2i+1}^{(n+1)} - a_{2i+1}^{(n)}\|_{L^{2p}(U_{2i+1})} \leq const \cdot \lambda^{(2+p)/2p}\frac{\delta_0}{2^n}.$$

We have the same estimates if n is odd, and

$$const \cdot \lambda^{(2+p)/2p} + const \cdot \lambda^{(2+p)/2p}\sum_{n\geq 0}\frac{\delta_0}{2^n} \leq const \cdot \lambda^{(2+p)/2p} \ll 1. \tag{15}$$

Hence we achieve the condition (ii) in Hypotheses 4.2 for any n if we take λ small enough. Consequently the iteration can continue indefinitely and each $\{a_i^{(n)}\}_{n=0}^{\infty}$ converges to a_i in $L^{2p}(U_i)$.

Proposition 3.1 (iii) and an argument similar to the above show that $\{a_i^{(n)}\}_{n=0}^{\infty}$ converges to a_i also in $L_1^p(U_i)$. From this and the geometric decay (10) we get an ASD connection

$$\mathbb{A}(\rho) = (A_i + a_i)_{i\in\mathbb{Z}}.$$

(15) implies that this is close to each A_i over U_i:

$$\|a_i\|_{L^{2p}(U_i)} \leq const \cdot \lambda^{(2+p)/2p}. \tag{16}$$

So we get a conclusion: this is the former part of Theorem 2.5.

Theorem 4.3. In the above situation, if we take $\lambda \ll 1$, we have an ASD connection $\mathbb{A}(\rho)$ on $\mathbb{E}(\rho)$ such that

$$\|\mathbb{A}(\rho) - A_i\|_{L^{2p}(U_i)} \leq const \cdot \lambda^{(2+p)/2p} \quad (\text{for any } i \in \mathbb{Z}).$$

Here ρ gives the identification between E_i and $\mathbb{E}(\rho)$ over U_i.

And the next is the latter part of Theorem 2.5.

Theorem 4.4. If there are infinitely many non-flat ASD connections A_i in the given sequence of gluing data, the L^2-energy of $\mathbb{A}(\rho)$ is infinite: $\|F(\mathbb{A}(\rho))\|_{L^2} = +\infty$.

Proof. We will prove this theorem by showing that the loss of energy over each U_i during the iteration is sufficiently small. At the n-th stage of the iteration, if n is even, we can apply Proposition 3.1 (iv) to (7) and get

$$\|F(A_{2i} + a_{2i}^{(n+1)}) - F(A_{2i} + a_{2i}^{(n)})\|_{L^2(U_{2i})} \leq const \cdot \lambda \cdot \delta_n \leq const \cdot \frac{\lambda}{2^n}.$$

Since $F(A_{2i+1} + a_{2i+1}^{(n+1)}) - F(A_{2i+1} + a_{2i+1}^{(n)})$ is supported in the overlaps and L^2-norm of 2-form is conformally invariant,

$$\|F(A_{2i+1} + a_{2i+1}^{(n+1)}) - F(A_{2i+1} + a_{2i+1}^{(n)})\|_{L^2(U_{2i+1})} \leq const \cdot \frac{\lambda}{2^n}.$$

We have the same estimates in the case that n is odd. Hence

$$\|F(A_i + a_i) - F(A_i + a_i^{(0)})\|_{L^2(U_i)} \leq const \cdot \lambda \sum_{n\geq 0} \frac{1}{2^n} \leq const \cdot \lambda.$$

Next we will show $\|F(A_i + a_i^{(0)}) - F(A_i)\|_{L^2(U_i)}$ is also small. A estimate similar to (13) shows

$$|F(A_i + a_i^{(0)}) - F(A_i)| \leq const \quad \text{(independent of } \lambda).$$

And this is supported in the overlaps, so we get

$$\|F(A_i + a_i^{(0)}) - F(A_i)\|_{L^2(U_i)} \leq const \cdot \lambda.$$

Hence

$$\|F(A_i + a_i) - F(A_i)\|_{L^2(U_i)} \leq const \cdot \lambda.$$

From the finite type condition, we have a positive constant M independent of λ such that $\|F(A_i)\|_{L^2(U_i)} \geq M$ if A_i is not flat.

Therefore if we take $\lambda \ll 1$,

$$\|F(A_i + a_i)\|_{L^2(U_i)} \geq \frac{1}{2} M$$

for non-flat A_i. Since there are infinitely many non-flat A_i, it follows that

$$\|F(\mathbb{A}(\rho))\|_{L^2} = +\infty$$

noting that X_i and $\sharp_{i\in\mathbb{Z}} X_i$ is conformally equivalent over U_i and that L^2-norm of 2-form is conformally invariant. $\square$

5. The Moduli Problem

We will prove Theorem 2.6 in this section. The proof is a generalization of [D2], (4.31) Lemma. We need one elementary Lemma:

Lemma 5.1. Let $\{\alpha_n\}, \{s_n\}$, $(n = 0, 1, 2, \cdots)$ be sequences of non-negative numbers which satisfy the following recurrence inequalities:

$$\alpha_{n+1} - \alpha_n \leq \varepsilon(s_n + \frac{1}{2^n}\alpha_n + \frac{1}{2^n}K),$$
$$s_{n+1} \leq \frac{1}{2}s_n + \frac{\varepsilon}{2^n}\alpha_n + \frac{1}{2^{n+1}}K.$$

Here K is a positive constant, and ε is a positive constant $\leq 1/100$.

Suppose that

$$\alpha_0 \leq \varepsilon \cdot K, \quad s_0 \leq K.$$

Then we get

$$\alpha_n \leq 10\varepsilon \cdot K \quad \text{(for all } n).$$

Proof. Put $t_n := 2^n s_n$. Then $t_0 \leq K$, and the above recurrence inequalities become

$$\alpha_{n+1} - \alpha_n \leq \frac{\varepsilon}{2^n}(t_n + \alpha_n + K),$$
$$t_{n+1} \leq t_n + 2\varepsilon\alpha_n + K.$$

We inductively prove $\alpha_n \leq 10\varepsilon \cdot K$. Suppose this is true if $n \leq n_0$ for some n_0. Then

$$t_n \leq K + 2\varepsilon \cdot (10\varepsilon K)n + Kn = K\{1 + (1 + 20\varepsilon^2)n\} \quad (n \leq n_0).$$

Hence

$$\begin{aligned}
\alpha_{n_0+1} - \alpha_0 &= \sum_{n=0}^{n_0} (\alpha_{n+1} - \alpha_n) \\
&\leq \sum_{n\geq 0} \frac{\varepsilon}{2^n}\left[K\{1 + (1 + 20\varepsilon^2)n\} + 10\varepsilon K + K\right] \\
&= \varepsilon K(6 + 20\varepsilon + 40\varepsilon^2) \leq 7\varepsilon K.
\end{aligned}$$

Thus

$$\alpha_{n_0+1} \leq K + 7\varepsilon K \leq 10\varepsilon K.$$

□

Proposition 5.2. For two *gluing parameters* $\rho = (\rho_i)_{i\in\mathbb{Z}}$ and $\rho' = (\rho'_i)_{i\in\mathbb{Z}}$, let $K := sup_{i\in\mathbb{Z}}|\rho_i - \rho'_i|$, then we have

$$\|a_i(\rho) - a_i(\rho')\|_{L^{2p}(U_i)} \leq const \cdot \lambda^{1/p} K.$$

Proof. Let $\bar{\rho}(t) = (\bar{\rho}_i(t))_{i\in\mathbb{Z}}$ $(0 \leq t \leq 1)$ be a path in GlP such that $\bar{\rho}_i(t)$ is a geodesic from ρ_i to ρ'_i in $SU(2)$. Then $|d\bar{\rho}_i/dt| = d(\rho_i, \rho'_i)$ (constant). Here note that $d(\rho_i, \rho'_i)$ is equivalent to $|\rho_i - \rho'_i|$:

$$C^{-1} \leq \frac{d(\rho_i, \rho'_i)}{|\rho_i - \rho'_i|} \leq C \quad \text{(for some constant } C\text{)}.$$

Hence $|d\bar{\rho}_i/dt| \leq C \cdot K$.

We consider $\mathbb{A}^{(n)}(\bar{\rho}(t)) = (A_i + a_i^{(n)}(\bar{\rho}(t)))_{i\in\mathbb{Z}}$ constructed as in Section 4; this is parametrized by t and we can apply Proposition 3.2. Put

$$\alpha_n := sup_{i\in\mathbb{Z}} \left\| \frac{\partial}{\partial t} a_i^{(n)} \right\|_{L^{2p}(X_i)}, \; s_n := \begin{cases} sup_{i\in\mathbb{Z}} \left\| \frac{\partial}{\partial t}\sigma_{2i}^{(n)} \right\|_{L^\infty(X_{2i})} & \text{if } n \text{ is even.} \\ sup_{i\in\mathbb{Z}} \left\| \frac{\partial}{\partial t}\sigma_{2i+1}^{(n)} \right\|_{L^\infty(X_{2i+1})} & \text{if } n \text{ is odd.} \end{cases}$$

Here we extend $a_i^{(n)}$ to X_i by zero and call it $a_i^{(n)}$ again.

From the definition of $a_i^{(0)}$, (11) and (12),

$$\left| \frac{\partial}{\partial t} a_i^{(0)} \right| \leq const \cdot \sqrt{\lambda} \cdot K$$

since the connection matrices in the exponential gauges are $O(\sqrt{\lambda})$. Noting that $\partial a_i^{(0)}/\partial t$ is supported in the overlaps, we get

$$\left\| \frac{\partial}{\partial t} a_i^{(0)} \right\|_{L^{2p}(X_i)} \leq const \cdot \lambda^{(2+p)/2p} K.$$

And also in a manner similar to the estimate of $\left\|\sigma_i^{(0)}\right\|_{L^\infty(X_i)}$ in (13), we have

$$\left\|\frac{\partial}{\partial t}\sigma_i^{(0)}\right\|_{L^\infty(X_i)} \le const\cdot K.$$

Consequently

$$\alpha_0 \le const\cdot\lambda^{(2+p)/2p}K, \quad s_0 \le const\cdot K. \tag{17}$$

We will establish recurrence inequalities about α_n and s_n similar to those in Lemma 5.1. First we consider the case that n is even. Proposition 3.1 and 3.2 give the bounds of $\|b_{2i}^{(n)}\|_{L^{2p}(X_{2i})}$ and $\|\partial b_{2i}^{(n)}/\partial t\|_{L^{2p}(X_{2i})}$. Then, from the inductive definitions (7), (8) and the geometric decay (10)

$$\left\|\frac{\partial}{\partial t}a_{2i}^{(n+1)}\right\|_{L^{2p}(X_{2i})} \le \alpha_n + const\cdot\lambda^{(2+p)/2p}(s_n+\frac{1}{2^n}\alpha_n),$$

$$\left\|\frac{\partial}{\partial t}a_{2i+1}^{(n+1)}\right\|_{L^{2p}(X_{2i+1})} \le \alpha_n + const\cdot\frac{\lambda^{(2+p)/2p}}{2^n}K + const\cdot\lambda^{(2+p)/2p}(s_n+\frac{1}{2^n}\alpha_n).$$

It follows that

$$\alpha_{n+1}-\alpha_n \le const\cdot\lambda^{(2+p)/2p}(s_n+\frac{1}{2^n}\alpha_n+\frac{K}{2^n}). \tag{18}$$

Similarly, from Proposition 3.2 (ii)' at the end of Section 3

$$\left\|\frac{\partial}{\partial t}\sigma_{2i}^{(n+1)}\right\|_{L^\infty(X_{2i})} \le \frac{N^4}{2}s_n + const\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n.$$

Because $\sigma_{2i+1}^{(n+1)} = \bar\rho_{2i}\sigma_{2i}^{(n+1)}\bar\rho_{2i}^{-1} + \bar\rho_{2i+1}^{-1}\sigma_{2i+2}^{(n+1)}\bar\rho_{2i+1}$, taking account of Remark 4.1, we get

$$\begin{aligned}\left\|\frac{\partial}{\partial t}\sigma_{2i+1}^{(n+1)}\right\|_{L^\infty(X_{2i+1})} &\le const\cdot\delta_{n+1}\cdot K + \frac{1}{2}s_n + const\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n\\ &\le const\cdot\frac{K}{2^{n+1}}+\frac{1}{2}s_n+const\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n.\end{aligned}$$

This implies that

$$s_{n+1} \le \frac{1}{2}s_n + const\cdot\frac{\lambda^{1/p}}{2^n}\alpha_n + const\cdot\frac{K}{2^{n+1}}. \tag{19}$$

The inequalities (18), (19) hold also in the case that n is odd. So we can apply Lemma 5.1 with initial values (17), and get

$$\alpha_n \le const\cdot\lambda^{1/p}K, \quad \text{that is,} \quad \left\|\frac{\partial}{\partial t}a_i^{(n)}\right\|_{L^{2p}(X_i)} \le const\cdot\lambda^{1/p}K.$$

From this inequality

$$\|a_i^{(n)}(\rho)-a_i^{(n)}(\rho')\|_{L^{2p}(U_i)} = \left\|\int_0^1\frac{\partial}{\partial t}a_i^{(n)}dt\right\|_{L^{2p}(U_i)} \le const\cdot\lambda^{1/p}K,$$

for any i and n. So we get a conclusion:

$$\|a_i(\rho)-a_i(\rho')\|_{L^{2p}(U_i)} \le const\cdot\lambda^{1/p}K.$$

□

The following is Theorem 2.6.

Theorem 5.3. For two *gluing parameters* ρ and ρ', $\mathbb{A}(\rho)$ is gauge equivalent to $\mathbb{A}(\rho')$ if and only if $[\rho] = [\rho']$ in $EGlP$.

Proof. Gauge transformation (-1) acts trivially on connections. Hence it is clear from the construction in Section 4 that $\mathbb{A}(\rho)$ is gauge equivalent to $\mathbb{A}(\rho')$ if $[\rho] = [\rho']$ in $EGlP$.

Conversely, suppose that there is a gauge transformation $g : \mathbb{E}(\rho) \to \mathbb{E}(\rho')$ such that $g.\mathbb{A}(\rho) = \mathbb{A}(\rho')$. This implies that there is a gauge transformation g_i over each U_i such that

$$d_{A_i} g_i = g_i a_i(\rho) - a_i(\rho') g_i.$$

$$g_{i+1}\rho_i = \rho'_i g_i.$$

Because A_i is irreducible, we have a constant C independent of λ such that

$$\min(\|g_i - 1\|_{C^0(U_i)}, \|g_i + 1\|_{C^0(U_i)}) \leq C \cdot \|d_{A_i} g_i\|_{L^{2p}(U_i)}.$$

The proof is standard; for example, see [D1] and [D2], (4.31) Lemma.

We can suppose that $\|g_i - 1\|_{C^0(U_i)} \leq C \cdot \|d_{A_i} g_i\|_{L^{2p}(U_i)}$ without loss of generality. Hence, from (16) and Proposition 5.2

$$\begin{aligned} \|g_i - 1\|_{C^0} &\leq C \cdot \|g_i a_i(\rho) - a_i(\rho') g_i\|_{L^{2p}}, \\ &\leq C \cdot \|g_i - 1\|_{C^0}(\|a_i(\rho)\|_{L^{2p}} + \|a_i(\rho')\|_{L^{2p}}) + C \cdot \|a_i(\rho) - a_i(\rho')\|_{L^{2p}}, \\ &\leq \mathit{const} \cdot \lambda^{(2+p)/2p} \|g_i - 1\|_{C^0} + \mathit{const} \cdot \lambda^{1/p} K. \end{aligned}$$

If λ is sufficiently small, we get

$$\|g_i - 1\|_{C^0} \leq \mathit{const} \cdot \lambda^{1/p} K.$$

But $g_{i+1}\rho_i = \rho'_i g_i$ implies that

$$\rho'_i - \rho_i = (g_{i+1} - 1)\rho_i g_i^{-1} + \rho_i g_i^{-1}(1 - g_i).$$

Hence

$$|\rho'_i - \rho_i| \leq \mathit{const} \cdot (\|g_{i+1} - 1\|_{C^0(U_{i+1})} + \|g_i - 1\|_{C^0(U_i)}) \leq \mathit{const} \cdot \lambda^{1/p} K.$$

This holds for any i, so it follows that

$$K \leq \mathit{const} \cdot \lambda^{1/p} K.$$

If we take $\lambda \ll 1$, this implies that $K = 0$, $\rho = \rho'$. □

Remark 5.4. So far we have discussed only acyclic connections (cf. Definition 2.1). But this is just for simplicity. The condition (iv) in Definition 2.1 can be replaced with:

(iv)' A is a regular ASD connection on E.

If we glue infinitely many regular ASD connections, the resulting instantons are parametrized by

$$EGlP \times \prod_{i \in \mathbb{Z}} B_i \subset EGlP \times \prod_{i \in \mathbb{Z}} H^1_{A_i}.$$

Here B_i is a small ball centered at the origin in $H^1_{A_i} := ker D_{A_i}$. This generalization is a simple application of the methods in this paper and [D2] (p.324 $\sim$ p.330). The proof is routine, so I omit the details.

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Masaki Tsukamoto, Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `tukamoto@math.kyoto-u.ac.jp`