THE ZERO-IN-THE-SPECTRUM CONJECTURE AND FINITELY PRESENTED GROUPS

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Abstract

We introduce an algorithm which transforms a discrete group G into another one G_Ψ which has some particular properties. For example when G is a non-amenable group, then G_Ψ does not satisfy an algebraic version of the zero-in-the-spectrum conjecture. Moreover when G is a finitely generated free group, then G_Ψ is also finitely presented and the p-th group homology of G_Ψ have infinite rank for all $p \geq 3$.

1 Introduction

In this paper we study some geometric properties of finitely presented groups. Here we will give an algorithm Ψ which transforms a discrete group G into another one G_Ψ. G_Ψ is given by successive procedures: taking infinite sums of G, a semi-direct product with $\mathbb{Z}$ and an HNN-extension (Section 2). We show that when G is a finitely presented group, then G_Ψ is also the same. In the case when G satisfies some conditions, then G_Ψ shows particular phenomena in the so-called zero-in-the-spectrum conjecture by Gromov ([3]), the Baum-Connes conjecture and the group homology.

Firstly we will give an application to the zero-in-the-spectrum conjecture. The conjecture claims that for a closed, aspherical and connected Riemannian manifold M there always exists some $p \geq 0$, such that zero belongs to the spectrum of the Laplace-Beltrami operator Δ_p acting on square integrable p-forms on the universal covering $\widetilde{M}$ of M. Let $H_p(G;\mathcal{N}(G))$ be the homology of G with coefficients in a group von Neumann algebra $\mathcal{N}(G)$. It is known that if BG is a closed manifold, then the conjecture is equivalent to an algebraic condition that for some $p \geq 0$, $H_p(G;\mathcal{N}(G)) \neq 0$ holds ([4, p.438]). Naturally we can generalize it to an algebraic version of the zero-in-the-spectrum conjecture.

Conjecture 1.1.
Let G be a discrete group. Then for some $p \geq 0$, $H_p(G;\mathcal{N}(G)) \neq 0$ holds.

Several counterexamples are known for finitely generated groups, but they are infinitely presented. In this paper we show that many G_Ψ are finitely presented groups which do not satisfy Conjecture 1.1. Actually, the following theorem is proved in Section 3.

Theorem 1.2.

Suppose that G is a non-amenable group, then G_Ψ satisfies $H_*\left(G_\Psi; \mathcal{N}(G_\Psi)\right) = 0$.

In particular when G is a finitely presented and non-amenable group, then G_Ψ is a finitely presented group which does not satisfy the algebraic version of the zero-in-the-spectrum conjecture.

Next we study the relation of G_Ψ to the Baum-Connes conjecture. The conjecture identifies G-equivariant K-homology with G-compact supports of the classifying space $\underline{E}G$ for proper actions of G and the K-theory of the reduced C^*-algebra $C^*_r(G)$ ([6]). The following theorem is proved in Section 3.

Theorem 1.3.

Suppose that G has Haagerup property, then G_Ψ satisfies the Baum-Connes conjecture.

It is known that the Baum-Connes conjecture implies the zero-in-the-spectrum conjecture in the case when BG are closed manifolds ([6, p.61]). On the other hand the situation is completely different when BG are far from being manifolds. Let G be finitely presented, non-amenable and has Haagerup property. For example, a free group of rank $m \geq 2$. Then G_Ψ satisfies the Baum-Connes conjecture, but does not satisfy the conjecture 1.1. Therefore,

Corollary 1.4.

The Baum-Connes conjecture does not imply the algebraic version of the zero-in-the-spectrum conjecture for finitely presented groups.

Finally we will calculate the group homology of G_Ψ coming from free groups in Section 4. Let $H_p\left(G; \mathbb{Z}\right)$ be the group homology of G.

Theorem 1.5.

Suppose that G is a free group of rank $m \geq 1$, then G_Ψ satisfies the following.

$$
\begin{array}{ccl}
H_p(G_\Psi; \mathbb{Z}) & has & infinite\ rank\ (\forall p \geq 3), \\
H_2(G_\Psi; \mathbb{Z}) & \cong & \mathbb{Z}^{2m+m^2}, \\
H_1(G_\Psi; \mathbb{Z}) & \cong & \mathbb{Z}^{m+1}, \\
H_0(G_\Psi; \mathbb{Z}) & \cong & \mathbb{Z}.
\end{array}
$$

In particular G_Ψ is a finitely presented group of infinite type and its rational cohomological dimension is infinite.

Moreover Ψ is injective on the class of free groups.

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2 Construction of the algorithm Ψ

Definition 2.1.

Let n be a non-negative integer or $n = \infty$. Define $\mathcal{F}_n$ to be the class of groups for which BG are CW-complexes which have a finite number of p-dimensional cells for $p \leq n$.

Example 2.2.

$$\begin{aligned}
G \in \mathcal{F}_0 &\Leftrightarrow G : a\ discrete\ group,\\
G \in \mathcal{F}_1 &\Leftrightarrow G : a\ finitely\ generated\ group,\\
G \in \mathcal{F}_2 &\Leftrightarrow G : a\ finitely\ presented\ group,\\
G \in \mathcal{F}_\infty &\Leftrightarrow G : a\ group\ of\ finite\ type.
\end{aligned}$$

Also we will use $[g,h] := g^{-1}h^{-1}gh, g^h := h^{-1}gh$ $(g,h \in G)$.

We will construct the algorithm

$$\Psi : \mathcal{F}_0 \to \mathcal{F}_0; G \mapsto G_\Psi$$

passing through three steps.

Construction of the algorithm Ψ.

Let $G^{(k)}$ $(k \in \mathbb{Z})$ be infinite copies of G. We identify $G^{(0)}$ with G. Let us put

$$G_0 := \bigoplus_{k\in\mathbb{Z}} G^{(k)}, H_0 := \bigoplus_{l\in\mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}, K_0 := \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}.$$

$G_1 := G_0 \rtimes \mathbb{Z}$ is an HNN-extension of $G_0 = \bigoplus_{k\in\mathbb{Z}} G^{(k)}$ by the isomorphism

$$G_0 \xrightarrow{\sim} G_0; g^{(k)} \mapsto g^{(k+1)}.$$

$H_1 := H_0 \rtimes \mathbb{Z}$ is an HNN-extension of $H_0 = \bigoplus_{l\in\mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}$ by the isomorphism

$$H_0 \xrightarrow{\sim} H_0; g^{(k)} \mapsto g^{(k+2)}.$$

$K_1 := K_0 \rtimes \mathbb{Z}$ is an HNN-extension of $K_0 = \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}$ by the isomorphism

$$K_0 \xrightarrow{\sim} K_0; g^{(k)} \mapsto g^{(k+3)}.$$

Then we have presentations as:

$$\begin{aligned}
G_1 &= \langle G, a \mid [G, G^{a^k}](0 \neq k \in \mathbb{Z})\rangle,\\
H_1 &= \left\langle G^{(0)}, G^{(1)}, b \;\middle|\; \begin{array}{l} [G^{(0)}, (G^{(1)})^{b^k}](k \in \mathbb{Z}),\\ [G^{(0)}, (G^{(0)})^{b^k}], [G^{(1)}, (G^{(1)})^{b^k}](0 \neq k \in \mathbb{Z}) \end{array} \right\rangle,\\
K_1 &= \left\langle G^{(0)}, G^{(1)}, c \;\middle|\; \begin{array}{l} [G^{(0)}, (G^{(1)})^{c^k}](k \in \mathbb{Z}),\\ [G^{(0)}, (G^{(0)})^{c^k}], [G^{(1)}, (G^{(1)})^{c^k}](0 \neq k \in \mathbb{Z}) \end{array} \right\rangle.
\end{aligned}$$

Let us regard H_1 and K_1 as subgroups of G_1 by

$$H_1 \hookrightarrow G_1; g^{(0)}, g^{(1)}, b \mapsto g, g^a, a^2,$$

$$K_1 \hookrightarrow G_1; g^{(0)}, g^{(1)}, c \mapsto g, g^a, a^3.$$

Definition 2.3.

G_Ψ is an HNN-extension of G_1 by the isomorphism

$$H_1 \xrightarrow{\sim} K_1; g^{(0)}, g^{(1)}, b \mapsto g^{(0)}, g^{(1)}, c.$$

Then we have a presentation as:

$$G_\Psi = \left\langle G, a, t \,\middle|\, \begin{array}{l} [G, G^{a^k}](0 \neq k \in \mathbb{Z}), \\ g^t = g, (g^a)^t = g^a(g \in G), (a^2)^t = a^3 \end{array} \right\rangle.$$

□

Here we claim the following.

Claim 2.4.

When

$$g^t = g, (g^a)^t = g^a(g \in G), (a^2)^t = a^3, 1 = [G, G^a],$$

then

$$1 = [G, G^{a^k}](0 \neq k \in \mathbb{Z}).$$

Proof. We have

$$1 = [G, G^a]^{ata^{-1}} = [G^a, G^{a^2}]^{ta^{-1}} = [G^a, G^{a^3}]^{a^{-1}} = [G, G^{a^2}]$$

and

$$1 = [G, G^{a^2}]^t = [G, G^{a^3}].$$

Suppose $1 = [G, G^{a^k}]$ for $1 \leq k \leq 3N$ $(N \geq 1)$. Then since $2N + 1 \leq 3N$,

$$1 = [G, G^{a^{2N+1}}]^t = [G, G^{a^{3N+1}}],$$

$$1 = [G, G^{a^{2N+1}}]^{ata^{-1}} = [G^a, G^{a^{2(N+1)}}]^{ta^{-1}} = [G^a, G^{a^{3(N+1)}}]^{a^{-1}} = [G, G^{a^{3N+2}}].$$

Then since $2(N + 1) \leq 3N + 1$,

$$1 = [G, G^{a^{2(N+1)}}]^t = [G, G^{a^{3(N+1)}}].$$

Hence $1 = [G, G^{a^k}]$ for $1 \leq k \leq 3(N + 1)$ $(N \geq 1)$. Consequently $1 = [G, G^{a^k}]$ for $k \geq 1$. Moreover

$$1 = ([G, G^{a^k}]^{a^{-k}})^{-1} = [G^{a^{-k}}, G]^{-1} = [G, G^{a^{-k}}]$$

for $k \geq 1$. Thus $1 = [G, G^{a^k}]$ for $0 \neq k \in \mathbb{Z}$. □

Corollary 2.5.

Let

$$G = \langle s_i(1 \leq i \leq m) \mid r_i(1 \leq i \leq n) \rangle$$

be a presentation. Then,

$$G_\Psi \quad = \quad \left\langle s_i(1 \leq i \leq m), a, t \,\middle|\, \begin{array}{l} r_i(1 \leq i \leq n), [s_i, s_j^a](1 \leq i, j \leq m), \\ s_i^t = s_i, (s_i^a)^t = s_i^a(1 \leq i \leq m), (a^2)^t = a^3 \end{array} \right\rangle.$$

In particular when G is finitely presented or generated, G_Ψ has the same property respectively.

Remark 2.6.

We can modify the algorithm. We fix an integer $q \geq 2$. Let us put

$$H_0^{'} := \bigoplus_{l \in \mathbb{Z}} G^{(ql)} \oplus G^{(ql+1)} \oplus \cdots \oplus G^{(ql+q-1)},$$

$$K_0^{'} := \bigoplus_{l \in \mathbb{Z}} G^{((q+1)l)} \oplus G^{((q+1)l+1)} \oplus \cdots \oplus G^{((q+1)l+q-1)}.$$

$H_1^{'} := H_0^{'} \rtimes \mathbb{Z}$ is an HNN-extension of $H_0^{'}$ by the isomorphism

$$H_0^{'} \xrightarrow{\sim} H_0; g^{(k)} \mapsto g^{(k+q)}.$$

$K_1^{'} := K_0^{'} \rtimes \mathbb{Z}$ is an HNN-extension of $K_0^{'}$ by the isomorphism

$$K_0^{'} \xrightarrow{\sim} K_0; g^{(k)} \mapsto g^{(k+q+1)}.$$

$G_\Psi^{'}$ is an HNN-extension of G_1 by the isomorphism

$$H_1^{'} \xrightarrow{\sim} K_1^{'}; g^{(0)}, g^{(1)}, \ldots, g^{(q-1)}, b^{'} \mapsto g^{(0)}, g^{(1)}, \ldots, g^{(q-1)}, c^{'}.$$

Then,

$$G_\Psi^{'} \quad = \quad \left\langle s_i(1 \leq i \leq m), a, t \,\middle|\, \begin{array}{l} r_i(1 \leq i \leq n), [s_i, s_j^{a^k}](1 \leq k \leq q-1, 1 \leq i, j \leq m), \\ (s_i^{a^k})^t = s_i^{a^k}(0 \leq k \leq q-1, 1 \leq i \leq m), (a^q)^t = a^{q+1} \end{array} \right\rangle.$$

However for simplicity we will deal with the only case of $q = 2$.

If G is a finitely generated free group of rank $m \geq 1$, then

$$H_1(G_\Psi; \mathbb{Z}) \cong G_\Psi / [G_\Psi, G_\Psi] = \langle s_i(1 \leq i \leq m), t \rangle \cong \mathbb{Z}^{m+1}.$$

Accordingly,

Corollary 2.7.

Ψ is injective on the class of free groups.

Here we will collect the groups appearing in the construction of the algorithm Ψ.

Notation 2.8.

$$\begin{aligned}
G &= \langle s_i(1 \le i \le m) \mid r_i(1 \le i \le n)\rangle, \\
G_0 &= \bigoplus_{k\in\mathbb{Z}} G^{(k)} \\
&= \left\langle s_i^{(k)}(1 \le i \le m, k \in \mathbb{Z}) \;\middle|\; \begin{array}{l} r_i^{(k)}(1 \le i \le n, k \in \mathbb{Z}), \\ [s_i^{(k)}, s_j^{(l)}](1 \le i, j \le m, k \ne l \in \mathbb{Z}) \end{array} \right\rangle, \\
H_0 &= \bigoplus_{l\in\mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)} = G_0, \\
K_0 &= \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}, \\
G_1 &= G_0 \rtimes \mathbb{Z} \\
&= \langle s_i(1 \le i \le m), a \mid r_i(1 \le i \le n), [s_i, s_j^{a^k}](1 \le i, j \le m, 0 \ne k \in \mathbb{Z})\rangle, \\
H_1 &= H_0 \rtimes \mathbb{Z} \\
&= \left\langle s_i^{(0)}, s_i^{(1)}(1 \le i \le m), b \;\middle|\; \begin{array}{l} r_i^{(0)}, r_i^{(1)}(1 \le i \le n), [s_i^{(0)}, (s_j^{(1)})^{b^k}](1 \le i, j \le m, k \in \mathbb{Z}), \\ [s_i^{(0)}, (s_j^{(0)})^{b^k}], [s_i^{(1)}, (s_j^{(1)})^{b^k}](1 \le i, j \le m, 0 \ne k \in \mathbb{Z}) \end{array} \right\rangle, \\
K_1 &= K_0 \rtimes \mathbb{Z} \\
&= \left\langle s_i^{(0)}, s_i^{(1)}(1 \le i \le m), c \;\middle|\; \begin{array}{l} r_i^{(0)}, r_i^{(1)}(1 \le i \le n), [s_i^{(0)}, (s_j^{(1)})^{c^k}](1 \le i, j \le m, k \in \mathbb{Z}), \\ [s_i^{(0)}, (s_j^{(0)})^{c^k}], [s_i^{(1)}, (s_j^{(1)})^{c^k}](1 \le i, j \le m, 0 \ne k \in \mathbb{Z}) \end{array} \right\rangle, \\
G_\Psi &= \left\langle s_i(1 \le i \le m), a, t \;\middle|\; \begin{array}{l} r_i(1 \le i \le n), [s_i, s_j^a](1 \le i, j \le m), \\ s_i^t = s_i, (s_i^a)^t = s_i^a(1 \le i \le m), (a^2)^t = a^3 \end{array} \right\rangle.
\end{aligned}$$

Proposition 2.9.

G_Ψ is torsion-free if and only if G is torsion-free.

Proof. G_Ψ is an HNN-extension of G_1 and G_1 is an HNN-extension of G_0. Thus this proposition is clear by the torsion theorem for HNN-extensions ([5, p.185]). □

Proposition 2.10.

The cohomological dimension of G_Ψ is infinite if and only if G is not trivial.

Proof. G has a torsion element if and only if G_Ψ has a torsion element by Proposition 2.9. Then the cohomological dimension of each is infinite. If G is torsion-free and not trivial, then $G \supset \mathbb{Z}$. Thus $G_\Psi \supset \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}$. Consequently the cohomological dimension of G_Ψ is infinite. If G is trivial, then $G_\Psi = \langle a, t \mid (a^2)^t = a^3\rangle$. Hence G_Ψ is a one-relator group. Therefore the cohomological dimension of G_Ψ is two. □

3 Counterexamples to the algebraic version of the zero-in-the-spectrum conjecture

We will get counterexamples to the algebraic version of the zero-in-the-spectrum conjecture for finitely presented groups.

Definition 3.1.

Let d be a non-negative integer or ∞. Define $\mathcal{Z}_d$ to be the class of groups for which $H_p(G;\mathcal{N}(G)) = 0$ hold for $p \leq d$.

Lemma 3.2.

Let d, e be a non-negative integer or ∞. Then
(1) *Let G be the directed union $\bigcup_{i\in I} G_i$ of subgroups $G_i \subset G$. Suppose that $G_i \in \mathcal{Z}_d$ for each $i \in I$. Then $G \in \mathcal{Z}_d$.*
(2) *If G contains a normal subgroup $H \subset G$ with $H \in \mathcal{Z}_d$, then $G \in \mathcal{Z}_d$.*
(3) *If $G \in \mathcal{Z}_d$ and $H \in \mathcal{Z}_e$, then $G \times H \in \mathcal{Z}_{d+e+1}$.*
(4) *$\mathcal{Z}_0$ is the class of non-amenable groups.*
(5) *Let $G = G_1 *_A G_2$ where $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$. Suppose that $G_1, G_2 \in \mathcal{Z}_d$ and $A \in \mathcal{Z}_{d-1}$. Then $G \in \mathcal{Z}_d$.*
(6) *Let $G = H*_A = \langle H, t \mid \theta(a) = a^t\rangle$ where $A \subset H$ and $\theta : A \hookrightarrow H$. Suppose that $H \in \mathcal{Z}_d$ and $A \in \mathcal{Z}_{d-1}$. Then $G \in \mathcal{Z}_d$.*

Proof. (1) $\sim$ (4) are proved in [4, p.448]. (5), (6) are clear by Mayer-Vietoris sequences ([1, p.178]). $\square$

Proof of Theorem 1.2.

When G is non-amenable, then $G_0, H_0 \in \mathcal{Z}_\infty$ by Lemma 3.2 (1), (3), (4). Moreover $G_1, H_1 \in \mathcal{Z}_\infty$ by Lemma 3.2 (2) or (6). Accordingly $G_\Psi \in \mathcal{Z}_\infty$ by Lemma 3.2 (6).

In particular when G is finitely presented and non-amenable, G_Ψ is a counterexample to the algebraic version of the zero-in-the-spectrum conjecture for finitely presented groups by Corollary 2.5.

$\square$

Proof of Theorem 1.3.

If G has Haagerup property, then $\bigoplus_{-K\leq k\leq K} G^{(k)}$ has Haagerup property, too. So $\bigoplus_{-K\leq k\leq K} G^{(k)}$ satisfies the Baum-Connes conjecture ([6, p.43]). G_0 and H_0 satisfy the Baum-Connes conjecture because G_0 and H_0 are directed unions of $\bigoplus_{-K\leq k\leq K} G^{(k)}$ for all $K \in \mathbb{Z}$ ([6, p.38]). G_1 and H_1 satisfy the Baum-Connes conjecture because G_1 and H_1 are HNN-extensions of G_0 and H_0 respectively ([6, p.40]). Therefore G_Ψ satisfies the Baum-Connes conjecture because G_Ψ is an HNN-extension of G_1 on H_1 ([6, p.40]).

$\square$

Remark 3.3.

Unfortunately any G_Ψ can not be a counterexample to the zero-in-the-spectrum conjecture in the case when BG are closed manifolds because if G is not trivial, then the cohomological dimension of G_Ψ is infinite and if G is trivial, G_Ψ satisfies the Baum-Connes conjecture.

4 The group homology of G_Ψ coming from a free group

In this section, we calculate the group homology of G_Ψ coming from a free group G. Let the generaters of G be $s_i (1 \leq i \leq m)$.

Proof of Theorem 1.5.

We will follow five steps.

Firstly we can decide the group homology of G_0, H_0 and K_0 by

$$\begin{aligned} H_n(G;\mathbb{Z}) &\cong 0\ (n \geq 2), \\ H_1(G;\mathbb{Z}) &= \langle s_i(1 \leq i \leq m) \rangle, \\ H_0(G;\mathbb{Z}) &\cong \mathbb{Z}. \end{aligned}$$

and Künneth formula. In fact

$$\begin{aligned} H_n(G_0 = H_0;\mathbb{Z}) &= \left\langle \begin{array}{l} s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)} \\ (1 \leq i_1, i_2, \ldots, i_n \leq m,\ k_1 < k_2 < \cdots < k_n) \end{array} \right\rangle (n \geq 1), \\ H_0(G_0 = H_0;\mathbb{Z}) &\cong \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H_n(K_0;\mathbb{Z}) &= \left\langle \begin{array}{l} s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)} (1 \leq i_1, i_2, \ldots, i_n \leq m, \\ k_1 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3) \end{array} \right\rangle (n \geq 1), \\ H_0(K_0;\mathbb{Z}) &\cong \mathbb{Z}. \end{aligned}$$

Secondly we will decide the group homology of G_1. $G_1 = G_0 \rtimes \mathbb{Z}$ is an HNN-extension of $G_0 = \bigoplus_{k \in Z} G^{(k)}$ by the isomorphism

$$\theta : G_0 \xrightarrow{\sim} G_0; s_i^{(k)} \mapsto s_i^{(k+1)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(G_0;\mathbb{Z}) \xrightarrow{\alpha_n} H_n(G_0;\mathbb{Z}) \to H_n(G_1;\mathbb{Z}) \to H_{n-1}(G_0;\mathbb{Z}) \to \cdots$$

where $\alpha_* := \theta_* - id_*$.

Claim 4.1. *α_n is injective for $n \geq 1$.*

Proof. Let us put $\mathbf{k} := (k_1, k_2, \ldots, k_n), \mathbf{1} := (1, 1, \ldots, 1), \mathbf{i} := (i_1, i_2, \ldots, i_n)$ and $s_{\mathbf{i}}^{\mathbf{k}} := s_{i_1}^{(k_1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$.

Now $\alpha_n(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+1}} - s_{\mathbf{i}}^{\mathbf{k}}$. If $\alpha_n(\sum \lambda_{\mathbf{k}}^{\mathbf{i}} s_{\mathbf{i}}^{\mathbf{k}}) = 0$, then $\sum(\lambda_{\mathbf{k-1}}^{\mathbf{i}} - \lambda_{\mathbf{k}}^{\mathbf{i}}) s_{\mathbf{i}}^{\mathbf{k}}) = 0$. Hence $\lambda_{\mathbf{k}}^{\mathbf{i}} = \lambda_{\mathbf{k\text{-}1}}^{\mathbf{i}}$. Because $H_n(G_0;\mathbb{Z})$ is finitely generated, $\lambda_{\mathbf{k}}^{\mathbf{i}} = 0$ ($\forall \mathbf{i}, \forall\ \mathbf{k}$). □

Because $\alpha_n(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+1}} - s_{\mathbf{i}}^{\mathbf{k}}$ and

$$H_n(G_1;\mathbb{Z}) \cong H_n(G_0;\mathbb{Z})/\alpha_n(H_n(G_0;\mathbb{Z}))$$

for $n \geq 2$,

$$\begin{array}{rcl} H_n(G_1;\mathbb{Z}) & \cong & \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}] \\ (1 \leq i_1, i_2, \ldots, i_n \leq m,\ 0 < k_2 < \cdots < k_n) \end{array} \right\rangle (n \geq 2), \\ H_1(G_1;\mathbb{Z}) & \cong & G_1/[G_1, G_1] = \langle s_i (1 \leq i \leq m), a \rangle, \\ H_0(G_1;\mathbb{Z}) & \cong & \mathbb{Z}, \end{array}$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]$ denotes the equivalence class of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(G_0;\mathbb{Z})/\alpha_n(H_n(G_0;\mathbb{Z}))$.

Thirdly we will decide the group homology of H_1. $H_1 := H_0 \rtimes \mathbb{Z}$ is an HNN-extension of $H_0 = \bigoplus_{l \in \mathbb{Z}} G^{(2l)} \oplus G^{(2l+1)}$ by the isomorphism

$$\theta' : H_0 \xrightarrow{\sim} H_0; s_i^{(k)} \mapsto s_i^{(k+2)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(H_0;\mathbb{Z}) \xrightarrow{\alpha_n'} H_n(H_0;\mathbb{Z}) \to H_n(H_1;\mathbb{Z}) \to H_{n-1}(H_0;\mathbb{Z}) \to \cdots$$

where $\alpha_*' := \theta_*' - id_*$. We have the following by the same argument as that in the proof of Claim 4.1.

Claim 4.2. *α_n' is injective for $n \geq 1$.*

Because $\alpha_n'(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+2}} - s_{\mathbf{i}}^{\mathbf{k}}$ and

$$H_n(H_1;\mathbb{Z}) \cong H_n(H_0;\mathbb{Z})/\alpha_n'(H_n(H_0;\mathbb{Z}))$$

for $n \geq 2$,

$$\begin{array}{rcl} H_n(H_1;\mathbb{Z}) & \cong & \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]' \\ (1 \leq i_1, i_2, \ldots, i_n \leq m,\ 0 < k_2 < \cdots < k_n) \\ [s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]' \\ (1 \leq i_1, i_2, \ldots, i_n \leq m,\ 1 < k_2 < \cdots < k_n) \end{array} \right\rangle (n \geq 2), \\ H_1(H_1;\mathbb{Z}) & \cong & H_1/[H_1, H_1] = \langle s_i^{(0)}, s_i^{(1)} (1 \leq i \leq m), b \rangle, \\ H_0(H_1;\mathbb{Z}) & \cong & \mathbb{Z}, \end{array}$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'$ and $[s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]'$ denote the equivalence classes of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ and $s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(H_0;\mathbb{Z})/\alpha_n'(H_n(H_0;\mathbb{Z}))$

respectively.

Fourthly we will decide the group homology of K_1. $K_1 := K_0 \rtimes \mathbb{Z}$ is an HNN-extension of $K_0 = \bigoplus_{l\in\mathbb{Z}} G^{(3l)} \oplus G^{(3l+1)}$ by the isomorphism

$$\theta'' : K_0 \xrightarrow{\sim} K_0; s_i^{(k)} \mapsto s_i^{(k+3)}.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(K_0;\mathbb{Z}) \xrightarrow{\alpha_n''} H_n(K_0;\mathbb{Z}) \to H_n(K_1;\mathbb{Z}) \to H_{n-1}(K_0;\mathbb{Z}) \to \cdots$$

where $\alpha_*'' := \theta_n'' - id_*$. We have the following by the same argument as that in the proof of Claim 4.1.

Claim 4.3. *α_n'' is injective for $n \geq 1$.*

Because $\alpha_n''(s_{\mathbf{i}}^{\mathbf{k}}) = s_{\mathbf{i}}^{\mathbf{k+3}} - s_{\mathbf{i}}^{\mathbf{k}}(k_1 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3)$ and

$$H_n(K_1;\mathbb{Z}) \cong H_n(K_0;\mathbb{Z})/\alpha_n''(H_n(K_0;\mathbb{Z}))$$

for $n \geq 2$,

$$\begin{aligned}
H_n(K_1;\mathbb{Z}) &\cong \left\langle \begin{array}{l} [s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''(1 \leq i_1, i_2, \ldots, i_n \leq m, \\ 0 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3) \\ [s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''(1 \leq i_1, i_2, \ldots, i_n \leq m, \\ 1 < k_2 < \cdots < k_n,\ k_j \equiv 0,\ 1 \mod 3) \end{array} \right\rangle (n \geq 2), \\
H_1(K_1;\mathbb{Z}) &\cong K_1/[K_1, K_1] = \langle s_i^{(0)}, s_i^{(1)}(1 \leq i \leq m), c\rangle, \\
H_0(K_1;\mathbb{Z}) &\cong \mathbb{Z},
\end{aligned}$$

where $[s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''$ and $[s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}]''$ denote the equivalence classes of $s_{i_1}^{(0)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ and $s_{i_1}^{(1)} \times s_{i_2}^{(k_2)} \times \cdots \times s_{i_n}^{(k_n)}$ in $H_n(K_0;\mathbb{Z})/\alpha_n''(H_n(K_0;\mathbb{Z}))$ respectively.

Finally we will calculate the group homology of G_2. G_Ψ is an HNN-extension of G_1 by the isomorphism

$$\phi : H_1 \xrightarrow{\sim} K_1; s_i^{(0)}, s_i^{(1)}, b \mapsto s_i^{(0)}, s_i^{(1)}, c.$$

Thus we can use a Mayer-Vietoris sequence

$$\cdots \to H_n(H_1;\mathbb{Z}) \xrightarrow{\beta_n} H_n(G_\Psi;\mathbb{Z}) \to H_n(G_\Psi;\mathbb{Z}) \to H_{n-1}(H_1;\mathbb{Z}) \to \cdots$$

where $\beta_* := \phi_* - i_*$. We use $\mathbf{l} := (0, l_2, \ldots, l_n), \mathbf{q} := (q_1, q_2, \ldots, q_n), (q_1, q_2, \ldots, q_n = 0,\ 1,\ q_1 < 2l_2 + q_2 < \ldots < 2l_n + q_n)$. Since $\beta_n([s_{\mathbf{i}}^{\mathbf{2l+q}}]') = [s_{\mathbf{i}}^{\mathbf{3l+q}}] - [s_{\mathbf{i}}^{\mathbf{2l+q}}]$, $\beta_n([s_{\mathbf{i}}^{\mathbf{2l}}]') =$

$\beta_n([s_{\mathbf{i}}^{\mathbf{2l+1}}]')$. Thus $\mathrm{Ker}\,\beta_n \supset \langle [s_{\mathbf{i}}^{\mathbf{2l+1}}]' - [s_{\mathbf{i}}^{\mathbf{2l}}]'(0 < 2l_2 < \ldots < 2l_n)\rangle$. Hence $\mathrm{Ker}\,\beta_n$ has infinite rank for $n \geq 2$. Thus $H_{n+1}(G_\Psi;\mathbb{Z})$ has infinite rank, too. Also since $\mathrm{Ker}\,\beta_1 = \langle s_i^{(0)}, s_i^{(1)}\rangle \cong \mathbb{Z}^{2m}$ and $H_2(G_1;\mathbb{Z})/\beta_2(H_2(H_1;\mathbb{Z})) \cong \langle [s_{i_1}^{(0)} \times s_{i_2}^{(1)}]\rangle \cong \mathbb{Z}^{m^2}$, $H_2(G_\Psi;\mathbb{Z}) \cong \mathbb{Z}^{2m+m^2}$. Hence

$$\begin{array}{lcl} H_n(G_\Psi;\mathbb{Z}) & has & infinite\ rank\ (\forall n \geq 3), \\ H_2(G_\Psi;\mathbb{Z}) & \cong & \mathbb{Z}^{2m+m^2}, \\ H_1(G_\Psi;\mathbb{Z}) & \cong & G_\Psi/[G_\Psi, G_\Psi] = \langle s_i(1 \leq i \leq m), t\rangle, \\ H_0(G_\Psi;\mathbb{Z}) & \cong & \mathbb{Z}. \end{array}$$

□

Let G_2 be G_Ψ. In this section, we proved that for $n = 0,1,2$, G_n coming from a free group of rank $m \geq 1$ is in $\mathcal{F}_n$ and the p-th group homology of G_n has infinite rank for any $p \geq n+1$. It is known when n is a non-negative integer, then $\mathcal{F}_n \supsetneq \mathcal{F}_{n+1}$ ([2]). Here we will formulate the following conjecture.

Conjecture 4.4.

When n is a non-negative integer, then there is $G \in \mathcal{F}_n$ of which the p-th group homology has infinite rank for any $p \geq n+1$.

The author does not know whether this is true or not except for the case $n = 0,1,2$.

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