The Elementary Transformation of vector bundles on regular schemes

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Abstract

We give a new definition of an elementary transformation of vector bundles (resp : reflexive sheaves) on regular schemes, by using Maximal Cohen-Macaulay sheaves (resp : torsion free sheaves) on their divisors. This definition is a natural extension of that given by Maruyama in [Mar] and has a connection with that given by Sumihiro in [Su-2] and [Su-3]. On nonsingular quasi-projective varieties over an algebraically closed field, we can construct, up to tensoring line bundles, all vector bundles and reflexive sheaves from trivial bundles by this new elementary transformation. As an application, we give a sufficient condition for a coherent sheaf of rank one on a hypersurface in a projective space to be locally free. As an example of this construction, we can show the explicit data to construct the Tango bundle, which is the only known indecomposable rank two bundle over $\mathbf{P}_k^5$ $(\mathrm{ch}(k) = 2)$.

0 Introduction

Originally, an elementary transformation is the theory on ruled surfaces, which enables one to construct a new ruled surface from given one. In [Mar], Maruyama generalized this method to apply to the construction theory of vector bundles. By using his idea and theory, we can construct a lot of interesting vector bundles on schemes, especially those on low dimensional projective varieties. On the other hand in [Su-2] and [Su-3], Sumihiro gave an another definition of an elementary transformation of vector bundles on schemes, which is related closely to the geometric characterization of the original elementary transformation. Let us review them.

The definition of an elementary transformation given by Maruyama is very useful to construct vector bundles and has a lot of applications and examples. However in higher dimensional cases, there is a disadvantage that not all vector bundles can be constructed from trivial bundles by this method.

The definition given by Sumihiro can be applied to the vector bundle construction on higher dimensional cases. i.e., by using this theory, we can construct, up to tensoring line bundles, all vector bundles on any dimensional nonsingular quasi-projective varieties over an algebraically closed field from trivial bundles. However, this elementary transformation needs a lot of geometric data and is hard to make examples. Note that the explicit relation between these two was not clear.

In this article, we give a new definition of an elementary transformation of vector bundles on regular schemes by using Maximal Cohen-Macaulay sheaves on thier divisors. This is a natural extension of Maruyama's definition and in the special case, it can be interpretated to Sumihiro's definition. i.e., by this theory, we can make it clear the relation between the two definitions of an elementary transformation. From this viewpoint, we can obtain several results not only on vector bundles but also on reflexive sheaves by using torsion free sheaves on divisors. Consequently, we can construct, up to tensoring line bundles, all the vector bundles and reflexive sheaves from trivial bundles on nonsingular quasi-projective varieties over an algebraically closed field of any characteristic by this method. This is one of the main results in this article and described in Theorem 1.2. As an application of this elementary transformation, we will consider the sufficient condition when a given coherent sheaf is locally free. The motivation to this problem is Horrocks' famous criterion ([OSS], Theorem 2.3.1), which tells us when a given vector bundle on $\mathbf{P}_k^n$ splits into the sum of line bundles. According to this criterion, the given vector bundle E splits if and only if $H^i(\mathbf{P}_k^n, E(k)) = 0$ for all integers $k \in \mathbb{Z}$ and $i = 1, \cdots, n-1$. Then it is natural to consider whether the same condition is sufficient for a coherent sheaf on some varieties to split into the sum of line bundles or be locally free. By using our elementary transformation, we can show that on a hypersurface Z in $\mathbf{P}_k^n$ $(n \geq 5)$ whose singular locus is of codimension more than or equal to 5, a rank one, Maximal Cohen-Macaulay sheaf F generated by two global sections is locally free if and only if it satisfies the Horrocks' condition. This is stated in Proposition 4.3. As an actual construction, we shall show explicit data to construct the Tango bundle, which is the only known indecomposable 2-bundle on $\mathbf{P}_k^5$ where k is an algebraically closed field of characteristic two. This is given in section five.

Contents of this article are as follows.

In section one, we give a new definition of an elementary transformation by introducing the concept of ET-data (Z, F), and investigate its basic properties. The main result is Theorem 1.2 as the above. The relation between a definition given here and Maruyama's one is described in Remark 1.2 and one between ours and Sumihiro's one in Proposition 1.5.

In section two, we relate the new definition to the result in [Su-3], which gives a geometric interpretation to our definition (Theorem 2.1). Explicitly, we can construct the data for our elementary transformation from a normal (or integral) divisor and Weil divisors on it which satisfy certain conditions. In other words, we can characterize Maximal Cohen-Macaulay modules on a divisor by Weil divisors on it. Moreover using our result, we can give the answer to the problem which was raised in [Su-3] (Corollary 2.9).

In section three, we consider the condition when the elementary transformation commutes with restrictions to a hyperplane or a closed subscheme (Proposition 3.1, 3.2). These results will play an important role in the next section.

In section four, we apply the new elementary transformation to investigate the freeness of a given coherent sheaf. We give a sufficient condition for a sheaf on a hypersurface Z in $\mathbf{P}_k^n$, which is of rank one, reflexive and Maximal Cohen-Macauley, genereated by two global sections to be locally free. It is the same as Horrocks' splitting criterion but it demands for the hypersurface Z to have its codimension of singular locus more than or equal to five. If not, i.e., when the codimension of singular locus is less than or equal to four, the given condition is not sufficient. That is described in Proposition 4.3 and the counter example on a hypersurface with higher dimensional singular locus is also described in this section.

In section five, we show the explicit data to construct the Tango bundle, which is the only known indecomposable rank two bundle on $\mathbf{P}_k^5$ ($\mathrm{ch}(k) = 2$). This is a new way to construct this bundle.

Notation. In this article, the term vartiey means an integral algebraic scheme over a field. We use the terms vector bundle and locally free sheaf interchangeably. We often denote a locally free sheaf of rank r by r-bundle and a reflexive sheaf of rank r by r-reflexive sheaf. We often consider the ideal sheaf of a Weil divisor W on Z, which is a divisor of a variety X. Then the ideal sheaf of W considered as the closed subscheme of X (resp : Z) is denoted by $I(W)$ (resp : $I_Z(W)$). $Gr(n, k)$ represents a Grassmannian which parametrizes k-dimensional linear subvarieties in $\mathbf{P}_k^n$. By $\mathrm{Ass}_X(F)$ for a coherent sheaf F on a noetherian scheme X, we denote associated points of F as an $\mathcal{O}_X$-module. We denote the sets of an $r \times r$ matrices with entries in a ring A by $M(r, A)$.

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1 A new definition of an elementary transformation and its properties

At first, let us give a new definition of an elementary transformation of vector bundles on regular schemes as follows. Here, we generalize it to apply to reflexive sheaves.

Definition 1.1
Let X be a regular scheme, E be an $r\,(\geq 2)$-bundle on X, and m be an integer such that $1 \leq m \leq r-1$. We say that the triple (Z, F, φ) is m-elementary-transformation-data for E (m-ET-data for E, in short) if Z is an effective reduced divisor on X, F is an $\mathcal{O}_Z$-module of rank $r-m$, which is Maximal Cohen-Macaulay (MCM, in short. For its definition, see the remark below) and there is a surjection $\varphi : E \to F$ as $\mathcal{O}_X$-modules.

Moreover, let X and m be the same as the above, E be an r-reflexive sheaf on X. Then we say that the triple (Z, F, φ) is m-weak-elementary-transformation-data for E (m-w-ET-data for E, in short) if Z is an effective reduced divisor on X, F is a torsion free $\mathcal{O}_Z$-module of rank $r-m$, and there is a surjection $\varphi : E \to F$ as $\mathcal{O}_X$-modules.

Note that we usually denote ET-data (or w-ET-data) (Z, F, φ) by (Z, F) if there is no confusing.

Definition 1.2
With the above notation, when (Z, F) is m-ET-data (resp : w-ET-data) for E, we say that $\ker(\varphi) =: \mathrm{elem}_F(E)$ is an elementary transformation of E by (m-)ET-data (Z, F) (resp : (m-)w-ET-data (Z, F)).

Remark 1.1
Let X be a Noetherian scheme and F be an $\mathcal{O}_X$-module. Then we say that F is a Maximal Cohen-Macaulay $\mathcal{O}_X$-module if for all $x \in X$, $\mathrm{depth}_{\mathcal{O}_{x,X}}(F_x) = \dim \mathcal{O}_{x,X}$.

In Maruyama's definition ([Mar]), the above F is not MCM but a vector bundle on Z. As we saw in the introduction, his definition and results are not fully applied to the higher dimensional cases (for details, see Remark 1.2). Here, we simply extend his definition (it is easy to see that vector bundles on a divisor of a regular scheme are MCM) to get a stronger result on higher dimensional cases and for reflexive sheaves.

The following lemma is easy to see from the discussion on the depth, using Auslander-Buchsbaum formula and results in [H3].

Lemma 1.1
If (Z, F) is ET-data for an r-bundle E, then $\mathrm{elem}_F(E)$ is a vector bundle on X of rank r. If (Z, F) is w-ET-data for an r-reflexive sheaf E, then $\mathrm{elem}_F(E)$ is an r-reflexive sheaf on X.

From this lemma, we can see that when given a data (Z, F), we can construct a new vector bundle or a new reflexive sheaf $\mathrm{elem}_F(E)$ from the given sheaf E. Now, let us show the main result of this article obtained by the extended definition of an elementary transformation.

Theorem 1.2
Let X be a nonsingular quasi-projective variety over an algebraically closed field k, $\mathcal{O}_X(1)$ be an ample line bundle on X, and E be an $r(>1)$-bundle on X (resp : r-reflexive sheaf on X). Then there is 1-ET-data (Z, F) (resp : 1-w-ET-data) for $\mathcal{O}_X^r$ such that Z is normal and $\mathrm{elem}_F(\mathcal{O}_X^r) \simeq E \otimes L$ for some line bundle $L \in \mathrm{Pic}(X)$. Moreover when $\dim X \geq 2$, we can take Z as an integral divisor.

Proof. At first, we prove this theorem when E is a vector bundle. Tensoring $\mathcal{O}_X(1)$ sufficiently many times, we may assume that E is very ample. Then there are global sections $s_1, \cdots, s_r \in H^0(X, E)$ such that if we denote the divisor in $\mathbf{P}(E)$ defined by the section s_i by D_i $(i = 1, \cdots, r)$, then the intersection $D_1 \cap \cdots \cap D_r$ is a smooth subscheme of pure codimension r in $\mathbf{P}(E)$. Then if we put $Z := Z(s_1 \wedge \cdots \wedge s_r)$, $W_i = Z(s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_r)$ $(i = 1, \cdots, r)$ and $U = Z \setminus B$ (where $B = \cap_{i=1}^r W_i$), from the calculation of its Jacobian we can see that $Z \cap U$ is a smooth divisor of U and $\mathrm{Sing}(Z) = B$. If B contains a point whose codimension is 1, then since $\dim \pi^{-1}(x) \geq 1$ for all $x \in B$ ($\pi : \mathbf{P}(E) \to X$ is a canonical projection), it holds that $\dim \pi^{-1}(B) \geq \dim Z$, this is a contradiction. Hence all the points which belong to B have codimensions more than one. This implies that Z is regular in codimension one. Moreover, since Z is Cohen-Macaulay, it satisfies Serre's criterion for normality. Hence Z is normal and if $\dim X \geq 2$, it follows that $Z \cap U \neq \phi$ and it is a nonempty smooth divisor in $U \neq \phi$. So we can also see that Z is integral in that case.

Now, let us see the exact sequence $\mathcal{O}_X^r \stackrel{(s_1, \cdots, s_r)}{\to} E \to F \to 0$ which is induced by the sections $s_1, \cdots, s_r$. If we put $e_1, \cdots, e_r$ as free basis of $\mathcal{O}_{x,X}^r$, $X_1, \cdots, X_r$ as free basis of E_x $(x \in X)$ and if we put $s_i = (s_{i1}, \cdots, s_{ir})$ at x, then the morphism $\mathcal{O}_X^r \to E$ can be written as the matrix ${}^tS = {}^t(s_{ij})_{i,j=1}^r$. If we denote the image of e_i by Y_i $(i = 1, \cdots, r)$, it can be written as $Y_i = s_{i1}X_1 + \cdots + s_{ir}X_r$. So locally $F = \oplus AX_i / \sum AY_i$. Then this is 0 if and only if there exists an $r \times r$ matrix T such that $TS = ST = I_r$. If we consider the adjoint matrix of S, it is easy to see that such T exists if and only if $s = \det S$

is a unit at that point. This is equivalent to $x \notin Z$. So $\mathrm{Supp}(F) = Z$ and easily we can see that F is an $\mathcal{O}_Z$-module. Next, we show that F is an MCM sheaf on Z of rank 1. Noting that s is a local equation of $Z = Z(s_1 \wedge \cdots \wedge s_r)$, this follows from the following lemma.

Lemma 1.3
Let A be a Cohen-Macaulay Noetherian local ring, $r(> 1)$ be an integer, and $S, S' \in M(r, A)$ be $r \times r$ matrices such that $SS' = S'S = sI_r$ for some nonzero-divisor $s \in A \setminus A^\times$. We denote the image of $a \in A$ in $A/sA =: \overline{A}$ by $\overline{a}$ and we also assume that $\operatorname{rank} \overline{S} \neq 0$ and $\operatorname{rank} \overline{S'} \neq 0$. Then $M := (A^r)/Im({}^tS)$ is a Maximal Cohen-Macaulay A/sA-module.

Proof of lemma. Put $\dim A = n \geq 1$. Clearly, A and $\overline{A}$ are Cohen-Macaulay local rings. Hence there is a regular sequence $(s = z_1, z_2 \cdots, z_n)$ for A. We shall prove that $(\overline{z_2}, \cdots, \overline{z_n})$ is a regular sequence for M by induction on its length. Assume that $(\overline{z_2}, \cdots, \overline{z_{l-1}})$ is a regular sequence for M (where $2 \leq l \leq n+1$). Take $\sum_{i=1}^r \overline{a_i} \overline{X_i} \in M$ (where $\{X_i\}_{i=1}^r$ are the free basis of A^r). Assume that $\overline{z_l} \sum \overline{a_i} \overline{X_i} = 0$ in $M/(\overline{z_2}, \cdots, \overline{z_{l-1}})M$. Then there are elements $b_i, c_{ki} \in A$ such that $z_l \sum_{i=1}^r a_i X_i = \sum_{i=1}^r b_i Y_i + \sum_{i=1}^r \sum_{k=2}^{l-1} c_{ki} z_k X_i$ (where $S = (s_{ij})$ and $Y_i = \sum_{j=1}^r s_{ij} X_j$). Then for all $j = 1, \cdots, n$, we have

$$z_l a_j = \sum_{i=1}^r b_i s_{ij} + \sum_{k=2}^{l-1} c_{kj} z_k. \tag{1}$$

Let us put $S' = (d_{ij})$. Multiplying d_{jp} to the equation (1) above and taking a sum on index j, we have

$$z_l \sum_{j=1}^r a_j d_{jp} = s b_p + \sum_{j=1}^r \sum_{k=2}^{l-1} z_k c_{kj} d_{jp} \tag{2}$$

for all $p = 1, \cdots, n$. Since $(z_1 = s, z_2, \cdots, z_l)$ is a regular sequence for A, there are elements $e_j, f_{kj} \in A$ $(j = 1, \cdots, r.\ k = 2, \cdots, l-1)$ such that

$$\sum_{i=1}^r a_i d_{ij} = s e_j + \sum_{k=2}^{l-1} z_k f_{kj} \tag{3}$$

for all j. On the other hand, we have $s \sum_{i=1}^r a_i X_i = \sum_{i,j=1}^r a_i d_{ij} Y_j$ since $SS' = S'S = sI_r$. Hence it holds that

$$s \sum_{i=1}^r a_i X_i = s \sum_{i=1}^r e_i Y_i + \sum_{i=1}^r \sum_{k=2}^{l-1} z_k f_{ki} Y_i. \tag{4}$$

Since A is local and Noetherian, the sequence $(z_2, \cdots, z_{l-1}, s)$ is also a regular sequence for A. Hence we have $\sum_{i=1}^r a_i X_i \in \sum_{i=1}^r AY_i + \sum_{k=2}^{l-1} \sum_{i=1}^r A(z_k X_i)$ and the lemma is proved.

Proof of theorem, continued.

At last, let us show $\mathrm{elem}_F(E) \simeq \mathcal{O}_X^r$. Let $\{U_i\}_{i \in I}$ be an affine open covering of X on which E (resp : $\mathrm{elem}_F(E)$) has G_{ij} (resp : H_{ij}) as a transition matrix on $U_i \cap U_j$. Then it is easy to see that ${}^tS_i = G_{ij}\, {}^tS_j$ by definitions of G_{ij} and $S_i = (s_1, \cdots, s_r)|_{U_i}$. On the other hand from the exact sequence

$$0 \to \mathrm{elem}_F(E) \to E \to F \to 0,$$

we see that ${}^tS_i H_{ij} = G_{ij}\, {}^tS_j$ for all i, j. This implies that $H_{ij} = I_r$. Hence we can see that $\mathrm{elem}_F(E) \simeq \mathcal{O}_X^r$. Thus we can get the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E(-Z) \arrow[r, equal] \arrow[d] & E(-Z) \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X^r \arrow[r, "{(s_1, \cdots, s_r)}"] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & E|_Z \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that $\mathrm{rank}_Z(S_i) = r - 1$ and this implies $\mathrm{rank}_Z F = 1$ and $\mathrm{rank}_Z F' = r - 1$. Then seeing the first column of this diagram, we can find the 1-ET-data (Z, F') for $\mathcal{O}_X^r$ such that $\mathrm{elem}_{F'}(\mathcal{O}_X^r) \simeq E(-Z)$, which is what we want in case that E is a vector bundle.

Next, we must prove the statement of the theorem when E is reflexive. From the result of [H3], there is a non empty open set $U \subset X$ such that $\mathrm{codim}_X(X \setminus U) \geq 3$ and $E|_U$ is locally free. Since U is nonsingular and quasi-projective, we can apply the same discussion of the vector bundle case to the vector bundle $E|_U$ on U. Note that by the above fact, we only have to consider the case when $\dim X \geq 3$ (Otherwise E is automatically locally free). Since all the points belonging to $Z \setminus U$ are of codimension ≥ 2 in Z, we can also use Serre's criterion and from the fact that $Z \cap U$ is integral and normal, we can see that Z is also integral and normal. Now, the statement follows immediately from the fact that if we put $j : U \to X$ as an open immersion, then $j_*(E|_U) \simeq E$ and that the direct image of MCM sheaves on $Z \cap U$ which have a surjection from a vector bundle on U is a torsion free $\mathcal{O}_Z$-module on Z. q.e.d.

Remark 1.2
Let us show the reason why we extended the data F from line bundles (defined by Maruyama) to MCM sheaves here. i.e., if F is a line bundle, we cannot apply the above discussion on $\mathbf{P}_k^n$ $(n \geq 4)$, hence for example, we cannot construct the Horrocks-Mumford bundle on $\mathbf{P}_k^4$ (Horrocks-Mumford bundle is an indecomposable 2-bundle on $\mathbf{P}_{\mathbb{C}}^4$. See [HM] for the construction of this bundle). This is shown by Sumihiro in [Su-2], and let us review here to see the difference of two definitions.

At first, we must remember the Grothendieck-Lefschetz theorem. That implies if Z is an effective divisor on $\mathbf{P}_k^n$ (where $n \geq 4$ and k is an algebraically closed field of characteristic zero), then it holds that $\mathrm{Pic}(\mathbf{P}_k^n) \simeq \mathrm{Pic}(Z) \simeq \mathbb{Z} \cdot \mathcal{O}_Z(1)$. Now, let us take arbitrary 1-ET-data (Z, F) for $\mathcal{O}_{\mathbf{P}_k^n}^2$ $(n \geq 4)$ such that F is locally free (i.e., F is a line bundle). Then we can see that the elementary transformation $\mathrm{elem}_F(\mathcal{O}_{\mathbf{P}_k^n}^2) =: E$ by these data always splits. In fact, Grothendieck-Lefschetz theorem implies that $\mathrm{Pic}(\mathbf{P}_k^n) \simeq \mathrm{Pic}(Z)$ in this case. Hence we can write $F \simeq \mathcal{O}_Z(k)$ for some integer $k \in \mathbb{Z}$. Considering the long exact sequence of the following exact sequence induced by our elementary transformation

$$0 \to E \to \mathcal{O}_{\mathbf{P}_k^n}^2 \to F \to 0,$$

we can see that $H^i(\mathbf{P}_k^n, E(l)) = 0$ $(\forall l \in \mathbb{Z},\ i = 1, 2, \cdots, n-1)$. Hence by Horrocks' splitting criterion (for example, see Theorem 2.3.1 in [OSS]), we can conclude that this E splits into the sum of line bundles. So if we want to construct an indecomposable 2-bundle on the higher dimensional projective space, we have to use MCM sheaves on divisors.

Next, let us consider the geometric characterization of this definition. i.e., we shall show the relation between projective bundles of a given vector bundle and its elementary transformation by using blowing up and blowing down.

Proposition 1.4
Let X be a nonsingular quasi-projective variety over an algebraically closed field, E be a locally free sheaf of rank r (> 1) on X, and (Z, F) be m-w-ET-data for E $(1 \leq m \leq r-1)$. Then for $\mathrm{elem}_F(E) = E'$, we have the following commutative diagram of exact sequences.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E(-Z) \arrow[r, equal] \arrow[d, "a'"'] & E(-Z) \arrow[d] & & \\
0 \arrow[r] & E' \arrow[r, "a"] \arrow[d] & E \arrow[r, "b"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & E|_Z \arrow[r, "\gamma"] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Then (Z, F') is $(r-m)$-w-ET-data for E'. Let us put $\mathbf{P}(F) =: Y \subset \mathbf{P}(E)$, $\mathbf{P}(F') =: Y' \subset \mathbf{P}(E')$, and $u : B \to \mathbf{P}(E)$ (resp : $u' : B' \to \mathbf{P}(E')$) be a blowing up of $\mathbf{P}(E)$ (resp : $\mathbf{P}(E')$) with the center Y (resp : Y'). Then there is an isomorphim $\varphi : B' \to B$ which makes the following diagram commutative.

$$\begin{tikzcd}
B' \arrow[rr, "\varphi"] \arrow[d, "u'"] & & B \arrow[d, "u"] \\
\mathbf{P}(E') \arrow[dr, "\pi'"] & & \mathbf{P}(E) \arrow[dl, "\pi"'] \\
 & X &
\end{tikzcd}$$

Proof. The proof is almost parallel to that in [Su-2], to which the reader should refer. When (Z, F) is ET-data for E, then this is just the special case of Theorem 1.5 in [Su-2], because we can consider the statement in terms of (2) in Proposition 1.5 we will show later.

Next, let us consider when (Z, F) is w-ET-data. We begin with the affine case, i.e., on $\mathrm{Spec}(A) = U \subset X$. Then E is the sheafification of the free module $\oplus_{i=1}^{r} AX_i$, (where $\{X_1, \ldots, X_r\}$ is the system of coordinates) and $E' := \mathrm{elem}_F(E)$ is the sheafification of the reflexive A-module M. Now let us fix the prime ideal $P \in \mathrm{Spec}(A)$ with $\mathrm{ht}(P) = 1$. Since the codimension of $\mathrm{Sing}(E')$ in X is more than or equal to three, M_P is a free module. Let $\{Y_1, \ldots, Y_r\}$ be the coordinate system of M_P such that $Y_i \in M$ for $1 \le i \le r$. Let S_P be the matrix such that

$$^t(Y_1, \ldots, Y_r) = S_P\ ^t(X_1, \ldots, X_r).$$

By the same way as above, there exists the matrix $S_Q \in M(r, A_Q)$ for each $Q \in \mathrm{Spec}(A)$ with $\mathrm{ht}(Q) = 1$ such that

$$^t(Y_1, \dots, Y_r) = S_Q\ ^t(X_1, \dots, X_r).$$

Since A is rexlexive, it holds that $S := S_P \in M(r, A)$ for all P with height one. By the same way, we may assume that $\det(S) = s$, where s is the defining equation of the divisor Z on U. Then we can find the elements $F_1, \dots, F_v$ such that $\{Y_1, \cdots, Y_r, F_1, \cdots, F_v\}$ is a generator of M over A. By the choice of $\{Y_1, \dots, Y_r\}$, each F_j is linearly dependent over A. So there are elements $\alpha_i, \beta_{ij} \in A$ $(i = 1, \cdots, v,\ j = 1, \cdots, r)$ such that

$$\alpha_j F_j = \sum_{i=1}^{r} \beta_{ji} Y_i \ (j = 1, \cdots, v).$$

Now, recall the exact sequence of given data. It is as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \oplus_{i=1}^r AX_i \arrow[r, equal] \arrow[d, "{}^tS'"'] & \oplus_{i=1}^r AX_i \arrow[d, "s"'] & & \\
0 \arrow[r] & M \arrow[r, "{}^tS"] \arrow[d] & \oplus_{i=1}^r AX_i \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & \oplus_{i=1}^r (A/sA)X_i \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $S = (s_{ij})$ is defined above and $S' = (s'_{ij})$ is its adjoint matrix. By the above consideration, we can write the image of F_j by $\ ^tS$ as $F_j = (1/\alpha_j)\sum_{i,k} \beta_{ji} s_{ik} X_k$. Now, we can write the blowing up B of $\mathbf{P}(E)$ with center Y as

$$\begin{aligned}
B &= \mathrm{Proj}\, A[X_1, \cdots, X_r][Y_1, \cdots, Y_r, F_1, \cdots, F_v] \\
&= \cup_{i=1,\cdots,r,\ j=1,\cdots,r} \mathrm{Spec}\, A[x_{i1}, \cdots, x_{ir}][\frac{y_{i1}}{y_{ij}}, \cdots, \frac{y_{ir}}{y_{ij}}, \frac{f_{i1}}{y_{ij}}, \cdots, \frac{f_{iv}}{y_{ij}}] \\
&\cup (\cup_{i=1,\cdots,r,\ j=1,\cdots,v} \mathrm{Spec}\, A[x_{i1}, \cdots, x_{ir}][\frac{y_{i1}}{f_{ij}}, \cdots, \frac{y_{ir}}{f_{ij}}, \frac{f_{i1}}{f_{ij}}, \cdots, \frac{f_{iv}}{f_{ij}}]),
\end{aligned}$$

where $x_{ij} = \dfrac{X_j}{X_i}, y_{ij} = \dfrac{Y_j}{X_i}$ and $f_{ij} = \dfrac{F_j}{X_i}$. Next we consider the blowing up B' of $\mathbf{P}(E')$ with center Y'. Let $\{X_1{}', \dots, X_r{}', F_1{}', \dots, F_v{}'\}$ be the generators of

M and $\{{Y_1}', \ldots, {Y_r}'\}$ be the coordinate system of $E(-Z)$ as we saw above. Then we can see that

$$\begin{aligned} B' &= \operatorname{Proj} A[X_1', \cdots, X_r', F_1', \cdots, F_v'][Y_1', \cdots, Y_r'] \\ &= \cup_{i,j=1,\cdots,r} \operatorname{Spec} A[x_{i1}', \cdots, x_{ir}', f_{i1}', \cdots, f_{iv}'][\frac{y_{i1}'}{y_{ij}'}, \cdots, \frac{y_{ir}'}{y_{ij}'}] \\ &\cup \quad (\cup_{i=1,\cdots,v,\ j=1,\cdots,r} \operatorname{Spec} A[\frac{X_1'}{F_i'}, \cdots, \frac{X_r'}{F_i'}, f_{i1}', \cdots, f_{iv}'][\frac{y_{i1}'}{y_{ij}'}, \cdots, \frac{y_{ir}'}{y_{ij}'}], \end{aligned}$$

where $x_{ij}' = \dfrac{X_j'}{X_i'}$, $y_{ij}' = \dfrac{Y_j'}{X_i'}$ and $f_{ij}' = \dfrac{F_j'}{X_i'}$. Let us put

$$A_{ij} := A[x_{i1}, \cdots, x_{ir}][\frac{y_{i1}}{y_{ij}}, \cdots, \frac{y_{ir}}{y_{ij}}, \frac{f_{i1}}{y_{ij}}, \cdots, \frac{f_{iv}}{y_{ij}}],$$

$$B_{ij} := A[x_{i1}, \cdots, x_{ir}][\frac{y_{i1}}{f_{ij}}, \cdots, \frac{y_{ir}}{f_{ij}}, \frac{f_{i1}}{f_{ij}}, \cdots, \frac{f_{iv}}{f_{ij}}],$$

$$A_{ij}' = A[x_{i1}', \cdots, x_{ir}', f_{i1}', \cdots, f_{iv}'][\frac{y_{i1}'}{y_{ij}'}, \cdots, \frac{y_{ir}'}{y_{ij}'}],$$

$$B_{ij}' := A[\frac{X_1'}{F_i'}, \cdots, \frac{X_r'}{F_i'}, f_{i1}', \cdots, f_{iv}'][\frac{y_{i1}'}{y_{ij}'}, \cdots, \frac{y_{ir}'}{y_{ij}'}],$$

and put

$$\begin{aligned} \operatorname{Spec} A_{ij} &= U_{ij}, \\ \operatorname{Spec} B_{ij} &= V_{ij}, \\ \operatorname{Spec} {A_{ij}}' &= {U_{ij}}', \\ \operatorname{Spec} {B_{ij}}' &= {V_{ij}}'. \end{aligned}$$

Let us define the field homomorphism (as A-algebras) $\varphi : Q(A[X_1, \cdots, X_r]) \to Q(A[X_1', \cdots, X_r'])$ by sending X_i to Y_i'. Since $SS' = S'S = sI_r$, this is an isomorphism. Moreover, it is easy to see that ring homomorphisms $\varphi_{ij} : A_{ij} \to A_{ji}'$ and $\eta_{ij} : B_{ij} \to B_{ji}'$, which are canonically induced from the field isomorphism φ, are ring isomorphisms and induce isomorphisms $U_{ij} \simeq U_{ji}'$ and $V_{ij} \simeq V_{ji}'$ for all i, j. Hence in the affine case, we can see that $B \simeq B'$ as desired.

Next, let us check the patch of two affine open sets. Take two affine open sets $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} A'$. On U, let us assume that Z is defined by $s \in A$, the homogeneous coordinates of $\mathbf{P}(E)$ is denoted by $\{X_1, \cdots, X_r\}$, $Y \subset \mathbf{P}(E)$ is defined by $Y_1 = Y_2 = \cdots = Y_r = F_1 =$

$\cdots = F_v = 0$, where $Y_i = \sum_{j=1}^r s_{ij}X_j$ and F_j is the same as the affine case. Simillarly on V, let us assume that Z is defined by $t \in A'$, the homogeneous coordinates of $\mathbf{P}(E)$ is denoted by $Z_1, \cdots, Z_r$, $Y \subset \mathbf{P}(E)$ is defined by $W_1 = W_2 = \cdots = W_r = G_1 = \cdots = G_u = 0$, where $W_i = \sum_{j=1}^r t_{ij}Z_j$ and $\{G_j\}$ corresponds to $\{F_j\}$ when considered on U. We put $S = (s_{ij})$ and $T = (t_{ij})$. Now we can choose a transition matrix $C \in M(r, H^0(U \cap V, \mathcal{O}_X))$ such that ${}^t(X_1, \cdots, X_r) = {}^tC\ {}^t(Z_1, \cdots, Z_r)$. Then there exists a matrix $H \in M(r, H^0(U \cap V, \mathcal{O}_X))$ such that $S\ {}^tC = HT$. Let us denote the other coordinates of $\mathbf{P}(E)$ or the equations of Y', as the same as the affine case, by X_i', $Y_i' = \sum s_{ij}'X_j'$, Z_i', $W_i' = \sum t_{ij}'Z_j'$, and put ${}^tC = (c_{ij})$, $H = (h_{ij})$, $S' = (s_{ij}')$ and $T' = (t_{ij}')$. Then we can see that since $S'H = (s/t)\ {}^tCT'$, it holds that $\sum_k s_{mk}'h_{kl} = (s/t)\sum_k c_{mk}t_{kl}'$. Then it holds that for $\varphi_U : Q(A[X_1, \cdots, X_r]) \to Q(A[X_1', \cdots, X_r'])$,

$$\begin{aligned}
\varphi_U(x_{im}) &= \frac{y_{jm}'}{y_{ji}'} = \sum_k s_{mk}'X_k' / \sum_k s_{ik}X_k' \\
&= \sum_k s_{mk}'(\sum_l h_{kl}Z_l') / \sum_k s_{ik}'(\sum_l h_{kl}Z_l') \\
&= \sum_{k,l} c_{mk}t_{kl}'Z_l' / \sum_{k,l} c_{ik}t_{kl}'Z_l' \\
&= \sum_k c_{mk}W_k' / \sum_k c_{ik}W_k'
\end{aligned}$$

On the other hand on V, for $\varphi_V : Q(A'[Z_1, \cdots, Z_r]) \to Q(A'[Z_1', \cdots, Z_r'])$ we have

$$\begin{aligned}
\varphi_V(x_{im}) &= \varphi_V(\sum_k c_{mk}Z_k / \sum_k c_{ik}Z_k) \\
&= \sum_k c_{mk}W_k' / \sum_k c_{ik}W_k' \qquad (5)
\end{aligned}$$

and so they are equal. Since the morphism from B to B' is completely determined locally by the free part of E, we can conclude that $\varphi_U|_{U\cap V} = \varphi_V|_{U\cap V}$ and hence the proposition is proved. q.e.d.

By the definition, we have a relation between algebraic data and geometric one as follows. The proof is trivial by the results in [Su-2].

Proposition 1.5
Let X be a regular scheme, E be a vector bundle of rank $r > 1$, and m be an integer such that $1 \leq m \leq r - 1$. Then there is a one to one correspondence between the following two sets.

(1) $(r-m)$-ET-data (Z,F) for E.
(2) Sumihiro's ET-data (Z,Y) for E. i.e., the pair (Z,Y) where Z is a reduced divisor on X and $Y \subset \mathbf{P}(E)$ is a closed subscheme satisfying $\pi(Y) = Z$ by the canonical projection $\pi : \mathbf{P}(E) \to X$. Moreover they satisfy for every $x \in Z$, there is an affine open neighborhood $U = \operatorname{Spec} A \subset X$ of x such that

1) $E|_U \simeq \oplus^r \mathcal{O}_U$.

2) Let $s \in A$ be a local equation of Z on U. Then on $\pi^{-1}(U) \simeq U \times \mathbf{P}^{r-1}$, Y is defined by the following linear equations

$$s_{i1}X_1 + \cdots + s_{ir}X_r = 0 \ (i = 1, 2, \cdots, r) \tag{6}$$

where $s_{ij} \in A$ and $X_1, \cdots, X_r$ are homogeneous coordinates of $\mathbf{P}^{r-1}$.

3) Put $S = (s_{ij})_{i,j=1}^r$. Then the rank *of S at every generic point of Z is m. Moreover, there exists a matrix $S' \in M(r, A)$ such that $SS' = S'S = sI_r$ and the* rank *of S' at every generic point of Z is $r-m$.*

Proof. From (1) to (2), it is sufficient to put $\mathbf{P}(F) =: Y \subset \mathbf{P}(E)$ and check properties by using Theorem 1.2. From (2) to (1), it is easy to see by using the results above. q.e.d.

This proposition tells us that our new definition shows the relation between Maruyama's one and Sumihiro's one. Next, let us investigate some properties of an elementary transformation.

Lemma 1.6
Let X and X' be regular schemes, $f : X' \to X$ be a morphism, E be an r-reflexive sheaf on X, and (Z,F) be m-w-ET-data for E. If f is flat, then (f^*Z, f^*F) is also m-w-ET-data for f^*E and it holds that $f^*(\mathrm{elem}_F(E)) \simeq \mathrm{elem}_{f^*F}(f^*E)$.

Lemma 1.7
Let X be a regular scheme, E be an r-reflexive sheaf on X, and (Z,F) be m-w-ET-data for E. If L is a line bundle on X, then it holds that $\mathrm{elem}_F(E) \otimes L \simeq \mathrm{elem}_{F\otimes L}(E \otimes L)$.

Lemma 1.8
Let X be a regular scheme, E_1 (resp : E_2) be an r_1-reflexive sheaf on X (resp : r_2-reflexive sheaf on X), and (Z,F_1) (resp : (Z,F_2)) be m_1-w-ET-data for E_1 (resp : m_2-w-ET-data for E_2). Then
(1) $\mathrm{elem}_{F_1}(E_1) \oplus \mathrm{elem}_{F_2}(E_2) \simeq \mathrm{elem}_{F_1\oplus F_2}(E_1 \oplus E_2)$.
(2) Assume that $m_1 = m_2 = m$ and $r_1 = r_2 = r$. If there exists an isomorphisms $\varphi : E_1 \to E_2$ and $\overline{\varphi} : F_1 \to F_2$ which makes the following diagram commutative.

$$\begin{array}{ccc} E_1 & \longrightarrow & F_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ E_2 & \longrightarrow & F_2 \end{array}$$

Then it holds that $\mathrm{elem}_{F_1}(E_1) \simeq \mathrm{elem}_{F_2}(E_2)$.

These lemma follow easily from usual discussions on exact sequences, so we left the proofs to the reader.

For an application of this aspect, let us show the simplest condition when two elementary transformation data commute.

Proposition 1.9
Let X be a regular scheme, E be a reflexive sheaf of rank $r(>1)$ on X, (Z, F, φ_1) (resp : (Z', F', φ_2)) be m (resp : m')-ET-data for E. Let us put $\mathrm{elem}_{F_i}(E) =: E_i$ and $f_i : E_i \to E$ $(i = 1, 2)$. If $\varphi_1 \circ f_2$ (or $\varphi_2 \circ f_1$) is surjective, then we can define $\mathrm{elem}_{F_1}(\mathrm{elem}_{F_2}(E)) =: E_{12}$ and $\mathrm{elem}_{F_2}(\mathrm{elem}_{F_1}(E)) =: E_{21}$. Moreover they are isomorphic.

Proof. We may assume that $\varphi_1 \circ f_2$ is surjective. Then we have the following exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{f_1} & E & \xrightarrow{\varphi_1} & F_1 & \longrightarrow & 0 \\ & & {\scriptstyle\varphi_2\circ f_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & F_2 & \xrightarrow{id} & F_2 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Then by the snake lemma, we have

$$\ker(\varphi_2) \to \ker(\gamma) \to \mathrm{coker}(\varphi_2 \circ f_1) \to 0,$$

this is equivalent to

$$E_2 \overset{\varphi_1 \circ f_2}{\to} F_1 \to 0.$$

Hence we can define E_{12}, E_{21} and moreover, we have $E_{12} \simeq \ker(\varphi_1 \circ f_2) \simeq E_2 \cap \ker(\varphi_1) \simeq E_1 \cap E_2 \simeq E_1 \cap \ker(\varphi_2) \simeq \ker(\varphi_2 \circ f_1) \simeq E_{21}$. q.e.d.

Remark 1.3
The geometric correspondence of this proposition in the view of Proposition 1.5 is as follows : If we put $Y_i := \mathbf{P}(F_i) \subset \mathbf{P}(E)$ $(i = 1, 2)$, then the assumption of this proposition corresponds to $Y_1 \cap Y_2 = \phi$. i.e., in Sumihiro's elementary transformation, $\mathrm{elem}_{Y_2}(\mathrm{elem}_{Y_1}(E)) \simeq \mathrm{elem}_{Y_1}(\mathrm{elem}_{Y_2}(E))$ if $Y_1 \cap Y_2 = \phi$.

2 Geometric ET-data and divisors

In this section, we shall investigate the relation between our ET-data for an r-bundle E on a nonsingular variety X and the set $(Z, W_1, \cdots, W_r)$, where Z is a normal divisor of X and each W_i is an effective Weil divisor of Z satisfying some conditions, which are called (semi-)invertible along Z. These data $(Z, W_1, \cdots, W_r)$ and the concept "invertible along Z" were introduced by Sumihiro in [Su-3]. By using this concept, we can understand a geometric characterization of our elementary transformation.

At first, let us begin with a definition of the property of sections of an MCM sheaf, which will play an important role in this section.

Definition 2.1
Let X be a nonsingular variety over an algebraically closed field k, $r(>1)$ be an integer, (Z, F) be 1-ET-data for $\mathcal{O}_X^r$, and put $E := \mathrm{elem}_F(\mathcal{O}_X^r)$. Then we say that the data (Z, F) is geometric ET-data if Z is normal and on any affine open set U of X which intersect with Z, no rows of the matrix which corresponds to the morphism $E \to \mathcal{O}_X^r$ over U vanish when restricted to Z.

Before the statement of the main theorem in this section, we review a definition of "invertible along Z" in [Su-3] and give an extended definition of it.

Definition 2.2
Let X be a noetherian scheme, Z be an effective normal Cartier divisor on X and $W_1, \cdots, W_r$ $(r \geq 2)$ be effective Weil divisors on Z such that for all $i, j = 1, \cdots, r$, W_i and W_j are rationally equivalent. Let us put $W_i = W_1 + (f_i)$ for $f_i \in k(Z)$. For each $x \in B = \cap_{i=1}^r W_i$, $m = \dim((I(W_i)_x/I(Z)_x) \otimes k(x))$ is independent of i. We assume that for these Weil divisors, it holds that $m \leq r$. Then we can choose elements $s_{ji} \in \mathcal{O}_{x,X}$ $(i = 1, \cdots, r.\ j = 1, \cdots, m)$ such that $I_Z(W_i)_x = I(W_i)_x/I(Z)_x = (\overline{s_{1i}}, \cdots, \overline{s_{mi}})$ and $\overline{s_{ji}} = f_i \overline{s_{j1}}$ (where $\overline{s_{ji}}$ is the image of s_{ji} by the morphism $\mathcal{O}_X \to \mathcal{O}_Z$). We say then that $W_1, \cdots, W_r$ are invertible along Z if there is a sequence of integers $\{i_1 < i_2 < \cdots < i_m\} \subset \{1, 2, \cdots, r\}$ and $T \in M(m, \mathcal{O}_{x,X})$ such that $AT = TA = sI_m$, where $A = (s_{ji})_{i,j=i_1,\cdots,i_m}$ and s is a local equation of Z at x.

Moreover, let X be a noetherian scheme, Z be an effective normal Cartier divisor on X and $W_1, \cdots, W_r$ $(r \geq 2)$ be effective Weil divisors on Z such that for each $i, j = 1, \cdots, r$, W_i and W_j are rationally equivalent. Let us put $W_i = W_1 + (f_i)$ for $f_i \in k(Z)$. Then we say that $W_1, \cdots, W_r$ are semi-invertible along Z if for each $x \in B = \cap_{i=1}^r W_i$, we can take the generators $(\overline{s_{1i}}, \cdots, \overline{s_{ri}})$ of $I_Z(W_i)_x$, not necessarily minimal one, which satisfy $\overline{s_{ji}} =$

$f_i\overline{s_{j1}}$ $(i, j = 1, \cdots, r)$ and there exists a matrix $T \in M(r, \mathcal{O}_{x,X})$ such that $ST = TS = sI_r$ (where $S = (s_{ji})_{i,j=1}^r$ and s is a local equation of Z at x).

We will see later that invertible along Z implies semi-invertible along Z. Now, the next theorem shows the connection between our elementary transformation and Sumihiro's geometric data $(Z, W_1, \cdots, W_r)$.

Theorem 2.1
Let X be a nonsingular variety over an algebraically closed field k, Z be a normal divisor of X and $r(> 1)$ be an integer. Then there is a one to one correspondence between the following two sets.
(1) $\{(Z, F)|$ geometric ET-data for $\mathcal{O}_X^r\}/\sim$.
(2) $\{(Z, W_1, \cdots, W_r)|$ the set of an effective normal divisor Z of X and effective Weil divisors $W_1, \cdots, W_r$ of Z such that for all $i, j = 1, \cdots, r,$ W_i and W_j are rationally equivalent and semi-invertible along $Z\}/\sim$.

Where $(Z, F, \varphi) \sim (Z, F', \varphi')$ in (1) if there are isomorphisms $f : F \to F'$ and $u : \mathcal{O}_X^r \to \mathcal{O}_X^r$ which make a following diagram commutative

$$\begin{tikzcd} \mathcal{O}_X^r \arrow[r, "\varphi"] \arrow[d, "u"'] & F \arrow[d, "f"] \\ \mathcal{O}_X^r \arrow[r, "\varphi'"] & F' \end{tikzcd}$$

and $(Z, W_1, \cdots, W_r) \sim (Z, W_1', \cdots, W_r')$ in (2) if there are isomorphisms $g : I_Z(W_1) \to I_Z(W_1')$ and $v : \mathcal{O}_Z^r \to \mathcal{O}_Z^r$ which make the following diagram commutative with respect to the rational functions $f_1, \cdots, f_r, f_1', \cdots, f_r' \in k(Z)$ such that $W_i = W_1 + (f_i)$ and $W_i' = W_1' + (f_i')$ $(i = 1, \cdots, r)$.

$$\begin{tikzcd} I_Z(W_1) \arrow[r, "\{f_i\}"] \arrow[d, "g"] & \mathcal{O}_Z^r \arrow[d, "v"] \\ I_Z(W_1') \arrow[r, "\{f_i'\}"] & \mathcal{O}_Z^r \end{tikzcd}$$

Proof. This proof proceeds in several steps. Note that since Z is a disjoint union of integral normal divisors, we may assume that Z is integral. At first, let us define the sets S_1 and S_2 as $S_1 = \{W_1, \cdots, W_r|$ effective Weil divisors on Z such that for all $i, j = 1, \cdots, r,$ W_i and W_j are rationally equivalent and semi-invertible along $Z\}/\sim$, and $S_2 = \{(Z, F)|$geometric ET-data for $\mathcal{O}_X^r\}/\sim$, where the equivalence relations are the same as the above. The way of our proof is to construct maps from S_1 to S_2 and its converse.

Step 1. Construction of the map $\delta : S_1 \to S_2$.

At first, we construct the map $\delta : S_1 \to S_2$. This is easy to construct by results in section one of this article. i.e., let us take the element $\{W_1, \cdots, W_r\} \in S_1$ and $f_i \in k(Z)$ as $W_i = W_1 + (f_i)$ for $i = 1, \cdots, r$. Then from the morphism $\gamma : I_Z(W_1) \to \mathcal{O}_Z^r$ defined by $(f_1, \cdots, f_r)$, we can construct the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_X^r(-Z) \arrow[r, equal] \arrow[d, "\alpha'"'] & \mathcal{O}_X^r(-Z) \arrow[d] & & \\
0 \arrow[r] & E(-Z) \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X^r \arrow[r, "\beta"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_Z(W_1) \arrow[r, "\gamma"] \arrow[d] & \mathcal{O}_Z^r \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The definition of being semi-invertible along Z and Lemma 1.3 imply that F is an MCM $\mathcal{O}_Z$-module of rank $r-1$ and (Z, F) is 1-ET-data for $\mathcal{O}_X^r$. So let us define $\delta(\{W_1, \cdots, W_r\}) := \{\mathcal{O}_X^r \xrightarrow{\beta} F \to 0\}$ in terms of the above diagram. Note that the class $\delta(\{W_1, \cdots, W_r\})$ is independent of the choice of rational functions $\{f_i\}_{i=1}^r$ from the defintion of the equivalence relation in S_2. The only nontrivial part is whether (Z, F) is geometric or not. To see that, from the above diagram and the definition of geometric ET-data, we can understand that in the middle row of the diagram $0 \to E(-Z) \xrightarrow{\alpha} \mathcal{O}_X^r \xrightarrow{\beta} F \to 0$, the assumtion that (Z, F) is geometric ET-data is equivalent to say that any rows of $\overline{\alpha}$ are not 0 on any affine open sets which intersect Z. On the other hand, it is an easy conclusion that all elements of the i-th row of $\overline{\alpha}$ generate the ideal sheaf of W_i over Z. So if the i-th row of $\overline{\alpha}$ is 0, then it is equivalent to say that $W_i \cap U = Z \cap U$ for some open set $U \subset X$ such that $U \cap Z \neq \phi$. Since Z is integral and W_i is a divisor on Z, this is impossible. So (Z, F) is geometric ET-data and δ is well defined.

Step 2. Construction of the converse map $\delta' : S_2 \to S_1$.

Let us consider the converse of Step 1. i.e., from geometric ET-data $(Z, F) \in S_2$, we want to construct $\{W_1, \cdots, W_r\} \in S_1$. At first, see the following diagram. This is canonically obtained from the geometric ET-data $\{\mathcal{O}_X^r \xrightarrow{\beta} F \to 0\} \in S_2$ and $F' = \ker(\gamma : \mathcal{O}_Z^r \to F)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_X^r(-Z) \arrow[r, equal] \arrow[d, "\alpha'"'] & \mathcal{O}_X^r(-Z) \arrow[d] & & \\
0 \arrow[r] & E(-Z) \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X^r \arrow[r, "\beta"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & \mathcal{O}_Z^r \arrow[r, "\gamma"] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

In Step 2 and Step 3, we shall use the notation in this diagram. Put $\alpha' = (s_1, \cdots, s_r)$, where $s_i \in H^0(X, E)$. We want to construct Z and W_i ($i = 1, \cdots, r$) by using these data. At first, let us prove the following.

Proposition 2.2
With the above notation, $Z = Z(s_1 \wedge \cdots \wedge s_r)$.

Proof. We may consider this locally. i.e., on an affine open set U=Spec(A) such that $U \cap Z \neq \phi$. We may assume that U is integral. Put $S' = \alpha'|_U = (s_{ij})_{i,j=1}^r$ and $S = \alpha|_U = (d_{ij})_{i,j=1}^r$. Since $SS' = S'S = sI_r$ (s : a local equation of Z on U), we have $\det S \neq 0$. By the assumption that $\mathrm{rank}_Z(F) = r-1$, it follows that $\mathrm{rank}\overline{S} = 1$. So there are elements $f_i = \overline{a_i}/\overline{b}$ ($a_i, b \in A$, $\overline{b} \neq 0$) such that $f_i(\overline{d_{11}}, \cdots, \overline{d_{1r}}) = (\overline{d_{i1}}, \cdots, \overline{d_{ir}})$ for all $i = 1, \cdots, r$ ($\overline{d_{ij}}$ is the image of d_{ij} by the morphism $A \to A/sA$). Notice that since (Z, F) is geometric data, $(\overline{d_{i1}}, \cdots, \overline{d_{ir}}) \neq (0, \cdots, 0)$ for all i. Hence $b^{r-1}\det S = s^{r-1}u' \neq 0$ ($u' \in A$). Since s is prime (if necessary, replace U by smaller one) and $\overline{b} \neq 0$, we see that $\det S = s^{r-1}u$ for some $u \in A$. Combining this equation with $SS' = sI_r$, we see that $u(\det S') = s$. If $s|u$, then $\det S'$ is a unit in A and so $\det\overline{S'}$ is also a unit. This contradicts the assumption that $\mathrm{rank}\overline{S'} = r-1$. Thus $s|\det S'$ and $\det S' = su^{-1}$, hence the proposition is proved. q.e.d.

By this proposition, we may assume that on any affine open sets $U \subset X$ which intersect with Z, the matrices S' defined in the above proof satisfy $\det S' = s$, where s is a local equation of Z on U.

Next, we prepare some facts about $B = \cap_{i=1}^r W_i$ and F'.

Lemma 2.3
If we put $W_i = Z(s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_r)$ ($i = 1 \cdots, r$) and $B = \cap_{i=1}^r W_i$, then $\mathrm{codim}_Z(B) \geq 2$. Moreover, if we put $V = Z \setminus B$ and $j : V \hookrightarrow Z$, then we have $j_* j^* F' \simeq F'$.

Proof. We use the same notation as in Proposition 2.2. Taking $x_1 \cdots, x_n$ as regular coordinates at $x \in X$ and using the Jacobian of S', it is easy to see that $B \subset \mathrm{Sing}(Z)$ since $\cap_{i=1}^{r} W_i$ is generated by all $r-1$ minors of S' and since a local equation of Z is $\det S'$ by Proposition 2.2. Then the normality of Z implies that $2 \leq \mathrm{codim}_Z(\mathrm{Sing}(Z)) \leq \mathrm{codim}_Z(B)$.

Next, it is obvious that F' is MCM. Hence it holds that $\mathrm{depth}_{\mathcal{O}_{x,Z}} F_x = \dim \mathcal{O}_{x,Z}$. In particular, we can see that F is reflexive on Z. Thus $j_* j^* F' \simeq F'$ follows from the general property of reflexive sheaves. q.e.d.

Now, we are able to prove that each W_i what we have constructed is an effective Weil divisor on Z.

Proposition 2.4
With the above notation, $W_i = \phi$ or it is an effective Weil divisor on Z for all i. For all $i, j = 1, \cdots, r$, W_i and W_j are linearly equivalent. Moreover, $I_Z(W_i) \simeq F'$ for all i.

Proof. At first, we prove that $F'|_V$ is an invertible sheaf on $V = Z \setminus B$. In the second place, we prove that $F'|_V \simeq I_Z(W_i)|_V$ for all i. So by Lemma 2.3 and the isomorphism $\varphi_i : j_*(I_Z(W_i)|_V) \simeq I_Z(W_i)$, we can see that they determine effective Weil divisors on Z. The latter isomorphism φ_i follows from the fact that Z is normal and the ideal sheaves of Weil divisors are determined on the open set whose complement has codimension more than or equal to 2. e.g., on $V \subset Z$.

In the previous diagram, put $\alpha' = (s_1, \cdots, s_r)$ where $s_i \in H^0(X, E)$. On an affine open set $U_a = \mathrm{Spec} A_a$, put $s_i = (s_{i1}^a, \cdots, s_{ir}^a)$, ${}^t S'_a = {}^t(s_{ij}^a) = \alpha'|_{U_a}$ and ${}^t S_a = {}^t(d_{ij}^a) = \alpha|_{U_a}$. $S_a S'_a = S'_a S_a = s I_r$ (where s is a local equation of Z on U_a) and $\mathrm{rank}_Z S_a = 1$ is obvious. In particular, the assumption that (Z, F) is geometric ET-data implies that all the rows of $\overline{\alpha_{U_a}}$, that is, all the columns of $\overline{S_a}$ are not zero vectors (where $\overline{\alpha}$ is the image of α by the morphism $\mathcal{O}_X \to \mathcal{O}_Z$). Hence if on $Z \cap U_a = \mathrm{Spec}\, \overline{A_a} = \mathrm{Spec}(A_a / s A_a)$ we put $V_{ij}^a = \mathrm{Spec}\, \overline{A_a}[1/\overline{d_{ij}^a}]$, then from the assumption we have

$$\begin{aligned} V_{ij}^a \neq \phi \iff \overline{d_{ij}^a} \neq 0 \quad &\iff \quad \overline{d_{ik}^a} \neq 0 \ (k = 1, \cdots, r) \\ &\iff \quad V_{ik}^a \neq \phi \ (k = 1, \cdots, r). \end{aligned} \tag{7}$$

Hence it suffices to prove the invertibility of F' on each open set V_{ij}^a since $U_a \cap V = \cup_{d_{ij}^a \neq 0} V_{ij}^a$. To see this, since $\sum_{k=1}^{r} \overline{s_{lk}^a} \overline{d_{kj}^a} = 0$ on $\phi \neq V_{ij}^a$, we have

$$\overline{s_{li}^a} = -\sum_{k \neq i} \frac{\overline{d_{kj}^a}}{\overline{d_{ij}^a}} \overline{s_{lk}^a} \quad (l = 1, \cdots, r). \tag{8}$$

On the other hand on V^a_{im} (Notice that this is not empty by (7)), we have

$$\overline{s^a_{li}} = -\sum_{k\neq i} \frac{\overline{d^a_{km}}}{\overline{d^a_{im}}}\overline{s^a_{lk}} \quad (l = 1, \cdots, r). \tag{9}$$

Using the fact that $\mathrm{rank}_Z S'_a = r-1$ on $V^a_{ij} \neq \phi$, the equations (8) and (9) imply that

$$\overline{d^a_{km}} = \frac{\overline{d^a_{kj}}}{\overline{d^a_{ij}}}\overline{d^a_{im}} \ (m = 1, \cdots, r). \tag{10}$$

As the image of $F'|_{U_a\cap Z} \hookrightarrow \oplus^r \mathcal{O}_Z|_{Z\cap U_a}$, F' is generated by all the column vectors of $\overline{\alpha|_{U_a}} = {}^t\overline{S_a}$, that is, $\{(\overline{d^a_{i1}}, \cdots, \overline{d^a_{ir}})\}$ for $i = 1, \cdots, r$. By (10), we see that on $V^a_{ij} \neq \phi$, F' is generated by one element $(\overline{d^a_{i1}}, \cdots, \overline{d^a_{ir}})$ and from (7) this is not a zero vector since $V^a_{ij} \neq \phi$. Hence $F'|_V$ is a line bundle and it has the transition matrix $\dfrac{\overline{d^a_{ij}}}{\overline{d^a_{kj}}}$ on $V^a_{kl} \cap V^a_{ij}$.

Next, for comparing F' with $I_Z(W_p)$, take an another open set U_b =SpecA_b such that $U_b\cap Z \neq \phi$. Let us denote the transition matrix of $E(-Z)$ on $U_b\cap U_a$ by G_{ba}. Then we see that $G_{ba}\,{}^tS'_a = {}^tS'_b$. Taking the cofactor and transposition of this equation, it holds that $S_b = (g_{ij})S_a$, where (g_{ij}) is the transposed cofactor of G_{ba}. Hence on $V^b_{kl} \cap V^a_{ij}$ it holds that

$$\begin{aligned} \overline{d^b_{km}} &= \sum_{n=1}^{r} \overline{g_{kn}}\,\overline{d^a_{nm}} \\ &= (\sum_{n=1}^{r} \overline{g_{kn}}\frac{\overline{d^a_{nj}}}{\overline{d^a_{ij}}})\overline{d^a_{im}} \quad (m = 1, \cdots, r). \end{aligned} \tag{11}$$

This implies that the transition matrix of $F'|_V$ on $V^b_{kl}\cap V^a_{ij}$ is $\dfrac{\overline{d^a_{ij}}}{\sum_{n=1}^r \overline{g_{kn}}\,\overline{d^a_{nj}}}$. On the other hand, $I_Z(W_p)$ is generated on V^a_{ij} by $\overline{d^a_{1p}}, \cdots, \overline{d^a_{rp}}$ by the definition. Again by (10), it is generated on $V^a_{ij} \neq \phi$ by one element $\overline{d^a_{ip}}$ and on $V^b_{kl} \neq \phi$ by $\overline{d^b_{kp}}$ (Notice that this is not zero since $V^a_{ij} \neq \phi$. This fact also implies that $W_p \neq Z$ for all p). Therefore $I_Z(W_p)$ is also a line bundle on V which has $\dfrac{\overline{d^a_{ip}}}{\overline{d^b_{kp}}} = \dfrac{\overline{d^a_{ij}}}{\sum_{n=1}^r \overline{g_{kn}}\,\overline{d^a_{nj}}}$ as the transition matrix on $V^b_{kl} \cap V^a_{ij}$. Since this is the same as that of $F'|_V$, we see that $F'|_V \simeq I_Z(W_p)|_V$ for all p and from the first consideration in this proof, we see that all W_i are effective Weil divisors on Z (including the case that some of $\{W_i\}^r_{i=1}$ are empty).

To prove that they are mutually linearly equivalent, we have to notice that since $\mathrm{codim}_Z(B) \geq 2$, it is enough to show that they are linearly equivalent on $V = Z \backslash B$. From the above proof, $\{W_i\}_{i=1}^r$ are all effective Cartier divisors on V. It is easy to see that $\mathcal{O}_V(W_1) \simeq \mathcal{O}_V(W_i)$ as invertible sheaves on V for all i. So they are linearly equivalent as effective Cartier divisors. Of course, this means that they are linearly equivalent as (locally principal) effective Weil divisors on V. q.e.d.

Consequently, we could have constructed $W_1, \cdots, W_r$ from an element (Z, F) of S_2. By Proposition 2.2, Lemma 2.3 and Proposition 2.4, we can see that $(Z, W_1, \cdots, W_r)$ is semi-invertible along Z and W_i and W_j are rationally equivalent for all $i, j = 1, \cdots, r$. i.e., $(W_1, \cdots, W_r) \in S_1$. So putting $\delta'(\{\mathcal{O}_X^r \overset{\beta}{\to} F \to 0\}) := \{W_1, \cdots, W_r\}$ with above terms, we can define $\delta' : S_2 \to S_1$ and it is easy to see this is well defined. Now from the construction of δ, its image is determined by the inclusion $F' \simeq I_Z(W_1) \hookrightarrow \oplus^r \mathcal{O}_Z$ and the next proposition implies the original data can be recovered. i.e., $\delta\delta' = id_{S_2}$.

Proposition 2.5
Let us fix an isomorphism $I_Z(W_1) \simeq F'$ the existence of which is proved in Proposition 2.4. Then the homomorphism $F' \to \oplus^r \mathcal{O}_Z$ coincides with the one $I_Z(W_1) \ni g \mapsto (gf_1, \cdots, gf_r) \in \oplus^r \mathcal{O}_Z$ (where $f_i \in k(Z)$ is the rational function such that $W_i = W_1 + (f_i)$).

Proof. We can define a homomorphism $\varphi : I_Z(W_1) \overset{(f_1,\cdots,f_r)}{\longrightarrow} \oplus^r \mathcal{O}_Z$. Thus we have only to check the coincidence locally, i.e., on an affine open set $U = \mathrm{Spec}A$ that intersects Z. Notice that because of the following diagram, this coincidence is enough to be checked on $V = Z \setminus B$ (the two columns are isomorphisms induced from the proof of Proposition 2.4 and from the normality of Z).

$$\begin{array}{ccc} H^0(U,\, I_Z(W_1)) & \xrightarrow{\varphi} & \oplus^r H^0(U,\, \mathcal{O}_Z) \\ \downarrow & & \downarrow \\ H^0(U \cap V,\, I_Z(W_1)|_V) & \xrightarrow{\varphi|_V} & \oplus^r H^0(U \cap V,\, \mathcal{O}_V) \end{array}$$

In the proof of Proposition 2.4, we know that on $V_{ij}^a \neq \phi$, F' is generated by $(\overline{d_{i1}^a}, \cdots, \overline{d_{ir}^a})$. Then considering the fact that $(\overline{d_{i1}^a}, \cdots, \overline{d_{ir}^a}) = \overline{d_{i1}^a}(f_1, \cdots, f_r)$ $(i = 1, \cdots, r)$ and that $I_Z(W_1)$ is generated by d_{i1}^a on V_{ij}^a, the proposition follows immediately. q.e.d.

Note that the class of $\delta\delta'(\{\mathcal{O}_X^r \to F \to 0\})$ as the element in S_2 is independent of the choice of an isomorphism $F' \simeq I_Z(W_1)$ in Proposition 2.5. This follows from the definition of the equivalence relation in S_1 and from the fact that differences between the isomorphisms above are only units since $I_Z(W_1)$ and F' are line bundles.

Step 3. The proof of $\delta'\delta = id_{S_1}$.

At last, we shall prove that $\delta'\delta = id_{S_1}$. Let us begin this step with the following lemma.

Lemma 2.6

Let us define $S_1' := \{W_1, \cdots, W_r |$ effective Weil divisors on Z such that W_i and W_j are rationally equivalent for all $i, j = 1, \cdots, r$, and invertible along $Z\}/\sim$ (where the equivalence relation is the same as that of S_1). Then $S_1' \subset S_1$.

Proof. Take $\{W_1, \cdots, W_r\} \in S_1'$ and fix a point $x \in B = \cap_{i=1}^r W_i$. Let us put m =dim$(I_Z(W_i)_x \otimes k(x)) \leq r$ and take $f_i \in k(Z)$ $(i = 1, \cdots, r)$ and $s_{ji} \in \mathcal{O}_{x,X}$ $(i = 1, \cdots, r,\ j = 1, \cdots, m)$ so that $I_Z(W_i)_x = (\overline{s_{1i}}, \cdots, \overline{s_{mi}})$ and $f_i\overline{s_{j1}} = \overline{s_{ji}}$ (where $\overline{s_{ji}}$ is the image of s_{ji} by the morphism $\mathcal{O}_X \to \mathcal{O}_Z$). We may assume that there is a matrix $T \in M(m, \mathcal{O}_{x,X})$ such that for $A = (s_{ji})_{i,j=1}^m$, $AT = TA = sI_m$, where s is a local equation of Z at x. Let us put for $m+1 \leq j \leq r$,

$$\begin{aligned} s_{ji} &= 0 \qquad (1 \leq i \leq m), \\ &= s\delta_{ji} \quad (m+1 \leq i \leq r), \end{aligned}$$

and put

$$S = (s_{ji}) = \begin{pmatrix} A & B \\ 0 & sI_{r-m} \end{pmatrix}$$

where $B = (s_{ji})_{1 \leq j \leq m,\ m+1 \leq i \leq r}$. Then it is obvious that $I_Z(W_i)_x = (\overline{s_{1i}}, \cdots, \overline{s_{ri}})$ and $f_i\overline{s_{j1}} = \overline{s_{ji}}$ for all i, j. Moreover, if we put

$$S' = \begin{pmatrix} T & -(1/s)TB \\ 0 & I_{r-m} \end{pmatrix}$$

then this is defined as the element of $M(r, \mathcal{O}_{x,X})$ by using the discussion in section one of [Su-3] and $SS' = S'S = sI_r$ is obvious. Hence the proposition follows. q.e.d.

Applying the process used in the proof of Lemma 2.6, we can regard all the elements in S_1' as in S_1. By this lemma, we can see that "semi-invertible along Z" is a natural extension of "invertible along Z". So we can use the simillar way to [Su-3] for the proof of this step.

Now, we shall show $\delta'\delta = 1_{S_1}$. To see this, we must check that whether for $\{W_1, \cdots, W_r\} \in S_1$, we can recover it from its image $\delta(\{W_1, \cdots, W_r\})$ through δ'. i.e., we must check whether the same statement of theorem in section one in [Su-3] holds when $\{W_1, \cdots, W_r\}$ are not invertible along Z but semi-invertible along Z. If this is proved, then by the construction of δ and δ', it follows that $\delta'\delta = 1_{S_1}$. So what we have to prove is the following.

Proposition 2.7
Let X be a Noetherian scheme and Z be a normal divisor of X. Take $W = \{W_1, \cdots, W_r\} \in S_1$ and put $\delta(W) = \{\mathcal{O}_X^r \to F \to 0\}$. If we put $\mathrm{elem}_F(\mathcal{O}_X^r) = E(-Z)$, then there are global sections $s_1, \cdots, s_r \in H^0(X, E)$ such that $Z = Z(s_1 \wedge \cdots \wedge s_r)$ and $W_i = Z(s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_r)$ for all i.

Proof. This can be proved by almost the same way as Theorem 1.2.3 in [Su-3] in case that $r = m$. Between the proofs of Theorem 1.2.3 in [Su-3] and this theorem, there is only one difference. That is, for each $x \in B = \cap_{i=1}^r W_i$, the generator of $I_Z(W_i)_x$ is minimal or not. However, in Sumihiro's proof, this condition only works when proving that $\det S = s^{r-1}u$ and $u \in \mathcal{O}_{x,X}^\times$, where $S = (s_{ji})_{i,j=1}^r$ are the generators of $I_Z(W_i)_x$ as Definition 2.2 and s is a local equation of Z at $x \in B$. So to finish this proof, it is sufficient to show the following proposition and if it is proved, Proposition 2.7 can be proved by the same way of [Su-3].

Proposition 2.8
With the above notation, $\det S = s^{r-1}u$ *and* $u \in \mathcal{O}_{x,X}^\times$.

Proof. Put $f_i = \overline{a_i}/\overline{b}$, $\overline{b} \neq 0$ (where $\overline{a_i}$ is the image of a_i by the morphism $\mathcal{O}_X \to \mathcal{O}_Z$). Then the $i(> 1)$-th column of $b(\det S)$ is ${}^t(a_i s_{11} + s h_{1i}, \cdots, a_i s_{r1} + s h_{ri})$ for some elements $h_{ij} \in \mathcal{O}_{x,X}$. So $b^{r-1}\det S = s^{r-1}u'$ for some $u' \in \mathcal{O}_{x,X}$. By the assumption that $SS' = S'S = sI_r$, we have $u(\det S') = s$ for some $u \in \mathcal{O}_{x,X}$. If $s|u$, then $\det\overline{S'}$ is a unit and so $\mathrm{rank}\overline{S'} = r$. On the other hand, since $\overline{S'S} = 0$ and $\mathrm{rank}(\overline{S}) \geq 1$ (since $\overline{S} \neq 0$), we see that $\dim(\ker \overline{S'}) \geq 1$. This is a contradiction. Hence $s|\det S'$ and the proposition is proved. q.e.d.

Now, let us finish the proof of Theorem 2.1. By Proposition 2.7, we can recover the data $W = \{W_1, \cdots, W_r\} \in S_1$ by using the map δ' from $\delta(W) = \{\mathcal{O}_X^r \to F \to 0\}$. Hence we can see that $\delta'\delta = 1_{S_1}$. Therefore, the one to one correspondence between S_1 and S_2 is gotton and Theorem 2.1 is proved. q.e.d.

Remark 2.1
It is obvious that not all ET-data $\{\oplus^r \mathcal{O}_X \xrightarrow{\beta} F \to 0\}$ (where $X,\ Z$ are as the above and F is an MCM $\mathcal{O}_Z$-module of rank $r-1>0$*) can be contained in S_2 but it is easy to see that if an MCM sheaf F is generated by r-sections, we can find r-generating sections which make (Z,F) geometric ET-data by changing bases. Hence we can construct all the MCM sheaves which are of* rank $r-1>0$ *and generated by r-sections from the data of S_1. Theorem 2.1 is, in some sense, a geometric characterization of MCM sheaves on divisors.*

Now, as in the remark of section one in [Su-3], when $W_1,\cdots,W_r$ are just only mutually linearly equivalent, we can construct from this data $E=E(Z,W_1,\cdots,W_r)$: a torsion free coherent sheaf of rank r on X, which is obtained as the kernel of the surjection $\mathcal{O}_X^r \to \mathrm{coker}(I_Z(W_1) \to \mathcal{O}_Z^r)$. In [Su-3], it is proved if $(W_1,\cdots,W_r)$ is invertible along Z, then E is locally free. Let us consider the necessary and sufficient condition that E becomes locally free in case that X is a regular algebraic scheme over a field $k=\overline{k}$, Z is an effective normal divisor on X and $r>1$ is an integer.

It is obvious from the above discussion that $E=E(Z,W_1\cdots,W_r)$ is locally free if and only if F is MCM of rank $r-1$. Looking at the diagram in Step 1, we see that if E is locally free, then $\oplus^r\mathcal{O}_X \to F$ is geometric ET-data. Hence we get the next conclusion.

Corollary 2.9
Let X be a nonsingular variety over an algebraically closed field and Z be an effective normal divisor on X. Let $W_1,\cdots,W_r$ $(r\geq 2)$ be effective Weil divisors on Z and assume that W_i and W_j are linearly equivalent for all $i,j=1,\cdots,r$. Then $E(Z,W_1,\cdots,W_r)$ is locally free if and only if $\{W_1,\cdots,W_r\}$ is semi-invertible along Z.

Proof. If $\{W_1,\cdots,W_r\}\in S_1$, then E is locally free by Lemma 1.1 and Lemma 1.3. Conversely, if E is locally free, then from the above discussion we see that $\{\oplus^r\mathcal{O}_X \to F \to 0\} \in S_2$ and if we send this element by δ', the image is obviously $\{W_1,\cdots,W_r\}$. Since the image of δ' is contained in S_1 (this follows from the proof of Theorem 2.1), we have the conclusion. q.e.d.

At last of this section, we shall show a way to construct MCM sheaves on divisors from geometric data $(Z,W_1,\cdots,W_r)$. The key point is that the assumption Z is normal in Theorem 2.1 is not necessary when we do not need geometric data.

Theorem 2.10
Let X be a noetherian scheme, Z be an effective integral Cartier divisor which has no embedded prime cycles. Let $W_1,\cdots,W_r$ $(r\geq 2)$ be effective Weil divisors on Z which satisfy the following conditions.

1) There are elements $f_i \in k(Z)$ $(i = 1, \cdots, r)$ which induce, by its multiplication, an $\mathcal{O}_Z$-module isomorphisms $f_i : I_Z(W_1) \to I_Z(W_i)$ for each i.

2) With respect to the isomorphisms of 1), they are (semi-)invertible along Z.

Then the cokernel of the $\mathcal{O}_Z$-module morphism $I_Z(W_1) \overset{(1,f_2,\cdots,f_r)}{\longrightarrow} \mathcal{O}_Z^r$ is an MCM $\mathcal{O}_Z$-module of rank $r-1$.

Proof. This follows from Lemma 1.3. For an explicit proof, see Theorem 4.3 in [A]. q.e.d.

In [Su-3], the normality of Z and invertible along Z are assumed. Here, we only need that Z is integral and semi-invertible along Z. This will be used in section five to construct the Tango bundle.

3 The commutativity of a closed immersion and an elementary transformation

In this section, let us consider when an elementary transformation and a closed immersion commute. They are used in the next section. The main result of this section is as follows.

Proposition 3.1
Let X be a nonsingular variety over an algebraically closed field k, E be an $r(>1)$-bundle on X, Z be an effective integral divisor, and (Z, F) be m-ET-data for E $(1 \leq m \leq r-1)$. Let us take a nonsingular effective divisor $D \subset X$. Assume that $Z' := Z \cap D \subset D$ is an effective integral divisor of D. Let us put $E' := \mathrm{elem}_F(E)$. Then $E'|_D \simeq \mathrm{elem}_{F|_D}(E|_D)$ if and only if $\mathrm{Ass}_X(F) \cap D = \phi$.

Proof. Let us consider the following exact sequence of the elementary transformation $0 \to E' \overset{f}{\to} E \to F \to 0$. Restricting this sequence to D, we obtain the sequence $E'|_D \overset{f|_D}{\to} E|_D \to F|_D \to 0$. What we must show is that $f|_D$ is injective precisely when $\mathrm{Ass}_X(F) \cap D = \phi$. This can be seen if we prove $\mathrm{Ass}_X(F) \cap D = \phi$ precisely when $\mathrm{Tor}^1_X(F, \mathcal{O}_D) = 0$. This is easy to see if we consider the projective resolution $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ of $\mathcal{O}_D$. Tensoring F to this resolution and considering when $F(-D) \to F$ is injective, it is precisely when $\mathrm{Ass}_X(F) \cap D = \phi$. q.e.d.

The next proposition requires more conditions but contains a more general result. i.e., we consider the restriction not only to divisors but also to closed subschemes of any dimension. Moreover, it has an application to the geometric data. i.e., when given a closed subscheme $H \subset X$, geometric ET-data (Z, F) for $\mathcal{O}_X^r$ which corresponds by Theorem 2.1 to $(Z, W_1, \cdots, W_r) \in S_1$, and the bundle $E := \mathrm{elem}_F(\mathcal{O}_X^r) = E(Z, W_1, \cdots, W_r)$ constructed by these data, we consider when the restrected vector bundle $E|_H$ and the bundle constructed by restricted data $E(Z \cap H, W_1 \cap H, \cdots, W_r \cap H)$ are isomorphic.

Proposition 3.2
Let X be a nonsingular variety over an algebraically closed field k of $\dim X = n \geq 1$, Z be an effective integral divisor, (Z, F) be m-ET-data for $\mathcal{O}_X^r$ $(r > 1,\ 1 \leq m \leq r-1)$. Let us take a nonsingular closed subscheme $H \subset X$ of dimension l $(1 \leq l \leq n-1)$. Assume that $H \cap Z$ is a locally complete intersection and $Z' := Z \cap H$ is an effective integral divisor of H. Let us put $E := \mathrm{elem}_F(\mathcal{O}_X^r)$. Then

(1) If $\mathrm{rank}(F|_H) = r - m$, then it holds that $E|_H = \mathrm{elem}_{F|_H}(\mathcal{O}_H^r)$.

(2) Assume moreover that (Z, F) is geometric ET-data which corresponds to $(Z, W_1, \cdots, W_r) \in S_1$. Let us put $E = E(Z, W_1, \cdots, W_r)$. If $Z' \not\subset W_i$ for all i, then $E|_H \simeq \mathrm{elem}_{F|_H}(\mathcal{O}_H^r) \simeq E(Z', W_1 \cap H, \cdots, W_r \cap H)$.

Proof. (1) Let us denote $F|_H$ by G. Pick and fix $x \in Z'$. Let us represent $\mathcal{O}_{x,Z'} = \mathcal{O}_{x,X}/(\alpha_1, \alpha_2, \cdots, \alpha_{l+1})$, where $(\alpha_1, \cdots, \alpha_{l+1})$ is a regular sequence in $\mathcal{O}_{x,X}$ and generate the defining ideal of $H \cap Z$ at x. If we prolong $(\alpha_1, \cdots, \alpha_{l+1})$ to the maximal regular sequence $(\alpha_1, \cdots, \alpha_{l+1}, \alpha_{l+2} \cdots, \alpha_n)$, then we can prove that $(\alpha_{l+2}, \cdots, \alpha_n)$ is a regular sequence for G_x by the same way of Lemma 1.3. Hence this is MCM on Z'. Moreover, since $\mathrm{rank}\, G = r - m$, we have $\mathrm{Supp}(G) = Z'$. So we have a surjection $E|_H \to \mathrm{elem}_G(\mathcal{O}_H^r)$ and this is an isomorphism since locally, they are both free modules of the same rank over a local Noetherian ring.

(2) From the proof of Theorem 2.1, it is sufficient to show that $\varphi : E|_H \to \mathcal{O}_H^r$ is geometric ET-data. For Theorem 2.1 states that if so, there exists the data $(Z', W_1', \cdots, W_r') \in S_1$ such that $E(Z', W_1', \cdots, W_r') \simeq E|_H$ and by the construction, it is obvious that $(Z', W_1', \cdots, W_r') = (Z', W_1 \cap H, \cdots, W_r \cap H)$. Now, let us put t_{ji} as an image of s_{ji} in $\mathcal{O}_{x,H}$, where $x \in B = \cap W_i$ and $\{s_{ji}\}_{j=1}^r$ represent the usual generators of $I_Z(W_i)_x$ $(i = 1, \cdots, r)$ as in section two. Let us put $\overline{t_{ji}}$ as its image in $\mathcal{O}_{Z'}$. What we must show is that no rows of ${}^t(\overline{t_{ji}})$ vanish. If $\overline{t_{1i}} = \cdots = \overline{t_{ri}} = 0$ for some i, then it implies that $(s_{1i}, \cdots, s_{ri})$ are contained in the defining ideal of $Z \cap H$. This means that there exists an open set $U \neq \phi$ which intersects with Z and $W_i \cap U \supset Z' \cap U \neq \phi$. This contradicts our assumption. q.e.d.

4 The Splitting Criterion of rank one coherent sheaves on a divisor in $\mathbf{P}_k^n$

In this section, we shall consider all the problems on $\mathbf{P}_k^n$, where $n \geq 3$ or 4, and k is an algebraically closed field. In Remark 1.2, we saw that in our elementary transformation, $\mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}_k^n})$ $(n \geq 4)$ splits if F is a line bundle. It is natural to consider that this converse is true or not. i.e., if $\mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}_k^n})$ splits, then is F locally free on Z? In this section, we consider this problem in a certain situation and give some answer and some counter-example.

At first, let us recall the result stated in section one, which gives a splitting condition for $\mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}_k^n})$.

Lemma 4.1 ([Su-2])
Let X be $\mathbf{P}_k^n$ $(n \geq 4)$, (Z, F) be 1-ET-data for $\mathcal{O}_X^2$, and $\mathrm{elem}_F(\mathcal{O}_X^2) =: E$. If F is a line bundle on Z, then E splits into the sum of line bundles.

Proof. See Remark 1.2. q.e.d.

Note that this lemma cannot be applied to the case of vector bundles on the projective line, plane, and 3-fold. This follows from the result in [Mar], i.e., all the bundles (including indecomposable bundles) on them can be constructed by Maruyama's elementary transformation.

Secondly, we need the following lemma, which is the splitting criterion of 2-bundles constructed by 1-ET-data (Z, F) for $\mathcal{O}^2_{\mathbf{P}_k^n}$ when $n \geq 3$.

Lemma 4.2
Let X be $\mathbf{P}_k^n$ $(n \geq 3)$, (Z, F) be 1-ET-data for $\mathcal{O}_X^2$ and $E := \mathrm{elem}_F(\mathcal{O}_X^2)$. Let us see the following diagram of our elementary transformation.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X^2(-Z) \arrow[r, equal] \arrow[d] & \mathcal{O}_X^2(-Z) \arrow[d] & & \\
0 \arrow[r] & E \arrow[r] \arrow[d] & \mathcal{O}_X^2 \arrow[r, "\varphi"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & \mathcal{O}_Z^2 \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Then E splits if and only if $H^i(Z, F'(k)) = 0$ for all k and $i = 1, \cdots, n-2$. (resp : $H^i(Z, F(k)) = 0$ for all k and $i = 1, \cdots, n-2$).

Proof. If E splits, then from the exact sequence

$$0 \to \mathcal{O}_X^2(-Z) \to E \to F' \to 0,$$

we have the statement on F'. Conversely if $H^i(Z, F'(k)) = 0$ for all $k \in \mathbb{Z}$ and $i = 1, \cdots, n-2$, then the same exact sequence implies $H^i(X, E(k)) = 0$ for all integers k and $i = 1, \cdots, n-2$. By Serre duality, $H^{n-1}(X, E(k))$ also vanishes for all $k \in \mathbb{Z}$. Hence by Horrocks' splitting criterion, E splits into the sum of line bundles. The result for F can be gained from the exact sequence

$$0 \to E \to \mathcal{O}_X^2 \to F \to 0$$

and repeating the same discussion as in the case of F'. q.e.d.

Note that this lemma cannot be applied to the vector bundles on the projective plane, since Z is a one dimensional variety in this case and so $H^1(Z, F'(k)) = 0$ for all $k \in \mathbb{Z}$ never happen.

Whether the converse of Lemma 4.1 hold or not is a natural question. i.e., given 1-ET-data (Z, F) for $\mathcal{O}^2_{\mathbf{P}^n_k}$ and if $E := \mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}^n_k})$ splits, then is the MCM sheaf F a line bundle? Or generalizing this problem from the viewpoint of Lemma 4.2, when given a coherent sheaf F on a projective variety X which satisfies $H^i(X, F(k)) = 0$ $(i = 1, \cdots, \dim X - 1)$, then is the sheaf F splits into the sum of line bundles? This is of course not true in general, and we will show a counter example in this section later. However, in a special situation, we can prove an affarmative result to this problem by using the elementary transformation.

Proposition 4.3
Let Z be an effective divisor of $\mathbf{P}^n_k$, where k is an algebraically closed field and $n \geq 5$. Assume that $\mathrm{codim}_X(\mathrm{Sing}(Z)) \geq 5$. Let us take an MCM $\mathcal{O}_Z$-module F of rank one, which is generated by two global sections. Then F is a line bundle if and only if $H^i(Z, F(k)) = 0$ for $i = 1, \cdots, n-2$ and all $k \in \mathbb{Z}$.

Proof. Note that in this situation, (Z, F) is 1-ET-data for $\mathcal{O}^2_{\mathbf{P}^5_k}$. Let us put $E := \mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}^5_k})$. Assume that F is a line bundle. Then by Lemma 4.1, E splits into the sum of line bundles. So Lemma 4.2 implies $H^i(Z, F(k)) = 0$ for $i = 1, \cdots, n-2$ and all $k \in \mathbb{Z}$. Hence if part is true. Let us assume the converse. We prove this direction by induction on n. At first, we prove when $n = 5$. By Bertini's theorem, there is a non empty open set $U' \subset Gr(n, n-1)$ such that for all $H \in U'$, a divisor $Z \cap H$ of $H \simeq \mathbf{P}^{n-1}$ is smooth. Since F is coherent, $\mathrm{Ass}(F)$ consists of only finitely many points. So the set of hyperplanes in $\mathbf{P}^n_k$ which contain points of $\mathrm{Ass}(F)$ becomes a closed set D

in $Gr(n, n-1)$. Let us put $U := U' \setminus D \neq \phi$. Then for each $H \in U$, $F|_H$ is an MCM $\mathcal{O}_{Z \cap H}$-module of rank 1 by Proposition 3.1. Hence for all $H \in U$, $(Z \cap U, F|_H)$ is 1-ET-data for $\mathcal{O}_H^2$ and it holds that $E|_H \simeq \mathrm{elem}_{F|_H}(\mathcal{O}_H^2)$. On the other hand, let us put for $x \in X$, $S_x = \{H \in Gr(n, n-1) \mid x \in H\}$. Then since $S_x \simeq \mathbf{P}^{n-1}$, we see that there are only finitely many points $\{x_i\}_{i \in I}$ such that $S_{x_i} \cap U = \phi$. Put $V = Z \setminus \{x_i\}_{i \in I}$. Then for all $x \in V$, there is at least one $H \in U$ which contains x. Now, let us fix for each $x \in V$ one $H_x \in U$ which contains x. Note that by the assumption, Grothendieck-Lefschetz theorem holds on $Z \cap H$. Combining this result with the above consideration, we can see that for all $x \in V$, there is an isomorphism $\varphi_x : \mathcal{O}_{H_x \cap Z}(k) \to F|_{H_x}$ (where k is an integer and independent of x, since we take H generally and the first chern class of F does not change. This follows immediately if we consider the locally free resolution of F, i.e., $0 \to E \to \mathcal{O}_X^2 \to F \to 0$ and by the definition of U). Let us extend the morphism φ_x to the one on Z. It is impossible for all $x \in \{x_i\}_{i \in I}$, but we have already omitted these points. So we can get the exact sequence

$$0 \to F(k) \to F(k+1) \to F|_{H_x}(k+1) \to 0$$

for all $x \in V$. Now, from the assumption and the following exact sequence

$$\begin{aligned} 0 \;&\to\; \mathrm{Hom}(\mathcal{O}_Z(k-1), F) \to \mathrm{Hom}(\mathcal{O}_Z(k), F) \to \mathrm{Hom}(\mathcal{O}_{Z \cap H_x}(k), F|_{H_x}) \\ &\to\; \mathrm{Ext}^1(\mathcal{O}_Z(k-1), F) = H^1(F(-k+1)) = 0, \end{aligned}$$

we can extend φ_x to the one on Z. Let us denote it by $f_x : \mathcal{O}_Z(k) \to F$. What we have proved is that for each $x \in V$, there exists $H_x \in Gr(n, n-1)$ such that $x \in H_x$ and on H_x, there is an isomorphism $\varphi_x : \mathcal{O}_{Z \cap H_x}(k) \to F|_{H_x}$ which extends to f_x on Z. Fix one such point $x \in Z$ and $f = f_x$. We shall prove that this is in fact an isomorphism. Let us restrict f to $H' = H_y$ for $y \in V$. Since $Z \cap H \neq \phi$, we also have $Z \cap H \cap H' \neq \phi$. Hence $f|_{H \cap H'} \neq 0$. Since the first chern class of $F|_{H'}$ is equal to k and $F|_{H'}$ is a line bundle on a smooth divisor of dimension more than three, Grothendieck-Lefschetz Theorem implies that $f|_{H'}$ is an isomorphism. Hence f is surjective on V by Nakayama's lemma. Since $\mathcal{O}_Z(k)$ and F are rank one and torsion free sheaves, we can see that f is injective. Then on V, f is an isomorphism and since F and $\mathcal{O}_Z(k)$ are reflexive sheaves and $\mathrm{codim}_Z(Z \setminus V) \geq 2$, we can extend this isomorphism to Z. Thus we have finished the proof when $n = 5$. For general n, we can apply the same discussion by using the inductive assumption. i.e., the general cutting of given data (Z, F) on $\mathbf{P}_k^n$ satisfies the same condition of the statement. Hence by induction, F is a line bundle when restricted to general hyperplanes. So the same discussion on $n = 5$ can be applied. q.e.d.

This proposition is not true on a divisor Z such that $\mathrm{codim}_{\mathbf{P}^n_k}(\mathrm{Sing}(Z)) \leq 4$. Let us make a counter example as follows.

Let us put $P = \mathbf{P}^n_k$, k as above, $n \geq 4$, and put $X, Y, Z, W, \cdots$ as coordinates of P. Consider a divisor $D \subset X$ defined by $XY - ZW = 0$ and Weil divisors $W_1 = (X = W = 0)$, $W_2 = (Y = Z = 0)$ in D. It is easy to see that $\mathrm{Sing}(D)$ is a linear space of codimension three in Z. For example when $n = 4$, $\mathrm{Sing}(D) = \{(0:0:0:0:1)\}$ and when $n = 5$, it is a projective line. In particular, D is normal. Of course, W_i is not a Cartier divisor. If we put $f = \overline{Z}/\overline{X} \in k(D)$, we can see that W_1 and W_2 are rationally equivalent and they are invertible along D by this rational function f . Hence from these data, we can construct a diagram of an elementary transformation (using Theorem 2.10) and from the easy calculation, we can see that $\mathrm{elem}_F(\mathcal{O}_X^2) \simeq \mathcal{O}_X(-1)^2$. We can also see that $B = W_1 \cap W_2 \neq \phi$ implies that F is not a line bundle. i.e., even if the elementary transformation of $\mathcal{O}^2_{\mathbf{P}^n_k}$ by the data (D, F) is a splitting bundle, F is not always locally free. This also gives an MCM $\mathcal{O}_D$-module of rank one, generated by two global sections and $H^i(D, F(k)) = 0$ for all $k \in \mathbb{Z}$ and $i = 1, 2, \cdots, n-2$, however not a line bundle.

Remark 4.1
In the above discussion, we used the fact that F is line bundle if and only if $B = \phi$ if and only if $\forall W_i$ is a Cartier divisor. This is easy to show.

5 An Example

In this section, we shall show an example of a vector bundle construction by an elementary transformation introduced in this article. What we are going to construct is the bundle, constructed by Tango in [Ta-1], which is the only known indecomposable rank two bundle on $\mathbf{P}^5_k$ where k is an algebraically closed field of characteristic two (for our convenience, let us call this bundle Tango bundle and denote by E_T). Here, we assume that the chern polynomial of E_T is adjusted as $c_t(E_T) = 1 - 6t + 12t^2$. Now, let us find 1-ET-data (Z, F) for $\mathcal{O}^2_{\mathbf{P}^5_k}$ such that $\mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}^5_k}) \simeq E_T$. This construction is as follows.

On $\mathbf{P}^5_k$, let us put $U_i = (\mathbf{P}^5_k)_{X_i}$ and define an ideal sheaf $I, J \subset \mathcal{O}_{P^5_k}$. For that purpose, let us define the local generators of I and J on each open set U_i as follows.

$I|_{U_0} = 1/X_0^6(X_5^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_2^2X_5^2)x,\ X_0^4X_1^2 + X_5^4X_4^2 + X_5^2X_0^2x + X_2^4X_0^2 + X_4^4X_3^2 + X_4^2X_2^2x)$ (where $x = X_0X_3 + X_1X_4 + X_2X_5$),

$I|_{U_1} = 1/X_1^6(X_1^6 + X_3^6,\ X_1^4X_2^2 + X_3^4X_5^2 + X_3^2X_1^2x),$

$I|_{U_2} = 1/X_2^6(X_2^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_1^2X_4^2 + X_2^2X_5^2)x,\ X_1^4X_2^2 + X_3^4X_5^2 + X_3^2X_1^2x),$

$I|_{U_3} = 1/X_3^6(X_1^4X_2^2 + X_3^4X_5^2 + X_3^2X_1^2x,\ X_1^6 + X_3^6),$

$I|_{U_4} = 1/X_4^6(X_0^4X_1^2 + X_5^4X_4^2 + X_5^2X_0^2x + X_2^4X_0^2 + X_4^4X_3^2 + X_4^2X_2^2x,\ X_2^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_1^2X_4^2 + X_2^2X_5^2)x),$

$I|_{U_5} = 1/X_5^6(X_1^4X_2^2 + X_3^4X_5^2 + X_3^2X_1^2x,\ X_5^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_2^2X_5^2)x),$

$J|_{U_0} = 1/X_0^6(X_5^4X_3^2 + X_1^4X_0^2 + X_1^2X_5^2x + X_4^4X_5^2 + X_0^4X_2^2 + X_0^2X_4^2x,\ X_4^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_1^2X_4^2)x),$

$J|_{U_1} = 1/X_1^6(X_3^4X_4^2 + X_2^4X_1^2 + X_2^2X_3^2x,\ X_1^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_1^2X_4^2 + X_2^2X_5^2)x),$

$J|_{U_2} = 1/X_2^6(X_3^4X_4^2 + X_2^4X_1^2 + X_2^2X_3^2x,\ X_2^6 + X_3^6)$

$J|_{U_3} = 1/X_3^6(X_2^6 + X_3^6,\ X_3^4X_4^2 + X_2^4X_1^2 + X_2^2X_3^2x),$

$J|_{U_4} = 1/X_4^6(X_4^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_1^2X_4^2)x,\ X_3^4X_4^2 + X_2^4X_1^2 + X_2^2X_3^2x),$

$J|_{U_5} = 1/X_5^6(X_1^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_1^2X_4^2 + X_2^2X_5^2)x,\ X_4^4X_5^2 + X_0^4X_2^2 + X_0^2X_4^2x + X_5^4X_3^2 + X_1^4X_0^2 + X_1^2X_5^2x).$

Now, let us define Z as the zero scheme of $(X_1^6 + X_2^6 + X_3^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_1^2X_4^2 + X_2^2X_5^2)x = 0)$, and take a rational function $f = (X_5^4X_3^2 + X_1^4X_0^2 + X_1^2X_5^2x + X_4^4X_5^2 + X_0^4X_2^2 + X_0^2X_4^2x)/(X_5^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_2^2X_5^2)x) \in k(Z)$. We have finished the preparation for constructing our (Z, F).

By the calculation, we can see that Z is integral (but not normal) and that $I_i|_{U_i \cap U_j} \simeq I_j|_{U_i \cap U_j},\ J_i|_{U_i \cap U_j} \simeq J_j|_{U_i \cap U_j}$ for all $i, j = 0, \cdots, 5$. Hence they define ideal sheaves $I, J \subset \mathcal{O}_{\mathbf{P}^5_k}$. Note that from this definition, the closed subschemes defined by I and J are contained in Z. From the calculation, we can see that f defines an isomorphism from I to J. Note that both data cannot be patched if the characteritc of the base field is not two. In the above notation, let us put the generator of I (resp : J) on U_i as (t^i_{11}, t^i_{21}) (resp : (t^i_{12}, t^i_{22})), where the order is the same as the above one. e.g.,

$$\begin{aligned} t^0_{11} &= 1/X_0^6(X_5^6 + X_0^2X_1^2X_2^2 + X_3^2X_4^2X_5^2 + (X_0^2X_3^2 + X_2^2X_5^2)x), \\ t^0_{21} &= 1/X_0^6(X_0^4X_1^2 + X_5^4X_4^2 + X_5^2X_0^2x + X_2^4X_0^2 + X_4^4X_3^2 + X_4^2X_2^2x). \end{aligned}$$

We can check that $(f\overline{t^i_{11}}, f\overline{t^i_{21}}) = (\overline{t^i_{12}}, \overline{t^i_{22}})$ for $i = 0, \cdots, 5$ (where $\overline{t_{ji}}$ is the image of t_{ji} by the morphism $\mathcal{O}_X \to \mathcal{O}_Z$). Then let us define a rank one $\mathcal{O}_Z$-module F as the cokernel of the morphism $I \stackrel{(1,f)}{\to} \mathcal{O}_Z^2$. If we denote the zero scheme of I (resp : J) by W_1 (resp : W_2), it is obvious that they are rationally equivalent and from the choice of generators, we can see that (W_1, W_2) are invertible along Z with respect to the isomorphism induced by f. Here, by using Theorem 2.10, we can see that (Z, F) is 1-ET-data for $\mathcal{O}^2_{\mathbf{P}^5_k}$. Let us put $E(-6) := \mathrm{elem}_F(\mathcal{O}^2_{\mathbf{P}^5_k})$ and we shall show that $E(-6) \simeq E_T$. To see this, put $T_k = (t^k_{ij})_{i,j=1,2}$. Then we can make the diagram of our elementary transformation as follows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}^2_{\mathbf{P}^5}(-6) \arrow[r, equal] \arrow[d, "\alpha'"'] & \mathcal{O}^2_{\mathbf{P}^5}(-6) \arrow[d] & & \\
0 \arrow[r] & E(-6) \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}^2_{\mathbf{P}^5} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & \mathcal{O}^2_Z \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that from the construction of I and J, the injection α is represented on U_i by ${}^tT_i$. Hence if we denote the transition matrices of $E(-6)$ on $U_i \cap U_j$ by N_{ij}, then we have

$$ {}^tT_i N_{ij} = {}^tT_j \quad (i, j = 0, \cdots, 5). \tag{12}$$

This follows from $id \circ \alpha|_{U_j} = \alpha|_{U_i} \circ N_{ij}$. Therefore we can calculate the transition matrix of $E(-6)$ from these data and comparing them with the one calculated in [Ta-2], we can conclude that $\{N_{ij}\}$ is the transition matrices of $(E_T(6))^*$. Since $(E_T(6))^* \simeq E_T$, we can conclude that $E(-6) \simeq E_T$.

Note that since this Z is integral but not normal, we cannot replace these ET-data by geometric ET-data. Of course, if we can find more general sections of E_T (and this is possible if tensored enough ample line bundles), it can be possible.

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