CANONICAL HEIGHT FUNCTIONS DEFINED ON THE AFFINE PLANE ASSOCIATED WITH REGULAR POLYNOMIAL AUTOMORPHISMS

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Abstract. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism (e.g., a Hénon map) defined over a number field K. We construct canonical height functions defined on $\mathbb{A}^2(\overline{K})$ associated with f. These functions satisfy the Northcott finiteness property, and an $\overline{K}$-valued point on $\mathbb{A}^2(\overline{K})$ is f-periodic if and only if its height is zero. As an application of canonical height functions, we give a refined estimate on the number of points with bounded height in an infinite f-orbit.

Introduction and the statement of the main results

One of the basic tools in Diophantine geometry is the theory of height functions. On Abelian varieties defined over a number field, Néron and Tate developed the theory of canonical height functions that behave well relative to the $[n]$-th map (cf. [8, Chap. 5]). On certain K3 surfaces with two involutions, Silverman [12] developed the theory of canonical height functions that behave well relative to the two involutions. For the theory of canonical height functions on some other projective varieties, see for example [1], [14], [6]. In this paper, we construct canonical height functions defined on the affine plane, which behave well relative to regular polynomial automorphisms, and in particular Hénon maps.

A Hénon map (also called a generalized Hénon map) is a polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ of the form

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}, \tag{0.1}$$

where $a \neq 0$ and p is a polynomial of degree $d \geq 2$. Hénon maps are basic objects in polynomial automorphisms of $\mathbb{A}^2$ in the sense that every polynomial automorphism of $\mathbb{A}^2$ of degree $d \geq 2$ over $\mathbb{C}$ is conjugate to either an elementary map, or a composite of Hénon maps (Friedland–Milnor [3]). A regular polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ is by definition a polynomial automorphism of $\mathbb{A}^2$ of degree greater than or equal to 2 such that the unique point of indeterminacy of $\overline{f}$ is different from the the unique point of indeterminacy of $\overline{f^{-1}}$, where the birational map $\overline{f} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ (resp. $\overline{f^{-1}} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$) is the extension of f (resp. f^{-1}). Hénon maps are examples of regular polynomial automorphisms. For more details, see the survey of Sibony [10] and the references therein. Over a number field, Silverman [13] studied arithmetic properties of quadratic Hénon maps, and then Denis [2] studied arithmetic properties of Hénon maps and some classes of polynomial automorphisms.

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Marcello [9] studied arithmetic properties of some other classes of polynomial automorphisms of the affine spaces, including regular polynomial automorphisms.

Our first result shows the existence of height functions that behave well relative to regular polynomial automorphisms of $\mathbb{A}^2$.

Theorem A. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$ defined over a number field K. Then there exists a function $\widehat{h} : \mathbb{A}^2(\overline{K}) \to \mathbb{R}$ with the following properties:*

(i) *$h_{nv} \gg\ll \widehat{h}$ on $\mathbb{A}^2(\overline{K})$ (Here h_{nv} is the logarithmic naive height function, and $h_{nv} \gg\ll \widehat{h}$ means that there are positive constants a_1, a_2 and constants b_1, b_2 such that $a_1 h_{nv} + b_1 \leq \widehat{h} \leq a_2 h_{nv} + b_2$) ;*

(ii) *$\widehat{h} \circ f + \widehat{h} \circ f^{-1} = \left(d + \frac{1}{d}\right) \widehat{h}$.*

Moreover, $\widehat{h}$ enjoys the following uniqueness property: if $\widehat{h}'$ is another function satisfying (i) and (ii) such that $\widehat{h}' = \widehat{h} + O(1)$, then $\widehat{h}' = \widehat{h}$. We call a function $\widehat{h}$ satisfying (i) and (ii) a canonical height function *associated with the regular polynomial automorphism of f.*

It follows from (i) that $\widehat{h}$ satisfies the Northcott finiteness property. Namely, for any positive number M and positive integer D, the set $\{x \in \mathbb{A}^2(\overline{K}) \mid [K(x) : K] \leq D,\ \widehat{h}(x) \leq M\}$ is finite. This leads to the following corollary, which shows that the set of $\overline{K}$-valued f-periodic points is not only the set of bounded height but also characterized as the set of height zero with respect to a canonical height function associated with f.

Corollary B. *Let $\widehat{h} : \mathbb{A}^2(\overline{K}) \to \mathbb{R}$ be a canonical height function associated with a regular polynomial automorphism f defined over a number field K. Then*

(1) *$\widehat{h}(x) \geq 0$ for all $x \in \mathbb{A}^2(\overline{K})$.*

(2) *$\widehat{h}(x) = 0$ if and only if x is f-periodic. (Here, $x \in \mathbb{A}^2(\overline{K})$ is said to be f-periodic if $f^m(x) = x$ for some positive integer m.)*

As an application of canonical height functions, we obtain an estimate on the number of points with bounded height in an infinite f-orbit. First we introduce some notation and terminology. For a canonical height function $\widehat{h}$ associated with f, we set

$$\widehat{h}^+(x) = \frac{d^2}{d^4 - 1}\left(d\widehat{h}(f(x)) - \frac{1}{d}\widehat{h}(f^{-1}(x))\right), \quad \widehat{h}^-(x) = \frac{d^2}{d^4 - 1}\left(d\widehat{h}(f^{-1}(x)) - \frac{1}{d}\widehat{h}(f(x))\right).$$

Then $\widehat{h}^+ \geq 0$ and $\widehat{h}^- \geq 0$, and $\widehat{h}^+(x) = 0$ if and only if $\widehat{h}^-(x) = 0$ if and only if x is f-periodic (cf. Lemma 5.1). For a point $x \in \mathbb{A}^2(\overline{K})$, let $O_f(x) := \{f^l(x) \mid l \in \mathbb{Z}\}$ denote the f-orbit of x. For a non f-periodic point $x \in \mathbb{A}^2(\overline{K})$, we set

$$\widehat{h}(O_f(x)) = \log_d\left(\widehat{h}^+(y)\widehat{h}^-(y)\right)$$

for any $y \in O_f(x)$. Then $\widehat{h}(O_f(x))$ is well-defined, i.e., $\widehat{h}(O_f(x))$ is independent of the choice of $y \in O_f(x)$. Moreover, as a function of x, we have $\widehat{h}(O_f(x)) \gg\ll \min_{y \in O_f(x)} \log_d \widehat{h}(y)$ on $\mathbb{A}^2(\overline{K}) \setminus \{f\text{-periodic points}\}$ (cf. Lemma 5.2).

For regular polynomial automorphisms of degree d, it is known that $\lim_{T\to\infty} \frac{\#\{y\in O_f(x) \mid h_{nv}(y)\le T\}}{\log_d T^2} = 1$, where $x \in \mathbb{A}^2(\overline{K})$ is not an f-periodic point ([13, Theorem C], [2, Théorème 2], and [9, Théorème A]). The next theorem gives its refinement.

Theorem C. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \ge 2$ defined over a number field K, and $\widehat{h}$ a canonical height function associated with f. Suppose $x \in \mathbb{A}^2(\overline{K})$ is not an f-periodic point. Then,*

$$\#\{y \in O_f(x) \mid h_{nv}(y) \le T\} = \log_d T^2 - \widehat{h}(O_f(x)) + O(1) \quad \text{as } T \to \infty, \tag{0.2}$$

where the $O(1)$ constant depends only on f and the choice of $\widehat{h}$.

We can construct a canonical height function $\widehat{h}_\circ$ starting from h_{nv} and considering iteration by f and f^{-1} (cf. Theorem 4.1 and its proof). Thus, if we take $\widehat{h}_\circ$ for $\widehat{h}$ in Theorem C, then the $O(1)$ constant in (0.2) depends only on f.

The contents of this paper is as follows. In §1 we briefly review the properties of height functions. In §2 and in §3 we show that if f is a regular polynomial automorphism of degree $d \ge 2$ then there is a constant c such that

$$h_{nv}(f(x)) + h_{nv}(f^{-1}(x)) \ge \left(d + \frac{1}{d}\right) h_{nv}(x) - c \tag{0.3}$$

for all $x \in \mathbb{A}^2(\overline{K})$. To show (0.3) when f is a Hénon map in §2, we use the results of Hubbard, Papadopol and Veselow [4, §2], which gives an explicit description of blow-ups of $\mathbb{P}^2$ such that $f : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ extends to a morphism $\varphi : W \to \mathbb{P}^2$, where W is the surface obtained from these blow-ups. To show (0.3) in general in §3, we give a similar explicit description of blow-ups for regular polynomial automorphisms, using the results of §2 and the classical results of Jung [5] and van der Kulk [7] about polynomial automorphisms of the affine plane. In §4 we prove Theorem A and Corollary B in a more general setting of polynomial automorphisms of $\mathbb{A}^n$ that satisfy an inequality similar to (0.3). In §5 we prove Theorem C in this more general setting. On certain K3 surfaces, Silverman counted the number of points with bounded height in a given infinite chain ([12, §3]). Our method of proof of Theorem C is inspired by his method.

1. Quick review on height theory

In this section, we briefly review the properties of height functions that we will use in this paper.

Let K be a number field and O_K its ring of integers. For $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$, the logarithmic naive height of x is defined by

$$h_{nv}(x) = \frac{1}{[K:\mathbb{Q}]}\left[\sum_{P \in \operatorname{Spec}(O_K)\setminus\{0\}} \max_{0\le i\le n}\{-\operatorname{ord}_P(x_i)\}\log \#(O_K/P) + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \max_{0\le i\le n}\{\log|\sigma(x_i)|\}\right].$$

This definition naturally extends to all points $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$ as to give the logarithmic naive height function $h_{nv} : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$.

We begin by the following two basic properties of height functions.

Theorem 1.1 (Northcott's finiteness theorem, [11] Corollary 3.4). *For any positive number M and positive integer D, the set*

$$\left\{x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(x) : \mathbb{Q}] \leq D,\ h_{nv}(x) \leq M\right\}$$

is finite.

Theorem 1.2 ([11] Theorem 3.3, [8] Chap. 4, Prop. 5.2). (1) (Height machine) *For any projective variety defined over $\overline{\mathbb{Q}}$, there exists a unique map*

$$h_X : \mathrm{Pic}(X) \longrightarrow \frac{\{\textit{real-valued functions on } X(\overline{\mathbb{Q}})\}}{\{\textit{real-valued bounded functions on } X(\overline{\mathbb{Q}})\}}, \quad L \mapsto h_{X,L}$$

with the following properties:

(i) $h_{X,L\otimes M} = h_{X,L} + h_{X,M} + O(1)$ *for any* $L, M \in \mathrm{Pic}(X)$;

(ii) *If* $X = \mathbb{P}^n$ *and* $L = \mathcal{O}_{\mathbb{P}^n}(1)$, *then* $h_{\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)} = h_{nv} + O(1)$;

(iii) *If* $f : X \to Y$ *is a morphism of projective varieties and L is a line bundle on X, then* $h_{X,f^*L} = h_{Y,L} \circ f + O(1)$.

(2) (Positivity of height) *Let X be projective variety defined over $\overline{\mathbb{Q}}$ and L a line bundle on X. We set* $B = \mathrm{Supp}(\mathrm{Coker}(H^0(X, L) \otimes \mathcal{O}_X \to L))$. *Then there exists a constant c_1 such that $h_{X,L}(x) \geq c_1$ for all $x \in (X \setminus B)(\overline{\mathbb{Q}})$.*

A rational map $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \cdots \to \mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$ is said to be of degree d if the F_i's are homogeneous polynomials of degree d over $\overline{\mathbb{Q}}$, with no common factors. Let $I_f \subset \mathbb{P}^n(\overline{\mathbb{Q}})$ denote the locus of indeterminacy.

Theorem 1.3 ([8] Chap. 4, Lemma 1.6). *Let $f : \mathbb{P}^n \cdots \to \mathbb{P}^n$ be rational map of degree d defined over $\overline{\mathbb{Q}}$. Then there exists a constant c_2 such that*

$$h_{nv}(f(x)) \leq d\, h_{nv}(x) + c_2$$

for all $x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus I_f$.

2. Geometric properties of Hénon maps

In this section, we will show (0.3) for Hénon maps. The results of this section are generalized in §3 for regular polynomial automorphisms.

Consider the Hénon map

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix},$$

where $a \neq 0$ and p is a polynomial of degree $d \geq 2$. Then f extends to the birational map $\overline{f} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ given in homogeneous coordinates as

$$\overline{f}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} Z^d p(X/Z) - aYZ^{d-1} \\ XZ^{d-1} \\ Z^d \end{bmatrix}. \tag{2.1}$$

Let H be the line at infinity on $\mathbb{P}^2$. Then $\overline{f}$ has the unique point of indeterminacy $\mathbf{p} = {}^t[0, 1, 0]$, and $\overline{f}$ maps $H \setminus \{\mathbf{p}\}$ to a point $\mathbf{q} = {}^t[1, 0, 0]$.

To show (0.3), as Silverman [13, §2] did for quadratic Hénon maps, we need an explicit description of blow-ups at (infinitely near) points on $\mathbb{P}^2$ that resolve the point of indeterminacy of $\overline{f}$. This was carried out by Hubbard–Papadopol–Veselov [4, §2] in their compactification of Hénon maps in $\mathbb{C}^2$ as dynamical systems. Let us put together their results in the following theorem. (Note that, for the next theorem, the field of definition of f can be any field, and $p(x)$ need not be monic.)

Theorem 2.1 ([4], §2)**.** (1) *The Hénon map* (2.1) *becomes well-defined after a sequence of* $2d-1$ *blow-ups. Explicitly, blow-ups are described as follows*:

- (i) *First blow-up at* $\mathbf{p}$;
- (ii) *Next blow up at the unique point of indeterminacy, which is given by the intersection of the exceptional divisor and the proper transform of* H;
- (iii) *For the next* $d-2$ *times after* (ii), *blow-up at the unique point of indeterminacy, which is given by the intersection of the last exceptional divisor and the proper transform of the first exceptional divisor*;
- (iv) *For the next* $d-1$ *times after* (iii), *blow-up at the unique point of indeterminacy, which lies on the last exceptional divisor but not on the proper transform of the other exceptional divisors.*

(2) *Let* $\overline{f_{2d-1}} : W \to \mathbb{P}^2$ *be the extension of the Hénon map after the sequence of* $2d-1$ *blow-ups. Let* $E_i^{'}$ *denote the proper transform of* i*-th exceptional divisor* $(i = 1, \cdots, 2d-1)$. *Then* $\overline{f_{2d-1}}$ *maps* $E_i^{'}$ $(i = 1, \cdots, 2d-2)$ *to* $\mathbf{q}$, *while* $E_{2d-1}^{'}$ *is mapped to* H *by an isomorphism.*

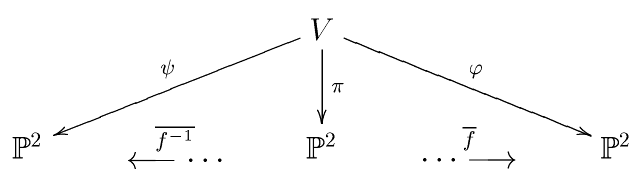
(3) ${E_1^{'}}^2 = -d$, ${E_i^{'}}^2 = -2$ $(i = 2, \cdots, 2d-2)$, *and* ${E_{2d-1}^{'}}^2 = -1$.

The inverse $\overline{f^{-1}}$ of the Hénon map $\overline{f}$ is given by

$$\overline{f^{-1}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} YZ^{d-1} \\ \frac{1}{a}\left(p(Y/Z)Z^d - XZ^{d-1}\right) \\ Z^d \end{bmatrix},$$

which has the unique point of indeterminacy $\mathbf{q}$. Let $\pi_W : W \to \mathbb{P}^2$ be the blow-ups of $\mathbb{P}^2$ given in Theorem 2.1. We will make blow-ups so that the birational map $\overline{f^{-1}} \circ \pi_W : W \cdots \to \mathbb{P}^2$ lifts to a morphism. Noting that π_W induces an isomorphism $\pi_W^{-1}(\mathbb{P}^2 \setminus \{\mathbf{p}\}) \to \mathbb{P}^2 \setminus \{\mathbf{p}\}$, we take $\mathbf{q}' \in W$ with $\pi_W(\mathbf{q}') = \mathbf{q}$. In a parallel way as for $\mathbf{p}$, $\overline{f^{-1}} \circ \pi_W : W \cdots \to \mathbb{P}^2$ extends to a morphism after $2d-1$ blow-ups starting at $\mathbf{q}'$.

To summarize, let V be the projective surface obtained by successive $2d-1$ blow-ups of $\mathbb{P}^2$ at $\mathbf{p}$ as in Theorem 2.1 and then successive $2d-1$ blow-ups at $\mathbf{q}$ in a parallel way as in Theorem 2.1. Let $\pi : V \to \mathbb{P}^2$ denote the morphism of blow-ups. Then $\overline{f} \circ \pi$ extends to a morphism $\varphi : V \to \mathbb{P}^2$, and $\overline{f^{-1}} \circ \pi$ extends to a morphism $\psi : V \to \mathbb{P}^2$.



Let E_i $(1 \leq i \leq 2d-1)$ be the proper transform of i-th exceptional divisor on V on the side of $\mathbf{p}$, and F_j $(1 \leq j \leq 2d-1)$ be the proper transform of j-th exceptional divisor on V

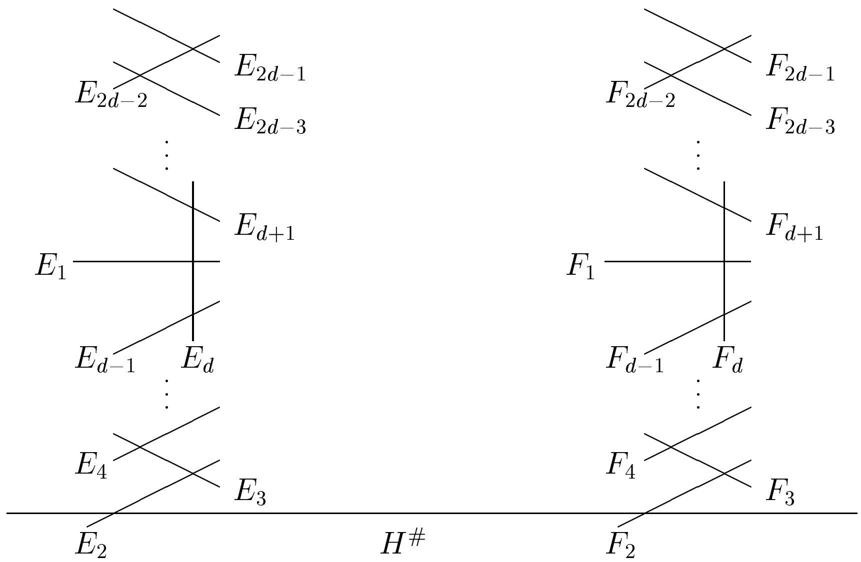


FIGURE 1. The configuration after blow-ups. The line $H^{\#}$ has the self-intersection number -3. The lines E_1 and F_1 have the self-intersection numbers $-d$. The lines $E_2, E_3, \cdots, E_{2d-2}$ and $F_2, F_3, \cdots, F_{2d-2}$ have the self-intersection numbers -2. The lines E_{2d-1} and F_{2d-1} have the self-intersection numbers -1.

on the side of $\mathbf{q}$. Let $H^{\#}$ be the proper transform of H. The configuration of $H^{\#}$, E_i and F_j is illustrated in Figure 1.

Proposition 2.2. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map of degree $d \geq 2$. Let the notation be as above.*

(1) *As divisors on V, we have*

$$\pi^* H = H^{\#} + \sum_{i=1}^{d} iE_i + \sum_{i=d+1}^{2d-1} dE_i + \sum_{j=1}^{d} jF_j + \sum_{j=d+1}^{2d-1} dF_j,$$

$$\varphi^* H = dH^{\#} + E_1 + \sum_{i=2}^{d} dE_i + \sum_{i=d+1}^{2d-1} (2d-i)E_i + \sum_{j=1}^{d} jdF_j + \sum_{j=d+1}^{2d-1} d^2 F_j,$$

$$\psi^* H = dH^{\#} + \sum_{i=1}^{d} idE_i + \sum_{i=d+1}^{2d-1} d^2 E_i + F_1 + \sum_{j=2}^{d} dF_j + \sum_{j=d+1}^{2d-1} (2d-j)F_j.$$

(2) *As a $\mathbb{Q}$-divisor on V, we have*

$$\varphi^* H + \psi^* H = \left(d + \frac{1}{d}\right) \pi^* H + D,$$

where D is the $\mathbb{Q}$-effective divisor given by

$$D = \frac{d^2-1}{d}H^{\#} + \frac{d-1}{d}E_1 + \sum_{i=2}^{d}\frac{d^2-i}{d}E_i + \sum_{i=d+1}^{2d-1}(2d-i-1)E_i + \frac{d-1}{d}F_1 + \sum_{j=2}^{d}\frac{d^2-j}{d}F_j + \sum_{j=d+1}^{2d-1}(2d-j-1)F_j.$$

Proof. We will show the expression for φ^*H. Since φ maps $H^{\#}$, E_i $(1 \le i \le 2d-2)$ and F_j $(1 \le j \le 2d-1)$ to the point $\mathbf{q}$, we have

$$\varphi^*H \cdot H^{\#} = 0, \qquad \varphi^*H \cdot E_i = 0, \qquad \varphi^*H \cdot F_j = 0$$

for $1 \le i \le 2d-2$ and $1 \le j \le 2d-1$. Since φ maps E_{2d-1} to H isomorphically, we have

$$\varphi^*H \cdot E_{2d-1} = 1.$$

Noting that the Picard group of V is generated by $H^{\#}, E_i, F_j$ $(1 \le i,j \le 2d-1)$, we set $\varphi^*H = aH^{\#} + \sum_{i=1}^{2d-1} b_iE_i + \sum_{j=1}^{2d-1} c_jF_j$. From the above information and the information of the configuration after blow-ups (cf. Figure 1), we have the system of linear equations

$$-3a + b_2 + c_2 = 0, \qquad \begin{cases} -db_1 + b_d = 0 \\ a - 2b_2 + b_3 = 0 \\ b_{i-1} - 2b_i + b_{i+1} = 0 \\ b_1 + b_{d-1} - 2b_d + b_{d+1} = 0 \\ b_{2d-2} - b_{2d-1} = 1, \end{cases} \qquad \begin{cases} -dc_1 + c_d = 0 \\ a - 2c_2 + c_3 = 0 \\ c_{j-1} - 2c_j + c_{j+1} = 0 \\ c_1 + c_{d-1} - 2c_d + c_{d+1} = 0 \\ c_{2d-2} - c_{2d-1} = 0, \end{cases}$$

where $i = 3, \cdots, d-1, d+1, \cdots, 2d-2$ and $j = 3, \cdots, d-1, d+1, \cdots, 2d-2$. By solving this system, we obtain the expression for φ^*H. Similarly we obtain the formula for ψ^*H. The formula for π^*H follows from the construction of V. (We can also show this by using $\pi^*H \cdot H^{\#} = 1$, $\pi^*H \cdot E_i = 0$ and $\pi^*H \cdot F_j = 0$ for all i and j.) The assertion (2) follows from (1). □

Theorem 2.3. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map of degree $d \ge 2$ defined over a number field K. Then, there exists a constant c such that*

$$h_{nv}(f(x)) + h_{nv}(f^{-1}(x)) \ge \left(d + \frac{1}{d}\right) h_{nv}(x) - c$$

for all $x \in \mathbb{A}^2(\overline{K})$.

Proof. We can prove Theorem 2.3 as in [13, Theorem 2.1]. Indeed, take $x \in \mathbb{A}^2(\overline{K})$. Since $\pi : V \to \mathbb{P}^2$ gives an isomorphism $\pi|_{\pi^{-1}(\mathbb{A}^2)} : \pi^{-1}(\mathbb{A}^2) \to \mathbb{A}^2$, there is a unique point $\widetilde{x} \in V$ with $\pi(\widetilde{x}) = x$. By Proposition 2.2, we have

$$h_{V,\mathcal{O}_V(\varphi^*H)}(\widetilde{x}) + h_{V,\mathcal{O}_V(\psi^*H)}(\widetilde{x}) = \left(d + \frac{1}{d}\right) h_{V,\mathcal{O}_V(\pi^*H)}(\widetilde{x}) + h_{V,\mathcal{O}_V(D)}(\widetilde{x}) + O(1).$$

It follows from Theorem 1.2(1) that $h_{V,\mathcal{O}_V(\varphi^* H)}(\widetilde{x}) = h_{\mathbb{P}^2,\mathcal{O}_V(H)}(\varphi(\widetilde{x}))+O(1) = h_{\mathbb{P}^2,\mathcal{O}_V(H)}(f(x))+O(1)$. We similarly have $h_{V,\mathcal{O}_V(\psi^* H)}(\widetilde{x}) = h_{\mathbb{P}^2,\mathcal{O}_V(H)}(f^{-1}(x)) + O(1)$ and $h_{V,\mathcal{O}_V(\pi^* H)}(\widetilde{x}) = h_{\mathbb{P}^2,\mathcal{O}_V(H)}(x) + O(1)$. On the other hand, since $\widetilde{x} \notin \mathrm{Supp}(D)$, we know from Theorem 1.2(2) that there is a constant c_2 independent of $\widetilde{x}$ such that $h_{V,\mathcal{O}_V(D)}(\widetilde{x}) \geq c_2$. Hence we get the assertion. □

3. Geometric properties of regular polynomial automorphisms

In this section, we show (0.3) for regular polynomial automorphisms of $\mathbb{A}^2$. First we recall the definition of regular polynomial automorphisms of $\mathbb{A}^2$. Consider a polynomial automorphism of degree $d \geq 2$ of the form

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix},$$

where $p(x,y)$ and $q(x,y)$ are polynomials in two variables, and d is the maximum of the degree of p and the degree of q. Let $\overline{f} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ be the extension of f given in homogeneous coordinates as

$$\overline{f} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} Z^d p(X/Z, Y/Z) \\ Z^d q(X/Z, Y/Z) \\ Z^d \end{bmatrix}.$$

As in §2, let H denote the line at infinity. Then $\overline{f}$ has a unique point of indeterminacy on H, denoted by $\mathbf{p}$. Let $f^{-1} : \mathbb{A}^2 \to \mathbb{A}^2$ be the inverse of f, and $\overline{f^{-1}} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ be its extension. Then $\overline{f^{-1}}$ has a unique point of indeterminacy on H, denoted by $\mathbf{q}$. A polynomial automorphism of $\mathbb{A}^2$ is said to be *regular* if $\mathbf{p} \neq \mathbf{q}$. Note that Hénon maps are regular, since $\mathbf{p} = {}^t[0,1,0]$ and $\mathbf{q} = {}^t[1,0,0]$ for Hénon maps.

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism (f need not be a regular polynomial automorphism for the moment). We will give an explicit description of blow-ups of $\mathbb{P}^2$ that resolve the (infinitely near) points of indeterminacy of $\overline{f}$. To this end, we use the classical results of Jung [5] and van der Kulk [7] as follows. For a field K, let

$$A = \left\{ f : \mathbb{A}^2 \to \mathbb{A}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by+s \\ cx+dy+t \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d,s,t \in K, \\ ad-bc \neq 0 \end{matrix} \right\}$$

be the group of affine automorphisms, and let

$$E = \left\{ f : \mathbb{A}^2 \to \mathbb{A}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+P(y) \\ by+c \end{pmatrix} \;\middle|\; \begin{matrix} a,b \in K^\times, c \in K \\ P(y) \in K[Y] \end{matrix} \right\}$$

be the group of elementary automorphisms (also called triangular automorphisms, or de Jonquères automorphisms).

Theorem 3.1 (Jung, van der Kulk, cf. [3], §2)**.** *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism of degree $d \geq 2$.*

(1) *There exist an integer $l \geq 1$, $\alpha_1, \cdots, \alpha_l, \alpha_{l+1} \in A$, and $\varepsilon_1, \cdots, \varepsilon_l \in E$ such that $\varepsilon_i \notin A$ for $1 \leq i \leq l$ and $\alpha_i \notin E$ for $2 \leq i \leq l$ and that*

$$f = \alpha_{l+1} \circ \varepsilon_l \circ \alpha_l \circ \cdots \circ \varepsilon_1 \circ \alpha_1.$$

(2) *If* $f = \alpha'_{m+1} \circ \varepsilon'_m \circ \alpha'_m \circ \cdots \circ \varepsilon'_1 \circ \alpha'_1$ *is another such decomposition, then* $m = l$ *and there exist* $\beta_1, \cdots, \beta_l, \lambda_1, \cdots, \lambda_l \in A \cap E$ *such that*

$$\alpha'_1 = \lambda_1 \circ \alpha_1, \qquad \alpha'_i = \lambda_i \circ \alpha_i \circ \beta_i^{-1} \quad (2 \le i \le l),$$
$$\alpha'_{l+1} = \alpha_l \circ \beta_{l+1}^{-1}, \qquad \varepsilon'_i = \beta_{i+1} \circ \varepsilon_i \circ \lambda_i^{-1} \quad (1 \le i \le l).$$

In other words, the group of polynomial automorphism of $\mathbb{A}^2$ *is the amalgamated product of* A *and* E *over* $A \cap E$.

If we set $d_i = \deg \varepsilon_i$ $(1 \le i \le l)$, then $d_i \ge 2$. It follows from Theorem 3.1(2) that the l-tuple of integers $(d_1, \cdots, d_l)$ is independent of the choice of decompositions of f. We call $(d_1, \cdots, d_l)$ the *polydegree* of f. (We note that the polydegree of f defined here is different from the one given in [3, §2], where the polydegree of f is defined as $(d_l, \cdots, d_1)$. However, for our purpose, the order $(d_1, \cdots, d_l)$ is convenient.) It follows from [3, Theorem 2.1] that

$$d = d_1 \cdots d_l. \tag{3.1}$$

Now we give an explicit description of blow-ups of $\mathbb{P}^2$ that resolve the points of indeterminacy of $\overline{f}$.

Theorem 3.2. *Let* $\mathbb{A}^2 \to \mathbb{A}^2$ *be a polynomial automorphism of degree* $d \ge 2$, *of polydegree* $(d_1, \cdots, d_l)$. *Let* $\overline{f} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ *be the extension of* f. *Then* $\overline{f}$ *becomes well-defined after* $(2d_1 - 1) + \cdots + (2d_l - 1)$ *blow-ups. Explicitly, blow-ups are described as follows*:

(1) (i) *First blow-up at the unique point of indeterminacy* $\mathbf{p}$ *of* $\overline{f}$;
(ii) *Next blow up at the unique point of indeterminacy, which is given by the intersection of the exceptional divisor and the proper transform of* H;
(iii) *For the next* $d_1 - 2$ *times after* (ii), *blow-up at the unique point of indeterminacy, which is given by the intersection of the last exceptional divisor and the proper transform of the first exceptional divisor*;
(iv) *For the next* $d_1 - 1$ *times after* (iii), *blow-up at the unique point of indeterminacy, which lies on the last exceptional divisor but not on the proper transform of the other exceptional divisors.*

(2) *Let* $\overline{f_{2d_1-1}} : W_1 \cdots \to \mathbb{P}^2$ *be the extension of* f *after the sequence of* $2d_1 - 1$ *blow-ups. Let* $E_i^{(1)'}$ *denote the proper transform of* i*-th exceptional divisor on* W_1 $(i = 1, \cdots, 2d_1 - 1)$. *Let* $\mathbf{p}_2$ *be the unique point of indeterminacy of* $\overline{f_{2d_1-1}}$. *Then* $\mathbf{p}_2 \in E_{2d_1-1}^{(1)'}$ *but* $\mathbf{p}_2 \notin E_i^{(1)'}$ *for* $i = 1, \cdots, 2d_1$.

(3) (iv) *Next blow-up at* $\mathbf{p}_2$: *This produces* $2d_1$*-th exceptional divisor*;
(v) *Next blow up at the unique point of indeterminacy, which is given by the intersection of the* $2d_1$*-th exceptional divisor and the proper transform of the* $(2d_1 - 1)$*-th exceptional divisor*;
(vi) *For the next* $d_2 - 2$ *times after* (ii), *blow-up at the unique point of indeterminacy, which is given by the intersection of the last exceptional divisor and the proper transform of the* $2d_1$*-th exceptional divisor*;
(vii) *For the next* $d_2 - 1$ *times after* (iii), *blow-up at the unique point of indeterminacy, which lies on the last exceptional divisor but not on the proper transform of the other exceptional divisors.*

(4) *Let* $\overline{f_{(2d_1-1)+(2d_2-1)}} : W_2 \cdots \to \mathbb{P}^2$ *be the composite of* $2d_2 - 1$ *blow-ups after* $\overline{f_{2d_1-1}} : W_1 \cdots \to \mathbb{P}^2$. *Let* $E_i^{(2)'}$ *denote the proper transform of* $((2d_1-1)+i)$*-th exceptional divisor on* W_2 $(i = 1, \cdots, 2d_2 - 1)$. *Let* $\mathbf{p}_3$ *be the unique point of indeterminacy of* $\overline{f_{(2d_1-1)+(2d_2-1)}}$. *Then* $\mathbf{p}_3 \in E_{2d_2-1}^{(2)'}$ *but not on the proper transform of the other exceptional divisors.*

(5) *We repeat this procedure for* $d_3, \cdots, d_l$. *Then after the sequence of* $(2d_1-1)+\cdots+(2d_l-1)$ *blow-ups, we obtain the morphism* $\overline{f_{(2d_1-1)\cdots+(2d_l-1)}} : W_l \to \mathbb{P}^2$, *which extends* f. *By slight abuse of notation, we also denote by* $E_i^{(s)'}$ $(s = 1, \cdots, l; i = 1, \cdots, 2d_s - 1)$ *the proper transform of* $((2d_1-1)\cdots+(2d_{s-1}-1)+i)$*-th exceptional divisor on* W_l. *Then, via* $\overline{f_{(2d_1-1)\cdots+(2d_l-1)}}$, $E_{2d_l-1}^{(l)'}$ *is mapped isomorphically to* H, *while* $E_i^{(s)'}$ $((s,i) \neq (l, 2d_l-1))$ *and the proper transform of* H *on* W_l *are mapped to* $\mathbf{q}$.

Proof. First we show the uniqueness of the point of indeterminacy in each step. Suppose W is a non-singular rational surface and $\overline{f_*} : W \cdots \to \mathbb{P}^2$ is a birational map that extends $f : \mathbb{A}^2 \to \mathbb{A}^2$. Then, if $\overline{f_*}$ is not a morphism, then $\overline{f_*}$ has the unique point of indeterminacy. Indeed, if $\mathbf{p}_*$ is a point of indeterminacy of $\overline{f_*}$, then there is a line L on $\mathbb{P}^2$ such that L is contracted to $\mathbf{p}_*$ by $\overline{f_*}^{-1}$. Since $\overline{f_*}$ extends to f, L must be equal to H. Hence $\mathbf{p}_* = \overline{f_*}^{-1}(L)$ and is unique.

Let $f = \alpha_{l+1} \circ \varepsilon_l \circ \alpha_l \circ \cdots \circ \varepsilon_1 \circ \alpha_1$ be a decomposition of f in Theorem 3.1(1). We set

$$g_1 = \varepsilon_1 \circ \alpha_1, \quad \cdots, \quad g_{l-1} = \varepsilon_{l-1} \circ \alpha_{l-1}, \quad g_l = \alpha_{l+1} \circ \varepsilon_l \circ \alpha_l.$$

Then $f = g_l \circ g_{l-1} \circ \cdots \circ g_1$ and each $g_i : \mathbb{A}^2 \to \mathbb{A}^2$ is a polynomial automorphism of degree $d_i \geq 2$. Let $\overline{g_i} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ be the extension of g_i. Let $\mathbf{p}_i'$ be the unique point of indeterminacy of $\overline{g_i}$. Let $\mathbf{q}_i'$ be the unique point of indeterminacy of $\overline{g_i}^{-1}$. The following claim is a key observation.

Claim 3.2.1. $\mathbf{p}_{i+1}' \neq \mathbf{q}_i'$ *for* $1 \leq i \leq l-1$.

Obviously, $\deg(g_{i+1} \circ g_i) \leq \deg(g_{i+1}) \deg(g_i)$. Moreover, $\deg(g_{i+1} \circ g_i) = \deg(g_i) \deg(g_{i+1})$ if and only if $\overline{g_i}(H \setminus \mathbf{p}_i') \neq \mathbf{p}_{i+1}'$ (cf. [10, Proposition 1.4.3]). Since $\overline{g_i}(H \setminus \mathbf{p}_i') = \mathbf{q}_i'$, this means $\deg(g_2 \circ g_1) = \deg(g_1)\deg(g_2)$ if and only if $\mathbf{q}_i' \neq \mathbf{p}_{i+1}'$. On the other hand, by (3.1) we have $\deg(g_l \circ \cdots \circ g_1) = \deg(g_l) \cdots \deg(g_1)$. Thus we get the claim.

Claim 3.2.2. $\mathbf{p}_1' = \mathbf{p}$.

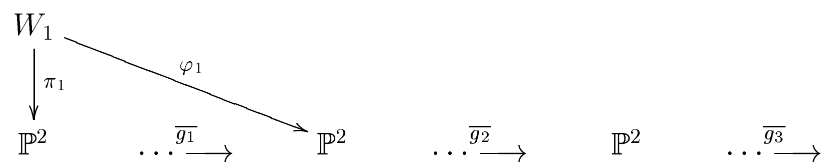
Suppose $\mathbf{p}_1' \neq \mathbf{p}$. Then $\overline{g_1}$ is defined at $\mathbf{p} \in H$, and $\overline{g_1}(\mathbf{p}) = \mathbf{q}_1' \in H$. It follows from Claim 3.2.1 that $\overline{g_2}$ is defined at $\mathbf{q}_1'$, and $\overline{g_2}(\mathbf{q}_1') = \mathbf{q}_2' \in H$. We can repeat this procedure to find that $\overline{f} = \overline{g_l} \circ \cdots \circ \overline{g_1}$ is defined at $\mathbf{p}$. This is a contradiction.

Let us prove (1). Since $g_1 = \varepsilon_1 \circ \alpha_1$, there are affine automorphisms α_1' and α_1'' such that $h := \alpha_1' g_1 \alpha_1''$ becomes a Hénon map. Since the extensions $\overline{\alpha_1'} : \mathbb{P}^2 \to \mathbb{P}^2$ and $\overline{\alpha_1''} : \mathbb{P}^2 \to \mathbb{P}^2$ are linear, the configuration of blow-ups of $\mathbb{P}^2$ that resolves the points of indeterminacy of $\overline{g_1}$ is the same as that of $\overline{h}$, where $\overline{h} : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ is the extension of h. Then it follows from Theorem 2.1 that after the $(2d_1 - 1)$ times successive blow-ups of the (infinitely near) points of indeterminacy, $\overline{g_1}$ becomes well-defined. Let $\pi_1 : W_1' \to \mathbb{P}^2$ be the successive $(2d_1 - 1)$ times blow-ups of $\mathbb{P}^2$.

Claim 3.2.3. *The* $(2d_1 - 1)$ *times blow-ups that resolves the points of indeterminacy of* $\overline{g_1}$ *coincide with the blow-ups in* (1). *In particular* $W_1 = W_1'$.

The first step is to show $\mathbf{p}_1^{'} = \mathbf{p}$, but this is just Claim 3.2.3. Let $\mathbf{p}_1^{''}$ (resp. $\mathbf{p}'$) be the unique point of indeterminacy of the composite of $\overline{g_1}$ (resp. $\overline{f}$) and the first blow-up. We denote this composite by $\widetilde{\overline{g_1}}$ (resp. $\widetilde{\overline{f}}$). The second step is to show $\mathbf{p}_1^{''} = \mathbf{p}'$. To lead a contradiction, suppose $\mathbf{p}_1^{''} \neq \mathbf{p}'$. Then $\widetilde{\overline{g_1}}$ is defined at $\mathbf{p}'$ and $\widetilde{\overline{g_1}}(\mathbf{p}') = \mathbf{q}_1^{'}$ by Theorem 2.1(2). Then as in the proof of Claim 3.2.2, we find that $\widetilde{\overline{f}} = \overline{g_l} \circ \cdots \circ \overline{g_2} \circ \widetilde{\overline{g_1}}$ is defined at $\mathbf{p}'$. This a contradiction. We can repeat this argument to obtain Claim 3.2.3.

Next we prove (2). We set $\varphi_1 = \overline{g_1} \circ \pi_1 : W_1 \to \mathbb{P}^2$. Then we have $\overline{f_{2d_1-1}} = \overline{g_l} \circ \cdots \circ \overline{g_2} \circ \varphi_1 : W_1 \cdots \to \mathbb{P}^2$. It follows from Theorem 2.1(2) that by φ_1, $E_i^{(1)'}$ $(1 \leq i \leq 2d_1 - 2)$ is contracted to the point $\mathbf{q}_1^{'}$. On the other hand, $\overline{g_l} \circ \cdots \circ \overline{g_2}$ is defined at $\mathbf{q}_1^{'}$. Thus $\overline{f_{2d_1-1}}$ is defined at every point on $E_i^{(1)'}$ $(1 \leq i \leq 2d_1 - 2)$. Hence, the unique point of indeterminacy $\mathbf{p}_2$ of $\overline{f_{2d_1-1}}$ lies on the last exceptional divisor $E_{2d_1-1}^{(1)'}$ but not on $E_i^{(1)'}$ $(1 \leq i \leq 2d_1 - 2)$. This shows (2).



Claim 3.2.4. $\varphi_1(\mathbf{p}_2) = \mathbf{p}_2^{'}$.

Indeed, suppose $\varphi_1(\mathbf{p}_2) \neq \mathbf{p}_2^{'}$. Then $\overline{g_2}$ is defined at $\varphi_1(\mathbf{p}_2)$, and then $\overline{f_{2d_1-1}} = \overline{g_l} \circ \cdots \circ \overline{g_2} \circ \varphi_1$ is defined at $\mathbf{p}_2$. This is a contradiction.

We prove (3). Let H' on W_1 be the proper transform of H by π_1. By Theorem 2.1(2), via φ_1, H' and $E_i^{(1)'}$ $(1 \leq i \leq 2d_1 - 2)$ are mapped to $\mathbf{q}_1^{'}$, while $E_{2d_1-1}^{(1)'}$ is mapped isomorphically to H. This shows that

$$\varphi_1|_{W_1 \setminus \left(H' \cup \bigcup_{i=1}^{2d_1-2} E_i^{(1)'}\right)} : W_1 \setminus \left(H' \cup \bigcup_{i=1}^{2d_1-2} E_i^{(1)'}\right) \longrightarrow \mathbb{P}^2 \setminus \{\mathbf{q}_1^{'}\}$$

is an isomorphism. On the other hand, it follows from Theorem 2.1 that the (infinitely near) points of indeterminacy of $\overline{g_2}$ are resolved after $(2d_2 - 1)$ times blow-ups, starting at the blow-up at $\mathbf{p}_2^{'}$. Since $\varphi_1(\mathbf{p}_2) = \mathbf{p}_2^{'}$ by Claim 3.2.4, $\varphi_1|_{W_1 \setminus \left(H' \cup \bigcup_{i=1}^{2d_1-2} E_i^{(1)'}\right)}$ is an isomorphism, and $\mathbf{q}_1^{'} \neq \mathbf{p}_2^{'}$, we find that $\overline{g_2} \circ \varphi_1 : W_1 \cdots \to \mathbb{P}^2$ is well-defined after $(2d_2 - 1)$ blow-ups starting at $\mathbf{p}_2$, and the configuration of blow-ups for $\overline{g_2} \circ \varphi_1$ is the same that for $\overline{g_2}$. Moreover, as in Claim 3.2.3, the $(2d_2 - 1)$ times blow-ups that resolves the points of indeterminacy of $\overline{g_2} \circ \varphi_1$ coincide with the blow-ups in (3).

We repeat these arguments to obtain (4) and (5). □

In what follows, we assume f is a regular polynomial automorphism. Let $\pi_{W_l} : W_l \to \mathbb{P}^2$ be the blow-ups of $\mathbb{P}^2$ given in Theorem 3.2. We will make blow-ups so that the birational map $\overline{f^{-1}} \circ \pi_{W_l} : W_l \cdots \to \mathbb{P}^2$ lifts to a morphism. Note that f^{-1} is of polydegree $(d_l, \cdots, d_1)$. Since π_{W_l} induces an isomorphism $\pi_{W_l}^{-1}(\mathbb{P}^2 \setminus \{\mathbf{p}\}) \to \mathbb{P}^2 \setminus \{\mathbf{p}\}$, we take $\mathbf{q}' \in W_l$ with $\pi_{W_l}(\mathbf{q}') = \mathbf{q}$.

In a parallel way as for $\mathbf{p}$, $\overline{f^{-1}} \circ \pi_{W_l} : W_l \cdots \to \mathbb{P}^2$ extends to a morphism after $(2d_l - 1) + \cdots + (2d_1 - 1)$ blow-ups starting at $\mathbf{q}'$.

Let V be the projective surface obtained by $(2d_1 - 1) + \cdots + (2d_l - 1)$ blow-ups of $\mathbb{P}^2$ starting at $\mathbf{p}$ as in Theorem 3.2 and then $(2d_l - 1) + \cdots + (2d_1 - 1)$ blow-ups starting at $\mathbf{q}$ in a parallel way as in Theorem 3.2. Let $\pi : V \to \mathbb{P}^2$ denote the morphism of blow-ups, $\varphi : V \to \mathbb{P}^2$ the composite $\overline{f} \circ \pi$, and $\psi : V \to \mathbb{P}^2$ the composite $\overline{f^{-1}} \circ \pi$.

Let $E_i^{(s)}$ $(s = 1, \cdots, l; i = 1, \cdots, 2d_s - 1)$ be the proper transform of $((2d_1 - 1) + \cdots + (2d_{s-1} - 1) + i)$-th exceptional divisor on V on the side of $\mathbf{p}$, and $F_j^{(t)}$ $(t = 1, \cdots, l; j = 1, \cdots, 2d_t - 1)$ the proper transform of $((2d_l - 1) + \cdots + (2d_{l+2-t} - 1) + j)$-th exceptional divisor on V on the side of $\mathbf{q}$. Let $H^{\#}$ on V be the proper transform of H.

We find from Theorem 3.2 the configuration of $H^{\#}$, $E_i^{(s)}$ and $F_j^{(t)}$.

Proposition 3.3. (1) *We have*

$$H^{\#\,2} = -3,$$

$$E_i^{(s)\,2} = \begin{cases} -d_s & (1 \le s \le l; i = 1) \\ -2 & (1 \le s \le l; 2 \le i \le 2d_s - 2) \\ -3 & (1 \le s \le l-1; i = 2d_s - 1) \\ -1 & (s = l; i = 2d_s - 1), \end{cases}$$

$$F_j^{(t)\,2} = \begin{cases} -d_{l+1-t} & (1 \le t \le l; j = 1) \\ -2 & (1 \le t \le l; 2 \le j \le 2d_{l+1-t} - 2) \\ -3 & (1 \le t \le l-1; j = 2d_{l+1-t} - 1) \\ -1 & (t = l; j = 2d_1 - 1). \end{cases}$$

(2) *The lines $H^{\#}$, $E_i^{(s)}$ and $F_j^{(t)}$ intersect as follows.*

(i) *The line $H^{\#}$ intersects with $E_2^{(1)}$ and $F_2^{(1)}$.*

(ii) *The line $E_1^{(s)}$ intersects with $E_{d_s}^{(s)}$ for $1 \le s \le l$. The line $E_2^{(s)}$ intersects with $E_{2d_s-1}^{(s-1)}$ and $E_3^{(s)}$ (resp. $H^{\#}$ and $E_3^{(s)}$) for $2 \le s \le l$ (resp. $s = 1$). The line $E_i^{(s)}$ intersects with $E_{i-1}^{(s)}$ and $E_{i+1}^{(s)}$ for $1 \le s \le l$ and $i = 3, \cdots, d_s - 1, d_s + 1, \cdots, 2d_s - 2$. The line $E_{d_s}^{(s)}$ intersects with $E_1^{(s)}$, $E_{d_s-1}^{(s)}$ and $E_{d_s+1}^{(s)}$ for $1 \le s \le l$. The line $E_{2d_s-1}^{(s)}$ intersects with $E_{2d_s-2}^{(s)}$ and $E_2^{(s+1)}$ (resp. only $E_{2d_s-2}^{(s)}$) for $1 \le s \le l-1$ (resp. $s = l$).*

(iii) *The same holds if we replace respectively $s, i, d_s, E_i^{(s)}$ by $t, j, d_{l+1-t}, F_j^{(t)}$ in* (ii).

Let us illustrate Proposition 3.3 when f is of polydegree $(2, 3)$.

Example 3.4. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the polynomial automorphism of degree 6 of the form

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x^2 - y)^3 - x \\ x^2 - y \end{pmatrix}.$$

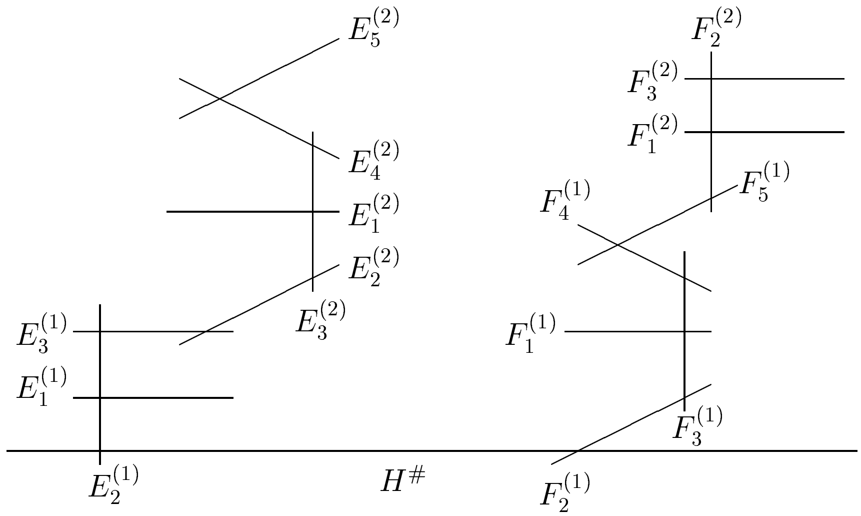


FIGURE 2. The configuration after blow-ups when f is of polydegree $(2,3)$. The line $H^\#$ has the self-intersection number -3. The lines $E_3^{(1)}$, $E_1^{(2)}$, $F_1^{(1)}$ and $F_5^{(1)}$ have the self-intersection numbers -3. The lines $E_5^{(2)}$ and $F_3^{(2)}$ have the self-intersection numbers -1. The other lines have self-intersection numbers -2.

Since f is the composite of two Hénon maps of the form $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y \\ x \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^3 - y \\ x \end{pmatrix}$, f is of polydegree $(2,3)$. The configuration after blow-ups on V is illustrated in Figure 2.

Proposition 3.5. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$, of polydegree $(d_1, d_2, \cdots, d_l)$. Let the notation be as above.*

(1) *As divisors on V, we have*

$$\pi^* H = H^\# + \sum_{s=1}^{l} d_1 d_2 \cdots d_{s-1} \left(\sum_{i=1}^{d_s} i E_i^{(s)} + \sum_{i=d_s+1}^{2d_s-1} d_s E_i^{(s)} \right)$$
$$+ \sum_{t=1}^{l} d_{l-t+2} d_{l-t+3} \cdots d_l \left(\sum_{j=1}^{d_{l-t+1}} j F_j^{(t)} + \sum_{i=d_{l-t+1}+1}^{2d_{l-t+1}-1} d_{l-t+1} F_j^{(t)} \right),$$

$$\varphi^* H = dH^\# + \sum_{s=1}^{l} d_{s+1} d_{s+2} \cdots d_l \left(E_1^{(s)} + \sum_{i=2}^{d_s} d_s E_i^{(s)} + \sum_{i=d_s+1}^{2d_s-1} (2d_s - i) E_i^{(s)} \right)$$
$$+ d \sum_{t=1}^{l} d_{l-t+2} d_{l-t+3} \cdots d_l \left(\sum_{i=1}^{d_{l-t+1}} j F_j^{(t)} + \sum_{i=d_{l-t+1}+1}^{2d_{l-t+1}-1} d_{l-t+1} F_j^{(t)} \right),$$

$$\psi^*H = dH^{\#} + d\sum_{s=1}^{l} d_1d_2\cdots d_{s-1}\left(\sum_{i=1}^{d_s} iE_i^{(s)} + \sum_{i=d_s+1}^{2d_s-1} d_sE_i^{(s)}\right)$$
$$+\sum_{t=1}^{l} d_1d_2\cdots d_{l-t}\left(F_1^{(t)} + \sum_{j=2}^{d_{l-t+1}} d_{l-t+1}F_j^{(t)} + \sum_{i=d_{l-t+1}+1}^{2d_{l-t+1}-1}(2d_{l-t+1}-j)F_j^{(t)}\right).$$

Here we set $d_1d_2\cdots d_i = 1$ *if* $i=0$, *and* $d_id_{i+1}\cdots d_l = 1$ *if* $i = l+1$.

(2) *As a* $\mathbb{Q}$*-divisor on* V, *we have*

$$\varphi^*H + \psi^*H = \left(d + \frac{1}{d}\right)\pi^*H + D,$$

where D *is the* $\mathbb{Q}$*-effective divisor given by*

$$D = \left(d - \frac{1}{d}\right)H^{\#} + \sum_{s=1}^{l}\left\{\left(d_{s+1}d_{s+2}\cdots d_l - \frac{1}{d_sd_{s+1}\cdots d_l}\right)E_1^{(s)}\right.$$
$$\left. + \sum_{i=2}^{d_s}\left(d_sd_{s+1}\cdots d_l - \frac{i}{d_sd_{s+1}\cdots d_l}\right)E_i^{(s)} + \sum_{i=d_s+1}^{2d_s-1}\left(d_{s+1}d_{s+2}\cdots d_l(2d_s-i) - \frac{1}{d_{s+1}d_{s+2}\cdots d_l}\right)E_i^{(s)}\right\}$$
$$+\sum_{t=1}^{l}\left\{\left(d_1d_2\cdots d_{l-t} - \frac{1}{d_1d_2\cdots d_{l-t+1}}\right)F_1^{(t)} + \sum_{j=2}^{d_{l-t+1}}\left(d_1d_2\cdots d_{l-t+1} - \frac{j}{d_1d_2\cdots d_{l-t+1}}\right)F_j^{(t)}\right.$$
$$\left. + \sum_{i=d_{l-t+1}+1}^{2d_{l-t+1}-1}\left(d_1d_2\cdots d_{l-t}(2d_{l-t+1}-j) - \frac{1}{d_1d_2\cdots d_{l-t}}\right)F_j^{(t)}\right\}.$$

Proof. We will show the expression for φ^*H. Since φ maps $H^{\#}$, $E_i^{(s)}$ $(\forall(s,i)\neq(l,2d_l-1))$ and $F_j^{(t)}$ $(\forall(t,j))$ to $\mathbf{q}$, we have

$$\varphi^*H\cdot H^{\#} = 0, \qquad \varphi^*H\cdot E_i^{(s)} = 0, \qquad \varphi^*H\cdot F_j^{(t)} = 0 \tag{3.2}$$

for every $(s,i)\neq(l,2d_l-1)$ and every (t,j). Since φ maps $E_{2d_l-1}^{(l)}$ to H isomorphically, we have

$$\varphi^*H\cdot E_{2d_l-1}^{(l)} = 1. \tag{3.3}$$

Since the Picard group of V is generated by $H^{\#}, E_i^{(s)}, F_j^{(t)}$ $(1\leq s,t\leq l, 1\leq i\leq 2d_s-1, 1\leq j\leq 2d_t-1)$, the expression for φ^*H follows from (3.2), (3.3) and Proposition 3.3. We leave the details for straightforward yet a little long calculation to the reader. Keeping in mind that f^{-1} is of polydegree $(d_l,\cdots,d_1)$, we obtain the expression for ψ^*H similarly. The expression for π^*H follows from the construction of V. (We can also show this by using $\pi^*H\cdot H^{\#}=1$, $\pi^*H\cdot E_i^{(s)}=0$ and $\pi^*H\cdot F_j^{(t)}=0$ for all (s,i) and (t,j).) The assertion (2) follows from (1). □

As in Theorem 2.3, we have the following theorem.

Theorem 3.6. *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$ defined over a number field K. Then, there exists a constant c such that*

$$h_{nv}(f(x)) + h_{nv}(f^{-1}(x)) \geq \left(d + \frac{1}{d}\right) h_{nv}(x) - c$$

for all $x \in \mathbb{A}^2(\overline{K})$.

4. Canonical height functions

In this section, we will prove Theorem A and Corollary B by showing Theorem 4.1. We first fix some notation and terminology. We refer to the survey [10] for more details about the dynamics of polynomial automorphisms.

Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial automorphism over a number field K of degree d. We use the notation $\overline{f}$ to denote the birational extension of f to $\mathbb{P}^n$. Let $f^{-1} : \mathbb{A}^n \to \mathbb{A}^n$ denote the inverse of f, and we use the notation $\overline{f^{-1}}$ to denote the birational extension of f^{-1} to $\mathbb{P}^n$. Let d_- be the degree of $\overline{f^{-1}}$. Note that d and d_- may not be the same (cf. [10, Chapitre 2]).

Let S be a set and T a subset of S. Two real-valued functions λ and λ' on S are said to be equivalent on T if there exist positive constants a_1, a_2 and constants b_1, b_2 such that $a_1\lambda(x) + b_1 \leq \lambda'(x) \leq a_2\lambda(x) + b_2$ for all $x \in T$. We use the notation $\lambda \gg\ll \lambda'$ to denote this equivalence. (Note that our notation $\gg\ll$ is different from that in [8, Chap. 4, §1] where $b_1 = b_2 = 0$.)

Theorem 4.1. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ a polynomial automorphism of degree $d \geq 2$ defined over a number field K. Let d_- denote the degree of $\overline{f^{-1}}$. We assume that there exists a constant c such that*

$$\frac{1}{d}h_{nv}(f(x)) + \frac{1}{d_-}h_{nv}(f^{-1}(x)) \geq \left(1 + \frac{1}{dd_-}\right) h_{nv}(x) - c \tag{4.1}$$

for all $x \in \mathbb{A}^n(\overline{K})$. Then there exists a function $\widehat{h} : \mathbb{A}^n(\overline{K}) \to \mathbb{R}$ with the following properties:

(i) $h_{nv} \gg\ll \widehat{h}$ *on* $\mathbb{A}^n(\overline{K})$;

(ii) $\frac{1}{d}\widehat{h} \circ f + \frac{1}{d_-}\widehat{h} \circ f^{-1} = \left(1 + \frac{1}{dd_-}\right)\widehat{h}$.

Moreover, $\widehat{h}$ enjoys the following uniqueness property: if $\widehat{h}'$ is another function satisfying (i) and (ii) such that $\widehat{h}' = \widehat{h} + O(1)$, then $\widehat{h}' = \widehat{h}$. Furthermore, $\widehat{h}(x) \geq 0$ for all $x \in \mathbb{A}^n(\overline{K})$, and $\widehat{h}(x) = 0$ if and only if x is f-periodic. We call a function $\widehat{h}$ satisfying (i) and (ii) a canonical height function *associated with f.*

Proof of Theorem A and Corollary B. It follows from Theorem 3.6 that regular polynomial automorphisms of the affine plane satisfy (4.1). Then Theorem A and Corollary B follows from Theorem 4.1. □

Proof of Theorem 4.1. For $x \in \mathbb{A}^n(\overline{K})$, we define

$$\widehat{h}^+_\circ(x) = \limsup_{l\to\infty} \frac{1}{d^l}h_{nv}(f^l(x)), \qquad \widehat{h}^-_\circ(x) = \limsup_{l\to\infty} \frac{1}{d^l_-}h_{nv}(f^{-l}(x)),$$

a priori in $\mathbb{R} \cup \{\infty\}$, but we will show in the next claim that this value is finite. We define

$$\widehat{h}_\circ(x) = \widehat{h}_\circ^+(x) + \widehat{h}_\circ^-(x).$$

Note that this definition of $\widehat{h}_\circ^\pm$ has some similarity to the definition of Green currents on $\mathbb{A}^n(\mathbb{C})$ associated with f (cf. [10, Définition 2.2.5]), and to Silverman's definition of canonical heights on certain K3 surfaces [12, §3]. Let us show $\widehat{h}_\circ$ satisfies the properties (i) and (ii).

Claim 4.1.1. *There exist constants $c^\pm$ such that $\widehat{h}_\circ^\pm(x) \leq h_{nv}(x) + c^\pm$ for all $x \in \mathbb{A}^n(\overline{K})$.*

Proof. By Theorem 1.3, there exists a constant c_2 such that $\frac{1}{d} h_{nv}(f(x)) \leq h_{nv}(x) + \frac{c_2}{d}$ for all $x \in \mathbb{A}^n(\overline{K})$. We show

$$\frac{1}{d^l} h_{nv}(f^l(x)) \leq h_{nv}(x) + \left(\sum_{i=1}^{l} \frac{1}{d^i} \right) c_2$$

by the induction on l. Indeed, since $\frac{1}{d} h_{nv}(f^{l+1}(x)) \leq h_{nv}(f^l(x)) + \frac{c_2}{d}$, we have

$$\frac{1}{d^{l+1}} h_{nv}(f^{l+1}(x)) \leq \frac{1}{d^l} h_{nv}(f^l(x)) + \frac{c_2}{d^{l+1}} \leq h_{nv}(x) + \left(\sum_{i=1}^{l+1} \frac{1}{d^i} \right) c_2.$$

By putting $c^+ = c_2 \sum_{i=1}^{\infty} \frac{1}{d^i} = \frac{c_2}{d-1}$, we obtain $\widehat{h}_\circ^+(x) = \limsup_{l \to \infty} \frac{1}{d^l} h_{nv}(f^l(x)) \leq h_{nv}(x) + c^+$. The estimate for $\widehat{h}_\circ^-$ is shown similarly. (Note that it follows from $d \geq 2$ that $d_- \geq 2$.) □

Claim 4.1.2. *We have*

$$\widehat{h}_\circ(x) \geq h_{nv}(x) - \frac{d d_-}{(d-1)(d_- - 1)} c$$

for all $x \in \mathbb{A}^n(\overline{K})$, where c is the constant given in (4.1).

Proof. We set $h' = h_{nv} - \frac{d d_-}{(d-1)(d_- -1)} c$. Then we have for all $x \in \mathbb{A}^n(\overline{K})$

$$\frac{1}{d} h'(f(x)) + \frac{1}{d_-} h'(f^{-1}(x)) \geq \left(1 + \frac{1}{d d_-}\right) h'(x). \tag{4.2}$$

Then we have $\frac{1}{d^2} h'(f^2(x)) + \frac{1}{d d_-} h'(x) \geq \left(1 + \frac{1}{d d_-}\right) \frac{1}{d} h'(f(x))$ and $\frac{1}{d d_-} h'(x) + \frac{1}{d_-^2} h'(f^{-2}(x)) \geq \left(1 + \frac{1}{d d_-}\right) \frac{1}{d_-} h'(f^{-1}(x))$. Adding these two inequalities and using (4.2) again, we obtain

$$\frac{1}{d^2} h'(f^2(x)) + \frac{1}{d_-^2} h'(f^{-2}(x)) \geq \left(1 + \frac{1}{(d d_-)^2}\right) h'(x).$$

Inductively, we obtain

$$\frac{1}{d^{2^l}} h'(f^{2^l}(x)) + \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x)) \geq \left(1 + \frac{1}{(d d_-)^{2^l}}\right) h'(x).$$

(Though not necessary for the proof, one can also show $\frac{1}{d^m}h'(f^m(x)) + \frac{1}{d_-^m}h'(f^{-m}(x)) \geq \left(1 + \frac{1}{(dd_-)^m}\right) h'(x)$ for every $m \in \mathbb{Z}$.) By letting $l \to \infty$, it follows that

$$(4.3) \quad \limsup_{l\to\infty} \frac{1}{d^{2^l}} h'(f^{2^l}(x)) + \limsup_{l\to\infty} \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x)) \\ \geq \limsup_{l\to\infty} \left(\frac{1}{d^{2^l}} h'(f^{2^l}(x)) + \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x)) \right) \geq h'(x).$$

Since

$$\widehat{h}_\circ^+(x) = \limsup_{m\to\infty} \frac{1}{d^m} h_{nv}(f^m(x)) \\ = \limsup_{m\to\infty} \frac{1}{d^m} \left(h'(f^m(x)) + \frac{dd_-}{(d-1)(d_- - 1)} c \right) \geq \limsup_{l\to\infty} \frac{1}{d^{2^l}} h'(f^{2^l}(x))$$

and similarly $\widehat{h}_\circ^-(x) \geq \limsup_{l\to\infty} \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x))$, the left-hand-side of (4.3) is less than or equal to $\widehat{h}_\circ(x)$, while the right-hand-side is $h_{nv}(x) - \frac{dd_-}{(d-1)(d_--1)}c$. Thus we get the desired inequality. □

The property (i) follows from Claim 4.1.1 and Claim 4.1.2. Indeed we have

$$h_{nv}(x) - \frac{dd_-}{(d-1)(d_- - 1)} c \leq \widehat{h}_\circ(x) \leq 2h_{nv}(x) + c^+ + c^-.$$

The property (ii) is checked by the following equations:

$$\widehat{h}_\circ^+(f(x)) = d\widehat{h}_\circ^+(x), \quad \widehat{h}_\circ^+(f^{-1}(x)) = \frac{1}{d}\widehat{h}_\circ^+(x);$$
$$\widehat{h}_\circ^-(f(x)) = \frac{1}{d_-}\widehat{h}_\circ^-(x), \quad \widehat{h}_\circ^-(f^{-1}(x)) = d_-\widehat{h}_\circ^-(x).$$

Thus $\widehat{h}_\circ : \mathbb{A}^n(\overline{K}) \to \mathbb{R}$ satisfies the properties (i) and (ii). This shows the existence of a canonical height function.

Next we will show some uniqueness property of $\widehat{h}$. In what follows, let $\widehat{h}$ denote a function with the properties (i) and (ii), not necessarily being equal to $\widehat{h}_\circ$.

Suppose $\widehat{h}'$ is another function with the properties (i) and (ii) such that $\gamma := \widehat{h}' - \widehat{h}$ is bounded on $\mathbb{A}^n(\overline{K})$. Set $M := \sup_{x\in\mathbb{A}^n(\overline{K})} |\gamma(x)|$. Then

$$\left(1 + \frac{1}{dd_-}\right) M = \left(1 + \frac{1}{dd_-}\right) \sup_{x\in\mathbb{A}^n(\overline{K})} |\gamma(x)| \\ = \sup_{x\in\mathbb{A}^n(\overline{K})} \left| \frac{1}{d}\gamma(f(x)) + \frac{1}{d_-}\gamma(f^{-1}(x)) \right| \leq \left(\frac{1}{d} + \frac{1}{d_-}\right) M.$$

Since $1 + \frac{1}{dd_-} - \frac{1}{d} - \frac{1}{d_-} = \frac{(d-1)(d_--1)}{dd_-} > 0$, we have $M = 0$, hence $h = h'$.

To show $\widehat{h} \geq 0$, we assume the contrary, so that there exists $x_0 \in \mathbb{A}^n(\overline{K})$ with $\widehat{h}(x_0) =: a < 0$. Then $\frac{1}{d}\widehat{h}(f(x_0)) + \frac{1}{d_-}\widehat{h}(f^{-1}(x_0)) = \left(1 + \frac{1}{dd_-}\right)\widehat{h}(x_0) = \left(1 + \frac{1}{dd_-}\right)a$. Thus we have

$$\widehat{h}(f(x_0)) \leq \frac{1+dd_-}{d+d_-}a \quad \text{or} \quad \widehat{h}(f^{-1}(x_0)) \leq \frac{1+dd_-}{d+d_-}a.$$

Since $\frac{1+dd_-}{d+d_-} > 1$, this shows that $\widehat{h}$ is not bounded from below. Since h_{nv} is bounded from below and $h_{nv} \gg\ll \widehat{h}$, this is a contradiction.

Finally we will show that $x \in \mathbb{A}^n(\overline{K})$ is f-periodic if and only if $\widehat{h}(x) = 0$.

Suppose $\widehat{h}(x_1) = 0$. Then by (4.1) and the non-negativity of $\widehat{h}$, we have $\widehat{h}(f(x_1)) = 0$ and $\widehat{h}(f^{-1}(x_1)) = 0$. Take an extension field L of K such that x_1 is defined over L. Since $\widehat{h} \gg\ll h_{nv}$, $\widehat{h}$ satisfies the Northcott finiteness property. Thus the set

$$\{f^l(x_1) \mid l \in \mathbb{Z}\} \quad \left(\subseteq \{x \in \mathbb{A}^n(L) \mid \widehat{h}(x) = 0\}\right)$$

is finite. Hence x_1 is f-periodic.

On the other hand, suppose $\widehat{h}(x_2) =: b > 0$. Then it follows from (ii) that

$$\widehat{h}(f(x_2)) \geq \frac{1+dd_-}{d+d_-}b \quad \text{or} \quad \widehat{h}(f^{-1}(x_2)) \geq \frac{1+dd_-}{d+d_-}b.$$

This shows that the set $\{f^l(x_2) \mid l \in \mathbb{Z}\}$ is not a set of bounded height. Thus x_2 cannot be f-periodic. □

In the remainder of this section, we would like to discuss the condition (4.1) in Theorem 4.1. The next proposition shows that the constant $(1 + \frac{1}{dd_-})$ in (4.1) is the largest number one can hope for.

Proposition 4.2. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ a polynomial automorphism of degree $d \geq 2$ over a number field K. Let d_- denote the degree of f^{-1}. Let $a \in \mathbb{R}$. Suppose there exists a constant c such that*

$$\frac{1}{d}h_{nv}(f(x)) + \frac{1}{d_-}h_{nv}(f^{-1}(x)) \geq a h_{nv}(x) - c$$

for all $x \in \mathbb{A}^n(\overline{K})$. Then $a \leq 1 + \frac{1}{dd_-}$

Proof. To lead a contradiction, we assume that $a > 1 + \frac{1}{dd_-}$. Noting $a > 1 + \frac{1}{dd_-} \geq \frac{1}{d} + \frac{1}{d_-}$, we set $c' := \left(a - \frac{1}{d} - \frac{1}{d_-}\right)^{-1} c$ and $h' := h_{nv} - c'$. Then h' satisfies

$$\frac{1}{d}h'(f(x)) + \frac{1}{d_-}h'(f^{-1}(x)) \geq a h'(x) \tag{4.4}$$

for all $x \in \mathbb{A}^n(\overline{K})$. Then we have $\frac{1}{d^2}h'(f^2(x)) + \frac{1}{dd_-}h'(x) \geq \frac{a}{d}h'(f(x))$ and $\frac{1}{dd_-}h'(x) + \frac{1}{d_-^2}h'(f^{-2}(x)) \geq \frac{a}{d_-}h'(f^{-1}(x))$. Adding these two inequalities and using (4.4) again, we get

$$\frac{1}{d^2}h'(f^2(x)) + \frac{1}{d_-^2}h'(f^{-2}(x)) \geq \left(a^2 - \frac{2}{dd_-}\right)h'(x).$$

We set $a_1 = a^2 - \frac{2}{dd_-}$. Since $a_1 - 1 - \frac{1}{(dd_-)^2} = a^2 - \frac{2}{dd_-} - 1 - \frac{1}{(dd_-)^2} > (1+\frac{1}{dd_-})^2 - \frac{2}{dd_-} - 1 - \frac{1}{(dd_-)^2} = 0$, we have $a_1 > 1 + \frac{1}{(dd_-)^2}$. Thus, if we define a sequence $\{a_l\}_{l=0}^{\infty}$ by $a_0 = a$ and $a_{l+1} = a_l^2 - \frac{2}{(dd_-)^{2^l}}$, then we get inductively

$$\frac{1}{d^{2^l}} h'(f^{2^l}(x)) + \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x)) \geq a_l h'(x).$$

On the other hand, it follows from Theorem 1.3 and the argument in Claim 4.1.1 that there is a constant c'' independent of $l \in \mathbb{Z}$ such that for all $x \in \mathbb{A}^2(\overline{K})$,

$$2h'(x) + c'' \geq \frac{1}{d^{2^l}} h'(f^{2^l}(x)) + \frac{1}{d_-^{2^l}} h'(f^{-2^l}(x)).$$

Thus $2h' + c''' \geq a_l h'$. Since $h' = h_{nv} - c'$ and $\lim_{l\to\infty} a_l = \infty$ follows from Lemma 4.3(1), this is a contradiction. □

Lemma 4.3. *Let $D \geq 4$. Let $\{a_l\}_{l=0}^{\infty}$ be a sequence defined by $a_0 = a$ and $a_{l+1} = a_l^2 - 2D^{-2^l}$.*

(1) *If $a > 1 + \frac{1}{D}$, then $\lim_{l\to\infty} a_l = \infty$.*
(2) *If $a = 1 + \frac{1}{D}$, then $\lim_{l\to\infty} a_l = 1$.*
(3) *If $1 \leq a < 1 + \frac{1}{D}$, then $\lim_{l\to\infty} a_l = 0$.*

Proof. We show (1). Set $\varepsilon_l = a_l - 1 - D^{-2^l}$. In particular $\varepsilon_0 = a - 1 - D^{-1} > 0$. Since $\varepsilon_{l+1} = a_{l+1} - 1 - 2D^{-2^{l+1}} = 2\varepsilon_l(1 + D^{-2^l}) + \varepsilon_l^2$, we get $\varepsilon_{l+1} > 2\varepsilon_l > \cdots > 2^{l+1}\varepsilon_0$. Hence $\lim_{l\to\infty} \varepsilon_l = \infty$ and thus $\lim_{l\to\infty} a_l = \infty$

We show (2). In this case, we have $a_l = 1 + D^{-2^l}$. Thus $\lim_{l\to\infty} a_l = 1$.

Finally we show (3). On one hand, we get by induction $a_l \geq 2D^{-2^{l-1}}$ for $l \geq 1$, and in particular $a_l \geq 0$ for $l \geq 1$. On the other hand, we claim for sufficiently large l that $a_l < 1$. Indeed, we assume the contrary and suppose $a_l \geq 1$ for all l. It follows from (2) that $a_l < 1 + D^{-2^l}$. We set $\lambda_l = 1 + D^{-2^l} - a_l$, and so $0 < \lambda_l \leq D^{-2^l}$. Then $a_{l+1} = a_l^2 - 2D^{-2^l} = (1 + D^{-2^l} - \lambda_l)^2 - 2D^{-2^l} = 1 + D^{-2^{l+1}} - 2\lambda_l(1 + D^{-2^l}) + \lambda_l^2$. Hence we get $\lambda_{l+1} = 2\lambda_l(1 + D^{-2^l}) - \lambda_l^2 \geq 2\lambda_l$, which says that $\lim_{l\to\infty} \lambda_l = \infty$. This is a contradiction. Hence there is an l_0 with $a_{l_0} < 1$. Since $(0 \leq)\ a_{l_0+k} \leq a_{l_0}^{2^k}$, we get $\lim_{l\to\infty} a_l = 0$. □

Let $a_{\sup}$ denote the supremum of $a \in \mathbb{R}$ that satisfies the inequality in Proposition 4.2. It follows from Theorem 3.6 that, if f is a regular polynomial automorphism of $\mathbb{A}^2$ of degree $d \geq 2$, then $d = d_-$ and $a_{\sup} = 1 + \frac{1}{d^2}$. We remark that Marcello [9, Théorème 3.1] showed that, if f is regular polynomial automorphism of $\mathbb{A}^n$ (this means the set of indeterminacy $I_{\overline{f}}$ and $I_{\overline{f^{-1}}}$ are disjoint, cf. [10, Définition 2.2.1]), then $a_{\sup} \geq 1$. It would be interesting to know what polynomial automorphisms on $\mathbb{A}^n$ satisfy (4.1).

5. The number of points with bounded height in an f-orbit

In this section, we will prove Theorem C. As in §4 we will show Theorem C in a more general setting. The arguments below are inspired by those of Silverman on certain K3 surfaces [12, §3].

Throughout this section, let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial automorphism of degree $d \geq 2$ over a number field K satisfying (4.1). By Theorem 4.1, there exists a canonical height function $\widehat{h} : \mathbb{A}^n(\overline{K}) \to \mathbb{R}$ associated with f. *Throughout this section, we also fix $\widehat{h}$.*

We define functions $\widehat{h}^{\pm} : \mathbb{A}^n(\overline{K}) \to \mathbb{R}$ to be

$$\widehat{h}^{+}(x) = \frac{dd_-}{(dd_-)^2 - 1}\left(d_-\widehat{h}(f(x)) - \frac{1}{d_-}\widehat{h}(f^{-1}(x))\right),$$
$$\widehat{h}^{-}(x) = \frac{dd_-}{(dd_-)^2 - 1}\left(d\widehat{h}(f^{-1}(x)) - \frac{1}{d}\widehat{h}(f(x))\right)$$

for $x \in \mathbb{A}^n(\overline{K})$.

Lemma 5.1. (1) $\widehat{h} = \widehat{h}^{+} + \widehat{h}^{-}$.
(2) $\widehat{h}^{+} \circ f = d\,\widehat{h}^{+}$, *and* $\widehat{h}^{-} \circ f^{-1} = d_-\,\widehat{h}^{-}$.
(3) $\widehat{h}^{+} \geq 0$ *and* $\widehat{h}^{-} \geq 0$.
(4) *For $x \in \mathbb{A}^n(\overline{K})$, $\widehat{h}^{+}(x) = 0$ if and only if $\widehat{h}^{-}(x) = 0$ if and only if $\widehat{h}(x) = 0$ if and only if x is f-periodic.*

Proof. By the property (ii) in Theorem 4.1, we readily see (1). Let us see (2). By the property (ii), we have $d_-\widehat{h}(f^2(x)) + d\widehat{h}(x) = (1 + dd_-)\widehat{h}(f(x))$ and $\left(\frac{1}{d_-} + d\right)\widehat{h}(x) = \widehat{h}(f(x)) + \frac{d}{d_-}\widehat{h}(f^{-1}(x))$ Taking the difference, we have

$$d_-\widehat{h}(f^2(x)) - \frac{1}{d_-}\widehat{h}(x) = d\left(d_-\widehat{h}(f(x)) - \frac{1}{d_-}\widehat{h}(f^{-1}(x))\right).$$

This shows $\widehat{h}^{+}(f(x)) = d\,\widehat{h}^{+}(x)$. Similarly we have $\widehat{h}^{+}(f^{-1}(x)) = d_-\,\widehat{h}^{-}(x)$. Next let us see (3). Since $\widehat{h} \geq 0$ by Theorem 4.1, we have $\widehat{h}^{+}(f^{l}(x)) + \widehat{h}^{-}(f^{l}(x)) = \widehat{h}(f^{l}(x)) \geq 0$ for any $l \in \mathbb{Z}$ and $x \in \mathbb{A}^n(\overline{K})$. This is equivalent to

$$\widehat{h}^{+}(x) \geq -\frac{1}{(dd_-)^{l}}\widehat{h}^{-}(x).$$

By letting $l \to \infty$, we have $\widehat{h}^{+}(x) \geq 0$. Similarly we have $\widehat{h}^{-}(x) \geq 0$.

Next we will show (4). The assertion that "$\widehat{h}(x) = 0$ if and only if x is f-periodic" is shown in Theorem 4.1. Since $\widehat{h}^{+} \geq 0$ and $\widehat{h}^{-} \geq 0$, $0 = \widehat{h}(x) = \widehat{h}^{+}(x) + \widehat{h}^{-}(x)$ implies $\widehat{h}^{+}(x) = 0$ and $\widehat{h}^{-}(x) = 0$. We will see that $\widehat{h}^{+}(x) = 0$ implies $\widehat{h}(x) = 0$. A key observation here is that $\widehat{h}$ satisfies Northcott's finiteness property, which is a consequence of the property (i) of $\widehat{h}$ in Theorem 4.1. Suppose $\widehat{h}^{+}(x) = 0$. Then

$$\widehat{h}(f^{l}(x)) = \widehat{h}^{+}(f^{l}(x)) + \widehat{h}^{-}(f^{l}(x)) = d^{l}\widehat{h}^{+}(x) + \frac{1}{d_-^{l}}\widehat{h}^{-}(x) = \frac{1}{d_-^{l}}\widehat{h}^{-}(x).$$

Let L be a finite extension of K over which x is defined. Then

$$\{f^{l}(x) \in \mathbb{A}^n(\overline{K}) \mid l \geq 0\} \quad \subseteq \quad \{y \in \mathbb{A}^n(\overline{K}) \mid \widehat{h}(y) \leq \widehat{h}^{-}(x)\}$$

is finite. Hence x is f-periodic. Similarly we see that $\widehat{h}^{-}(x) = 0$ implies $\widehat{h}(x) = 0$. $\square$

For $x \in \mathbb{A}^n(\overline{K})$, we define the *$f$-orbit* of x to be

$$O_f(x) := \{f^l(x) \mid l \in \mathbb{Z}\}.$$

Note that $O_f(x)$ is a finite set if and only if x is f-periodic.

For an f-orbit $O_f(x)$, we define the *canonical height of $O_f(x)$* to be

$$\widehat{h}(O_f(x)) = \frac{\log \widehat{h}^+(y)}{\log d} + \frac{\log \widehat{h}^-(y)}{\log d_-} \qquad \in \mathbb{R} \cup \{-\infty\}$$

for any $y \in O_f(x)$.

Lemma 5.2. (1) *$\widehat{h}(O_f(x))$ is well-defined, i.e., $\widehat{h}(O_f(x))$ is independent of the choice of $y \in O_f(x)$. Moreover, $\widehat{h}(O_f(x)) = -\infty$ if and only if $O_f(x)$ is a finite set.*

(2) *Assume $\#O_f(x) = \infty$. Then we have*

$$\widehat{h}(O_f(x)) + \epsilon_1 \leq \left(\frac{1}{\log d} + \frac{1}{\log d_-}\right) \min_{y \in O_f(x)} \log \widehat{h}(y) \leq \widehat{h}(O_f(x)) + \epsilon_2,$$

where the constants ϵ_1 and ϵ_2 are given by

$$\epsilon_1 = \frac{1}{\log d} \log\left(1 + \frac{\log d}{\log d_-}\right) + \frac{1}{\log d_-} \log\left(1 + \frac{\log d_-}{\log d}\right),$$

$$\epsilon_2 = \epsilon_1 + \left(\frac{1}{\log d} + \frac{1}{\log d_-}\right) \log \max\{d, d_-\}.$$

Proof. (1) follows from Lemma 5.1. To prove (2), set

$$p = 1 + \frac{\log d}{\log d_-} \quad \text{and} \quad q = 1 + \frac{\log d_-}{\log d}.$$

Then $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$\begin{aligned}\widehat{h}(y) = \widehat{h}^+(y) + \widehat{h}^-(y) &= \frac{1}{p}\left(p^{\frac{1}{p}} \widehat{h}^+(y)^{\frac{1}{p}}\right)^p + \frac{1}{q}\left(q^{\frac{1}{q}} \widehat{h}^-(y)^{\frac{1}{q}}\right)^q \\ &\geq p^{\frac{1}{p}} q^{\frac{1}{q}} \widehat{h}^+(y)^{\frac{1}{p}} \widehat{h}^-(y)^{\frac{1}{q}}\end{aligned}$$

Hence, $\frac{1}{p}\log p + \frac{1}{q}\log q + \frac{1}{p}\log \widehat{h}^+(y) + \frac{1}{q}\log \widehat{h}^-(y) \leq \log \widehat{h}(y)$. Since

$$\frac{1}{p}\log \widehat{h}^+(y) + \frac{1}{q}\log \widehat{h}^-(y) = \left(\frac{1}{\log d} + \frac{1}{\log d_-}\right)^{-1} \widehat{h}(O_f(x)),$$

we obtain $\widehat{h}(O_f(x)) + \epsilon_1 \leq \left(\frac{1}{\log d} + \frac{1}{\log d_-}\right) \min_{y \in O_f(x)} \log \widehat{h}(y)$.

On the other hand, we have $\widehat{h}(f^l(x)) = d^l \widehat{h}^+(x) + d_-^{-l} \widehat{h}^-(x)$ for $l \in \mathbb{Z}$. We set $g(t) = d^t \widehat{h}^+(x) + d_-^{-t} \widehat{h}^-(x)$ for $t \in \mathbb{R}$. Noting that $d^{\frac{1}{p}} = d_-^{\frac{1}{q}}$, we have

$$\begin{aligned}g(t) &= d^t \widehat{h}^+(x) + d_-^{-t} \widehat{h}^-(x) \\ &= \frac{1}{p}\left(p^{\frac{1}{p}} d^{\frac{t}{p}} \widehat{h}^+(x)^{\frac{1}{p}}\right)^p + \frac{1}{q}\left(q^{\frac{1}{q}} d_-^{\frac{-t}{q}} \widehat{h}^-(x)^{\frac{1}{q}}\right)^q \\ &\geq p^{\frac{1}{p}} q^{\frac{1}{q}} d^{\frac{t}{p}} d_-^{\frac{-t}{q}} \widehat{h}^+(x)^{\frac{1}{p}} \widehat{h}^-(x)^{\frac{1}{q}} = p^{\frac{1}{p}} q^{\frac{1}{q}} \widehat{h}^+(x)^{\frac{1}{p}} \widehat{h}^-(x)^{\frac{1}{q}},\end{aligned}$$

where the equality holds if and only if $\left(p^{\frac{1}{p}} d^{\frac{t}{p}} \widehat{h}^{+}(x)^{\frac{1}{p}}\right)^{p-1} = q^{\frac{1}{q}} d_{-}^{\frac{-t}{q}} \widehat{h}^{-}(x)^{\frac{1}{q}}$. We set

$$t_0 = \frac{\log(\widehat{h}^{-}(x)\log d_{-}) - \log(\widehat{h}^{+}(x)\log d)}{\log d + \log d_{-}}.$$

Then g takes its minimum at t_0, with $g(t_0) = p^{\frac{1}{p}} q^{\frac{1}{q}} \widehat{h}^{+}(x)^{\frac{1}{p}} \widehat{h}^{-}(x)^{\frac{1}{q}}$. Consequently as a function of $l \in \mathbb{Z}$, $\widehat{h}(f^l(x))$ takes its minimum at $l = [t_0]$ or $l = [t_0]+1$, where $[t_0]$ denotes the largest integer less than or equal to t_0. Then we get

$$\begin{aligned}\widehat{h}(f^{[t_0]}(x)) &= d^{[t_0]}\widehat{h}^{+}(x) + d_{-}^{-[t_0]}\widehat{h}^{-}(x) = d^{-(t_0-[t_0])} d^{t_0}\widehat{h}^{+}(x) + d_{-}^{t_0-[t_0]} d_{-}^{-t_0}\widehat{h}^{-}(x) \\ &< \max\{d, d_{-}\}\left(d^{t_0}\widehat{h}^{+}(x) + d_{-}^{-t_0}\widehat{h}^{-}(x)\right) = \max\{d, d_{-}\} p^{\frac{1}{p}} q^{\frac{1}{q}} \widehat{h}^{+}(x)^{\frac{1}{p}} \widehat{h}^{-}(x)^{\frac{1}{q}}.\end{aligned}$$

Similarly we get

$$\begin{aligned}\widehat{h}(f^{[t_0]+1}(x)) &= d^{1+[t_0]-t_0} d^{t_0}\widehat{h}^{+}(x) + d^{-(1+[t_0]-t_0)} d_{-}^{-t_0}\widehat{h}^{-}(x) \\ &< \max\{d, d_{-}\} p^{\frac{1}{p}} q^{\frac{1}{q}} \widehat{h}^{+}(x)^{\frac{1}{p}} \widehat{h}^{-}(x)^{\frac{1}{q}}.\end{aligned}$$

This shows $\left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right) \min_{y \in O_f(x)} \log \widehat{h}(y) \leq \widehat{h}(O_f(x)) + \epsilon_2$. □

Theorem 5.3. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial automorphism of degree $d \geq 2$ over a number field K satisfying (4.1), and $\widehat{h} : \mathbb{A}^n(\overline{K}) \to \mathbb{R}$ a canonical height function associated with f. Let x be an element of $\mathbb{A}^n(\overline{K})$ such that $\#O_f(x) = \infty$. Then we have the following.*

(1) *If $\left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right) \log T \geq \widehat{h}(O_f(x))$, then*

$$\left|\#\{y \in O_f(x) \mid \widehat{h}(y) \leq T\} - \left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right)\log T + \widehat{h}(O_f(x))\right| \leq \frac{\log 2}{\log d} + \frac{\log 2}{\log d_{-}} + 1.$$

Note that if $\left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right) \log T \leq \widehat{h}(O_f(x))$, it follows from Lemma 5.2(2) that $\#\{y \in O_f(x) \mid \widehat{h}(y) \leq T\} = \emptyset$.

(2) $\#\{y \in O_f(x) \mid h_{nv}(y) \leq T\} = \left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right)\log T - \widehat{h}(O_f(x)) + O(1)$ *as $T \to \infty$, where the $O(1)$ constant depends only on f and the choice of $\widehat{h}$.*

Proof. Since $\#O_f(x) = \infty$, the map $\mathbb{Z} \ni l \mapsto f^l(x) \in \mathbb{A}^n(\overline{K})$ is one-to-one. Then

$$\begin{aligned}\#\{y \in O_f(x) \mid \widehat{h}(y) \leq T\} &= \#\{l \in \mathbb{Z} \mid \widehat{h}(f^l(x)) \leq T\} \\ &= \#\{l \in \mathbb{Z} \mid d^l\widehat{h}^{+}(x) + d_{-}^{-l}\widehat{h}^{-}(x) \leq T\}.\end{aligned}$$

Then it follows from Lemma 5.4 that

$$-1 + \frac{\log \frac{T}{2\widehat{h}^{+}(x)}}{\log d} + \frac{\log \frac{T}{2\widehat{h}^{-}(x)}}{\log d_{-}} \leq \#\{y \in O_f(x) \mid \widehat{h}(y) \leq T\} \leq 1 + \frac{\log \frac{T}{\widehat{h}^{+}(x)}}{\log d} + \frac{\log \frac{T}{\widehat{h}^{-}(x)}}{\log d_{-}},$$

for $T \geq \widehat{h}^{+}(x)^{\frac{\log d_{-}}{\log d + \log d_{-}}} \widehat{h}^{-}(x)^{\frac{\log d}{\log d + \log d_{-}}}$ or equivalently $\left(\frac{1}{\log d} + \frac{1}{\log d_{-}}\right)\log T \geq \widehat{h}(O_f(x))$.

On the other hand, we have

$$-1+\frac{\log\frac{T}{2\widehat{h}^+(x)}}{\log d}+\frac{\log\frac{T}{2\widehat{h}^-(x)}}{\log d_-}=-1-\frac{\log 2}{\log d}-\frac{\log 2}{\log d_-}+\left(\frac{1}{\log d}+\frac{1}{\log d_-}\right)\log T-\widehat{h}(O_f(x)),$$

$$1+\frac{\log\frac{T}{\widehat{h}^+(x)}}{\log d}+\frac{\log\frac{T}{\widehat{h}^-(x)}}{\log d_-}=1+\left(\frac{1}{\log d}+\frac{1}{\log d_-}\right)\log T-\widehat{h}(O_f(x)).$$

Thus we obtain (1). Next, we will show (2). Since $h_{nv} \gg\ll \widehat{h}$ by the property (i) of Theorem A, there exist a positive constant a_2 and a constant b_2 such that $\widehat{h} \le a_2 h_{nv} + b_2$. Then we have

$$\begin{aligned}
&\#\{y \in O_f(x) \mid h_{nv}(y) \le T\}\\
&\quad\le \#\{y \in O_f(x) \mid \widehat{h}(y) \le a_2T + b_2\}\\
&\quad\le \left(\frac{1}{\log d}+\frac{1}{\log d_-}\right)\log(a_2T+b_2)-\widehat{h}(O_f(x))+1+\frac{\log 2}{\log d}+\frac{\log 2}{\log d_-}\\
&\quad\le \left(\frac{1}{\log d}+\frac{1}{\log d_-}\right)\log T-\widehat{h}(O_f(x))+O(1) \qquad \text{as } T\to\infty.
\end{aligned}$$

Using $a_1h_{nv} + b_1 \le \widehat{h}$ for some positive constant a_1 and constant b_1, we have $\#\{y \in O_f(x) \mid h_{nv}(y) \le T\} \ge \left(\frac{1}{\log d}+\frac{1}{\log d_-}\right)\log T - \widehat{h}(O_f(x)) + O(1)$ as $T \to \infty$. □

Lemma 5.4. *Let $A, B, T > 0$ be positive numbers. If $T \ge A^{\frac{\log d_-}{\log d+\log d_-}} B^{\frac{\log d}{\log d+\log d_-}}$, then we have*

$$-1+\frac{\log\frac{T}{2A}}{\log d}+\frac{\log\frac{T}{2B}}{\log d_-}\le \#\{l\in\mathbb{Z}\mid d^lA+d_-^{-l}B\le T\}\le 1+\frac{\log\frac{T}{A}}{\log d}+\frac{\log\frac{T}{B}}{\log d_-}.$$

Proof. If $l \in \mathbb{Z}$ satisfies $d^lA + d_-^{-l}B \le T$, then $d^lA \le T$ and $d_-^{-l}B \le T$. Note that $\frac{\log\frac{B}{T}}{\log d_-} \le \frac{\log\frac{T}{A}}{\log d}$ is equivalent to $T \ge A^{\frac{\log d_-}{\log d+\log d_-}} B^{\frac{\log d}{\log d+\log d_-}}$. Then, for $T \ge A^{\frac{\log d_-}{\log d+\log d_-}} B^{\frac{\log d}{\log d+\log d_-}}$, we have

$$\#\{l\in\mathbb{Z}\mid d^lA+d_-^{-l}B\le T\}\le \#\left\{l\in\mathbb{Z}\;\middle|\;\frac{\log\frac{B}{T}}{\log d_-}\le l\le\frac{\log\frac{T}{A}}{\log d}\right\}\le 1+\frac{\log\frac{T}{A}}{\log d}+\frac{\log\frac{T}{B}}{\log d_-}.$$

On the other hand, if $l \in \mathbb{Z}$ satisfies $d^lA \le \frac{T}{2}$ and $d_-^{-l}B \le \frac{T}{2}$, then $d^lA + d_-^{-l}B \le T$. Thus,

$$\#\{l\in\mathbb{Z}\mid d^lA+d_-^{-l}B\le T\}\ge \#\left\{l\in\mathbb{Z}\;\middle|\;\frac{\log\frac{2B}{T}}{\log d_-}\le l\le\frac{\log\frac{T}{2A}}{\log d}\right\}\ge -1+\frac{\log\frac{T}{2A}}{\log d}+\frac{\log\frac{T}{2B}}{\log d_-}.$$

□

Proof of Theorem C. It follows from Theorem 3.6 that regular polynomial automorphisms of the affine plane satisfy (4.1). Then Theorem C follows from Theorem 5.3. □

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