CANONICAL HEIGHTS, INVARIANT CURRENTS, AND DYNAMICAL SYSTEMS OF MORPHISMS ASSOCIATED WITH LINE BUNDLES

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ABSTRACT. We construct canonical heights of subvarieties for dynamical systems of several morphisms associated with line bundles defined over a number field, and study some of their properties. We also construct invariant currents for such systems over $\mathbb{C}$.

INTRODUCTION

Let X be a projective variety over a field K and $f_i : X \to X$ $(i = 1, \cdots, k)$ morphisms over K. Let L be a line bundle on X, and $d > k$ a real number. We say that a pair $(X; f_1, \cdots, f_k)$ is a *dynamical system of* k *morphisms over* K *associated with* L *of degree* d if $\bigotimes_{i=1}^{k} f_i^*(L) \simeq L^{\otimes d}$ holds. The purpose of this paper to construct canonical heights of subvarieties for such systems when K is a number field and study some of their properties, and construct invariant currents for such systems when K is $\mathbb{C}$. We also remark on distribution of points of small heights for Lattès examples, which are certain endomorphisms of $\mathbb{P}^N$.

Firstly, we explain canonical heights. Weil heights play one of the key roles in Diophantine geometry, and particular Weil heights that enjoy nice properties (called "canonical" heights) are sometimes of great use.

Over abelian varieties A defined over a number field K, Néron and Tate constructed height functions (called canonical height functions or Néron–Tate height functions) $\widehat{h}_L : A(\overline{K}) \to \mathbb{R}$ with respect to symmetric ample line bundles L which enjoys nice properties. More generally, in [7] Call and Silverman constructed canonical height functions on projective varieties X defined over a number field which admit a morphism $f : X \to X$ with $f^*(L) \simeq L^{\otimes d}$ for some line bundle L and some $d > 1$. In another direction, Silverman [20] constructed canonical height functions on certain K3 surfaces S with two involutions σ_1, σ_2 (called Wheler's K3 surfaces, cf. [13]) and developed an arithmetic theory analogous to the arithmetic theory on abelian varieties.

Regarding canonical heights of subvarieties of projective varieties, Philippon [17], Gubler [8] and Kramer [8] constructed canonical heights of subvarieties of abelian varieties. In [26], Zhang constructed canonical heights of subvarieties of projective varieties X which admit a morphism $f : X \to X$ with $f^*(L) \simeq L^{\otimes d}$ for some line bundle L and some $d > 1$. For Wheler's K3 surfaces, however, canonical heights of subvarieties seem not to have been constructed.

Our first results are construction of canonical heights of subvarieties for dynamical system of morphisms associated with line bundles.

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Theorem A (cf. Theorem 1.2.1, Proposition 1.3.1, Theorem 2.3.1, Theorem 4.3.1)**.** *Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over a number field K associated with a line bundle L of degree $d > k$. Then*

(1) *there exists a unique real-valued function*

$$\widehat{h}_{L,\{f_1,\cdots,f_k\}} : X(\overline{K}) \to \mathbb{R}$$

with the following properties:

(i) *$\widehat{h}_{L,\{f_1,\cdots,f_k\}}$ is a Weil height corresponding to L, i.e., $\widehat{h}_{L,\{f_1,\cdots,f_k\}} = h_L + O(1)$;*

(ii) *$\sum_{i=1}^{k} \widehat{h}_{L,\{f_1,\cdots,f_k\}}(f_i(x)) = d\, \widehat{h}_{L,\{f_1,\cdots,f_k\}}(x)$ for all $x \in X(\overline{K})$.*

(2) *Assume L is ample. Then*

(a) *$\widehat{h}_{L,\{f_1,\cdots,f_k\}}(x) \geq 0$ for all $x \in X(\overline{K})$.*

(b) *Let $C(x) := \{f_{i_1} \circ \cdots \circ f_{i_l}(x) \mid l \geq 0, 1 \leq i_1, \cdots, i_l \leq k\}$ denote the forward orbit of x under $\{f_1, \cdots, f_k\}$. Then $\widehat{h}_{L,\{f_1,\cdots,f_k\}}(x) = 0$ if and only if $C(x)$ is finite.*

(3) *Assume that L is ample, X is normal and $f_1, \cdots, f_k$ are all surjective. Then one can define the canonical height $\widehat{h}_{L,\{f_1,\cdots,f_k\}}(Y) \geq 0$ for any subvariety $Y \subset X_{\overline{K}}$.*

(4) *Assume X is normal. Then the function $\widehat{h}_{L,\{f_1,\cdots,f_k\}}$ decomposes into the sum of local canonical height functions.*

Consider Wheler's K3 surface S. It is shown that $(S; \sigma_1, \sigma_2)$ becomes a dynamical system of two morphisms associated with a certain ample line bundle of degree 4. Applying Theorem A to $(S; \sigma_1, \sigma_2)$, one obtains a height function $\widehat{h}_{L,\{\sigma_1,\sigma_2\}} : S(\overline{K}) \to \mathbb{R}$. It will be shown that $(1+\sqrt{3})\widehat{h}_{L,\{\sigma_1,\sigma_2\}}$ coincides with the the canonical height function defined by Silverman. In this way, Theorem A generalizes Call–Silverman and Zhang's construction of canonical heights for dynamical systems of one morphism to dynamical systems of k morphisms, incorporating Silverman's canonical height functions on Wheler's K3 surfaces. Moreover, one has canonical heights of subvarieties of Wheler's K3 surfaces.

Using Theorem A, we can similarly construct canonical height functions (and canonical heights of subvarieties) on other K3 surfaces: K3 surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the intersection of two hypersurfaces of bidegrees $(1, 2)$ and $(2, 1)$; and K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by hypersurfaces of tridegree $(2, 2, 2)$. For arithmetic properties of the last surfaces, see Mazur [13] and Wang [25]. Another example of a dynamical system is the projective space $\mathbb{P}^N$ and several surjective morphisms $f_i : \mathbb{P}^N \to \mathbb{P}^N$ such that $\deg f_i \geq 2$ for some i. For their properties over $\mathbb{C}$ in the case $N = 1$ (dynamics of finitely generated rational semigroups), we refer to Hinkkanen and Martin [9], which studies the theory of the dynamics of semigroups of rational functions on $\mathbb{P}^1$. See §1.4 for other examples.

Secondly, we explain invariant currents. In [10], Hubbard and Papadopol introduced an invariant current, called the Green current, for a morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$, which is a generalization of the Brodin measure (also called the equilibrium measure) of $N = 1$. (More strongly, the Green current of f is defined when $f : \mathbb{P}^N \cdots \to \mathbb{P}^N$ is an algebraically stable rational map, cf. [18], Théorème 1.6.1.) Properties of the Green current of f have since been deeply studied by many authors, see for example the survey of Sibony [18] and the references therein.

Our second results are the existence of $(1, 1)$-currents with nice properties, which we call invariant currents, for dynamical systems of morphisms over $\mathbb{C}$ associated with line bundles.

Theorem B (cf. Theorem 3.1.1, Theorem 3.2.1)**.** *Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over $\mathbb{C}$ associated with a line bundle L of degree $d > k$. We assume that X is smooth and L is ample.*

(1) *Let $\eta_0 \in A^{1,1}(X(\mathbb{C}))$ be a closed C^∞ $(1,1)$-form whose cohomology class coincides with $\mathrm{c}_1(L)$. We inductively define $\eta_n \in A^{1,1}(X(\mathbb{C}))$ by*
$$\eta_{n+1} = \frac{1}{d}(f_1^* \eta_n + \cdots + f_k^* \eta_n).$$
Then, $[\eta_n]$ converges to a closed positive $(1,1)$-current T. Moreover, T is independent of the choice of η_0, and satisfies $f_1^* T + \cdots + f_k^* T = dT$.

(2) *The current T admits a locally continuous potential. Indeed, there is a unique continuous metric $\| \cdot \|_\infty$, called the admissible metric (cf. [26]), on L with $\| \cdot \|_\infty^d = \varphi^*(f_1^* \| \cdot \|_\infty \cdots f_k^* \| \cdot \|_\infty)$, and T is given by $\mathrm{c}_1(L, \| \cdot \|_\infty)$.*

When $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree $d \geq 2$, the current T for $(\mathbb{P}^N; f)$ is nothing but the Green current of f. Consider Wheler's K3 surface S. In this case, $\sigma_2 \circ \sigma_1$ (resp. $\sigma_1 \circ \sigma_2$) has a strictly positive topological entropy, and by the results of Cantat [5] there exists a closed positive $(1,1)$-current T^+ (resp. T^-), unique up to scale, such that $(\sigma_2 \circ \sigma_1)^*(T^+) = (7 + 4\sqrt{3}) T^+$ (resp. $(\sigma_1 \circ \sigma_2)^*(T^-) = (7 + 4\sqrt{3}) T^-$). It will be shown that the current T for the dynamical system $(S; \sigma_1, \sigma_2)$ is equal to $T^+ + T^-$ with a certain normalization of a scale.

Since Szpiro, Ullmo and Zhang [23] proved equidistribution of small points for abelian varieties, and since heights of subvarieties and currents both with nice properties are constructed for $(X; f_1, \cdots, f_k)$, it is tempting to pose a question for $(X; f_1, \cdots, f_k)$ how the Galois orbit of x_n is distributed with respect to $\mu := \frac{T \wedge \cdots \wedge T}{\mathrm{c}_1(L)^{\dim X}}$ ($\dim X$-times) whenever there exists a sequence $\{x_n\}_{n=1}^\infty$ of $X(\overline{K})$ of small points. Using the methods of [23] and [6], we remark that this question is true for Lattès examples, which are endomorphisms of $\mathbb{P}^N$ for which the Green currents are on some non-empty (analytic) open subset smooth and strictly positive.

The organization of this paper is as follows. In §1, we construct canonical heights, study some of their properties, and give some examples. In §2, we construct canonical heights of subvarieties of X using adelic intersection theory developed by Zhang. In §3, we construct invariant currents on $X(\mathbb{C})$. Then, we remark on distribution of points of small height for Lattès's examples. In §4, we construct canonical local heights and see that the canonical height in §1 decomposes into the sum of these canonical local heights. The case $k = 1$ is studied by Call and Silverman. In the appendix, we show finiteness of $\{f_1, \cdots, f_k\}$-periodic points of bounded degree in a more general setting.

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1. Canonical heights

1.1. **Quick review of heights.** In this subsection, we briefly review some part of height theory. We formulate it in terms of $\mathbb{R}$-line bundles (not just line bundles), since $\mathbb{R}$-line bundles are used in [20]. For details of height theory, we refer to [11] and [19].

Let K be a number field and O_K its ring of integers. For $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$, the logarithmic naive height of x is defined by

$$h_{nv}(x) = \frac{1}{[K:\mathbb{Q}]}\left[\sum_{P\in \mathrm{Spec}(O_K)\setminus\{0\}} \max_{0\le i\le n}\{-\mathrm{ord}_P(x_i)\}\log\#(O_K/P) + \sum_{\sigma:K\hookrightarrow\mathbb{C}}\max_{0\le i\le n}\{\log|\sigma(x_i)|\}\right].$$

By the product formula, $h_{nv}(x)$ is independent of the homogeneous coordinates of x. It is also independent of the choice of fields over which x is defined. Thus, we have the logarithmic naive height function $h_{nv} : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Let X be a projective variety over $\overline{\mathbb{Q}}$. Let $\mathrm{Pic}(X)$ denote the group of isomorphic classes of line bundles on X. For $L, M \in \mathrm{Pic}(X)$, we will often denote the tensor product additively, i.e., $L + M$ in place of $L \otimes M$. Let $F(X) := \{f : X(\overline{\mathbb{Q}}) \to \mathbb{R}\}$ be the space of real-valued functions on $X(\overline{\mathbb{Q}})$, and $B(X) := \{f : X(\overline{\mathbb{Q}}) \to \mathbb{R} \mid \sup_{x\in X(\overline{\mathbb{Q}})} |f(x)| < \infty\}$ the space of real-valued bounded functions on $X(\overline{\mathbb{Q}})$. We use the notation $h_1 = h_2 + O(1)$ if $h_1, h_2 \in F(X)$ satisfies $h_1 - h_2 \in B(X)$.

Theorem 1.1.1 (cf. [19])**.** *For any projective variety X over $\overline{\mathbb{Q}}$, there exists a unique map*

$$h_X : \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R} \longrightarrow F(X) \ \textit{modulo}\ B(X), \quad L \mapsto h_{X,L}$$

with the following properties:

(1) *$h_{X,L}$ is $\mathbb{R}$-linear;*
(2) *If $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, then $h_{\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)} = h_{nv} + O(1)$;*
(3) *If $f : X \to Y$ is a morphism of projective varieties and L is an element of $Pic(X)\otimes_{\mathbb{Z}}\mathbb{R}$, then $h_{X,f^*L} = h_{Y,L}\circ f + O(1)$.*

Proof. See [19], §3 for a proof when $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ is replaced by $\mathrm{Pic}(X)$, and $\mathbb{R}$-linear by $\mathbb{Z}$-linear. (The theorem there holds without the assumption of the smoothness of X.) Let $h_X : \mathrm{Pic}(X) \to F(X)$ modulo $B(X)$ be the map defined in [19], §3. By tensoring $\mathbb{R}$ over $\mathbb{Z}$, we have a map

$$h_X : \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R} \longrightarrow F(X) \text{ modulo } B(X).$$

Obviously, h_X satisfies (1) and (2). To see h_X satisfies (3), let L be an element of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$. We write $L \cong r_1L_1 + \cdots + r_nL_n$ for $r_i \in \mathbb{R}$ and $L_i \in \mathrm{Pic}(X)$. By [19], §3 it follows that $h_{X,f^*L_i} = h_{Y,L_i}\circ f$, and by (1) it follows that $h_{X,\sum_i r_if^*L_i} = h_{Y,\sum_i r_iL_i}\circ f$. Thus we have $h_{X,f^*L} = h_{Y,L}\circ f$. The uniqueness of h_X follows from [19], §3 and $\mathbb{R}$-linearity. □

By slight abuse of notation, we will use the same notation $h_{X,L} \in F(X)$ for a representative of $h_{X,L} \in F(X)$ modulo $B(X)$. We will often abbreviate $h_{X,L}$ by h_L. The following Northcott's finiteness theorem is useful in the study of rational points on projective varieties.

Theorem 1.1.2 (Northcott's finiteness theorem)**.** *Let X be a projective variety over $\overline{\mathbb{Q}}$, and $L \in \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ an ample $\mathbb{R}$-line bundle. Then for any positive number M and a positive integer D, the set*

$$\left\{x \in X(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(x):\mathbb{Q}] \le D,\ h_L(x) \le M\right\}$$

is finite.

Proof. See [19], §3 for a proof when $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ is replaced by $\mathrm{Pic}(X)$. In general, write $L \cong r_1L_1 + \cdots + r_nL_n$ for $r_i > 0$ and ample L_i. By changing representatives if necessary, we can take $h_{L_i} \in F(X)$ such that $h_{L_i} \geq 0$ and $h_L = \sum_i r_i h_{L_i} \in F(X)$. Then, the assertion follows from

$$\left\{ x \in X(\overline{\mathbb{Q}}) \;\middle|\; \begin{matrix} [\mathbb{Q}(x):\mathbb{Q}] \leq D, \\ h_L(x) \leq M \end{matrix} \right\} \quad \subseteq \quad \left\{ x \in X(\overline{\mathbb{Q}}) \;\middle|\; \begin{matrix} [\mathbb{Q}(x):\mathbb{Q}] \leq D, \\ h_{L_1}(x) \leq \dfrac{M}{r_1} \end{matrix} \right\}.$$

□

1.2. **Construction of canonical heights.** Let X be a projective variety defined over a number field K, and $f_1, \cdots, f_k : X \to X$ be morphisms over K. Assume that there exists an $\mathbb{R}$-line bundle L on X such that $\sum_{i=1}^k f_i^*(L) \cong dL$ for some positive real number $d > k$ in $\mathrm{Pic}(X) \otimes \mathbb{R}$.

We set $\mathcal{F}_0 := \{\mathrm{id}\}$ and $\mathcal{F}_l := \{f_{i_1} \circ \cdots \circ f_{i_l} \mid 1 \leq i_1, \cdots, i_l \leq k\}$ for $l \geq 1$. We set $\mathcal{F} := \mathcal{F}_1 (= \{f_1, \cdots, f_k\})$.

Theorem 1.2.1. *Let X be a projective variety over a number field K, and $f_1, \cdots, f_k : X \to X$ morphisms over K. Assume that there exists an $\mathbb{R}$-line bundle L on X such that $\sum_{i=1}^k f_i^*(L) \cong dL$ for some positive real number $d > k$ in* $\mathrm{Pic}(X) \otimes \mathbb{R}$. *Then there exists a unique real-valued function,*

$$\widehat{h}_{L,\mathcal{F}} : X(\overline{K}) \to \mathbb{R}$$

with the following properties:

(i) $\widehat{h}_{L,\mathcal{F}} = h_L + O(1)$.
(ii) $\sum_{i=1}^k \widehat{h}_{L,\mathcal{F}}(f_i(x)) = d\widehat{h}_{L,\mathcal{F}}(x)$ *for all $x \in X(\overline{K})$. We call $\widehat{h}_{L,\mathcal{F}}$ the* canonical height function *associated with L and $\mathcal{F}$.*

Proof. First let us construct $\widehat{h}_{L,\mathcal{F}}$. We take any representative $h_L \in F(X)$ of $h_L \in F(X)$ modulo $B(X)$ and fix it. Since $\sum_{i=1}^k f_i^*(L) \cong dL$, there exists a constant $C > 0$ such that

$$\left| \sum_{i=1}^k h_L(f_i(x)) - dh_L(x) \right| \leq C$$

for all $x \in X(\overline{K})$. Set $a_0(x) := h_L(x)$ and

$$a_l(x) := \frac{1}{d^l} \sum_{f \in \mathcal{F}_l} h_L(f(x)) \in \mathbb{R}.$$

We claim $\{a_l(x)\}_{l=0}^{\infty}$ is a Cauchy sequence. Indeed, we have

$$\begin{aligned}|a_{l+1}(x) - a_l(x)| &= \left| \frac{1}{d^{l+1}} \sum_{f \in \mathcal{F}_{l+1}} h_L(f(x)) - \frac{1}{d^l} \sum_{f \in \mathcal{F}_l} h_L(f(x)) \right| \\ &= \frac{1}{d^{l+1}} \left| \sum_{f \in \mathcal{F}_l} \left\{ \sum_{i=1}^{k} h_L(f_i \circ f(x)) - d h_L(f(x)) \right\} \right| \\ &\leq \frac{\#\mathcal{F}_l}{d^{l+1}} C = \frac{k^l C}{d^{l+1}}.\end{aligned}$$

Since $\sum_{l=0}^{\infty} \frac{k^l C}{d^{l+1}} = \frac{C}{d-k} < \infty$, $\{a_l(x)\}_{l=0}^{\infty}$ is a Cauchy sequence.

We define $\widehat{h}_{L,\mathcal{F}}(x) := \lim_{l \to \infty} a_l(x)$. we need to show $\widehat{h}_{L,\mathcal{F}}$ satisfies (i) and (ii). Note that $|a_l(x) - a_0(x)| \leq \sum_{\alpha=0}^{l-1} |a_{\alpha+1}(x) - a_\alpha(x)| \leq \frac{C}{d-k}$. Since $a_0(x) = h_L(x)$, by letting $l \to \infty$ we obtain

$$\left| \widehat{h}_{L,\mathcal{F}}(x) - h_L(x) \right| \leq \frac{C}{d-k}.$$

Since $\sum_{i=1}^{k} a_l(f_i(x)) = d a_{l+1}(x)$, we also obtain (ii) by letting $l \to \infty$.

To see the uniqueness of $\widehat{h}_{L,\mathcal{F}}$, suppose both $\widehat{h}_1$ and $\widehat{h}_2$ satisfy (i) and (ii). By (i) there exists a constant C' with $|\widehat{h}_1(x) - \widehat{h}_2(x)| \leq C'$ for all $x \in X(\overline{K})$. By (ii) it follows that $\widehat{h}_j(x) = \frac{1}{d^l} \sum_{f \in \mathcal{F}_l} \widehat{h}_j(f(x))$. Since

$$|\widehat{h}_1(x) - \widehat{h}_2(x)| \leq \frac{\#\mathcal{F}_l}{d^l} C' = \frac{k^l C'}{d^l},$$

we obtain $\widehat{h}_1(x) = \widehat{h}_2(x)$ by letting $l \to \infty$. □

1.3. **Some properties of canonical heights.** Let the notation and assumption be as in §1.2. For $x \in X(\overline{K})$, we define the forward orbit of x under $\mathcal{F}$ to be

$$C(x) := \{f(x) \mid f \in \mathcal{F}_l \text{ for some } l \geq 0\}.$$

Note that when there is only one morphism f_1, i.e., $k = 1$, $C(x)$ is finite if and only if x is preperiodic with respect to f_1.

Proposition 1.3.1. *Let the notation and assumption be as in Theorem 1.2.1. Assume* $L \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ *is ample. Then*

(1) $\widehat{h}_{L,\mathcal{F}}(x) \geq 0$ *for all* $x \in X(\overline{K})$.

(2) $\widehat{h}_{L,\mathcal{F}}(x) = 0$ *if and only if* $C(x)$ *is finite.*

Proof. (1): Take a height function $h_L \in F(X)$ corresponding to L. Since L is ample, there exists a constant C such that $h_L(x) \geq C$ for all $x \in X(\overline{K})$. Then we have

$$\begin{aligned}\widehat{h}_{L,\mathcal{F}}(x) &= \lim_{l \to \infty} \frac{1}{d^l} \sum_{f \in \mathcal{F}_l} h_L(f(x)) \\ &\geq \limsup_{l \to \infty} \frac{1}{d^l} k^l C = 0\end{aligned}$$

(2): We first show the "only if" part. Let K' be a finite extension of K such that $x \in X(K')$. Since $\widehat{h}_{L,\mathcal{F}} \geq 0$ and $\sum_{i=1}^{k} \widehat{h}_{L,\mathcal{F}}(f_i(x)) = d\widehat{h}_{L,\mathcal{F}}(x)$, it follows that $\widehat{h}_{L,\mathcal{F}}(f(x)) = 0$ for all $f \in \bigcup_{l \geq 0} \mathcal{F}_l$. Then by Northcott's finiteness theorem (Theorem 1.1.2), $C(x) \subseteq \{y \in X(K') \mid \widehat{h}_{L,\mathcal{F}}(y) = 0\}$ is finite.

We next show the "if" part. Suppose $a := \widehat{h}_{L,\mathcal{F}}(x) > 0$. Since $\sum_{i=1}^{k} \widehat{h}_{L,\mathcal{F}}(f_i(x)) = d\widehat{h}_{L,\mathcal{F}}(x)$, it follows that $\widehat{h}_{L,\mathcal{F}}(f_{i_1}(x)) \geq \frac{d}{k}a$ for some i_1. Since $\sum_{i=1}^{k} \widehat{h}_{L,\mathcal{F}}(f_i(f_{i_1}(x))) = d\widehat{h}_{L,\mathcal{F}}(f_{i_1}(x))$, it follows that $\widehat{h}_{L,\mathcal{F}}(f_{i_2} \circ f_{i_1}(x)) \geq (\frac{d}{k})^2 a$ for some i_2. Similarly, for any $l \in \mathbb{Z}_{>0}$, there exists $f \in \mathcal{F}_l$ with $\widehat{h}_{L,\mathcal{F}}(f(x)) \geq (\frac{d}{k})^l a$. Since $\frac{d}{k} > 1$, $C(x)$ cannot be finite. □

Corollary 1.3.2. *Let the notation and assumption be as in Theorem 1.2.1. Assume $L \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is ample. Then for any $D \in \mathbb{Z}_{>0}$,*

$$\left\{x \in X(\overline{K}) \ \mid [K(x) : K] \leq D, \ \ C(x) \textit{ is finite}\right\}$$

is finite.

Proof. The assertion follows from Proposition 1.3.1(2) and Theorem 1.1.2. □

1.4. **Examples.**

1.4.1. *Abelian varieties.* Let A be an abelian variety over $\overline{\mathbb{Q}}$, and L a symmetric ample line bundle on A. Since $[2]^*(L) \cong 4L$, the canonical height function $\widehat{h}_{L,[2]} : A(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$, called Néron-Tate's height function, is defined so that $\widehat{h}_{L,[2]}$ is a height corresponding to L and $\widehat{h}_L \circ [2] = 4\widehat{h}_L$.

1.4.2. *Projective spaces.* Let $F_0, \cdots, F_N \in \overline{\mathbb{Q}}[X_0, \cdots, X_N]$ be homogeneous polynomials of degree $d \geq 1$ such that 0 is the only common zero of $F_0, \cdots, F_N$. Then $f = (F_0 : \cdots : F_N)$ defines a morphism of $\mathbb{P}^N$ to $\mathbb{P}^N$. Assume $d \geq 2$. Since $f^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^N}(d)$, the canonical height function $\widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),f}$ associated with $\mathcal{O}_{\mathbb{P}^N}(1)$ and f is defined.

More generally, for $i = 1, \cdots, k$, let $f_i : \mathbb{P}^N \to \mathbb{P}^N$ be morphisms of degree $d_i \geq 1$. Assume $d_i \geq 2$ for some i. Since $\bigotimes_{i=1}^{k} f_i^*(\mathcal{O}_{\mathbb{P}^N}(1)) \simeq \mathcal{O}_{\mathbb{P}^N}(\sum_{i=1}^{k} d_i)$, the canonical height function $\widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),\{f_1,\cdots,f_k\}}$ associated with $\mathcal{O}_{\mathbb{P}^N}(1)$ and $\{f_1, \cdots, f_k\}$ is defined. We note that the dynamical system $(\mathbb{P}^N; f_1, \cdots, f_k)$ over $\mathbb{C}$ in the case $N = 1$ (dynamics of finitely generated rational semigroups) is studied in [9].

1.4.3. *Toric varieties.* Let $\mathbb{P}(\Delta)$ be a smooth projective toric variety $\overline{\mathbb{Q}}$, and D an ample torus-invariant Cartier divisor on $\mathbb{P}(\Delta)$. Let $p \geq 2$ be an integer, and $[p] : \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$ the morphism associated with the multiplication of Δ by p. Since $[p]^*(\mathcal{O}_{\mathbb{P}(\Delta)}(D)) \simeq \mathcal{O}_{\mathbb{P}(\Delta)}(D)^{\otimes p}$, the canonical height function $\widehat{h}_{\mathcal{O}_{\mathbb{P}(\Delta)}(D),[p]} : \mathbb{P}(\Delta)(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated with $\mathcal{O}_{\mathbb{P}(\Delta)}(D)$ and $[p]$ is defined. For details, we refer Maillot [12], Theorem 3.3.3.

1.4.4. *K3 surfaces, I.* Let S be a K3 surface in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the intersection of two hypersurfaces of bidegrees $(1,1)$ and $(2,2)$ over $\overline{\mathbb{Q}}$. Such K3 surfaces are called Wheler's K3 surfaces (cf. [13]). Silverman [20] constructed a canonical height function on S and developed an arithmetic theory. Here, we will show we can construct a height function on S by using Theorem 1.2.1, which turns out to coincide with the one Silverman constructed up to a constant multiple.

Let $p_i : S \to \mathbb{P}^2$ be the projection to the i-th factor for $i = 1, 2$. Since p_i is a double cover, it gives an involution $\sigma_i \in \mathrm{Aut}(S)$. Set $L_i := p_i^* \mathcal{O}_{\mathbb{P}^2}(1)$. By [20], Lemma 2.1, we have

$$\sigma_1^*(L_1) \cong L_1, \quad \sigma_2^*(L_1) \cong -L_1 + 4L_2,$$
$$\sigma_1^*(L_2) \cong 4L_1 - L_2, \quad \sigma_2^*(L_2) \cong L_2.$$

Thus the ample line bundle $L := L_1 + L_2$ on S satisfies $\sigma_1^*(L) + \sigma_2^*(L) \cong 4L$. Applying Theorem 1.2.1 to S, L and $\{\sigma_1, \sigma_2\}$, we obtain a height function $\widehat{h}_{L,\{\sigma_1,\sigma_2\}} : S(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$.

Let us recall Silverman's construction of a height function. Setting $E^+ := (2+\sqrt{3})L_1 - L_2$ and $E^- := -L_1 + (2+\sqrt{3})L_2$, we have $(\sigma_2 \circ \sigma_1)^{\pm 1 *}(E^\pm) = (7+4\sqrt{3})E^\pm$. Then we obtain two height functions $\widehat{h}^\pm$ which are characterized by being a height corresponding to $E^\pm$ and $\widehat{h}^\pm \circ (\sigma_2 \circ \sigma_1)^{\pm 1} = (7+4\sqrt{3})\widehat{h}^\pm$. (In the notation of Theorem 1.2.1, $\widehat{h}^+ = \widehat{h}_{E^+, \sigma_2 \circ \sigma_1}$ and $\widehat{h}^- = \widehat{h}_{E^-, \sigma_1 \circ \sigma_2}$.) Silverman defined a canonical height function on S to be $\widehat{h}^+ + \widehat{h}^-$, which is denoted by $\widehat{h}_{Sil}$ in the following.

The next proposition relates $\widehat{h}_{Sil}$ with $\widehat{h}_{L,\{\sigma_1,\sigma_2\}}$.

Proposition 1.4.1. $\widehat{h}_{Sil} = (1+\sqrt{3})\ \widehat{h}_{L,\{\sigma_1,\sigma_2\}}$.

Proof. It suffices to show $\frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}$ satisfies the two conditions of Theorem 1.2.1. Since $\widehat{h}_{Sil}$ is by construction a height corresponding to $E^+ + E^- \cong (1+\sqrt{3})L$, it follows that $\frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}$ satisfies the first condition. By [20], Theorem 1.1(ii), we have

$$\widehat{h}^\pm \circ \sigma_1 = (2+\sqrt{3})^{\mp 1}\widehat{h}^\mp, \quad \widehat{h}^\pm \circ \sigma_2 = (2+\sqrt{3})^{\pm 1}\widehat{h}^\mp.$$

Then we have

$$\begin{aligned}
&\frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}(\sigma_1(x)) + \frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}(\sigma_2(x)) \\
=&\frac{1}{1+\sqrt{3}}\left\{\widehat{h}^+(\sigma_1(x)) + \widehat{h}^-(\sigma_1(x)) + \widehat{h}^+(\sigma_2(x)) + \widehat{h}^-(\sigma_2(x))\right\} \\
=&\frac{1}{1+\sqrt{3}}\left((2+\sqrt{3}) + (2+\sqrt{3})^{-1}\right)(\widehat{h}^+(x) + \widehat{h}^-(x)) = 4\frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}.
\end{aligned}$$

Thus $\frac{1}{1+\sqrt{3}}\widehat{h}_{Sil}$ also satisfies the second condition of Theorem 1.2.1. □

1.4.5. *K3 surfaces, II.* Let S be a K3 surface in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the intersection of two hypersurfaces of bidegrees $(1,2)$ and $(2,1)$ over $\overline{\mathbb{Q}}$. Let $p_i : S \to \mathbb{P}^2$ be the projection to the i-th factor for $i = 1, 2$. Since p_i is a double cover, it gives an involution $\sigma_i \in \mathrm{Aut}(S)$. Set

$L_i := p_i^*\mathcal{O}_{\mathbb{P}^2}(1)$. By similar computations as above, we have

$$\sigma_1^*(L_1) \cong L_1, \quad \sigma_2^*(L_1) \cong -L_1 + 5L_2,$$
$$\sigma_1^*(L_2) \cong 5L_1 - L_2, \quad \sigma_2^*(L_2) \cong L_2.$$

Thus the ample line bundle $L := L_1 + L_2$ on S satisfies $\sigma_1^*(L) + \sigma_2^*(L) \cong 5L$. Applying Theorem 1.2.1 to S, L and $\{\sigma_1, \sigma_2\}$, we obtain a height function $\widehat{h}_{L,\{\sigma_1,\sigma_2\}} : S(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$.

1.4.6. *K3 surfaces, III.* Let S be a hypersurface of tridegrees $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ over $\overline{\mathbb{Q}}$ (cf. [13]). For $i = 1,2,3$, let $p_i : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the (j,k)-th factor with $\{i,j,k\} = \{1,2,3\}$. Since p_i is a double cover, it gives an involution $\sigma_i \in \mathrm{Aut}(S)$. Let $q_i : S \to \mathbb{P}^1$ be the projection to the i-th factor, and set $L_i := q_i^*\mathcal{O}_{\mathbb{P}^1}(1)$. By similar computations as above, we have

$$\sigma_i^*(L_i) \cong -L_i + 2L_j + 2L_k \quad \text{for } \{i,j,k\} = \{1,2,3\},$$
$$\sigma_j^*(L_i) \cong L_i \quad \text{for } i \neq j.$$

Thus the ample line bundle $L := L_1 + L_2 + L_3$ on S satisfies $\sigma_1^*(L) + \sigma_2^*(L) + \sigma_3^*(L) \cong 5L$. Applying Theorem 1.2.1 to S, L and $\{\sigma_1, \sigma_2, \sigma_3\}$, we obtain a height function $\widehat{h}_{L,\{\sigma_1,\sigma_2,\sigma_3\}} : S(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$. We remark that Wang [25] constructed a height function on S. It may be interesting to compare it with $\widehat{h}_{L,\{\sigma_1,\sigma_2,\sigma_3\}}$.

To illustrate Proposition 1.3.1(2), suppose now S is defined by the affine equation

$$x(1-x) + y(1-y) + z(1-z) - xyz = 0.$$

Then S is smooth. It is easy to check that the orbit of $(0,0,0)$ under $\{\sigma_1, \sigma_2, \sigma_3\}$ is

$$\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1)\}$$

and thus finite. In particular, $\widehat{h}_{L,\{\sigma_1,\sigma_2,\sigma_3\}}((0,0,0)) = 0$.

1.4.7. *Product of a projective space and an abelian variety.* Let A be an abelian variety over $\overline{\mathbb{Q}}$, and L a symmetric ample line bundle on A. Let $f = (F_0 : \cdots : F_N) : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined by the homogeneous polynomials $F_0, \cdots, F_N$ of degree $d > 1$ such that 0 is the only common zero of $F_0, \cdots, F_N$. Set $X = A \times \mathbb{P}^N$, $g_1 = [2] \times \mathrm{id}_{\mathbb{P}^N}$, and $g_2 = \mathrm{id}_A \times f$. Put $M = p_1^*L \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$, where p_1 and p_2 are the obvious projections. Then

$$\overbrace{g_1^*(M) \otimes \cdots \otimes g_1^*(M)}^{(d-1) \text{ times}} \otimes g_2^*(M) \otimes g_2^*(M) \otimes g_2^*(M) \simeq M^{\otimes(4d-1)}.$$

Thus we have the canonical height function $\widehat{h}_{M,\{g_1,\cdots,g_1,g_2,g_2,g_2\}}$ on $A \times \mathbb{P}^N(\overline{\mathbb{Q}})$. It is easy to see $\widehat{h}_{M,\{g_1,\cdots,g_1,g_2,g_2,g_2\}} = \widehat{h}_L \circ p_1 + \widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),f} \circ p_2$.

2. Canonical heights of subvarieties

In this section, first we briefly review the adelic intersection theory due to Zhang [26]. Then we construct an adelic sequence for a given dynamical system of k morphisms associated with a line bundle, and finally define canonical heights for its subvarieties. We refer to [26] and [16] for details of adelic intersection theory, to [22] and [1] for details of Gillet-Soulé's arithmetic intersection theory and height theory of subvarieties in general.

2.1. **Adelic sequence.** Let K be a number field and O_K its ring of integers. By a projective arithmetic variety, we mean a flat and projective integral scheme over O_K. A pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ is called a C^∞*-hermitian* $\mathbb{Q}$*-line bundle* over a projective arithmetic variety $\mathcal{X}$ if $\mathcal{L}$ is a $\mathbb{Q}$-line bundle over $\mathcal{X}$ and $\|\cdot\|$ is a C^∞-hermitian metric on $\mathcal{L}\otimes_{\mathbb{Z}}\mathbb{C}$ that is invariant under the complex conjugation. Recall that $\|\cdot\|$ is said to be C^∞ if, for any analytic morphism $h : M \to \mathcal{X}\otimes_{\mathbb{Z}}\mathbb{C}$ from any complex manifold M, $h^*(\|\cdot\|)$ is a C^∞-hermitian metric on $h^*(\mathcal{L}\otimes_{\mathbb{Z}}\mathbb{C})$.

Following [15], we say that $\overline{\mathcal{L}}$ is *nef* if the first Chern form $\mathrm{c}_1(\mathcal{L}\otimes_{\mathbb{Z}}\mathbb{C}, \|\cdot\|)$ is semipositive on $X(\mathbb{C})$, and for all one-dimensional integral closed subschemes Γ of $\mathcal{X}$, $\widehat{\deg}(\overline{\mathcal{L}}|_\Gamma) \geq 0$. We say that $\overline{\mathcal{L}}$ is $\mathbb{Q}$*-effective* (denoted by $\overline{\mathcal{L}} \gtrsim 0$) if there is a positive integer n and a non-zero section $s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ with $\|s\|_{\sup} \leq 1$. For two C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$, we use the notation $\overline{\mathcal{L}} \gtrsim \overline{\mathcal{M}}$ if $\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \gtrsim 0$. We quote [15], Proposition 2.3 here.

Proposition 2.1.1 ([15]). *Set* $\dim X = e$.

(1) *If* $\overline{\mathcal{L}_1}, \cdots, \overline{\mathcal{L}_{e+1}}$ *are nef, then* $\widehat{\deg}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_1})\cdots\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_{e+1}})) \geq 0$.
(2) *If* $\overline{\mathcal{L}_1}, \cdots, \overline{\mathcal{L}_e}$ *are nef and* $\overline{\mathcal{M}}$ *is* $\mathbb{Q}$*-effective, then* $\widehat{\deg}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_1})\cdots\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_e})\cdot\widehat{\mathrm{c}}_1(\overline{\mathcal{M}})) \geq 0$.

Let X a projective variety over K and L a $\mathbb{Q}$-line bundle over X. A pair $(\mathcal{X}, \overline{\mathcal{L}})$ of a projective arithmetic variety and a C^∞-hermitian $\mathbb{Q}$-line bundle is called a C^∞*-model of* (X, L) if one has $\mathcal{X}\otimes_{O_K} K = X$ and $\mathcal{L}\otimes_{O_K} K = L$.

Following [26] and [16], we make following definitions.

Definition 2.1.2. (1) Let $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ be a sequence of C^∞-models of (X, L). We say that $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ is an *adelic sequence* of (X, L) if it satisfies the following three conditions:

(i) For every $n \geq 0$, $\overline{\mathcal{L}_n}$ is nef;
(ii) There exists a Zariski dense open set $U \subseteq \mathrm{Spec}(O_K)$ such that, for every $n \geq 0$, $\mathcal{X}_n|_U = \mathcal{X}_{n+1}|_U$ and $\mathcal{L}_n|_U = \mathcal{L}_{n+1}|_U$; We set $\mathrm{Spec}\, O_K \setminus U = \{\mathfrak{P}_1, \cdots, \mathfrak{P}_r\}$;
(iii) For every $n \geq 0$, there exist a projective arithmetic variety $\mathcal{X}_n'$, birational morphisms $\mu_n : \mathcal{X}_n' \to \mathcal{X}_n$ and $\nu_n : \mathcal{X}_n' \to \mathcal{X}_{n+1}$, and positive rational numbers $c_{n1}, \cdots, c_{nr} \in \mathbb{Q}_{>0}$ and positive real numbers $c_{n\sigma} \in \mathbb{R}_{>0}$ for every $\sigma : K \hookrightarrow \mathbb{C}$ such that

$$\pi_{\mathcal{X}_n'}^*\left(\left(-\sum_{\alpha=1}^r c_{n\alpha}[\mathfrak{P}_\alpha], -\sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}[\sigma]\right)\right) \lesssim \mu_n^*(\overline{\mathcal{L}_n}) - \nu_n^*(\overline{\mathcal{L}_{n+1}})$$
$$\lesssim \pi_{\mathcal{X}_n'}^*\left(\left(\sum_{\alpha=1}^r c_{n\alpha}[\mathfrak{P}_\alpha] \sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}[\sigma]\right)\right),$$

$$\sum_{n=0}^\infty c_{n\alpha} < \infty \quad (\alpha = 1, \cdots, r), \qquad \sum_{n=0}^\infty c_{n\sigma} < \infty \quad (\sigma : K \hookrightarrow \mathbb{C})$$

where $\pi_{\mathcal{X}_n'} : \mathcal{X}_n' \to \mathrm{Spec}(O_K)$ is the structure morphism.

A open set U of $\mathrm{Spec}(O_K)$ satisfying (ii) and (iii) is called a common base of $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$. Note that if U is a common base of $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ and U' is a non-empty Zariski open set of U, then U' is also a common base of $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$.

(2) Two adelic sequences $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ and $\{(\mathcal{Y}_n, \overline{\mathcal{M}_n})\}_{n=0}^{\infty}$ of (X, L) are said to be *equivalent* if there exists a non-empty Zariski open set $\widetilde{U}$ of $\operatorname{Spec}(O_K)$ with the following properties:
  (i) $\widetilde{U}$ is a common base of both $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ and $\{(\mathcal{Y}_n, \overline{\mathcal{M}_n})\}_{n=0}^{\infty}$.
  (ii) For every $n \geq 0$, there exist a projective arithmetic variety $\mathcal{Z}_n$, birational morphisms $\widetilde{\mu}_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\widetilde{\nu}_n : \mathcal{Z}_n \to \mathcal{Y}_n$, and positive rational numbers $\widetilde{c}_{n1}, \cdots, \widetilde{c}_{n\widetilde{r}}$ and positive real numbers $\widetilde{c}_{n\sigma}$ for every $\sigma : K \hookrightarrow \mathbb{C}$ such that

$$\pi_{\mathcal{Z}_n}^{*}\left(\left(-\sum_{\alpha=1}^{\widetilde{r}} \widetilde{c}_{n\alpha}[\widetilde{\mathfrak{P}}_\alpha], -\sum_{\sigma:K\hookrightarrow\mathbb{C}} \widetilde{c}_{n\sigma}[\sigma]\right)\right) \lesssim \widetilde{\mu}_n^{*}(\overline{\mathcal{L}_n}) - \widetilde{\nu}_n^{*}(\overline{\mathcal{M}_n})$$
$$\lesssim \pi_{\mathcal{Z}_n'}^{*}\left(\left(\sum_{\alpha=1}^{\widetilde{r}} \widetilde{c}_{n\alpha}[\widetilde{\mathfrak{P}}_\alpha], \sum_{\sigma:K\hookrightarrow\mathbb{C}} \widetilde{c}_{n\sigma}[\sigma]\right)\right),$$

$$\sum_{n=0}^{\infty} \widetilde{c}_{n\alpha} < \infty \quad (\alpha = 1, \cdots, \widetilde{r}), \qquad \sum_{n=0}^{\infty} \widetilde{c}_{n\sigma} < \infty \quad (\sigma : K \hookrightarrow \mathbb{C})$$

where $\operatorname{Spec} O_K \setminus \widetilde{U} = \{\widetilde{\mathfrak{P}}_1, \cdots, \widetilde{\mathfrak{P}}_{\widetilde{r}}\}$ and $\pi_{\mathcal{Z}_n} : \mathcal{Z}_n \to \operatorname{Spec}(O_K)$ is the structure morphism.

The next lemma follows directly from Definition 2.1.2.

Lemma 2.1.3. *Let $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ be an adelic sequence of (X, L).*

(1) *Let Y be a subvariety of $X_{\overline{K}}$ such that Y is defined over K. Let $\mathcal{Y}_n$ be the Zariski closure of Y in $\mathcal{X}_n$. Then $\{(\mathcal{Y}_n, \overline{\mathcal{L}_n}|_{\mathcal{Y}_n})\}_{n=0}^{\infty}$ is an adelic sequence of $(Y, L|_Y)$.*
(2) *Let K' be a finite extension field of K and $O_{K'}$ its ring of integers. We set $\mathcal{X}_n' = \mathcal{X}_n \times_{\operatorname{Spec}(O_K)} \operatorname{Spec}(O_K')$ and $\overline{\mathcal{L}_n'} = \overline{\mathcal{L}_n'} \otimes_{O_K} O_K'$. Then $\{(\mathcal{X}_n', \overline{\mathcal{L}'_n})\}_{n=0}^{\infty}$ is an adelic sequence of $(X_{K'}, L_{K'})$.*

The next theorem (a special case of [26], Theorem (1.4) and [16], Proposition 4.1.1) asserts the existence of the adelic intersection number.

Theorem 2.1.4 (A special case of adelic intersection number [26])**.** *Let $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ be an adelic sequence of (X, L). Then the arithmetic intersection number*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}_n})^{\dim X+1}\right)$$

converges as n goes to ∞. Moreover, the limit is independent of the choice of equivalent adelic sequences. We call the limit the adelic intersection number of $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$.

Proof. Since we adopt a slightly different definition for an adelic sequence, we sketch a proof here. Let U be a common base and $\mu_n : \mathcal{X}_n' \to \mathcal{X}_n$ and $\nu_n : \mathcal{X}_n' \to \mathcal{X}_{n+1}$ birational morphisms as in Definition 2.1.2. Then

$$\begin{aligned}
&\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_n})^{\dim X+1} - \widehat{\mathrm{c}}_1(\overline{\mathcal{L}_{n+1}})^{\dim X+1} \\
&\qquad = \widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}))^{\dim X+1} - \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_{n+1}}))^{\dim X+1} \\
&\qquad = \widehat{\mathrm{c}}_1\left(\mu_n^*(\overline{\mathcal{L}_n}) - \nu_n^*(\overline{\mathcal{L}_{n+1}})\right)\left\{\sum_{i=0}^{\dim X} \widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}))^i\, \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_n}))^{\dim X - i}\right\},
\end{aligned}$$

where we use the projection formula in the first equality. Since

$$\begin{aligned}
\pi_{\mathcal{X}_n'}^*\left(\left(-\sum_{\alpha=1}^{r} c_{n\alpha}[\mathfrak{P}_\alpha], -\sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}[\sigma]\right)\right) &\lesssim \mu_n^*(\overline{\mathcal{L}_n}) - \nu_n^*(\overline{\mathcal{L}_{n+1}}) \\
&\lesssim \pi_{\mathcal{X}_n'}^*\left(\left(\sum_{\alpha=1}^{r} c_{n\alpha}[\mathfrak{P}_\alpha], \sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}[\sigma]\right)\right),
\end{aligned}$$

it follows from the projection formula and Proposition 2.1.1 that

$$\begin{aligned}
&\left|\widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}))^{\dim X+1} - \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_{n+1}}))^{\dim X+1}\right| \\
&\leq \pi_{\mathcal{X}_n}^*\left(\left(\sum_{\alpha=1}^{r} c_{n\alpha}[\mathfrak{P}_\alpha], \sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}[\sigma]\right)\right)\left\{\sum_{i=0}^{\dim X} \widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}))^i\, \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_n}))^{\dim X - i}\right\} \\
&= \left(\sum_{\alpha=1}^{r} c_{n\alpha}\log\#(O_K/\mathfrak{P}_\alpha) + \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}} c_{n\sigma}\right)\mathrm{c}_1(L)^{\dim X}.
\end{aligned}$$

Since $\sum_{n=0}^{\infty} c_{n\alpha} < \infty$ and $\sum_{n=0}^{\infty} c_{n\sigma} < \infty$, $\widehat{\deg}\left(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_n})^{\dim X+1}\right)$ converges as n goes to ∞.

Next let $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ and $\{(\mathcal{Y}_n, \overline{\mathcal{M}_n})\}_{n=0}^{\infty}$ be equivalent adelic sequences for (X, L). Take birational morphisms $\widetilde{\mu}_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\widetilde{\nu}_n : \mathcal{Z}_n \to \mathcal{Y}_n$ as in Definition 2.1.2. Note that by the projection formula $\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_n})^{\dim X+1} = \widehat{\mathrm{c}}_1(\widetilde{\mu}_n^*(\overline{\mathcal{L}_n}))^{\dim X+1}$ and $\widehat{\mathrm{c}}_1(\overline{\mathcal{M}_n})^{\dim X+1} = \widehat{\mathrm{c}}_1(\widetilde{\nu}_n^*(\overline{\mathcal{M}_n}))^{\dim X+1}$. A similar argument as above yields

$$\left|\widehat{\mathrm{c}}_1(\widetilde{\mu}_n^*(\overline{\mathcal{L}_n}))^{\dim X+1} - \widehat{\mathrm{c}}_1(\widetilde{\nu}_n^*(\overline{\mathcal{M}_n}))^{\dim X+1}\right| \leq \left(\sum_{\alpha=1}^{\widetilde{r}} \widetilde{c}_{n\alpha}\log\#(O_K/\widetilde{\mathfrak{P}}_\alpha) + \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}} \widetilde{c}_{n\sigma}\right)\mathrm{c}_1(L)^{\dim X}.$$

Thus $\widehat{\deg}\left(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_n})^{\dim X+1}\right)$ and $\widehat{\deg}\left(\widehat{\mathrm{c}}_1(\overline{\mathcal{M}_n})^{\dim X+1}\right)$ have the same limit as n goes to ∞. $\square$

2.2. **Adelic sequence arising from several morphisms.** Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over K associated with L of degree $d > k$. *In §2.2 we assume that X is normal, $f_1, \cdots f_k$ are surjective and L is an ample $\mathbb{Q}$-line bundle.*

Since X is normal and L is ample, we can take a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) such that $\mathcal{X}$ is normal and $\overline{\mathcal{L}}$ is nef. We take a non-empty Zariski open subset $U \subseteq \operatorname{Spec}(O_K)$ such that each $f_i : X \to X$ extends to $f_i : \mathcal{X}_U \to \mathcal{X}_U$ and that $f_1^*(\mathcal{L}_U) \otimes \cdots \otimes f_k^*(\mathcal{L}_U) \simeq \mathcal{L}_U^{\otimes d}$ in $\operatorname{Pic}(\mathcal{X}_U) \otimes \mathbb{Q}$. By fixing an isomorphism, we use the notation $f_1^*(\mathcal{L}_U) \otimes \cdots \otimes f_k^*(\mathcal{L}_U) = \mathcal{L}_U^{\otimes d}$ in the following.

Let us construct another C^∞-model of (X, L) from $(\mathcal{X}, \overline{\mathcal{L}})$. Let $\mathcal{X}^{(i)}$ be the normalization of

$$\mathcal{X}_U \xrightarrow{f_i} \mathcal{X}_U \hookrightarrow \mathcal{X}.$$

We denote the induced morphism by $f^{(i)} : \mathcal{X}^{(i)} \to \mathcal{X}$. Let $\mathcal{X}_1$ be the Zariski closure of

$$\mathcal{X}_U \xrightarrow{\Delta} \overbrace{\mathcal{X}_U \times_U \cdots \times_U \mathcal{X}_U}^{k} \hookrightarrow \mathcal{X}^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}^{(k)},$$

where Δ is the diagonal map. Let $p_i : \mathcal{X}^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}^{(k)} \to \mathcal{X}^{(i)}$ denote the projection to the i-th factor. We set

$$\overline{\mathcal{L}_1} = \left[\left((f^{(1)} \circ p_1)^* \overline{\mathcal{L}} \otimes \cdots \otimes (f^{(k)} \circ p_k)^* \overline{\mathcal{L}} \right)\big|_{\mathcal{X}_1} \right]^{\otimes \frac{1}{d}}.$$

Then we get the C^∞-model $(\mathcal{X}_1, \overline{\mathcal{L}_1})$ of (X, L) from $(\mathcal{X}_0, \overline{\mathcal{L}_0}) := (\mathcal{X}, \overline{\mathcal{L}})$. Inductively we obtain the C^∞-model $(\mathcal{X}_{n+1}, \overline{\mathcal{L}_{n+1}})$ of (X, L) from $(\mathcal{X}_n, \overline{\mathcal{L}_n})$.

Theorem 2.2.1. (1) *$\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ is an adelic sequence of (X, L). We call $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ the adelic sequence of (X, L) associated with $(\mathcal{X}_0, \overline{\mathcal{L}_0})$ and $\{f_1, \cdots, f_k\}$.*

(2) *Let $(\mathcal{X}'_0, \overline{\mathcal{L}'_0})$ be another C^∞-model of (X, L) such that $\mathcal{X}'_0$ is normal and $\mathcal{L}'_0$ is nef. Let $\{(\mathcal{X}'_n, \overline{\mathcal{L}'_n})\}_{n=0}^\infty$ be the adelic sequence associated with $(\mathcal{X}'_0, \overline{\mathcal{L}'_0})$ and $\{f_1, \cdots, f_k\}$. Then $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ and $\{(\mathcal{X}'_n, \overline{\mathcal{L}'_n})\}_{n=0}^\infty$ are equivalent.*

Proof. (1) By construction, $\overline{\mathcal{L}_n}$ is nef for all $n \geq 0$. Moreover, over U, $\mathcal{X}_n|_U = \mathcal{X}_U$ and $\mathcal{L}_{n+1}|_U = [f_1^*(\mathcal{L}_U) \otimes \cdots \otimes f_k^*(\mathcal{L}_U)]^{\otimes \frac{1}{d}} = \mathcal{L}_U$. Thus, we have only to check the condition (iii) of Definition 2.1.2.

Let us first estimate the difference between $(\mathcal{X}_1, \overline{\mathcal{L}_1})$ and $(\mathcal{X}_2, \overline{\mathcal{L}_2})$, using the difference between $(\mathcal{X}_0, \overline{\mathcal{L}_0})$ and $(\mathcal{X}_1, \overline{\mathcal{L}_1})$. Let $\mathcal{X}_0^{(i)}$ (resp. $\mathcal{X}_1^{(i)}$) be the normalization of $\mathcal{X}_U \xrightarrow{f_i} \mathcal{X}_U \hookrightarrow \mathcal{X}_0$ (resp. $\mathcal{X}_U \xrightarrow{f_i} \mathcal{X}_U \hookrightarrow \mathcal{X}_1$). We denote the induced morphism by $f_0^{(i)} : \mathcal{X}_0^{(i)} \to \mathcal{X}_0$ (resp. $f_1^{(i)} : \mathcal{X}_1^{(i)} \to \mathcal{X}_1$). By the definition of $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$, $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) is the Zariski closure of $\mathcal{X}_U \xrightarrow{\Delta} \mathcal{X}_U \times_U \cdots \times_U \mathcal{X}_U \hookrightarrow \mathcal{X}_0^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_0^{(k)}$, (resp. $\mathcal{X}_U \xrightarrow{\Delta} \mathcal{X}_U \times_U \cdots \times_U \mathcal{X}_U \hookrightarrow \mathcal{X}_1^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_1^{(k)}$. Let $p_{0i} : \mathcal{X}_0^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_0^{(k)} \to \mathcal{X}_0^{(i)}$ (resp. $p_{1i} : \mathcal{X}_1^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_1^{(k)} \to \mathcal{X}_1^{(i)}$ denote the projection to the i-th factor.

Take a projective arithmetic variety $\mathcal{Y}$ such that there exist birational morphisms $\rho_0 : \mathcal{Y} \to \mathcal{X}_0$ and $\rho_1 : \mathcal{Y} \to \mathcal{X}_1$ that are the identity maps over U. For example, we may take $\mathcal{Y}$ as the integral closure of the image of the diagonal map from X to $\mathcal{X}_0 \times_{O_K} \mathcal{X}_1$. Let $\mathcal{Y}^{(i)}$ be the normalization of

$$\mathcal{Y}_U = \mathcal{X}_U \xrightarrow{f_i} \mathcal{Y}_U = \mathcal{X}_U \hookrightarrow \mathcal{Y}.$$

We denote the induced morphism by $g^{(i)} : \mathcal{Y}^{(i)} \to \mathcal{Y}$. Let $\mathcal{Z}$ be the Zariski closure of

$$\mathcal{Y}_U \xrightarrow{\Delta} \overbrace{\mathcal{Y}_U \times_U \cdots \times_U \mathcal{Y}_U}^{k} \hookrightarrow \mathcal{Y}^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{Y}^{(k)},$$

where Δ is the diagonal map. Then there are biratinal morphisms $\mu_0 : \mathcal{Z} \to \mathcal{X}_1$ and $\mu_1 : \mathcal{Z} \to \mathcal{X}_2$ that are identity maps over U. Let $q_i : \mathcal{Y}^{(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{Y}^{(k)} \to \mathcal{Y}^{(i)}$ denote the projection to the i-th factor.

Claim 2.2.1.1. (1) $\left.\left(\bigotimes_{i=1}^{k}(\rho_0\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_0}\right)\right|_{\mathcal{Z}} = \mu_0^*\left(\left.\bigotimes_{i=1}^{k}(f_0^{(i)}\circ p_{0i})^*\overline{\mathcal{L}_0}\right|_{\mathcal{X}_1}\right)$.

(2) $\left.\left(\bigotimes_{i=1}^{k}(\rho_1\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_1}\right)\right|_{\mathcal{Z}} = \mu_1^*\left(\left.\bigotimes_{i=1}^{k}(f_1^{(i)}\circ p_{1i})^*\overline{\mathcal{L}_1}\right|_{\mathcal{X}_2}\right)$.

Indeed, for each $i = 1,\cdots,k$, we consider two morphisms:

$$\mathcal{Z}\xrightarrow{\Delta}\mathcal{Y}^{(1)}\times_{O_K}\cdots\times_{O_K}\mathcal{Y}^{(k)}\xrightarrow{g^{(i)}\circ q_i}\mathcal{Y}\xrightarrow{\rho_0}\mathcal{X}_0 \qquad \text{and}$$
$$\mathcal{Z}\xrightarrow{\mu_0}\mathcal{X}_1\xrightarrow{\Delta}\mathcal{X}^{(1)}\times_{O_K}\cdots\times_{O_K}\mathcal{X}^{(k)}\xrightarrow{f_0^{(i)}\circ p_{0i}}\mathcal{X}_0.$$

These two morphisms coincide with each other over U, and hence over O_K. Thus we get the first assertion. The second assertion follows by the same argument.

Claim 2.2.1.2. $\mu_0^*(\overline{\mathcal{L}_1})-\mu_1^*(\overline{\mathcal{L}_2}) = \dfrac{1}{d}\left.\left(\bigotimes_{i=1}^{k}(\rho_0\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_0}-(\rho_1\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_1}\right)\right|_{\mathcal{Z}}$

Indeed, it follows from Claim 2.2.1.1 that

$$\begin{aligned}\left.\left(\sum_{i=1}^{k}(\rho_0\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_0}\right)\right|_{\mathcal{Z}} &= \mu_0^*\left(\left.\sum_{i=1}^{k}(f_0^{(i)}\circ p_{0i})^*\overline{\mathcal{L}_0}\right|_{\mathcal{X}_1}\right)\\ &= \mu_0^*(d\overline{\mathcal{L}_1}) = d\mu_0^*(\overline{\mathcal{L}_1})\end{aligned}$$

and similarly

$$\left.\left(\sum_{i=1}^{k}(\rho_1\circ g^{(i)}\circ q_i)^*\overline{\mathcal{L}_1}\right)\right|_{\mathcal{Z}} = d\mu_1^*(\overline{\mathcal{L}_2}).$$

Since $\rho_0:\mathcal{Y}\to\mathcal{X}_0$ and $\rho_1:\mathcal{Y}\to\mathcal{X}_1$ give the same morphism over U, there are positive rational numbers $c_{01},\cdots,c_{0r}$ and positive real numbers $c_{0\sigma}$ ($\sigma:K\to\mathbb{C}$) such that

$$\begin{aligned}\pi_{\mathcal{Y}}^*\left(\left(-\sum_{\alpha=1}^{r}c_{0\alpha}[\mathfrak{P}_\alpha],-\sum_{\sigma:K\to\mathbb{C}}c_{0\sigma}[\sigma]\right)\right) &\lesssim \rho_0^*(\overline{\mathcal{L}_0})-\rho_1^*(\overline{\mathcal{L}_1})\\ &\lesssim \pi_{\mathcal{Y}}^*\left(\left(\sum_{\alpha=1}^{r}c_{0\alpha}[\mathfrak{P}_\alpha],\sum_{\sigma:K\to\mathbb{C}}c_{0\sigma}[\sigma]\right)\right).\end{aligned}$$

Then it follows from Claim 2.2.1.2 that

$$\begin{aligned}\pi_{\mathcal{Z}}^*\left(\left(-\sum_{\alpha=1}^{r}\frac{k}{d}c_{0\alpha}[\mathfrak{P}_\alpha],-\sum_{\sigma:K\to\mathbb{C}}\frac{k}{d}c_{0\sigma}[\sigma]\right)\right) &\lesssim \mu_0^*(\overline{\mathcal{L}_1})-\mu_1^*(\overline{\mathcal{L}_2})\\ &\lesssim \pi_{\mathcal{Z}}^*\left(\left(\sum_{\alpha=1}^{r}\frac{k}{d}c_{0\alpha}[\mathfrak{P}_\alpha],\sum_{\sigma:K\to\mathbb{C}}\frac{k}{d}c_{0\sigma}[\sigma]\right)\right).\end{aligned}$$

Inductively we find that there exist a projective arithmetic variety $\mathcal{W}$, and birational morphisms $\nu_n : \mathcal{W} \to \mathcal{X}_n$ and $\nu_n : \mathcal{W} \to \mathcal{X}_{n+1}$ such that

$$\pi_{\mathcal{W}}^*\left(\left(-\sum_{\alpha=1}^{r}\frac{k^n}{d^n}c_{0\alpha}[\mathfrak{P}_\alpha], -\sum_{\sigma:K\to\mathbb{C}}\frac{k^n}{d^n}c_{0\sigma}[\sigma]\right)\right) \lesssim \mu_n^*(\overline{\mathcal{L}_n}) - \nu_n^*(\overline{\mathcal{L}_{n+1}})$$
$$\lesssim \pi_{\mathcal{W}}^*\left(\left(\sum_{\alpha=1}^{r}\frac{k^n}{d^n}c_{0\alpha}[\mathfrak{P}_\alpha], \sum_{\sigma:K\to\mathbb{C}}\frac{k^n}{d^n}c_{0\sigma}[\sigma]\right)\right).$$

Since $\sum_{n=0}^{\infty}\left(\frac{k}{d}\right)^n < \infty$, $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ satisfies the condition (iii) of Definition 2.1.2.

(2) Take a non-empty Zariski open set $\widetilde{U}$ of $\mathrm{Spec}(O_K)$ that is a common base of both $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^{\infty}$ and $\{(\mathcal{X}_n^{'}, \overline{\mathcal{L}_n^{'}})\}_{n=0}^{\infty}$. Let $\mathcal{X}_0^{''}$ be a projective arithmetic variety such that there exist birational morphisms $\widetilde{\mu}_0 : \mathcal{X}_0^{''} \to \mathcal{X}_0$ and $\widetilde{\nu}_0 : \mathcal{X}_0^{''} \to \mathcal{X}_0^{'}$ that are the identity maps over $\widetilde{U}$. Then there are positive rational numbers $\widetilde{c}_{01}, \cdots, \widetilde{c}_{0\widetilde{r}}$ and positive real numbers $\widetilde{c}_{0\sigma}$ $(\sigma : K \to \mathbb{C})$ such that

$$\pi_{\mathcal{X}_0^{''}}^*\left(\left(-\sum_{\alpha=1}^{\widetilde{r}}\widetilde{c}_{0\alpha}[\widetilde{\mathfrak{P}}_\alpha], -\sum_{\sigma:K\to\mathbb{C}}\widetilde{c}_{0\sigma}[\sigma]\right)\right) \lesssim \widetilde{\mu}_0^*(\overline{\mathcal{L}_0}) - \widetilde{\nu}_0^*(\overline{\mathcal{L}_0^{'}})$$
$$\lesssim \pi_{\mathcal{X}_0^{''}}^*\left(\left(\sum_{\alpha=1}^{\widetilde{r}}\widetilde{c}_{0\alpha}[\widetilde{\mathfrak{P}}_\alpha], \sum_{\sigma:K\to\mathbb{C}}\widetilde{c}_{0\sigma}[\sigma]\right)\right),$$

where $\mathrm{Spec}\, O_K \setminus \widetilde{U} = \{\widetilde{\mathfrak{P}}_1, \cdots, \widetilde{\mathfrak{P}}_{\widetilde{r}}\}$ and $\pi_{\mathcal{X}_0^{''}} : \mathcal{X}_0^{''} \to \mathrm{Spec}(O_K)$ is the structure morphism. Let $\mathcal{X}_0^{''(i)}$ be the normalization of $\mathcal{X}_{0\widetilde{U}}^{''} \xrightarrow{f_i} \mathcal{X}_{0\widetilde{U}}^{''} \hookrightarrow \mathcal{X}_0^{''}$. Let $\mathcal{X}_1^{''}$ be the Zariski closure of $\mathcal{X}_{0\widetilde{U}}^{''} \xrightarrow{\Delta} \mathcal{X}_{0\widetilde{U}}^{''} \times_{\widetilde{U}} \cdots \times_{\widetilde{U}} \mathcal{X}_{0\widetilde{U}}^{''} \hookrightarrow \mathcal{X}_0^{''(1)} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_0^{''(k)}$. Then there are biratinal morphisms $\widetilde{\mu}_1 : \mathcal{X}_1^{''} \to \mathcal{X}_1$ and $\widetilde{\nu}_1 : \mathcal{X}_1^{''} \to \mathcal{X}_1^{'}$ that are identity maps over $\widetilde{U}$. By the same argument as in (1), we have

$$\pi_{\mathcal{X}_1^{''}}^*\left(\left(-\sum_{\alpha=1}^{\widetilde{r}}\frac{k}{d}\widetilde{c}_{0\alpha}[\widetilde{\mathfrak{P}}_\alpha], -\sum_{\sigma:K\to\mathbb{C}}\frac{k}{d}\widetilde{c}_{0\sigma}[\sigma]\right)\right) \lesssim \widetilde{\mu}_1^*(\overline{\mathcal{L}_1}) - \widetilde{\nu}_1^*(\overline{\mathcal{L}_1^{'}})$$
$$\lesssim \pi_{\mathcal{X}_1^{''}}^*\left(\left(\sum_{\alpha=1}^{\widetilde{r}}\frac{k}{d}\widetilde{c}_{0\alpha}[\widetilde{\mathfrak{P}}_\alpha], \sum_{\sigma:K\to\mathbb{C}}\frac{k}{d}\widetilde{c}_{0\sigma}[\sigma]\right)\right).$$

Inductively we find that there exist a projective arithmetic variety $\mathcal{X}_n^{''}$, birational morphisms $\widetilde{\mu}_n : \mathcal{X}_n^{''} \to \mathcal{X}_n$ and $\widetilde{\nu}_n : \mathcal{X}_n^{''} \to \mathcal{X}_n^{'}$ such that

$$\pi_{\mathcal{X}_n^{''}}^*\left(\left(-\sum_{\alpha=1}^{\widetilde{r}}\frac{k^n}{d^n}\widetilde{c}_{n\alpha}[\widetilde{\mathfrak{P}}_\alpha], -\sum_{\sigma:K\to\mathbb{C}}\frac{k^n}{d^n}\widetilde{c}_{0\sigma}[\sigma]\right)\right) \lesssim \widetilde{\mu}_n^*(\overline{\mathcal{L}_n}) - \widetilde{\nu}_n^*(\overline{\mathcal{L}_n^{'}})$$
$$\lesssim \pi_{\mathcal{X}_n^{''}}^*\left(\left(\sum_{\alpha=1}^{\widetilde{r}}\frac{k^n}{d^n}\widetilde{c}_{0\alpha}[\widetilde{\mathfrak{P}}_\alpha], \sum_{\sigma:K\to\mathbb{C}}\frac{k^n}{d^n}\widetilde{c}_{0\sigma}[\sigma]\right)\right).$$

Since $\sum_{n=0}^{\infty}\frac{k^n}{d^n} < \infty$, we get the assertion. $\square$

2.3. **Adelic intersection and heights of subvarieties.** As in §2.2, let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over K associated with L of degree $d > k$, and we assume that X is normal, $f_1, \cdots, f_k$ are surjective, and L is an ample $\mathbb{Q}$-line bundle. Fix a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L).

Let $Y \subset X_{\overline{K}}$ be a subvariety. Take a finite extension field K' of K over which Y is defined. Let $\mathcal{Y}$ denote the Zariski closure of Y in $\mathcal{X} \otimes_{O_K} O_{K'}$. We define the height of Y with respect to $(\mathcal{X}, \overline{\mathcal{L}})$ to be

$$h_{(\mathcal{X},\overline{\mathcal{L}})}(Y) = \frac{\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{L}} \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}}\right)^{\dim Y+1}\right)}{[K' : \mathbb{Q}](\dim Y + 1)\deg(L \otimes_K K'|_Y^{\cdot \dim Y})}.$$

In particular, if Y is a closed point of $X_{\overline{K}}$, then

$$h_{(\mathcal{X},\overline{\mathcal{L}})}(Y) = \frac{\widehat{\deg}\left(\overline{\mathcal{L}} \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}}\right)}{[K' : \mathbb{Q}]}.$$

Then we have the following theorem.

Theorem 2.3.1. *Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over K associated with L of degree $d > k$. We assume that X is normal, $f_1, \cdots, f_k$ are surjective, and L is an ample $\mathbb{Q}$-line bundle.*

(1) *Let $(\mathcal{X}_0, \overline{\mathcal{L}_0})$ be a C^∞-model of (X, L) such that $\mathcal{X}_0$ is normal and $\overline{\mathcal{L}_0}$ is nef. Let $\{(\mathcal{X}_n, \overline{\mathcal{L}_n})\}_{n=0}^\infty$ be the adelic sequence of (X, L) associated with $(\mathcal{X}_0, \overline{\mathcal{L}_0})$ and $\mathcal{F} = \{f_1, \cdots, f_k\}$. Then, for any subvariety $Y \subset X_{\overline{K}}$,*

$$\widehat{h}_{L,\mathcal{F}}(Y) := \lim_{n\to\infty} h_{(\mathcal{X}_n,\overline{\mathcal{L}_n})}(Y)$$

converges. The limit $\widehat{h}_{L,\mathcal{F}}(Y)$ is independent of the choice of C^∞-models $(\mathcal{X}_0, \overline{\mathcal{L}_0})$ of (X, L), and $\widehat{h}_{L,\mathcal{F}_1}(Y) \geq 0$ for all $Y \subset X_{\overline{K}}$. We call this limit the canonical height *of Y associated with L and $\mathcal{F}$.*

(2) *Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a C^∞-model of (X, L). Then there exists a constant C such that*

$$\left|\widehat{h}_{L,\mathcal{F}}(Y) - h_{(\mathcal{X},\overline{\mathcal{L}})}(Y)\right| \leq C$$

for any subvariety $Y \subset X_{\overline{K}}$.

(3) *When Y is a closed point of $X_{\overline{K}}$, $\widehat{h}_{L,\mathcal{F}}(Y)$ coincides with the canonical height in Theorem 1.2.1.*

Proof. (1) Let $\mathcal{Y}_n$ denote the Zariski closure of Y in $\mathcal{X}_n \otimes_{O_K} O_{K'}$. By Lemma 2.1.3, $\{(\mathcal{Y}_n, \overline{\mathcal{L}_n} \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}_n})\}_{n=0}^\infty$ is an adelic sequence of $(Y, L|_Y)$. Then it follows from Theorem 2.1.4 that $h_{(\mathcal{X}_n,\overline{\mathcal{L}_n})}(Y)$ converges as n goes to ∞. It follows from Theorem 2.1.4 and Theorem 2.2.1(2) that this limit is independent of the choice of $(\mathcal{X}_0, \overline{\mathcal{L}_0})$. Moreover, since $\overline{\mathcal{L}_n}$ is nef, $h_{(\mathcal{X}_n,\overline{\mathcal{L}_n})}(Y) \geq 0$ by Proposition 2.1.1. Thus $\widehat{h}_L(Y) = \lim_{n\to\infty} h_{(\mathcal{X}_n,\overline{\mathcal{L}_n})}(Y) \geq 0$.

(2) Take a C^∞-model $(\mathcal{X}', \overline{\mathcal{L}'})$ of (X, L) such that $\mathcal{X}'$ is normal and $\overline{\mathcal{L}'}$ is nef. By [1], Proposition 3.2.2, there exists a constant C_1 such that

$$\left|h_{(\mathcal{X},\overline{\mathcal{L}})}(Y) - h_{(\mathcal{X}',\overline{\mathcal{L}'})}(Y)\right| \leq C_1$$

for any subvariety $Y \subset X_{\overline{K}}$. Thus, to prove (2), we may assume that $(\mathcal{X}, \overline{\mathcal{L}})$ is nef.

We take birational morphisms $\mu_n : \mathcal{X}_n^{'} \to \mathcal{X}_n$ and $\nu_n : \mathcal{X}_n^{'} \to \mathcal{X}_{n+1}$ as in Definition 2.1.2. Let $\mathcal{Y}'$ be the Zariski closure of Y in $\mathcal{X}_n^{'}$. Then by a similar argument as in the proof of Theorem 2.1.4, we have

$$\left|\widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}) \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}'})^{\dim Y+1} - \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_{n+1}}) \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}'})^{\dim Y+1}\right| \\ \leq \frac{k^n}{d^n}[K' : K]\left(\sum_{\alpha=1}^{r} c_{0\alpha}\log\#(O_K/\mathfrak{P}_\alpha) + \frac{1}{2}\sum_{\sigma:K\to\mathbb{C}} c_{0\sigma}\right).$$

Since by the projection formula $\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_n} \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}_n})^{\dim Y+1} = \widehat{\mathrm{c}}_1(\mu_n^*(\overline{\mathcal{L}_n}) \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}'})^{\dim Y+1}$ and $\widehat{\mathrm{c}}_1(\overline{\mathcal{L}_{n+1}} \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}_{n+1}})^{\dim Y+1} = \widehat{\mathrm{c}}_1(\nu_n^*(\overline{\mathcal{L}_{n+1}}) \otimes_{O_K} O_{K'}\big|_{\mathcal{Y}'})^{\dim Y+1}$, we have

$$\left|h_{(\mathcal{X}_n,\mathcal{L}_n)}(Y) - h_{(\mathcal{X}_{n+1},\mathcal{L}_{n+1})}(Y)\right| \leq \frac{k^n\left(\sum_{\alpha=1}^{r} c_{0\alpha}\log\#(O_K/\mathfrak{P}_\alpha) + \frac{1}{2}\sum_{\sigma\to\mathbb{C}} c_{0\sigma}\right)}{d^n[K:\mathbb{Q}](\dim Y+1)}.$$

Putting $C = \frac{d\left(\sum_{\alpha=1}^{r} c_{0\alpha}\log\#(O_K/\mathfrak{P}_\alpha) + \frac{1}{2}\sum_{\sigma\to\mathbb{C}} c_{0\sigma}\right)}{(d-k)[K:\mathbb{Q}]}$, we obtain the desired estimate.

(3) Let us check that if $y \in X(\overline{K})$ is a closed point, $\widehat{h}_{L,\mathcal{F}_1}(y)$ defined via adelic intersection satisfies (i) and (ii) of Theorem 1.2.1. Note that $h_{(\mathcal{X}_0,\overline{\mathcal{L}_0})} = h_L + O(1)$. Then it follows from (2) that $\widehat{h}_{L,\mathcal{F}} = h_L + O(1)$. To show (ii), let K' be a finite extension field of K over which y is defined. Let $\Delta_y^{(n)}$ denote the Zariski closure of y in $\mathcal{X}_n \times_{\mathrm{Spec}(O_K)} \mathrm{Spec}(O_{K'})$. Put

$$b_n(y) := \frac{\widehat{\deg}\left(\overline{\mathcal{L}_n} \otimes_{O_K} O_{K'}\big|_{\Delta_y^{(n)}}\right)}{[K':\mathbb{Q}]}.$$

Then $b_n(y)$ is independent of the choice of K', and by definition $\widehat{h}_{L,\mathcal{F}}(y) = \lim_{n\to\infty} b_n(y)$.

Claim 2.3.1.1. $\sum_{i=1}^{k} b_n(f_i(y)) = d\, b_{n+1}(y)$.

Let us show the assertion for $n = 0$. The assertion for $n \geq 1$ can be shown by the same argument. We follow the notation in the beginning of §2.2. By the projection formula, we have

$$\begin{aligned}
&\sum_{i=1}^{k} b_0(f_i(y)) - d\, b_1(y) \\
=&\sum_{i=1}^{k} \frac{\widehat{\deg}\left(\overline{\mathcal{L}_0} \otimes_{O_K} O_{K'}\big|_{\Delta_{f_i(y)}^{(0)}}\right)}{[K':\mathbb{Q}]} - d\frac{\widehat{\deg}\left(\overline{\mathcal{L}_1} \otimes_{O_K} O_{K'}\big|_{\Delta_y^{(1)}}\right)}{[K':\mathbb{Q}]} \\
=&\sum_{i=1}^{k} \frac{\widehat{\deg}\left((f^{(i)} \circ p_i)^*(\overline{\mathcal{L}_0} \otimes_{O_K} O_{K'})\big|_{\Delta_y^{(1)}}\right)}{[K':\mathbb{Q}]} - d\frac{\widehat{\deg}\left(\overline{\mathcal{L}_1} \otimes_{O_K} O_{K'}\big|_{\Delta_y^{(1)}}\right)}{[K':\mathbb{Q}]} \\
=&\frac{\widehat{\deg}\left(\sum_{i=1}^{k} (f^{(i)} \circ p_i)^*(\overline{\mathcal{L}_0} \otimes_{O_K} O_{K'}) - d(\overline{\mathcal{L}_1} \otimes_{O_K} O_{K'})\big|_{\Delta_y^{(1)}}\right)}{[K':\mathbb{Q}]} = 0
\end{aligned}$$

By letting n to ∞ in Claim 2.3.1.1, we get $\sum_{i=1}^{k} \widehat{h}_{L,\mathcal{F}}(f_i(y)) = d\widehat{h}_{L,\mathcal{F}}(y)$. $\square$

3. Admissible metrics and invariant currents

In this section, we introduce an invariant current for a dynamical system of k morphisms over $\mathbb{C}$ associated with a line bundle L of degree $d > k$. If $X = \mathbb{P}^N$, $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree $d \geq 2$, and $L = \mathcal{O}_{\mathbb{P}^N}(1)$, then this invariant current coincides with the Green current of f introduced by Hubbard–Papadopol [10]. When $(X; \sigma_1, \sigma_2)$ is Wheler's K3 surface, we will relate the invariant current with the currents T^+ and T^- corresponding respectively to $\sigma_2 \circ \sigma_1$ and $\sigma_1 \circ \sigma_2$ introduced by Cantat [5]. Finally we pose a question regarding distribution of small points, and show that this question is true for Lattés examples, which are certain endomorphisms of $\mathbb{P}^n$.

3.1. **Admissible metrics.** Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over $\mathbb{C}$ associated with a line bundle L of degree $d > k$. We fix an isomorphism $\varphi : L^{\otimes d} \overset{\sim}{\to} f_1^*L \otimes \cdots \otimes f_k^*L$. The following theorem is a generalization of [26], Theorem(2.2) of one morphism to k morphisms over $\mathbb{C}$.

Theorem 3.1.1. (1) *Let $\|\cdot\|_0$ be a continuous metric on L. We inductively define a metric $\|\cdot\|_n$ on L by*

$$\|\cdot\|_n^d = \varphi^*(f_1^*\|\cdot\|_{n-1} \cdots f_k^*\|\cdot\|_{n-1}).$$

Then $\|\cdot\|_n$ converges uniformly to a metric, which we denote by $\|\cdot\|_\infty$.

(2) *$\|\cdot\|_\infty$ is the unique continuous metric on L with*

$$\|\cdot\|_\infty^d = \varphi^*(f_1^*\|\cdot\|_\infty \cdots f_k^*\|\cdot\|_\infty).$$

(3) *If φ is replaced with $c\varphi$ ($c \neq 0 \in \mathbb{C}$), then $\|\cdot\|_\infty$ is replaced with $|c|^{\frac{1}{d-k}}\|\cdot\|_\infty$.*

We call the metric $\|\cdot\|_\infty$ the admissible metric *for L.*

Proof. Set $H = \frac{\|\cdot\|_1}{\|\cdot\|_0}$. Then H is a continuous function on $X(\mathbb{C})$. We have

$$\begin{aligned} \frac{\|\cdot\|_{n+1}}{\|\cdot\|_n}(x) &= \left(\frac{\varphi^*(f_1^*\|\cdot\|_n \cdots f_k^*\|\cdot\|_n)}{\varphi^*(f_1^*\|\cdot\|_{n-1} \cdots f_k^*\|\cdot\|_{n-1})}\right)^{\frac{1}{d}}(x) \\ &= \left(\frac{\|\cdot\|_n}{\|\cdot\|_{n-1}}(f_1(x)) \cdots \frac{\|\cdot\|_n}{\|\cdot\|_{n-1}}(f_k(x))\right)^{\frac{1}{d}} \\ &= \cdots = \left(\prod_{f \in \mathcal{F}_n} \frac{\|\cdot\|_1}{\|\cdot\|_0}(f(x))\right)^{\frac{1}{d^n}} = \left(\prod_{f \in \mathcal{F}_n} H \circ f\right)^{\frac{1}{d^n}}(x). \end{aligned}$$

Thus we get $\|\cdot\|_n = \|\cdot\|_0 \cdot \prod_{l=0}^{n-1} \left(\prod_{f \in \mathcal{F}_l} H \circ f\right)^{\frac{1}{d^l}}$. We set $H_n = \prod_{l=0}^{n-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}}$.

Claim 3.1.1.1. *The function H_n converges uniformly as $n \to \infty$.*

For positive integers m and n, we have

$$H_{n+m} - H_n = \left(\prod_{l=0}^{n-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}}\right) \left(\prod_{l=n}^{m-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}} - 1\right).$$

We set $M = \max\{1, \sup_{x \in X(\mathbb{C})} H(x)\}$ and $m = \min\{1, \inf_{x \in X(\mathbb{C})} H(x)\}$. Then $0 < m \le M$. It follows from $\prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}} \le M^{(\frac{k}{d})^l}$ that

$$\prod_{l=0}^{n-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}} \le M^{\sum_{l=0}^{n-1} (\frac{k}{d})^l} \le M^{\sum_{l=0}^{\infty} (\frac{k}{d})^l} = M^{\frac{d}{d-k}}.$$

We also have $\prod_{l=n}^{m-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}} \le M^{\sum_{l=n}^{m-1} (\frac{k}{d})^l} \le M^{(\frac{k}{d})^n \frac{d}{d-k}}$ and $\prod_{l=n}^{m-1} \prod_{f \in \mathcal{F}_l} (H \circ f)^{\frac{1}{d^l}} \ge m^{\sum_{l=n}^{m-1} (\frac{k}{d})^l} \ge m^{(\frac{k}{d})^n \frac{d}{d-k}}$. Then we have

$$\|H_{n+m} - H_n\|_{\sup} \le M^{\frac{d}{d-k}} \max\left\{M^{(\frac{k}{d})^n \frac{d}{d-k}} - 1, 1 - m^{(\frac{k}{d})^n \frac{d}{d-k}}\right\} \to 0 \quad (n, m \to \infty).$$

This shows H_n converges uniformly. Let H_∞ denote the limit of H_n.

Since

$$\inf_{x \in X(\mathbb{C})} H_\infty(x) = \inf_{x \in X(\mathbb{C})} \prod_{l=0}^{\infty} \prod_{f \in \mathcal{F}_l} (H(f(x)))^{\frac{1}{d^l}} \ge m^{\sum_{l=0}^{\infty} (\frac{k}{d})^l} \ge m^{\frac{d}{d-k}},$$

we find that H_∞ is a positive function. Thus $\|\cdot\|_n = \|\cdot\|_0 H_n$ converges uniformly to a metric $\|\cdot\|_\infty := \|\cdot\|_0 H_\infty$. Thus we have shown (1).

Next we show (2). Letting $n \to \infty$ in $\|\cdot\|_{n+1}^d = \varphi^*(f_1^*\|\cdot\|_n \cdots f_k^*\|\cdot\|_n)$, we get $\|\cdot\|_\infty^d = \varphi^*(f_1^*\|\cdot\|_\infty \cdots f_k^*\|\cdot\|_\infty)$. To show the uniqueness of $\|\cdot\|_\infty$, suppose continuous metrics $\|\cdot\|_\infty$ and $\|\cdot\|_\infty'$ both satisfy the equality in (2). We set $A = \frac{\|\cdot\|_\infty'}{\|\cdot\|_\infty}$. Then A is a positive continuous function on $X(\mathbb{C})$, and satisfies $A^d = \prod_{i=1}^k A \circ f_i$. Then it follows from $\sup_{x \in X(\mathbb{C})} A(x) = \sqrt[d]{\sup_{x \in X(\mathbb{C})} \prod_{i=1}^k A(f_i(x))} \le \left(\sup_{x \in X(\mathbb{C})} A(x)\right)^{\frac{k}{d}}$ and $\frac{k}{d} < 1$ that $\sup_{x \in X(\mathbb{C})} A(x) \le 1$. It follows from $\inf_{x \in X(\mathbb{C})} A(x) = \sqrt[d]{\inf_{x \in X(\mathbb{C})} \prod_{i=1}^k A(f_i(x))} \ge \left(\inf_{x \in X(\mathbb{C})} A(x)\right)^{\frac{k}{d}}$ and $\frac{k}{d} < 1$ that $\inf_{x \in X(\mathbb{C})} A(x) \ge 1$. Hence A is identically 1. We have shown (2).

To show (3), set $\widetilde{\varphi} = c\varphi$. Let $\widetilde{\|\cdot\|_\infty}$ denote the metric with respect to $c\varphi$. Set $\alpha\|\cdot\|_\infty = \widetilde{\|\cdot\|_\infty}$. Since $\widetilde{\varphi}(f_1^*\widetilde{\|\cdot\|_\infty} \cdots f_k^*\widetilde{\|\cdot\|_\infty}) = \widetilde{\|\cdot\|_\infty}^d$, we have $|c|\alpha^k = \alpha^d$. Thus $\alpha = |c|^{\frac{1}{d-k}}$. □

3.2. **Invariant currents.** Let M be a complex manifold. A closed $(1,1)$-current S on M is said to admit a locally continuous potential if, for every $m \in M$, there exists an open neighborhood U of m and a continuous function u on U such that $S = dd^c u$.

Let $\pi : N \to M$ be a morphism of complex manifolds. If a closed $(1,1)$-current S on M admits a locally continuous potential, we can define its pull-back π^*S by $\pi^*S|_{\pi^{-1}(U)} = dd^c \pi^* u$. Since $\pi^* u$ is continuous and unique up to a pluriharmonic function, this formula is well-defined and defines a global current π^*S on N. (cf. [10], p330.)

We denote by $A^{p,p}(M)$ the space of C^∞ (p,p)-forms on X, and by $D^{p,p}(M)$ the space of (p,p)-currents on X. For $\eta \in A^{p,p}(M)$, we denote by $[\eta] \in D^{p,p}(X)$ the current corresponding to η.

The next theorem is the main theorem of this section.

Theorem 3.2.1. *Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over $\mathbb{C}$ associated with a line bundle L of degree $d > k$. We assume that X is smooth and L is ample.*

(1) *Let $\|\cdot\|_\infty$ be the admissible metric of L. We set $T = \mathrm{c}_1(L, \|\cdot\|_\infty) := dd^c[-\log\|s\|^2_\infty] + \delta_{\mathrm{div}(s)} \in D^{1,1}(X(\mathbb{C}))$, where s is a non-zero rational section of L. (This formula does not depend on s and defines a closed $(1,1)$-current on X.) Then T is positive.*

(2) *Since T admits a locally continuous potential by (1), we have the pull-back $f_i^*T \in D^{1,1}(X(\mathbb{C}))$. Then we have*
$$f_1^*T + \cdots + f_k^*T = dT.$$

(3) *Let $\eta_0 \in A^{1,1}(X(\mathbb{C}))$ be any closed C^∞ $(1,1)$-form whose cohomology class coincides with $\mathrm{c}_1(L)$. We inductively define $\eta_n \in A^{1,1}(X(\mathbb{C}))$ by*
$$\eta_{n+1} = \frac{1}{d}(f_1^*\eta_n + \cdots + f_k^*\eta_n).$$
Then, $[\eta_n]$ converges to T as currents.

Proof. (1) Since X is a smooth projective variety and L is an ample line bundle, there exists a C^∞ metric $\|\cdot\|_0$ on L such that its Chern form $\omega_0 := \mathrm{c}_1(L, \|\cdot\|_\infty) \in A^{1,1}(X)$ is everywhere positive.

By fixing $\varphi : L^{\otimes d} \xrightarrow{\sim} f_1^*L_1 \otimes \cdots \otimes f_k^*L_k$, as in the proof of Theorem 3.1.1, we inductively define a metric $\|\cdot\|_n$ on L by
$$\|\cdot\|_{n+1} = \left\{\varphi^*(f_1^*\|\cdot\|_n \cdots f_k^*\|\cdot\|_n)\right\}^{\frac{1}{d}}.$$
As in Theorem 3.1.1, let $H = \frac{\|\cdot\|_1}{\|\cdot\|_0}$. Then H is a C^∞ positive function on $X(\mathbb{C})$. By replacing φ with $c\varphi$ with $c > 0$ if necessary, we may assume that $H \geq 1$. (Indeed we may take $c = (\min_{x \in X(\mathbb{C})} H(x))^{\frac{1}{d}}$.)

We define a C^∞ $(1,1)$-form ω_n by $\omega_n = \frac{1}{d^n}\sum_{f \in \mathcal{F}_n} f^*\omega_0$.

For $x \in X(\mathbb{C})$, we take a non-zero rational section s of L such that $s(x) \neq 0$. We set $U = X \setminus \mathrm{Supp}(\mathrm{div}(s))$. We define a C^∞ map $G_{n,s} : U \to \mathbb{R}$ by $G_{n,s} = -\log\|s\|_n^2$.

Claim 3.2.1.1. *$G_{n,s}$ is non-increasing with respect to n. Moreover, $dd^cG_{n,s} = \omega_n|_U$, and thus $dd^cG_{n,s}$ is everywhere positive.*

Indeed, by the proof of Theorem 3.1.1, we have $\|\cdot\|_{n+1} = \|\cdot\|_n \cdot \left(\prod_{f \in \mathcal{F}_n} H \circ f\right)^{\frac{1}{d^n}}$. Thus we find
$$-\log\|s\|^2_{n+1} = -\log\|s\|^2_n + \frac{1}{d^n}\sum_{f \in \mathcal{F}_n} -2\log H \circ f.$$
Since $H \geq 1$, $\frac{1}{d^n}\sum_{f \in \mathcal{F}_n} -2\log H \circ f \leq 0$. Thus $G_{n,s}$ is non-increasing.

On the other hand, we have $dd^cG_{n,s} = \mathrm{c}_1(L, \|\cdot\|_n)|_U$. Since
$$\begin{aligned}\mathrm{c}_1(L, \|\cdot\|_n) &= \frac{1}{d}\left\{f_1^*\,\mathrm{c}_1(L, \|\cdot\|_{n-1}) + \cdots + f_k^*\,\mathrm{c}_1(L, \|\cdot\|_{n-1})\right\} \\ &= \cdots = \frac{1}{d^n}\sum_{f \in \mathcal{F}_n} f^*\,\mathrm{c}_1(L, \|\cdot\|_0) = \omega_n,\end{aligned}$$

we have $dd^cG_{n,s} = \omega_n|_U$, which is everywhere positive.

Since $G_{n,s}$ is a C^∞-function with $dd^cG_{n,s} \geq 0$ by Claim 3.2.1.1, $G_{n,s}$ is a psh (plurisubharmonic) function. Since $G_{n,s}$ is non-increasing and bounded from below, $G_{\infty,s} := \lim_{n\to\infty} G_{n,s}$ is also a psh function. By Theorem 3.1.1, we have $G_{\infty,s} = -\log\|s\|_\infty$. Thus $G_{\infty,s}$ is a continuous psh function, and so $dd^c[-\log\|s\|^2_\infty]$ is a positive current on U.

Since positivity of a current is a local condition and we see that $\mathrm{c}_1(L,\|\cdot\|_\infty)$ does not depend on s, we find that the closed $(1,1)$-current $T = \mathrm{c}_1(L,\|\cdot\|_\infty)$ admits a locally continuous potential and is positive. We note $\lim_{n\to\infty}[\omega_n] = T$ as currents.

(2) Since T admits a locally continuous potential, the currents f_i^*T are defined. Locally, we have

$$\begin{aligned} f_1^*T + \cdots + f_1^*T &= dd^c[f_1^*(-\log\|s\|^2_\infty) + \cdots + f_k^*(-\log\|s\|^2_\infty)] \\ &= dd^c[-\log\|s\|^{2d}_\infty] = dT. \end{aligned}$$

Thus we get (2).

(3) Let $\eta_0 \in A^{1,1}(X)$ be a closed $(1,1)$-form whose cohomology class coincides with $\mathrm{c}_1(L)$. Since the cohomology class of ω_0 is the same as that of η_0, by dd^c-lemma, there exists $u \in C^\infty(X(\mathbb{C}))$ with $\eta_0 - \omega_0 = dd^cu$. Then we have

$$\begin{aligned} \eta_n &= \left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(\eta_0) \\ &= \left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(\omega_0) + dd^c\left(\left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(u)\right) = \omega_n + dd^c\left(\left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(u)\right). \end{aligned}$$

Here u is a bounded function, and $\left\|\left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(u)\right\|_{\mathrm{sup}} \leq \left(\frac{k}{d}\right)^n \|u\|_{\mathrm{sup}} \to 0$ as $n \to \infty$. Thus, as currents, $\left[\left(\frac{1}{d}\right)^n \sum_{f\in\mathcal{F}_n} f^*(u)\right]$ goes to zero as $n\to\infty$. Since $[\omega_n] \to T$ as currents, we get $\lim_{n\to\infty}[\eta_n] = T$. □

3.3. **Examples.**

3.3.1. *Projective spaces.* We remark that, for the projective space, the invariant current in Theorem 3.2.1 coincides with the Green current.

Proposition 3.3.1. *Let $f : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ be a morphism over $\mathbb{C}$ of degree $d \geq 2$, and let T_G denote the Green current of f. We regard $(\mathbb{P}^N_{\mathbb{C}}, f)$ as a dynamical system associated with $\mathcal{O}_{\mathbb{P}^N}(1)$ of degree d, and we denote by T the invariant current in Theorem 3.2.1. Then $T = T_G$,*

Proof. By definition, T_G is the limit of $\frac{1}{d^n}(f^n)^*\omega_{FS}$ as $d\to\infty$, where ω_{FS} is the Fubini-Study metric. On the other hand, by Theorem 3.2.1(3), $\frac{1}{d^n}(f^n)^*\omega_{FS}$ converges to T. Thus $T = T_G$. □

Remark 3.3.2. We remark that our assumption f being a morphism is rather strong, and that, to define the Green current, f suffices to be an algebraically stable rational map (cf. [18], Théorème 1.6.1).

3.3.2. *Wheler's K3 surfaces.* Let $(S; \sigma_1, \sigma_2)$ be Wheler's K3 surface as in §1.4.4.

Lemma 3.3.3. *Let $(\sigma_2 \circ \sigma_1)^* : H^{1,1}(S, \mathbb{R}) \to H^{1,1}(S, \mathbb{R})$ be the $\mathbb{R}$-linear map induced by $\sigma_2 \circ \sigma_1 : S \to S$, and let $\lambda = \max\{|t| \mid \det(tI - (\sigma_2 \circ \sigma_1)^*) = 0\}$ be the spectral radius of $\sigma_2 \circ \sigma_1$. Then $\lambda = 7 + 4\sqrt{3}$.*

Proof. With the notation in §1.4.4, $(\sigma_2 \circ \sigma_1)^*(E^+) = (7+4\sqrt{3})E^+$. Thus $\lambda \geq 7+4\sqrt{3}$. On the other hand, it follows from [14], Theorem 3.2 that $(\sigma_2 \circ \sigma_1)^*$ has at most one eigenvalue λ with $|\lambda| > 1$. Thus $\lambda = 7 + 4\sqrt{3}$. □

It is known that the topological entropy of $\sigma_2 \circ \sigma_1$, denoted by $h_{\mathrm{top}}(\sigma_2 \circ \sigma_1)$, is equal to $\log \lambda$ (cf. [5], Théorème 2.1). Cantat [5] proved the following theorem about K3 automorphisms with strictly positive topological entropy.

Theorem 3.3.4 ([5], §2, cf. [14], Theorem 11.3)**.** *Let X be a K3 surface with an automorphism ϕ whose topological entropy $h_{\mathrm{top}}(\phi) = \log \lambda$ is strictly positive. Then there exists a closed positive $(1,1)$-current T^+, unique up to scale, such that $f^*(T^+) = \lambda T^+$. Moreover, for any other closed positive $(1,1)$-current ω, there is a constant c^+ such that*

$$\frac{1}{\lambda^n}(f^*)^n(\omega) \to c^+ T^+ \quad (n \to \infty)$$

as $(1,1)$-currents.

Considering ϕ^{-1}, we have a closed positive $(1,1)$-current T^-. The next proposition relates T^+ and T^- with the invariant current T in Theorem 3.2.1 for $(S; \sigma_1, \sigma_2)$.

Proposition 3.3.5. *Let the notation be as in §1.4.4. Let $\lambda = 7 + 4\sqrt{3}$. Let $(S; \sigma_1, \sigma_2)$ be Wheler's K3 surface over $\mathbb{C}$, which we regard as a dynamical system of two morphisms associated with L of degree 4. Let T be the invariant current in Theorem 3.2.1. Let η be a positive C^∞ $(1,1)$-form on S whose cohomology class is equal to $\mathrm{c}_1(L)$. We set*

$$T^+ = \lim_{n\to\infty} \frac{1}{\lambda^n}(\sigma_2 \circ \sigma_1)^{*n}(\eta), \qquad T^- = \lim_{n\to\infty} \frac{1}{\lambda^n}(\sigma_2 \circ \sigma_1)^{-1*n}(\eta),$$

whose convergence is assured by Theorem 3.3.4. Then $T = T^+ + T^-$.

Proof. We set

$$S_n = \frac{1}{\lambda^n}(\sigma_2 \circ \sigma_1)^{*n}(\eta) + \frac{1}{\lambda^n}(\sigma_2 \circ \sigma_1)^{-1*n}(\eta),$$
$$T_n = \frac{1}{4^n} \sum_{f \in \mathcal{F}_n} f^*(\eta),$$

where $\mathcal{F}_n = \{\sigma_{i_1} \circ \cdots \circ \sigma_{i_n} \mid i_j = 1, 2,\ 1 \leq j \leq n\}$. Since $\sigma_1^2 = \sigma_2^2 = \mathrm{id}$, we have

$$T_{2n} = \frac{1}{4^{2n}} \sum_{i=0}^{n} \binom{2n}{i} \left\{(\sigma_2 \circ \sigma_1)^{*(n-i)}(\eta) + (\sigma_2 \circ \sigma_1)^{-1*(n-i)}(\eta)\right\} = \frac{1}{4^{2n}} \sum_{i=0}^{n} \binom{2n}{i} \lambda^{n-i} S_{n-i}.$$

From the definition, as n goes to ∞, S_n converges to $T^+ + T^-$ and T_{2n} converges to T. Thus it is sufficient to show the following claim to prove the proposition.

Claim 3.3.5.1. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence with $s_n \in \mathbb{C}$ that converges to s_∞ as $n \to \infty$. We set $t_{2n} = \frac{1}{4^{2n}} \sum_{i=0}^{n} \binom{2n}{i} \lambda^{n-i} s_{n-i}$. Then $\lim_{n\to\infty} t_{2n} = s_\infty$.*

Note that $\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}\lambda^{n-i}$. Since $\sqrt{\lambda} = 2+\sqrt{3}$, this shows $\frac{1}{4^{2n}}\sum_{i=0}^{2n}\binom{2n}{i}\lambda^{n-i} = 1$. On the other hand, for fixed i $(0 \le i \le n)$, we have $\frac{1}{4^{2n}}\binom{2n}{i}\lambda^{n-i} = \left(\frac{\lambda}{4^2}\right)^n\binom{2n}{i}\lambda^{-i}$ goes to zero as $n \to \infty$. Moreover, we have $\frac{1}{4^{2n}}\sum_{i=n+1}^{2n}\binom{2n}{i}\lambda^{n-i} < \frac{1}{4^{2n}}\sum_{i=n+1}^{2n}\binom{2n}{i} < \frac{1}{4^{2n}}(1+1)^{2n}$, and thus $\frac{1}{4^{2n}}\sum_{i=n+1}^{2n}\binom{2n}{i}\lambda^{n-i}$ goes to zero as $n \to \infty$. Combining these estimates, we obtain the claim. □

3.4. **Question about distribution of small points.** Let $(X; f_1, \cdots, f_k)$ be a dynamical system of k morphisms over a number field K associated with a line bundle L of degree $d > k$.

In §2, we have defined the canonical heights of subvarieties, and in §3 we have defined the invariant current T on $X(\mathbb{C})$. Here, we would like to pose a question regarding distribution of small points.

A sequence $\{x_n\}_{n=1}^{\infty}$ of $X(\overline{K})$ is said to be generic if for any closed subscheme $Y \subsetneq X$ there exists n_0 such that $x_n \notin Y$ for $n \ge n_0$. A sequence $\{x_n\}_{n=1}^{\infty}$ of $X(\overline{K})$ is called a sequence of small points if $\lim_{n\to\infty}\widehat{h}_{L,\mathcal{F}}(x_n) = 0$. For $x \in X(\overline{K})$, let $O(x)$ denote the Galois orbit of x over K. Since T admits a locally continuous potential by Theorem 3.2.1, we can define a probability measure $\mu := \dfrac{1}{\mathrm{c}_1(L)^{\dim X}} T \wedge \cdots \wedge T$ ($\dim X$-times).

Question 3.4.1 (distribution of small points)**.** Suppose there exists a generic sequence $\{x_n\}_{n=1}^{\infty}$ of $X(\overline{K})$ of small points. Then does $\frac{1}{\#O(x_n)}\sum_{y\in O(x_n)}\delta_y$ converge weakly to μ as n goes to ∞?

We note that Szpiro, Ullmo and Zhang [23] proved equiditribution of small points for abelian varieties, Bilu [2] for algebraic tori, and Chambert-Loir [6] for certain semi-abelian varieties.

We would like to point out a similarity between the above question and the theorem by Briend and Duval for morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \ge 2$ over $\mathbb{C}$. (Here we list only a part of their results.)

Theorem 3.4.2 (Briend and Duval)**.** (1) (*Distribution of repelling periodic points of f.*) *The measure* $\frac{1}{d^{kn}}\sum_{f^n(y)=y,\, y \text{ is repelling}}\delta_y$ *converge weakly to* μ *as* n *goes to* ∞.

(2) (*Distribution of preimages of points outside the exceptional set.*) *The exceptional set* E *of* f *is defined as the biggest proper algebraic set of* $\mathbb{P}^N$ *that is totally invariant by* f*. Then for any* $a \in \mathbb{P}^N(\mathbb{C}) \setminus E$, $\frac{1}{d^{kn}}\sum_{y\in\mathbb{P}^N(\mathbb{C}), f^n(y)=a}\delta_y$ *converge weakly to* μ *as* n *goes to* ∞.

Suppose f is defined over a number field K. Then, for any periodic point x, $\widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),f}(x) = 0$. Also for $a \in \mathbb{P}^N(\overline{K}) \setminus E$, consider the set $B = \{x \in X(\overline{K}) \mid f^n(x) = a \text{ for some } n \ge 0\}$. Note $\widehat{h}_{\mathcal{O}_{\mathbb{P}^n}(1),f}(x) = \frac{1}{d^n}\widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),f}(a)$ if $f^n(x) = a$. Since $a \notin E$, we can choose a generic sequence $\{x_n\}$ of $\mathbb{P}^N(\overline{K})$ of small points with $x_n \in B$ for every $n \ge 0$. Thus Question 3.4.1 might be seen as an arithmetic analogue of Theorem 3.4.2 for $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \ge 2$.

Remark 3.4.3. When $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial map, Baker and Hsia [3] have recently showed the distribution theorem for small points on $\mathbb{P}^1(\overline{K})$.

3.5. **Distribution of small points for Lattès examples.** In this subsection, using the methods of [23] and [6], we would like to briefly remark that Question 3.4.1 is true for Lattès examples.

3.5.1. *Quick Review of Lattès examples.* Let $U(N)$ be the unitary group of size N, and $E(N) = \{(U, a) \mid U \in U(N),\ a \in \mathbb{C}^N\}$ the complex motion group acting on $\mathbb{A}^N_{\mathbb{C}}$. A subgroup G of $E(N)$ is called a *complex crystallographic group* if G is a discrete subgroup of $E(N)$ with compact quotient.

Berteloot and Loeb [4] obtained a geometric characterization of morphisms of $\mathbb{P}^N_{\mathbb{C}}$ whose Green current is smooth and strictly positive on some non-empty (analytic) open subset.

Recall that, by Proposition 3.3.1, for a morphism $f : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ of degree $d \geq 2$, the invariant current in Theorem 3.2.1 is nothing but the Green current of f.

Theorem 3.5.1 ([4], Théorème 1.1, Proposition 4.1)**.** *Let $f : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ be a morphism over $\mathbb{C}$ of degree $d \geq 2$, and T the Green current of f. Assume that T is smooth and strictly positive on some non-empty (analytic) open subset of $\mathbb{P}^N_{\mathbb{C}}$. Then there exists a complex crystallographic group G, and an affine transformation $D : \mathbb{A}^N_{\mathbb{C}} \to \mathbb{A}^N_{\mathbb{C}}$ whose linear part is $\sqrt{d}U$ with $U \in U(N)$ such that the ramified covering $\sigma : \mathbb{A}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ satisfies $\sigma \circ f = D \circ \sigma$ and that G acts on the fibers of σ transitively. Moreover, $\sigma^*(T) = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^N dz_i \wedge \overline{dz_i}$, where z_i's are the standard coordinate of $\mathbb{A}^N_{\mathbb{C}}$.*

Endomorphisms $f : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ that satisfy the assumption of Theorem 3.5.1 are called *Lattès examples.*

Example 3.5.2 (Some Lattès examples)**.** (1) Let $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Set $G_1 = \{(\pm 1, m + n\tau) \mid m, n \in \mathbb{Z}\} \subset E(1)$, and let $D : \mathbb{A}^1 \to \mathbb{A}^1$ be the $[d]$-th map. Then the quotient $\mathbb{A}^1/G_1$ is isomorphic to $\mathbb{P}^1$ and D descends a morphism $f_1 : \mathbb{P}^1 \to \mathbb{P}^1$.

(2) (Ueda [24]) Consider the morphism $f_1 \times \cdots \times f_1 : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (N-times). The N-th symmetric group S_N acts on $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by $\sigma \cdot (x_1, \cdots, x_n) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for $\sigma \in S_N$. Observing that the quotient $(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)/S_N$ is isomorphic to $\mathbb{P}^N_{\mathbb{C}}$, one finds that $f_1 \times \cdots \times f_1$ descends a morphism $f_N : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$.

We need the following proposition in the next subsection.

Proposition 3.5.3. *Let $f : \mathbb{P}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ be a Lattès example, and T the Green current of f.*

(1) *There exists $c > 0$ such that $T \geq c\,\omega_{FS}$ as a current.*

(2) *Let $v : \mathbb{P}^N(\mathbb{C}) \to \mathbb{R}$ be a C^∞-function. Then there exists $\varepsilon_0 > 0$ such that, for all ε with $0 < \varepsilon \leq \varepsilon_0$, $T + \varepsilon dd^c v$ is positive as a current.*

Proof. Let $\sigma : \mathbb{A}^N_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$ be the ramified covering as in Theorem 3.5.1. Since $\sigma^*\omega_{FS}$ is a C^∞ bounded $(1,1)$-form on $\mathbb{A}^N(\mathbb{C})$, there exists $c > 0$ such that $\frac{\sqrt{-1}}{2\pi} \sum_{i=1}^N dz_i \wedge \overline{dz_i} - c\,\sigma^*\omega_{FS}$ is a strictly positive $(1,1)$-form. Then considering the pull-back by σ and observing $\sigma^* T = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^N dz_i \wedge \overline{dz_i}$, we find $T \geq c\,\omega_{FS}$. The assertion (2) follows from (1). $\square$

3.5.2. *Distribution of small points for Lattès examples.* We follow [23] and [6]. In this subsection, we use the terminology in [26] freely.

Theorem 3.5.4 ([23], Thèoréme 3.1, [6], Proposition 6.2). *Let X be a projective variety defined over a number field $K \subset \mathbb{C}$, and L an ample line bundle on X. Let $\|\cdot\|$ be an ample adelic metric on L. Assume that the height of X is zero with respect to $(L, \|\cdot\|)$. Let $\{x_n\}_{n=1}^{\infty}$ be a generic sequence of small points in $X(\overline{K})$. Let $v : X(\mathbb{C}) \to \mathbb{R}$ is a continuous function such that, for all sufficiently small $\varepsilon > 0$, $c_1(L, \|\cdot\|) + \varepsilon dd^c v$ is positive as a current on $X(\mathbb{C})$. Then we have*

$$\liminf_{n\to\infty} \frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} v(y) \geq \int_X v \frac{c_1(L, \|\cdot\|)^{\dim X}}{c_1(L)^{\dim X}}.$$

Using Theorem 3.5.4, we find that Question 3.4.1 is true for Lattès examples.

Theorem 3.5.5. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a Lattès example defined over a number field $K \subset \mathbb{C}$, T the Green current of f, and $\mu = \wedge^N T$. Let $\{x_n\}_{n=1}^{\infty}$ be a generic sequence of small points in $X(\overline{K})$. Then $\frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} \delta_y$ converges weakly to μ.*

Proof. We fix an isomorphism $\varphi : \mathcal{O}_{\mathbb{P}^N}(d) \simeq f^*(\mathcal{O}_{\mathbb{P}^N}(1))$, where d is the degree of f. Fix a C^∞ model $(\mathcal{X}, \overline{\mathcal{L}})$ of $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ such that $\mathcal{L}$ is ample and $c_1(\overline{\mathcal{L}_{\mathbb{C}}})$ is strictly positive on $\mathbb{P}^N(\mathbb{C})$. Then, by [26], (2.3), we inductively obtain a sequence of C^∞ models $(\mathcal{X}_n, \overline{\mathcal{L}_n})$ of $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, considering the pull-backs by φ (cf. Theorem 2.2.1 and its proof). By [26], (2.3), the models $(\mathcal{X}_n, \overline{\mathcal{L}_n})$ induces an ample adelic metric on $\mathcal{O}_{\mathbb{P}^N}(1)$.

In particular, by [26], Theorem(2.2), the induced metric $\|\cdot\|_\infty$ at infinity is characterized by a continuous metric on $\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)$ with $\|\cdot\|_\infty^d = \varphi^* f^* \|\cdot\|_\infty$. By Proposition 3.3.1, $c_1(\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1), \|\cdot\|_\infty)$ is equal to the Green current T.

To apply Theorem 3.5.4, we will show that the height of $\mathbb{P}^N$ is zero with respect to $(\mathcal{O}_{\mathbb{P}^N}(1), \|\cdot\|)$. On one hand, it follows from [26], Theorem(2.4) that $h_{\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|}(\mathbb{P}^N) \geq 0$. One the other hand, since $(\mathcal{O}_{\mathbb{P}^N}(1), \|\cdot\|)$ is ample metrized line bundle, if follows from [26], Thereom(1.10) that

$$\sup_{Y \subsetneq \mathbb{P}^N} \inf_{x \in \mathbb{P}^N \setminus Y(\overline{K})} h_{(\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|)}(x) \geq h_{(\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|)}(\mathbb{P}^N),$$

where Y runs through the algebraic subsets of X. (In the notation of Theorem 2.3.1, $h_{(\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|)} = \widehat{h}_{\mathcal{O}_{\mathbb{P}^N}(1),f}$.) If $x \in \mathbb{P}^N(\overline{K})$ is a periodic point, then $h_{(\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|)}(x) = 0$. Since the support of T is $\mathbb{P}^N$ (cf. [18], Théorème 1.6.5), the set of periodic points are Zariski dense in $\mathbb{P}^N(\overline{K})$ (cf. Theorem 3.4.2(1)). Hence the left-hand-side of the above inequality is zero, and so $h_{(\mathcal{O}_{\mathbb{P}^N}(1),\|\cdot\|)}(\mathbb{P}^N) = 0$.

Let $v : \mathbb{P}^N(\mathbb{C}) \to \mathbb{R}$ be a continuous function. We need to show

$$\lim_{n\to\infty} \frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} v(y) = \int_X v\mu.$$

Since the C^∞-functions are dense in the set of continuous functions on $\mathbb{P}^N(\mathbb{C})$, we may assume v is C^∞. Since, by Proposition 3.5.3, $T + \varepsilon dd^c v$ is positive for all sufficiently small $\varepsilon > 0$, it follows from Theorem 3.5.4 that $\liminf_{n\to\infty} \frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} v(y) \geq \int_X v\mu$. Considering $-v$ instead of v, we find $\limsup_{n\to\infty} \frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} v(y) \leq \int_X v\mu$. Thus we get the assertion. $\square$

4. Local canonical heights

In this section, for normal projective varieties, we see that there exists canonical local heights for closed points and that the canonical height in §1.2 decomposes into a sum of these canonical local heights. In particular, this gives yet another construction of the canonical heights for closed points. The argument goes similarly as in [11], Chap. 10 and [7], §2 which treat the case $k = 1$.

4.1. **Quick review of local heights.** In this subsection, we briefly review local heights and their properties. For details, we refer to Lang's book [11], Chap. 10.

Let X be a projective variety over a number field K, and U a non-empty Zariski open set of X. In this subsection, we assume X is normal. Let M_K denote the set of absolute values of K, and M denote the set of absolute values on $\overline{K}$ extending those of K.

Definition 4.1.1. (1) A function $\lambda : U(\overline{K}) \times M$ is said to be *M_K-continuous* if, for every $v \in M$, $\lambda_v : U(\overline{K}) \to \mathbb{R}$, $x \mapsto \lambda(x, v)$ is continuous with respect to v-topology.

(2) A function $\gamma : M_K \to \mathbb{R}$ is said to be *M_K-constant* if $\gamma(v) = 0$ for all but finitely many $v \in M_K$. For $v' \in M$ that is an extension of $v \in M_K$, we set $\gamma(v') := \gamma(v)$. Then γ is extended to a function $\gamma : M \to \mathbb{R}$, which is also said to be M_K-constant.

(3) A function $\lambda : U(\overline{K}) \times M$ is said to be *M_K-bounded* if there is a M_K-constant function γ such that $|\alpha(x, v)| \leq \gamma(v)$ for all $(x, v) \in U(\overline{K}) \times M$.

(4) Let $D \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. A function

$$\lambda_{X,D} : (X \setminus \mathrm{Supp}(D))(\overline{K}) \times M \to \mathbb{R}$$

is said to be a *local height* (or a *Weil local height function*) associated with D if it has the following property: There are an affine covering $\{U_i\}$ of X, a Cartier divisor $\{(U_i, s_i)\}$ representing D such that $\alpha_i(x, v) := \lambda_{X,D}(x, v) - v \circ s_i(x)$ for $x \in (U_i \setminus \mathrm{Supp}(D))(\overline{K})$ and $v \in M$ is M_K-bounded and M_K-continuous.

Note that, for every $s \in K(X)^*$,

$$\lambda_{X,\mathrm{div}(s)} : (X \setminus \mathrm{Supp}(\mathrm{div}(s)))(\overline{K}) \times M \to \mathbb{R}, \quad (x, v) \mapsto v \circ s(x)$$

is a local height function associated with $\mathrm{div}(s)$.

We list some properties of local heights that we will use later.

Theorem 4.1.2 ([11]). (1) *For every $D \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, there exists a local canonical height function associated with D.*

(2) *Let $\lambda_{X,D}$ be a local height function associated with D. Suppose there exists a proper Zariski closed set Z containing $\mathrm{Supp}(D)$ such that $\lambda_{X,D}$ is M_K-bounded on $(X \setminus \mathrm{Supp}(Z))(\overline{K}) \times M$. Then $D = 0$, and $\lambda_{X,D}$ extends uniquely to a M_K-bounded and M_K-continuous function on $X(\overline{K}) \times M$.*

(3) *If $\lambda_{X,D}$ and $\lambda_{X,D'}$ are local height functions associated with D and D' respectively, then $\lambda_{X,D} + \lambda_{X,D'}$ is a local height function associated with $D + D'$.*

(4) *Let $g : Y \to X$ be a morphism of normal projective varieties, D an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\lambda_{X,D}$ a local height function associated with D. Assume that $\phi(Y)$ is not contained in $\mathrm{Supp}(D)$. Then*

$$\lambda_D \circ (g \times \mathrm{id}_M) : (Y \setminus \mathrm{Supp}(\phi^* D))(\overline{K}) \times M \to \mathbb{R}$$

is a local height function associated with $\phi^* D$.

Proof. For (1) see [11], Chap. 10, Theorem 3.5. For (2) see [ibid.], Prop 2.3, Prop 1.5, Cor 2.4. Note that the assumption that X is normal is enough to show (2). For (3) and (4), see [ibid.], Prop 2.1 and Prop 2.6. These proofs can apply for $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. □

In the remainder of this subsection, we recall how local height functions are related to global height functions.

Let $\lambda_{X,D}$ be a local height function associated with a divisor D. Then, for any point $x \in (X \setminus \mathrm{Supp}(D))(\overline{K})$, the *associated height* is defined by

$$h_{\lambda_{X,D}}(x) = \frac{1}{[L:K]} \sum_{w \in M_L} [L_w : K_v] \lambda_{X,D}(x, w),$$

where L is an extension field of K with $x \in X(L)$ and w is an extension of v. The height $\lambda_{X,D}$ is defined for all $x \in X(\overline{K})$. Indeed, for any point $x \in X(\overline{K})$, there exists a rational function s such that $x \notin \mathrm{Supp}(D - \mathrm{div}(s))$: Then we define

$$h_{\lambda_{X,D}}(x) := h_{\lambda_{X,D-\mathrm{div}(s)}}(x).$$

By the product formula, this value does not depend on the choice of s.

Theorem 4.1.3 ([11]). *Let* $h_{X,\mathcal{O}_X(D)} : X(\overline{K}) \to \mathbb{R}$ *be a height function corresponding to* $\mathcal{O}_X(D)$ *defined in §1.1. Then* $h_{\lambda_{X,D}} = h_{X,\mathcal{O}_X(D)} + O(1)$.

Proof. For its proof, we refer to [11], Chap. 10, §4. □

4.2. **Construction of local canonical heights.** Let X be a normal projective variety over a number field K, and $f_1, \cdots, f_k : X \to X$ be morphisms over K. Set $\mathcal{F} = \{f_1, \cdots, f_k\}$ as before. Let E an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Assume that $f_i(X)$ is not contained in $\mathrm{Supp}(E)$ for each i, and that $f_1^* E + \cdots + f_k^* E \sim dE$ with $d > k$. Take $s \in \mathrm{Rat}(X) \otimes \mathbb{R}$ with

$$f_1^* E + \cdots + f_k^* E = dE + \mathrm{div}(s) \quad (\in \mathrm{Rat}(X) \otimes \mathbb{R}).$$

Theorem 4.2.1. *Let the notation and assumption be as above. Then there exists a unique function*

$$\widehat{\lambda}_{X,E,\mathcal{F},s} : (X \setminus \mathrm{Supp}(E))(\overline{K}) \times M \to \mathbb{R}$$

with the following properties:

(1) $\widehat{\lambda}_{X,E,\mathcal{F},s}$ *is a Weil local height associated with* E.
(2) *For any* $x \in (X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(f_1^* E_1) \cup \cdots \cup \mathrm{Supp}(f_k^* E)))$ *and any* $v \in M$,

$$\sum_{i=1}^{k} \widehat{\lambda}_{X,E,\mathcal{F},s}(f_i(x), v) = d \widehat{\lambda}_{X,E,\mathcal{F},s}(x, v) + v(s(x)).$$

We first prove the following lemma.

Lemma 4.2.2. *Let $\mathfrak{X}$ be a topological space, $f_1, \cdots, f_k : \mathfrak{X} \to \mathfrak{X}$ continuous maps, and $\gamma : \mathfrak{X} \to \mathbb{R}$ a bounded continuous function. Fix a real number d larger than k. Then, there exists a unique bounded continuous function $\widehat{\gamma} : \mathfrak{X} \to \mathbb{R}$ such that*

$$\gamma(x) = \sum_{i=1}^{k} \widehat{\gamma}(f_i(x)) - d\widehat{\gamma}(x)$$

for any $x \in \mathfrak{X}$. Moreover,

$$\|\widehat{\gamma}\|_{\sup} \leq \left(\frac{2d+1}{d-k}\right) \|\gamma\|_{\sup}$$

Proof. We can prove the lemma as in [11], Chap. 11 Lemma 1.2 and [7], Lemma 2.1. Namely, for a continuous function $\delta : \mathfrak{X} \to \mathbb{R}$, define $S\delta : \mathfrak{X} \to \mathbb{R}$ by $S\delta(x) = \frac{1}{d}\left[\sum_{i=1}^{k} \delta(f_i(x)) - \gamma(x)\right]$.

Claim 4.2.2.1. *$\{S^l\gamma\}_{l=0}^{\infty}$ is a Cauchy sequence with respect to the sup norm.*

Note that for bounded continuous functions $\delta_1, \delta_2 : \mathfrak{X} \to \mathbb{R}$, $\|S\delta_1 - S\delta_2\|_{\sup} \leq \frac{k}{d}\|\delta_1 - \delta_2\|_{\sup}$. Then $\|S^l\delta_1 - S^l\delta_2\|_{\sup} \leq \left(\frac{k}{d}\right)^l \|\delta_1 - \delta_2\|_{\sup}$. Since $\sum_{l=1}^{\infty} \left(\frac{k}{d}\right)^l < \infty$, $\{S^l\gamma\}_{l=0}^{\infty}$ is a Cauchy sequence.

Set $\widehat{\gamma}(x) := \lim_{l\to\infty} S^l\gamma(x)$ for $x \in \mathfrak{X}$. Then $\widehat{\gamma} : \mathfrak{X} \to \mathbb{R}$ is a continuous function. It follows from $S\widehat{\gamma}(x) = \widehat{\gamma}(x)$ that $\gamma(x) = \sum_{i=1}^{k} \widehat{\gamma}(f_i(x)) - d\widehat{\gamma}(x)$. On the other hand, we have

$$\begin{aligned}\|S^l\gamma\|_{\sup} &\leq \|S^l\gamma - \gamma\|_{\sup} + \|\gamma\|_{\sup} \leq \sum_{\alpha=0}^{l-1} \|S^{\alpha+1}\gamma - S^{\alpha}\gamma\|_{\sup} + \|\gamma\|_{\sup} \\ &\leq \sum_{\alpha=0}^{l-1} \left(\frac{k}{d}\right)^{\alpha} \|S\gamma - \gamma\|_{\sup} + \|\gamma\|_{\sup} \leq \frac{d}{d-k}\|S\gamma - \gamma\|_{\sup} + \|\gamma\|_{\sup}.\end{aligned}$$

Since

$$\begin{aligned}\|S\gamma - \gamma\|_{\sup} &= \sup_{x\in\mathfrak{X}} \left| \frac{1}{d}\sum_{i=1}^{k} \gamma(f_i(x)) - \left(\frac{1}{d} + 1\right)\gamma(x) \right| \\ &\leq \left(\frac{k}{d} + \frac{1}{d} + 1\right) \|\gamma\|_{\sup} = \frac{d+k+1}{d}\|\gamma\|_{\sup},\end{aligned}$$

we have $\|S^l\gamma\|_{\sup} \leq \left(\frac{2d+1}{d-k}\right)\|\gamma\|_{\sup}$. By letting $l \to \infty$, we get $\|\widehat{\gamma}\|_{\sup} \leq \left(\frac{2d+1}{d-k}\right)\|\gamma\|_{\sup}$. Finally we show the uniqueness of $\widehat{\gamma}$. Indeed suppose $\widehat{\gamma}_1, \widehat{\gamma}_2$ satisfy $\gamma(x) = \sum_{i=1}^{k} \widehat{\gamma}_j(f_i(x)) - d\widehat{\gamma}_j(x)$ for $j = 1, 2$. Then $\sum_{i=1}^{k} (\widehat{\gamma}_1 - \widehat{\gamma}_2)(f_i(x)) = d(\widehat{\gamma}_1 - \widehat{\gamma}_2)(x)$. Then we have $\|\widehat{\gamma}_1 - \widehat{\gamma}_2\|_{\sup} \leq \frac{k}{d}\|\widehat{\gamma}_1 - \widehat{\gamma}_2\|_{\sup}$. Since $d > k$, we have $\widehat{\gamma}_1 = \widehat{\gamma}_2$. □

Proof of Theorem 4.2.1. Take a local height function $\lambda_{X,E}$. Since, by the assumption, $f_i(X)$ is not contained in $\mathrm{Supp}(E)$ for each i, $\lambda_{X,E} \circ (f_i \times \mathrm{id}_M)$ is a local height function associated with $f_i^*(E)$ by Theorem 4.1.2(4). We put $\lambda_{X,f_i^*E} := \lambda_{X,E} \circ (f_i \times \mathrm{id}_M)$. Set $Z := \mathrm{Supp}(E) \cup \mathrm{Supp}(f_1^*E) \cup \cdots \cup \mathrm{Supp}(f_k^*E)$. Since $f_1^*E + \cdots + f_k^*E = dE + \mathrm{div}(s)$,

$$\gamma(x, v) := \sum_{i=1}^{k} \lambda_{X,f_i^*E}(x, v) - d\lambda_{X,E}(x, v) - v(s(x))$$

is an M_K-bounded and M_K-continuous function on $(X \setminus Z)(\overline{K}) \times M$. By Theorem 4.1.2(2), γ extends to an M_K-bounded and M_K-continuous function on $X(\overline{K}) \times M$. Then, by Lemma 4.2.2, there exists an M_K-bounded and M_K-continuous function $\widehat{\gamma} : X(\overline{K}) \times M \to \mathbb{R}$ such that

$$\gamma(x,v) = \sum_{i=1}^{k} \widehat{\gamma}(f_i(x), v) - d\widehat{\gamma}(x,v)$$

for $x \in X(\overline{K})$ and $v \in M$.

We define a function $\widehat{\lambda}_{X,E,\mathcal{F},s} : (X \setminus \mathrm{Supp}(E))(\overline{K}) \times M \to \mathbb{R}$ to be

$$\widehat{\lambda}_{X,E,\mathcal{F},s}(x,v) := \lambda_{X,E}(x,v) - \widehat{\gamma}(x,v).$$

Let us see $\widehat{\lambda}_{X,E,\mathcal{F},s}$ enjoys the properties (1) and (2). Since $\widehat{\gamma}$ is an M_K-bounded and M_K-continuous function, $\widehat{\lambda}_{X,E,\mathcal{F},s}$ is a local height function associated with E. Moreover, for $x \in (X \setminus Z)(\overline{K})$ and $v \in M$, we have

$$\begin{aligned}
\sum_{i=1}^{k} \widehat{\lambda}_{X,E,\mathcal{F},s}(f_i(x), v) &= \left(\sum_{i=1}^{k} \lambda_{X,E}(f_i(x), v) \right) - \left(\sum_{i=1}^{k} \widehat{\gamma}(f_i(x), v) \right) \\
&= (d\lambda_{X,E}(x,v) + v(s(x)) + \gamma(x,v)) - (d\widehat{\gamma}(x,v) + \gamma(x,v)) \\
&= d\left(\lambda_{X,E}(x,v) - \widehat{\gamma}(x,v)\right) + v(s(x)) \\
&= d\widehat{\lambda}_{X,E,\mathcal{F},s}(x,v) + v(s(x)).
\end{aligned}$$

Next we see the uniqueness of $\widehat{\lambda}_{X,E,\mathcal{F},s}$. Suppose $\widehat{\lambda}_{X,E,\mathcal{F},s}$ and $\widehat{\lambda}'_{X,E,\mathcal{F},s}$ are two functions satisfying (1) and (2). Set $\delta = \widehat{\lambda}_{X,E,\mathcal{F},s} - \widehat{\lambda}'_{X,E,\mathcal{F},s}$. By Theorem 4.1.2(2), δ extends to an M_K-bounded and M_K-continuous function on $X(\overline{K}) \times M$. Moreover,

$$\sum_{i=1}^{k} \delta(f_i(x), v) = d\delta(x,v)$$

holds for $x \in (X \setminus Z)(\overline{K})$ and $v \in M$, and hence for all $x \in X(\overline{K})$ and $v \in M$ by M_K-continuity. Take M_K-constant $\gamma : M \to \mathbb{R}$ with $|\delta(x,v)| \le \gamma(v)$. Then

$$|\delta(x,v)| \le \left| \frac{1}{d^l} \sum_{f \in \mathcal{F}_l} \delta(f(x), v) \right| \le \frac{k^l}{d^l} \gamma(v) \to 0 \quad (l \to \infty),$$

hence $\delta(x,v) = 0$ for all $x \in X(\overline{K})$ and $v \in M$. This shows the uniqueness. □

4.3. **Decomposition of canonical heights into local canonical heights.** In this subsection, we show that the associate height of $\widehat{\lambda}_{X,E,\mathcal{F},s}$ is the canonical height $\widehat{h}_{\mathcal{O}_X(E),\mathcal{F}}$ constructed in §4.2 coincides with the canonical height constructed in §1.2. In particular, this gives another construction of the canonical heights when X is a normal projective variety.

Theorem 4.3.1. *Let the notation and assumption be the same as in Theorem 4.2.1. Let $h_{\widehat{\lambda}_{X,E,\mathcal{F},s}}$ be the associated height of $\widehat{\lambda}_{X,E,\mathcal{F},s}$. Let $\widehat{h}_{\mathcal{O}_X(E),\mathcal{F}}$ be the canonical height constructed in Theorem 1.2.1. Then $h_{\widehat{\lambda}_{X,E,\mathcal{F},s}} = \widehat{h}_{\mathcal{O}_X(E),\mathcal{F}}$.*

Proof. Since $\widehat{\lambda}_{X,E,\mathcal{F},s}$ is a local height function associated with E, $h_{\widehat{\lambda}_{X,E,\mathcal{F},s}} = h_{\mathcal{O}_X(E)} + O(1)$ by Theorem 4.1.3. Thus in virtue of Theorem 1.2.1, we have only to show

$$h_{\widehat{\lambda}_{X,E,\mathcal{F},s}}(f_i(x)) = dh_{\widehat{\lambda}_{X,E,\mathcal{F},s}}(x) \tag{4.3.1.0}$$

for any $x \in X(\overline{K})$. First suppose $x \notin Z = \mathrm{Supp}(E) \cup \mathrm{Supp}(f_1^*E) \cup \cdots \cup \mathrm{Supp}(f_k^*(E))$. Then, the equation (4.3.1.0) follows from

$$\sum_{i=1}^{k} \widehat{\lambda}_{X,E,\mathcal{F},s}(f_i(x), v) = d\widehat{\lambda}_{X,E,\mathcal{F},s}(x, v) + v(s(x))$$

and the product formula. Next, suppose $z \in Z$. By taking a suitable rational function t and setting $E' := E - \mathrm{div}(t)$, we have $x \notin \mathrm{Supp}(E') \cup \mathrm{Supp}(f_1^*(E')) \cup \cdots \cup \mathrm{Supp}(f_k^*(E'))$. Put $s' = st^d \prod_{i=1}^{k} f_i^*(t^{-1})$ in $\mathrm{Rat}(X) \otimes \mathbb{R}$. Then, since $f_1^*E' + \cdots + f_k^*E = dE' + \mathrm{div}(s')$, we similarly obtain $h_{\widehat{\lambda}_{X,E',\mathcal{F},s'}}(f_i(x)) = dh_{\widehat{\lambda}_{X,E,\mathcal{F},s'}}(x)$. On the other hand, $h_{\widehat{\lambda}_{X,E',\mathcal{F},s'}} = h_{\widehat{\lambda}_{X,E,\mathcal{F},s}}$ by the product formula. Thus the equation (4.3.1.0) holds for all $x \in X(\overline{K})$. □

Appendix: Finiteness results for $\mathcal{F}$-periodic points

Let K be a number field, and X a projective variety defined over K. Let $f_1, f_2, \cdots, f_k : X \cdots \to X$ be rational maps. As in §1.2, we set $\mathcal{F}_0 = \{\mathrm{id}\}$, $\mathcal{F}_l = \{f_{i_1} \circ \cdots \circ f_{i_l} \mid 1 \le i_1, \cdots, i_l \le k\}$ for $l \ge 1$. We set $\mathcal{F} := \mathcal{F}_1 (= \{f_1, \cdots, f_k\})$. For $f \in \bigcup_{l \ge 0} \mathcal{F}_l$, let $I(f)$ denote the set of indeterminacy of f. Set $V = X \setminus \bigcup_{f \in \bigcup_{l\ge 0} \mathcal{F}_l} I(f)$.

We say that $x \in X(\overline{K})$ is *$\mathcal{F}$-periodic* if the following conditions are satisfied: (1) $x \in V(\overline{K})$; (2) $C(x) = \{f(x) \mid f \in \bigcup_{l \ge 0} \mathcal{F}_l\}$ is finite; (3) If $C' \subseteq C(x)$ satisfies $f_i(C') \subseteq C'$ for every $i = 1, \cdots, k$, then $C' = C(x)$. Note that when $k = 1$ and $f_1 : X \to X$ is a morphism, x is $\mathcal{F}$-periodic if and only if x is periodic with respect to f_1.

In Corollary 1.3.2, for dynamical systems $(X; f_1, \cdots, f_k)$ associated with ample line bundles, we show finiteness of the number of finite forward-orbits of bounded degree, whence finiteness of the number of $\mathcal{F}$-periodic points of bounded degree. Here we would like to show finiteness of $\mathcal{F}$-periodic points of bounded degree under a milder condition.

Theorem App. 1. *Let K be a number field, X a projective variety over K, $f_1, f_2, \cdots, f_k : X \cdots \to X$ rational maps. Let S be a subset of $V(\overline{K})$ such that $f_i(S) \subseteq S$ for every $i = 1, \cdots, k$. (For example, we can take S as $V(\overline{K})$ itself.) Assume there exists an ample line bundle $L \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ over X and a positive number $\epsilon > 0$ such that*

$$\sum_{i=1}^{k} h_L(f_i(x)) \ge (k + \epsilon) h_L(x) + O(1)$$

for all $x \in S$. Then for any $D \in \mathbb{Z}_{>0}$,

$$\left\{ x \in S \;\middle|\; \begin{array}{c} [K(x) : K] \le D, \\ x \text{ is } \mathcal{F}\text{-periodic} \end{array} \right\}$$

is finite.

Proof. Take a height function $h_L \in F(X)$ corresponding to L and fix it. By the assumption, there exist a constant C such that, for all $x \in S$,

$$\sum_{i=1}^{k} h_L(f_i(x)) \geq (k+\epsilon)h_L(x) - C. \tag{App. 1.0}$$

Suppose $x \in S$ has the finite forward orbit $C(x) = \{x_1, \cdots, x_n\}$. For each x_j, consider the set $\{f_1(x_j), \cdots, f_k(x_j)\}$ and let $a_{ij} \in \mathbb{Z}_{\geq 0}$ denote the number such that x_i appears a_{ij} times in $\{f_1(x_j), \cdots, f_k(x_j)\}$ for $i = 1, \cdots, n$. Thus we have

$$h_L(f_1(x_j)) + \cdots + h_L(f_k(x_j)) = a_{1j}h_L(x_1) + \cdots + a_{nj}h_L(x_n).$$

Since $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and $a_{1j} + \cdots + a_{nj} = k$ for each $j = 1, \cdots, n$, by Lemma App. 2 below, there exist $c_1, \cdots, c_n \geq 0$ such that $(c_1, \ldots, c_n) \neq 0$ and $\sum_{j=1}^{k} a_{ij}c_j = kc_i$.

Substituting $x = x_j$ in (App. 1.0), multiplying it by c_j, and taking the sum with respect to $j = 1, \cdots, n$, we get

$$k\left(c_1h_L(x_1) + \cdots + c_nh_L(x_n)\right) \geq (k+\epsilon)\left(c_1h_L(x_1) + \cdots + c_nh_L(x_n)\right) - C\left(\sum_{j=1}^{n} c_j\right).$$

Thus we get $c_1h_L(x_1) + \cdots + c_nh_L(x_n) \leq \frac{C}{\epsilon}\left(\sum_{j=1}^{n} c_j\right)$.

Take x_α for which $h_L(x_\alpha) = \min\{h_L(x_1), \cdots, h_L(x_n)\}$. Since $c_1, \cdots, c_n \geq 0$ and $\sum_{j=1}^{n} c_j > 0$, we have $h_L(x_\alpha) \leq \frac{C}{\epsilon}$. By Northcott's finiteness theorem (Theorem 1.1.2),

$$\left\{ x \in X(\overline{K}) \;\middle|\; [K(x):K] \leq D \text{ and } h_L(x) \leq \frac{C}{\epsilon} \right\}$$

is finite. By the definition of $\mathcal{F}$-periodic points, $x \in C(x_\alpha)$. Thus

$$\left\{ x \in S \;\middle|\; \begin{array}{c} [K(x):K] \leq D \\ x \text{ is } \mathcal{F}\text{-periodic} \end{array} \right\} \subset \left\{ C(y) \;\middle|\; \begin{array}{c} y \in S,\ [K(y):K] \leq D, \\ \text{and } h_L(y) \leq \dfrac{C}{\epsilon} \\ C(y) \text{ is finite} \end{array} \right\}$$

is finite. $\square$

Lemma App. 2. *Let k, n be positive integers.*

1. *Let $A = (a_{ij}) \in M(n, n, \mathbb{R})$ be a real valued (n, n) matrix. Assume that $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} a_{ij} = k$ for every $j = 1, \cdots, n$. Then there exists a non-zero vector $c = {}^t(c_1, \cdots, c_n) \in \mathbb{R}^n$ such that $c_i \geq 0$ for every $i = 1, \cdots, n$ and $Ac = kc$.*
2. *Assume further that $a_{ij} > 0$ for all $1 \leq i, j \leq n$. Then $c \in \mathbb{R}^n$ as above is unique up to positive scalars.*

Proof. We only sketch a proof. Since (1) follows from (2) by taking a suitable limit, we will prove (2). Since $\det(A - kI_n) = 0$, there exists a non-zero vector $c = {}^t(c_1, \cdots, c_n)$ with $(A - kI_n)c = 0$. Without loss of generality, we may assume $c_n = 1$. Define a $(n-1, n-1)$ matrix $B = (b_{ij})$ by $b_{ii} = k - a_{ii}$ $(1 \leq i \leq n-1)$ and $b_{ij} = -a_{ij}$ $(1 \leq i, j \leq n-1, i \neq j)$. Then by the following claim, which can be shown by the induction on the size of matrix, B is invertible and every component of B^{-1} is non-negative. Since ${}^t(c_1, \cdots, c_{n-1}) = B^{-1}\, {}^t(a_{1n}, \cdots, a_{n-1n})$, we have $c_i \geq 0$ for $i = 1, \cdots, n-1$.

Claim App. 2.1. *Let $B' = (b'_{ij})$ be a real valued $(n-1, n-1)$ matrix. Assume that $b'_{ii} > 0$ $(1 \leq i \leq n-1)$, $b'_{ij} \leq 0$ $(1 \leq i, j \leq n-1, i \neq j)$ and $\sum_{i=1}^{n} b'_{ij} > 0$ $(1 \leq i \leq n-1)$. Then B' is invertible and every component of ${B'}^{-1}$ is non-negative.*

□

Example App. 3 ([21])**.** For $a \in \overline{\mathbb{Q}} \setminus \{0\}$ and $b \in \overline{\mathbb{Q}}$, a Hénon map $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ is defined by $\phi(x, y) = (y, y^2 + b + ax)$. The morphism ϕ is an affine automorphism with the inverse $\phi^{-1} : \mathbb{A}^2 \ni (x, y) \mapsto \left(-\frac{1}{a}x^2 - \frac{b}{a} + \frac{1}{a}y, x\right) \in \mathbb{A}^2$. The morphisms ϕ and ϕ^{-1} extend to birational maps of $\mathbb{P}^2$, which we also denote by ϕ and ϕ^{-1} respectively. Silverman has shown in [21], equation (13) that

$$h_{\mathcal{O}_{\mathbb{P}^2(1)}}(\phi(x)) + h_{\mathcal{O}_{\mathbb{P}^2(1)}}(\phi^{-1}(x)) \geq \frac{5}{2} h_{\mathcal{O}_{\mathbb{P}^2(1)}}(x) + O(1)$$

for all $x \in \mathbb{A}^2(\overline{\mathbb{Q}})$. Note that $x \in \mathbb{A}^2(\overline{\mathbb{Q}})$ is ϕ-periodic if and only if x is $\{\phi, \phi^{-1}\}$-periodic. Then, by Silverman's argument, or by Theorem App. 1, we obtain finiteness of the number of ϕ-periodic points in $\mathbb{A}^2(\overline{\mathbb{Q}})$ with bounded degree ([21], Theorem 4.1.)

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