# A CHARACTERIZATION OF SYMMETRIC TUBE DOMAINS BY CONVEXITY OF CAYLEY TRANSFORM IMAGES

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ABSTRACT. In this paper, we show that a homogeneous tube domain is symmetric if and only if its Cayley transform image as well as the dual Cayley transform image of the dual tube domain is convex. In this case, the parameters of these Cayley transforms reduce to specific ones, so that they are essentially the usual Cayley transforms defined in terms of Jordan algebra structure.

## 1. INTRODUCTION

In one complex variable case, the Cayley transform

$$w \mapsto \frac{w-1}{w+1} = 1 - 2(w+1)^{-1} \tag{1.1}$$

maps the right half plane to the open unit disc. This is generalized to the case of symmetric tube domains by introducing a Jordan algebra structure. To define a Cayley transform for such domains, it suffices to interpret (1.1) in the Jordan algebra terminology: 1 is the unit element in the Jordan algebra, and $(w + 1)^{-1}$ is the Jordan algebra inverse of $w + 1$. Then, the image of the symmetric tube domain is the open unit ball with respect to a certain norm (see [3, Chapter X] and §4 of this paper). In particular, the image is a convex set. The situation is further generalized to non-tube symmetric Siegel domains in [5]. The Cayley transform in that paper maps the Harish-Chandra realization of non-compact Hermitian symmetric space to a Siegel domain, and the Harish-Chandra realization is known to be equal to the open unit ball for a certain norm defined in terms of Jordan triple system structure just as in the tube case (see [11, Chapter II] or [6, §4], for example). The purpose of the present paper is to show that this convexity property of Cayley transform images is characteristic of symmetric domains among homogeneous tube domains. The case of non-tube Siegel domains will be treated separately in a future paper.

Cayley transforms for general homogeneous Siegel domains have been introduced by [2], [10], [7] and [9]. These are now included in a single family of Cayley transforms introduced by the second author in [8]. The Cayley transforms in this paper are precisely this family of Cayley transforms specialized to tube domains. For symmetric tube domains, this Cayley transform with a specific parameter essentially coincides with the above-mentioned Cayley transform defined in terms of Jordan algebra structure.

In order to write down a precise formula for our Cayley transforms, we need to fix the notation. Let $\Omega$ be a homogeneous convex cone in a finite-dimensional real vector space $V$. In this paper, we always assume that $\Omega$ is irreducible for simplicity. By [12],

there exists a split solvable subgroup $H$ in the linear automorphism group $G(\Omega)$ of $\Omega$ such that $H$ acts on $\Omega$ simply transitively. We fix any reference point $E \in \Omega$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Since the infinitesimal orbit map $\mathfrak{h} \ni T \mapsto TE \in V$ is a linear isomorphism, we denote by $x \mapsto L_x$ its inverse map. Then the ambient vector space $V$ has an algebra structure by $x \triangle y := L_x y$ with unit element $E$. This (non-associative) algebra is called a *clan* associated with $\Omega$. Let $E_1, \ldots, E_r$ be a complete system of primitive idempotents in $V$. Then $L_{E_1}, \ldots, L_{E_r}$ form a commutative Lie subalgebra $\mathfrak{a}$ of $\mathfrak{h}$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we denote by $\chi_{\mathbf{s}}$ the one-dimensional representation of $A := \exp \mathfrak{a}$ defined by $\chi_{\mathbf{s}}(\exp \sum t_j L_{E_j}) = \exp \sum s_j t_j$ $(t_j \in \mathbb{R})$. This $\chi_{\mathbf{s}}$ can be extended canonically to $H$, and we transfer it to a function $\Delta_{\mathbf{s}}$ on the cone $\Omega$ through the orbit map: $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ for $h \in H$. Suppose that $s_j > 0$ for all $j = 1, \ldots, r$ (we simply write $\mathbf{s} > 0$ in this case), and let $\langle \cdot | \cdot \rangle_{\mathbf{s}}$ be the inner product defined by the admissible linear form $E_{\mathbf{s}}^*$ (see the end of section 2).

Now, for $x \in \Omega$, the *pseudoinverse* $\mathcal{I}_{\mathbf{s}}(x)$ is defined as

$$\langle \mathcal{I}_{\mathbf{s}}(x) | y \rangle_{\mathbf{s}} = -\frac{d}{dt} \log \Delta_{-\mathbf{s}}(x + ty)\Big|_{t=0} \qquad (y \in V).$$

The pseudoinverse map $\mathcal{I}_{\mathbf{s}} : x \mapsto \mathcal{I}_{\mathbf{s}}(x)$ gives a bijection of $\Omega$ onto $\Omega^{\mathbf{s}}$, where $\Omega^{\mathbf{s}}$ stands for the dual cone of $\Omega$ realized in $V$ by means of $\langle \cdot | \cdot \rangle_{\mathbf{s}}$:

$$\Omega^{\mathbf{s}} := \{x \in V; \langle x | y \rangle_{\mathbf{s}} > 0 \text{ for any } y \in \overline{\Omega} \setminus \{0\}\}.$$

If $\Omega$ is a symmetric cone and if $\mathbf{s}$ is a positive number multiple of the parameter $\mathbf{d}$ (see (2.5) for the definition of $\mathbf{d}$), then the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$ is a positive constant multiple of the Jordan algebra inverse map. The pseudoinverse map $\mathcal{I}_{\mathbf{s}}$ extends to a rational map $W \to W$, and we make $\mathcal{I}_{\mathbf{s}}(x)$ serve as a denominator. Thus we define a Cayley transform $\mathcal{C}_{\mathbf{s}}$ ($\mathbf{s} > 0$) for $\Omega + iV$ by

$$\mathcal{C}_{\mathbf{s}}(w) := E - 2\mathcal{I}_{\mathbf{s}}(w + E) \qquad (w \in \Omega + iV).$$

Starting with the dual cone $\Omega^{\mathbf{s}}$, we get a similar map $\mathcal{I}_{\mathbf{s}}^*$. We know that $\mathcal{I}_{\mathbf{s}}^*$ extends to a rational map $W \to W$ and that $\mathcal{I}_{\mathbf{s}}^* = \mathcal{I}_{\mathbf{s}}^{-1}$. We also need the dual Cayley transforms $\mathcal{C}_{\mathbf{s}}^*$ ($\mathbf{s} > 0$) for $\Omega^{\mathbf{s}} + iV$ defined by

$$\mathcal{C}_{\mathbf{s}}^*(w) := E - 2\mathcal{I}_{\mathbf{s}}^*(w + E) \qquad (w \in \Omega^{\mathbf{s}} + iV).$$

By [8, Theorem 4.20] the images $\mathcal{C}_{\mathbf{s}}(\Omega + iV)$ and $\mathcal{C}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ are both bounded domains.

We are now ready to state our main theorem.

**Theorem 1.1.** Let $\Omega + iV$ be an irreducible homogeneous tube domain and suppose that $\mathbf{s} > 0$. Then the following are equivalent:

(A) Both $\mathcal{C}_{\mathbf{s}}(\Omega + iV)$ and $\mathcal{C}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ are convex.
(B) The parameter $\mathbf{s}$ is a positive number multiple of $\mathbf{d}$, and $\Omega + iV$ is symmetric.

We remark that (A) of Theorem 1.1 implies that $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ and $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ are both convex (see the discussion made just after Lemma 5.3). This means that Theorem 1.1 gives another proof to our previous result [4, Theorem 1.2] though not easier at all.

We organize this paper as follows. In Section 2, we summarize basic facts about clans associated with homogeneous convex cones. Section 3 is the introduction of the pseudoinverse maps and the Cayley transforms. We prove (B) $\Rightarrow$ (A) of Theorem 1.1 in Section 4. In Section 5, proof of (A) $\Rightarrow$ (B) is given. Our way of proof is parallel to that of [4].

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## 2. Preliminaries

We summarize here some of basic properties of clan. Our reference is Vinberg's classical paper [12] (see also our previous paper [4]). Let $V$ be a finite dimensional vector space over $\mathbb{R}$. An open convex cone $\Omega \subset V$ is called *a homogeneous convex cone* if the following conditions are satisfied:

(i) $\Omega$ is *regular*. In other words, $\Omega$ does not contain any straight line (not necessarily passing through the origin).

(ii) The linear automorphism group $G(\Omega)$ of $\Omega$ defined by

$$G(\Omega) := \{g \in GL(V) \; ; \; g(\Omega) = \Omega\}$$

acts transitively on $\Omega$.

A homogeneous convex cone $\Omega$ is said to be *irreducible* if it does not split into a direct sum of non-empty homogeneous convex cones.

Let $\Omega \subset V$ be an irreducible homogeneous convex cone. By [12, Theorem 1], there exists a split solvable subgroup $H$ of $G(\Omega)$ acting simply transitively on $\Omega$. Fix any point $E \in \Omega$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Since the orbit map $H \ni h \mapsto hE \in \Omega$ is a diffeomorphism, differentiation at the unit element of $H$ gives a linear isomorphism $\mathfrak{h} \ni T \mapsto TE \in V$. The inverse map is denoted as $V \ni x \mapsto L_x \in \mathfrak{h}$. We introduce a multiplication $\triangle$ on $V$ by the following formula:

$$x \triangle y := L_x y \qquad (x, y \in V).$$

Then we have

$$[L_x, L_y] = L_{x \triangle y - y \triangle x}, \tag{2.1}$$

$$\operatorname{Tr} L_{x \triangle x} > 0 \text{ for any non-zero } x \in V, \tag{2.2}$$

$$\text{the operator } L_x \ (x \in V) \text{ has only real eigenvalues.} \tag{2.3}$$

Thus, having the properties $(2.1) \sim (2.3)$, the vector space $V$ with the multiplication $\triangle$ becomes a *clan* after Vinberg [12]. We know that $E$ is the unit element of $V$. Vinberg's theory [12] tells us that there is a one-to-one correspondence between the set of isomorphic classes of homogeneous convex cones and the set of isomorphic classes of clans with unit element.

Now let $V$ be a clan with unit element $E$, and $\Omega$ the homogeneous convex cone associated to the clan $V$. Then there exist positive integer $r$ and idempotents

$E_1, \ldots, E_r$ such that

$$V = \sum_{i=1}^{r} \mathbb{R}E_i \oplus \sum_{k>j} V_{kj}, \qquad E = E_1 + \cdots + E_r,$$

where we put for $1 \leq j < k \leq r$,

$$V_{kj} := \left\{ x \in V \; ; \; \forall c = \sum \lambda_i E_i, \; c \triangle x = \tfrac{1}{2}(\lambda_j + \lambda_k)x, \; x \triangle c = \lambda_j x \right\}.$$

The integer $r$ is called the *rank* of $V$. Setting $V_{kk} := \mathbb{R}E_k$ for $k = 1, \ldots, r$, we have the following multiplication rule:

$$\begin{aligned} V_{lk} \triangle V_{kj} &\subset V_{lj}, \\ \text{if } k \neq i, j, \text{ then } V_{lk} \triangle V_{ij} &= 0, \\ V_{lk} \triangle V_{mk} \subset V_{lm} \text{ or } V_{ml} \text{ according to } l \geq m \text{ or } m \geq l. \end{aligned} \tag{2.4}$$

Let us define linear forms $E_i^*$ $(i = 1, \ldots, r)$ by

$$\left\langle \sum_{j=1}^{r} x_j E_j + \sum_{k>j} X_{kj}, E_i^* \right\rangle := x_i \qquad (x_j \in \mathbb{R}, \; X_{kj} \in V_{kj}).$$

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we set $E_{\mathbf{s}}^* := \sum s_i E_i^*$. If $s_i > 0$ for $i = 1, \ldots, r$, we say that $\mathbf{s}$ is *positive*, and we write $\mathbf{s} > 0$. A linear form $f$ on $V$ is said to be *admissible* if the bilinear form $\langle x | y \rangle_f := \langle x \triangle y, f \rangle$ defines a positive definite inner product on $V$. We know by [4, Proposition 2.1] that the linear forms $E_{\mathbf{s}}^*$ $(\mathbf{s} > 0)$ represent all the admissible linear forms on $V$. An example of admissible linear forms is given by $\langle x, g \rangle := \operatorname{Tr} L_x$ $(x \in V)$ as shown by (2.1) and (2.2). Putting

$$n_{kj} := \dim V_{kj} \qquad (k > j),$$
$$d_i := 1 + \tfrac{1}{2} \sum_{\alpha > i} n_{\alpha i} + \tfrac{1}{2} \sum_{\beta < i} n_{i\beta} \qquad (i = 1, \ldots, r), \tag{2.5}$$

and $\mathbf{d} := (d_1, \ldots, d_r)$, we see that $\langle x | y \rangle_g = \langle x \triangle y, E_{\mathbf{d}}^* \rangle$. In what follows, we put $\langle \cdot | \cdot \rangle_{\mathbf{s}} := \langle \cdot | \cdot \rangle_{E_{\mathbf{s}}^*}$ for simplicity.

## 3. Family of Cayley transforms

Keeping the notation established in the previous section, we introduce, in this section, Cayley transforms for the tube domain $\Omega + iV$ originally defined in [8] for general homogeneous Siegel domains. To do so, we need pseudoinverse maps which serve as the "denominator". We refer the readers to [4, Section 3] or [8] for details.

Put $\mathfrak{a} := \sum_{j=1}^{r} \mathbb{R}L_{E_j}$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{h}$ such that $\operatorname{ad} \mathfrak{a}$ is a commutative family of diagonalizable operators on $\mathfrak{h}$. In fact, setting $\mathfrak{n}_{kj} := \{L_x \; ; \; x \in V_{kj}\}$ $(k > j)$, we see that the spaces $\mathfrak{n}_{kj}$ are simultaneous eigenspaces of $\operatorname{ad} \mathfrak{a}$. Put $\mathfrak{n} := \sum_{k>j} \mathfrak{n}_{kj}$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{h}$ and $\mathfrak{h}$ is written as a semidirect product $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}$.

Let $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. We have $H = A \ltimes N$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we define a one-dimensional representation of $A$ by

$$\chi_{\mathbf{s}}\big(\exp\big(\sum t_j L_{E_j}\big)\big) := \exp\big(\sum s_j t_j\big).$$

We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of $H$ by defining $\chi_{\mathbf{s}}|_N \equiv 1$. Recalling that $H$ acts on $\Omega$ simply transitively, we introduce functions $\Delta_{\mathbf{s}}$ on $\Omega$ by

$$\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h) \qquad (h \in H).$$

Let $\mathbf{s} > 0$. The *pseudoinverse* $\mathcal{I}_{\mathbf{s}}(x)$ of $x \in \Omega$ is given by

$$\langle v | \mathcal{I}_{\mathbf{s}}(x) \rangle_{\mathbf{s}} = -\frac{d}{dt} \log \Delta_{-\mathbf{s}}(x + tv)\Big|_{t=0} \qquad (v \in V).$$

We call $\mathcal{I}_{\mathbf{s}} : \Omega \to V$ the pseudoinverse map. Let $\Omega^{\mathbf{s}}$ denote the dual cone of $\Omega$ realized in $V$ by means of $\langle \cdot | \cdot \rangle_{\mathbf{s}}$:

$$\Omega^{\mathbf{s}} := \{ x \in V ; \ \langle x | y \rangle_{\mathbf{s}} > 0, \text{ for } \forall y \in \overline{\Omega} \setminus \{0\} \}.$$

By [8, Proposition 3.12], $\mathcal{I}_{\mathbf{s}}$ is a bijection of $\Omega$ onto $\Omega^{\mathbf{s}}$. $H$ acts also on $V$ by the coadjoint action: $x \mapsto {}^{\mathbf{s}}h^{-1}x$, where ${}^{\mathbf{s}}h$ stands for the adjoint operator of $h$ relative to $\langle \cdot | \cdot \rangle_{\mathbf{s}}$. We see that $\mathcal{I}_{\mathbf{s}}$ is $H$-equivariant: $\mathcal{I}_{\mathbf{s}}(hx) = {}^{\mathbf{s}}h^{-1}\mathcal{I}_{\mathbf{s}}(x)$ $(h \in H)$, and we have $\mathcal{I}_{\mathbf{s}}(E) = E$. Moreover, the action of $H$ on $\Omega^{\mathbf{s}}$ is also simply transitive.

Put $W := V_{\mathbb{C}}$, the complexification of $V$. We extend both the multiplication $\triangle$ and the inner product $\langle \cdot | \cdot \rangle_{\mathbf{s}}$ to $W$ by complex bilinearity. Put $W_{kj} := (V_{kj})_{\mathbb{C}}$ $(k > j)$. Then the properties similar to (2.4) hold:

$$W_{lk} \triangle W_{kj} \subset W_{lj},$$
$$\text{if } k \neq i, j, \text{ then } W_{lk} \triangle W_{ij} = 0, \qquad (3.1)$$
$$W_{lk} \triangle W_{mk} \subset W_{lm} \text{ or } W_{ml} \text{ according to } l \geq m \text{ or } m \geq l.$$

We know by [8, Lemma 3.17] that the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$ can be continued analytically to a rational map $W \to W$. Let $H_{\mathbb{C}}$ be the complexification of the Lie group $H$. By analytic continuation, we see easily that $\mathcal{I}_{\mathbf{s}}$ is $H_{\mathbb{C}}$-equivariant: $\mathcal{I}_{\mathbf{s}}(hx) = {}^{\mathbf{s}}h^{-1}\mathcal{I}_{\mathbf{s}}(x)$ $(h \in H_{\mathbb{C}})$. Let $w \mapsto \overline{w}$ be the conjugation in $W$ relative to the real form $V$. Clearly we have $\mathcal{I}_{\mathbf{s}}(\overline{w}) = \overline{\mathcal{I}_{\mathbf{s}}(w)}$.

Starting with the dual cone $\Omega^{\mathbf{s}}$, we get a similar map $\mathcal{I}_{\mathbf{s}}^* : \Omega^{\mathbf{s}} \to V$, called the dual pseudoinverse map. $\mathcal{I}_{\mathbf{s}}^*$ gives a bijection of $\Omega^{\mathbf{s}}$ onto $\Omega$, and is extended to a rational map $W \to W$ which is $H_{\mathbb{C}}$-equivariant: $\mathcal{I}_{\mathbf{s}}^*({}^{\mathbf{s}}h^{-1}x) = h\mathcal{I}_{\mathbf{s}}^*(x)$ $(h \in H_{\mathbb{C}})$. We have $\mathcal{I}_{\mathbf{s}}^*(E) = E$. Furthermore, it turns out that $\mathcal{I}_{\mathbf{s}}$ and $\mathcal{I}_{\mathbf{s}}^*$ are inverse to each other by [8, Proposition 3.16]. Thus $\mathcal{I}_{\mathbf{s}}$ is a birational map with $\mathcal{I}_{\mathbf{s}}^{-1} = \mathcal{I}_{\mathbf{s}}^*$. We remark that $\mathcal{I}_{\mathbf{s}}$ is holomorphic on $\Omega + iV$, and $\mathcal{I}_{\mathbf{s}}^*$ on $\Omega^{\mathbf{s}} + iV$. Finally we know that $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is contained in the holomorphic domain of $\mathcal{I}_{\mathbf{s}}^*$, and $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ in the holomorphic domain of $\mathcal{I}_{\mathbf{s}}$.

Once we have the pseudoinverse maps at hand, our Cayley transforms are defined as in Introduction. For $\mathbf{s} > 0$, the Cayley transforms $\mathcal{C}_{\mathbf{s}}$ are

$$\mathcal{C}_{\mathbf{s}}(w) := E - 2\mathcal{I}_{\mathbf{s}}(w + E) \qquad (w \in \Omega + iV).$$

It should be noted here that if $w \in \Omega + iV$, then $w + E \in \Omega + iV$, so that $\mathcal{I}_{\mathbf{s}}(w + E)$ is well-defined. Similarly we define dual Cayley transforms $\mathcal{C}_{\mathbf{s}}^*$ by

$$\mathcal{C}_{\mathbf{s}}^*(w) := E - 2\mathcal{I}_{\mathbf{s}}^*(w + E) \qquad (w \in \Omega^{\mathbf{s}} + iV).$$

We emphasize that unlike the pair $\mathcal{I}_{\mathbf{s}}$ and $\mathcal{I}_{\mathbf{s}}^*$, the pair $\mathcal{C}_{\mathbf{s}}$ and $\mathcal{C}_{\mathbf{s}}^*$ are no longer inverse to each other.

## 4. Proof (B) $\Rightarrow$ (A) of the main theorem

We assume that (B) of Theorem 1.1 holds. Then $\Omega$ is a symmetric cone which is irreducible. Moreover Theorem 1.2 in [4] tells us that $\Omega = \Omega^{\mathbf{s}}$. Let $\varphi$ be the characteristic function of $\Omega$:

$$\varphi(x) := \int_{\Omega} e^{-\langle x | y \rangle_{\mathbf{s}}} \, dy \qquad (x \in \Omega).$$

Let $\Omega \ni x \mapsto x^* \in V$ be Vinberg's $*$-map defined in a usual way by

$$\langle x^* | y \rangle_{\mathbf{s}} = -\frac{d}{dt} \log \varphi(x + ty) \Big|_{t=0} \qquad (y \in V).$$

By [3, Proposition I.3.5], the $*$-map has a unique fixed point $e$. Then $V$ has a Euclidean Jordan algebra structure with unit element $e$, so that $W = V_{\mathbb{C}}$ is a complex semisimple Jordan algebra. By assumption, we have $\mathbf{s} = p\mathbf{d}$ for some $p > 0$. In this situation, Lemma 5.2 in [4] gives $x^* = x^{-1}$ for invertible $x \in V$, where $x^{-1}$ is the Jordan algebra inverse of $x$. Moreover we know by [4, Subsection 5.2] that $\mathcal{I}_{\mathbf{s}}(x) = px^{-1}$. Hence $pE^{-1} = \mathcal{I}_{\mathbf{s}}(E) = E$, so that $(p^{-1/2}E)^{-1} = p^{-1/2}E$. This together with $x^* = x^{-1}$ gives $(p^{-1/2}E)^* = p^{-1/2}E$. Since the fixed point of the $*$-map is unique, we get

$$e = p^{-1/2}E. \tag{4.1}$$

Let us denote by $\mathcal{C}$ the Cayley transform defined in terms of the Jordan algebra structure:

$$\mathcal{C}(x) := (x - e)(x + e)^{-1} = e - 2(x + e)^{-1}.$$

Here we need to introduce *the spectral norm* on $W$ to describe the image $\mathcal{C}(\Omega + iV)$. We denote by $L(x)$ the multiplication by $x$ in the Jordan algebra $W$. For $x, y \in W$, let $x \square y$ denote the linear operator on $W$ defined by

$$x \square y := L(xy) + [L(x), L(y)].$$

and set $|w| := \|w \square \overline{w}\|^{1/2}$ for $w \in W$, the square root of the operator norm of the operator $w \square \overline{w}$. By [3, Proposition X.4.1], $|\cdot|$ is a norm on $W$, called *the spectral norm*. Let $B$ be the open unit ball for the spectral norm $|\cdot|$. By [3, Theorem X.4.3] we see easily that $\mathcal{C}(\Omega + iV) = B$, which shows, in particular, that $\mathcal{C}(\Omega + iV)$ is convex. We note here that the tube domains used in [3, Chapter X] are upper half planes $V + i\Omega$, while ours are right half planes $\Omega + iV$.

On the other hand, (4.1) together with $\mathcal{I}_{\mathbf{s}}(x + E) = p(x + E)^{-1}$ gives

$$\mathcal{C}_{\mathbf{s}}(x) = E - 2p(x + E)^{-1} = p^{1/2}e - 2p^{1/2}(p^{-1/2}x + e)^{-1} = p^{1/2}\mathcal{C}(p^{-1/2}x),$$

so that $\mathcal{C}_{\mathbf{s}}(\Omega + iV) = p^{1/2}B$. Therefore, $\mathcal{C}_{\mathbf{s}}(\Omega + iV)$ is convex.

Since $\mathcal{I}_{\mathbf{s}}^*(x) = \mathcal{I}_{\mathbf{s}}^{-1}(x) = px^{-1} = \mathcal{I}_{\mathbf{s}}(x)$, we also have $\mathcal{C}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV) = p^{1/2}B$, so that $\mathcal{C}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ is convex. Now the proof of (B) $\Rightarrow$ (A) is complete. $\square$

## 5. PROOF OF (A) $\Rightarrow$ (B)

Throughout this section, we assume that the integers $j, k, l$ always satisfy $j < k < l$, even though we do not mention it explicitly. In addition, we let $w_{kj} \in W_{kj}$, $w_{lj} \in W_{lj}$ and $w_{lk} \in W_{lk}$ without any explicit references. In order to simplify the description, we write $\langle \cdot | \cdot \rangle$ instead of $\langle \cdot | \cdot \rangle_{\mathbf{s}}$ and put $\nu[w] := \langle w | w \rangle$ for $w \in W$. Note that $\nu[iw] = -\nu[w]$.

We collect here some formulas needed in this section which hold without any restrictions on the clan. Given $w_{lj}, w_{kj}$, we set

$$S_{lk} := \tfrac{1}{2}(w_{lj} \triangle w_{kj} + w_{kj} \triangle w_{lj}). \tag{5.1}$$

We have $S_{lk} \in W_{lk}$ by (3.1). The following two propositions will be used to compute pseudoinverses.

**Proposition 5.1** ([4, Proposition 4.2]). Let $t_j, t_k, t_l \in \mathbb{R}$. Then one has

$$\begin{aligned}
\exp\left(L_{w_{lj}} + L_{w_{kj}}\right) &\exp\left(L_{w_{lk}}\right) \exp\left(t_j H_j + t_k H_k + t_l H_l\right) E \\
&= \sum_{m \neq j,k,l} E_m + e^{t_j} E_j + \left(e^{t_k} + (2s_k)^{-1} e^{t_j} \nu[w_{kj}]\right) E_k \\
&\quad + \left(e^{t_l} + (2s_l)^{-1} e^{t_k} \nu[w_{lk}] + (2s_l)^{-1} e^{t_j} \nu[w_{lj}]\right) E_l \\
&\quad + e^{t_j} w_{lj} + e^{t_j} w_{kj} + \left(e^{t_j} S_{lk} + e^{t_k} w_{lk}\right).
\end{aligned}$$

**Proposition 5.2** ([4, Proposition 4.6]). One has

$$\begin{aligned}
{}^{\mathbf{s}}\left(\exp(L_{w_{lj}} + L_{w_{kj}}) \exp(L_{w_{lk}}) \exp\left(t_j H_j + t_k H_k + t_l H_l\right)\right)^{-1} E \\
= \sum_{m \neq j,k,l} E_m + \left(e^{-t_j} + (2s_j)^{-1}\left(e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]\right) \nu[w_{kj}]\right. \\
\left. + (2s_j)^{-1} e^{-t_l} \nu[w_{lj}] - s_j^{-1} e^{-t_l} \langle S_{lk} | w_{lk} \rangle\right) E_j \\
+ \left(e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]\right) E_k + e^{-t_l} E_l \\
+ \left(e^{-t_l}\, {}^{\mathbf{s}}L_{w_{lj}} w_{lk} - (e^{-t_k} + (2s_k)^{-1} e^{-t_l} \nu[w_{lk}]) w_{kj}\right) \\
+ e^{-t_l}\left({}^{\mathbf{s}}L_{w_{kj}} w_{lk} - w_{lj}\right) - e^{-t_l} w_{lk}.
\end{aligned}$$

Let $v_{lk} \in V_{lk}$ and $v_{kj} \in V_{kj}$. Then, by [4, Lemma 4.7], we have the following norm equality:

$$\|v_{lk} \triangle v_{kj}\|^2 = (2s_k)^{-1} \|v_{lk}\|^2 \|v_{kj}\|^2. \tag{5.2}$$

This leads us to the following lemma (see [4, Lemma 4.8]).

**Lemma 5.3.** (1) If $n_{kj} \neq 0$, then one has $n_{lj} \geq n_{lk}$.
(2) If $n_{lk} \neq 0$, then one has $n_{lj} \geq n_{kj}$.

Now, let us begin the proof of (A) $\Rightarrow$ (B). We assume that (A) of Theorem 1.1 holds. First of all we show that $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ and $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ are convex. Let $\delta > 0$. Since $\Omega$ is a cone, we have

$$\mathcal{C}_{\mathbf{s}}(\Omega + iV) = E - 2\mathcal{I}_{\mathbf{s}}(\Omega + E + iV) = E - 2\delta\mathcal{I}_{\mathbf{s}}(\Omega + \delta E + iV).$$

This means that $\mathcal{I}_{\mathbf{s}}(\Omega + \delta E + iV)$ is convex for any $\delta > 0$. Let us take two points $\mathcal{I}_{\mathbf{s}}(z_1), \mathcal{I}_{\mathbf{s}}(z_2)$ in $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ with $z_1, z_2$ in $\Omega + iV$, and denote by $\ell$ the line segment with endpoints $\mathcal{I}_{\mathbf{s}}(z_1)$ and $\mathcal{I}_{\mathbf{s}}(z_2)$. Since $\Omega$ is open, we have $z_1, z_2 \in \Omega + \delta E + iV$ for sufficiently small $\delta > 0$. The convexity of $\mathcal{I}_{\mathbf{s}}(\Omega + \delta E + iV)$ just shown implies $\ell \subset \mathcal{I}_{\mathbf{s}}(\Omega + \delta E + iV) \subset \mathcal{I}_{\mathbf{s}}(\Omega + iV)$, so that $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is convex. Similarly, $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$ is convex, too.

5.1. **First step.** The purpose of this subsection is to show that $s_1 = \cdots = s_r$.

**Lemma 5.4.** If $n_{kj} \neq 0$, then $s_k \leq s_j$.

*Proof.* Take any non-zero $v_{kj} \in V_{kj}$ and set

$$p := \log(1 + (2s_k)^{-1} \|v_{kj}\|^2). \tag{5.3}$$

We consider two points $z_1 := \mathcal{I}_{\mathbf{s}}(E + iv_{kj})$ and $z_2 := \mathcal{I}_{\mathbf{s}}(E - iv_{kj}) = \overline{z}_1$ in $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$. We know by the proof of [4, Lemma 7.1] that (note that $p$ is written as $t_k$ there)

$$z_1 = \sum_{m \neq j,k,l} E_m + \left(1 - (2s_j)^{-1} e^{-p} \|v_{kj}\|^2\right) E_j + e^{-p} E_k + E_l - ie^{-p} v_{kj}.$$

Let $w := (1 - u)z_1 + uz_2$, where $0 \leq u \leq 1$, and we require that $u$ satisfies

$$1 - 2s_j^{-1} e^{-p} \|v_{kj}\|^2 u(1 - u) > 0. \tag{5.4}$$

Put $\delta := 4u(1 - u)$ for simplicity. Then $0 \leq \delta \leq 1$, and using (5.3), we see that (5.4) is equivalent to

$$\delta < \frac{s_j}{s_k} \frac{2s_k + \|v_{kj}\|^2}{\|v_{kj}\|^2}. \tag{5.5}$$

By definition we have

$$w = \sum_{m \neq j,k,l} E_m + \left(1 - (2s_j)^{-1} e^{-p} \|v_{kj}\|^2\right) E_j + e^{-p} E_k + E_l + i(2u - 1)e^{-p} v_{kj}.$$

In Proposition 5.2, we put

$$\begin{aligned} t_j = -\log\left(1 - (2s_j)^{-1}\delta e^{-p} \|v_{kj}\|^2\right), \quad t_k = p, \quad t_l = 0, \\ w_{kj} = -i(2u - 1)v_{kj}, \quad w_{lk} = 0, \quad w_{lj} = 0, \end{aligned} \tag{5.6}$$

and $\eta := \exp\left(L_{w_{kj}}\right) \exp\left(t_j H_j + t_k H_k\right)$. The condition (5.4) guarantees that $t_j$ is a real number. Then the formula in Proposition 5.2 becomes ${}^{\mathbf{s}}\eta^{-1}E = w$. Since $\mathcal{I}_{\mathbf{s}}^*(w) = \eta\mathcal{I}_{\mathbf{s}}^*(E) = \eta E$, we see by Proposition 5.1 that

$$\mathrm{Re}\left(\mathcal{I}_{\mathbf{s}}^*(w)\right) = \sum_{m \neq j,k,l} E_m + e^{t_j} E_j + \left(e^p - (2s_k)^{-1}(1 - \delta)e^{t_j} \|v_{kj}\|^2\right) E_k + E_l. \tag{5.7}$$

Since $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is convex, we must have $w \in \mathcal{I}_{\mathbf{s}}(\Omega + iV)$ for any $u$ with $0 \leq u \leq 1$. Since $\mathcal{I}_{\mathbf{s}}^{-1} = \mathcal{I}_{\mathbf{s}}^{*}$, it turns out that $\operatorname{Re}\mathcal{I}_{\mathbf{s}}^{*}(w)$ must be in $\Omega$. Therefore, under the condition (5.5), the coefficient of $E_k$ in (5.7) should be positive, that is,

$$e^p - (2s_k)^{-1}(1 - \delta)e^{t_j}\|v_{kj}\|^2 > 0.$$

By a computation using (5.3) and (5.6), this becomes

$$2s_j s_k > \delta \cdot (s_k - s_j)\|v_{kj}\|^2. \tag{5.8}$$

Let us assume, contrary to the conclusion, that $s_k > s_j$. Then, if $\|v_{kj}\|$ is sufficiently large, we have

$$\frac{s_j}{s_k}\frac{2s_k + \|v_{kj}\|^2}{\|v_{kj}\|^2} < 1.$$

Thus if $\delta > 0$ satisfies (5.5), we have $0 < \delta < 1$ automatically, so that we can find $u$ with $0 < u < 1$ satisfying (5.4). The requirement that (5.8) is always true for any $\delta > 0$ satisfying (5.5) forces

$$\frac{s_j}{s_k}\frac{2s_k + \|v_{kj}\|^2}{\|v_{kj}\|^2} \leq \frac{2s_j s_k}{(s_k - s_j)\|v_{kj}\|^2}.$$

But, since $s_j > 0$, this is impossible for large $\|v_{kj}\|$ as can be seen by the limiting procedure $\|v_{kj}\| \to \infty$. This contradiction shows $s_k \leq s_j$. □

**Lemma 5.5.** If $n_{kj} \neq 0$, then $s_k \geq s_j$.

*Proof.* Though the argument is completely parallel to the previous lemma, we write down the proof for completeness. Take any non-zero $v_{kj} \in V_{kj}$ and put

$$p := -\log(1 + (2s_j)^{-1}\|v_{kj}\|^2). \tag{5.9}$$

Consider $z_1 := \mathcal{I}_{\mathbf{s}}^{*}(E + iv_{kj})$ and $z_2 := \mathcal{I}_{\mathbf{s}}^{*}(E - iv_{kj}) = \overline{z}_1$, which are in $\mathcal{I}_{\mathbf{s}}^{*}(\Omega^{\mathbf{s}} + iV)$. By the proof of [4, Lemma 7.2] we have (note that $p$ is written as $t_j$ there)

$$z_1 = \sum_{m \neq j,k,l} E_m + e^p E_j + \left(1 - (2s_k)^{-1}e^p\|v_{kj}\|^2\right)E_k + E_l - ie^p v_{kj}.$$

Let $w := (1 - u)z_1 + uz_2$, where $0 \leq u \leq 1$, and we require that $u$ satisfies

$$1 - 2s_k^{-1}e^p\|v_{kj}\|^2 u(1 - u) > 0. \tag{5.10}$$

We put $\delta := 4u(1 - u)$ as before. Then $0 \leq \delta \leq 1$. By using (5.9), the requirement (5.10) is seen to be equivalent to

$$\delta < \frac{s_k}{s_j}\frac{2s_j + \|v_{kj}\|^2}{\|v_{kj}\|^2}. \tag{5.11}$$

Now we have by definition

$$w = \sum_{m \neq j,k,l} E_m + e^p E_j + \left(1 - (2s_k)^{-1}e^p\|v_{kj}\|^2\right)E_k + E_l + i(2u - 1)e^p v_{kj}.$$

In Proposition 5.1, we put

$$t_j = p, \quad t_k = \log\left(1 - (2s_k)^{-1}\delta e^p \|v_{kj}\|^2\right), \quad t_l = 0,$$
$$w_{kj} = i(2u-1)v_{kj}, \quad w_{lj} = w_{lk} = 0, \tag{5.12}$$

and $\eta := \exp\left(L_{w_{kj}}\right)\exp\left(t_j H_j + t_k H_k\right)$. The inequality (5.10) assures that $t_k$ is a real number. Then the formula in Proposition 5.1 becomes $\eta E = w$. Since $\mathcal{I}_{\mathbf{s}}(w) = {}^{\mathbf{s}}\eta^{-1}\mathcal{I}_{\mathbf{s}}(E) = {}^{\mathbf{s}}\eta^{-1}E$, we see by Proposition 5.2 that

$$\operatorname{Re}\mathcal{I}_{\mathbf{s}}(w) = \sum_{m \neq j,k,l} E_m + \left(e^{-p} - (2s_j)^{-1}(1-\delta)e^{-t_k}\|v_{kj}\|^2\right)E_j + e^{-t_k}E_k + E_l. \tag{5.13}$$

As in the previous case, we must have $w \in \mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$. Hence $\operatorname{Re}\mathcal{I}_{\mathbf{s}}(w)$ should be an element of $\Omega^{\mathbf{s}}$. Therefore, under the condition (5.11), the coefficient of $E_j$ in (5.13) must be positive, that is,

$$e^{-p} - (2s_j)^{-1}(1-\delta)e^{-t_k}\|v_{kj}\|^2 > 0.$$

Rewriting this by using (5.9) and (5.12), we arrive at

$$2s_j s_k > \delta \cdot (s_j - s_k)\|v_{kj}\|^2. \tag{5.14}$$

Now we assume $s_k < s_j$ contrary to the conclusion of the lemma. Then, if $\|v_{kj}\|$ is sufficiently large, we have

$$\frac{s_k}{s_j}\frac{2s_j + \|v_{kj}\|^2}{\|v_{kj}\|^2} < 1.$$

Thus, if $\delta > 0$ satisfies (5.11), then we have $0 < \delta < 1$ automatically, so that we can find $u$ with $0 < u < 1$ satisfying (5.10). With the $\delta$ chosen in this manner, the condition (5.14) compels

$$\frac{s_k}{s_j}\frac{2s_j + \|v_{kj}\|^2}{\|v_{kj}\|^2} \leq \frac{2s_j s_k}{(s_j - s_k)\|v_{kj}\|^2},$$

which is absurd for large $\|v_{kj}\|$. Hence we get $s_k \geq s_j$.  $\square$

Lemmas 5.4 and 5.5 give

**Proposition 5.6.** If $n_{kj} \neq 0$, then $s_k = s_j$.

Here we need the following proposition due to Asano:

**Proposition 5.7** ([1, Theorem 4]). The homogeneous convex cone $\Omega$ is irreducible if and only if for each pair $(j, k)$ of integers with $1 \leq j < k \leq r$, there exists a series $j_0, \ldots, j_m$ of distinct positive integers such that $j_0 = k, j_m = j$ and $n_{j_{\lambda-1}j_\lambda} \neq 0$ for $\lambda = 1, \ldots, m$, where if $j_{\lambda-1} < j_\lambda$, then one puts $n_{j_{\lambda-1}j_\lambda} := n_{j_\lambda j_{\lambda-1}}$.

Therefore we get by Propositions 5.6 and 5.7

**Proposition 5.8.** The numbers $s_m$ $(m = 1, \ldots, r)$ are independent of $m$.

5.2. **Second step.** We shall show here that if $n_{lk} \neq 0$, then $n_{kj} \geq n_{lj}$. In view of Proposition 5.8, we put $s := s_m$ from now on.

**Lemma 5.9.** If $n_{lk} \neq 0$, then $n_{kj} \geq n_{lj}$.

*Proof.* If $n_{lj} = 0$, then the conclusion of the lemma is obviously true. Hence we assume $n_{lj} \neq 0$. Take any non-zero $v_{lj} \in V_{lj}$, $v_{lk} \in V_{lk}$. We put $z_{kj} := -\,{}^{\mathbf{s}}L_{v_{lj}} v_{lk}$. By [4, Lemma 4.4] we have $z_{kj} \in V_{kj}$, because $v_{lj}, v_{lk}$ are real vectors. We set

$$
\begin{aligned}
p &:= -\log\left(1 + (2s)^{-1}\left\|z_{kj}\right\|^2 + (2s)^{-1}\left\|v_{lj}\right\|^2\right), \\
q &:= -\log\left(1 + (2s)^{-1}\left\|v_{lk}\right\|^2\right).
\end{aligned}
\tag{5.15}
$$

We consider two points

$$
z_1 := \mathcal{I}_{\mathbf{s}}^*\left(E + i\left(v_{lk} - {}^{\mathbf{s}}L_{z_{kj}} v_{lk} + v_{lj}\right)\right), \quad z_2 := \overline{z}_1
$$

in $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}} + iV)$. It is shown in the proof of [4, Lemma 7.8] that (note that $p, q, z_{kj}$ are written as $t_j, t_k, w_{kj}$ respectively there)

$$
\begin{aligned}
z_1 = \sum_{m \neq j,k,l} E_m &+ e^p E_j + \left((2s)^{-1} e^p \|z_{kj}\|^2 + e^q\right) E_k - (f-1)E_l + e^p z_{kj} \\
&- i(e^p v_{lj} + e^q v_{lk} + e^p T_{lk}),
\end{aligned}
$$

where, for simplicity, we have put

$$
f := (2s)^{-1}(e^p\|v_{lj}\|^2 + e^q\|v_{lk}\|^2), \qquad T_{lk} := \tfrac{1}{2}(v_{lj} \triangle z_{kj} + z_{kj} \triangle v_{lj}).
$$

Let $w := (1-u)z_1 + u z_2$ for $0 \leq u \leq 1$, where we require that $u$ satisfies the inequality $1 - 4u(1-u)f > 0$. Put $\delta := 4u(1-u)$ as before. Then we have $0 \leq \delta \leq 1$, and the requirement is equivalent to

$$
\delta < f^{-1}.
\tag{5.16}
$$

By definition we have

$$
\begin{aligned}
w = \sum_{m \neq j,k,l} E_m &+ e^p E_j + \left((2s)^{-1} e^p \|z_{kj}\|^2 + e^q\right) E_k - (f-1)E_l + e^p z_{kj} \\
&+ i(2u-1)(e^p v_{lj} + e^q v_{lk} + e^p T_{lk}).
\end{aligned}
$$

In Proposition 5.1, we set

$$
t_j = p, \quad t_k = q, \quad t_l = \log(1 - \delta f),
$$

$$
w_{kj} = z_{kj}, \quad w_{lj} = i(2u-1)v_{lj}, \quad w_{lk} = i(2u-1)v_{lk},
$$

and put $\eta := \exp\left(L_{w_{lj}} + L_{w_{kj}}\right)\exp\left(L_{w_{lk}}\right)\exp\left(t_j H_j + t_k H_k + t_l H_l\right)$. The requirement (5.16) assures that $t_l$ is a real number. Then the formula in Proposition 5.1 becomes $\eta E = w$ (note $S_{lk} = i(2u-1)T_{lk}$ in the current situation). Before proceeding, we show

**Claim 1.** $\langle T_{lk} | v_{lk} \rangle = -\|z_{kj}\|^2$.

Indeed, we have, by definition

$$
\langle v_{lj} \triangle z_{kj} | v_{lk} \rangle = \left\langle z_{kj} \right| {}^{\mathbf{s}}L_{v_{lj}} v_{lk} \right\rangle = -\|z_{kj}\|^2.
\tag{5.17}
$$

On the other hand, since $L_{v_{lk}}v_{lj} = 0$ and $L_{(z_{kj}\triangle v_{lk})} = 0$ by (2.4), we have

$$v_{lk}\triangle(z_{kj}\triangle v_{lj}) = [L_{v_{lk}}, L_{z_{kj}}]v_{lj} = L_{(v_{lk}\triangle z_{kj} - z_{kj}\triangle v_{lk})}v_{lj} = (v_{lk}\triangle z_{kj})\triangle v_{lj},$$

where we used (2.1) for the second equality. This gives

$$\langle z_{kj}\triangle v_{lj}|v_{lk}\rangle = \langle v_{lk}\triangle(z_{kj}\triangle v_{lj}), E_{\mathbf{s}}^*\rangle = \langle (v_{lk}\triangle z_{kj})\triangle v_{lj}, E_{\mathbf{s}}^*\rangle$$
$$= \langle v_{lk}\triangle z_{kj}|v_{lj}\rangle = \langle z_{kj}|\,^{\mathbf{s}}L_{v_{lk}}v_{lj}\rangle.$$

Lemma 7.7 in [4] shows that the last term equals $\langle z_{kj}|\,^{\mathbf{s}}L_{v_{lj}}v_{lk}\rangle$, so that we obtain

$$\langle z_{kj}\triangle v_{lj}|v_{lk}\rangle = -\|z_{kj}\|^2. \tag{5.18}$$

Claim 1 follows from (5.17) and (5.18). $\qquad\qquad\square$

Now, since $\mathcal{I}_{\mathbf{s}}(w) = \mathcal{I}_{\mathbf{s}}(\eta E) = {}^{\mathbf{s}}\eta^{-1}E$, Proposition 5.2 and Claim 1 yield

$$\operatorname{Re}\mathcal{I}_{\mathbf{s}}(w) = \sum_{m\neq j,k,l} E_m + \Big(e^{-p} + (2s)^{-1}\big(e^{-q} - (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lk}\|^2\big)\|z_{kj}\|^2$$
$$- (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lj}\|^2 - s^{-1}(1-\delta)e^{-t_l}\|z_{kj}\|^2\Big)E_j$$
$$+ \big(e^{-q} - (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lk}\|^2\big)E_k + e^{-t_l}E_l$$
$$+ \big((1-\delta)e^{-t_l} - e^{-q} + (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lk}\|^2\big)z_{kj}. \tag{5.19}$$

Since $\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}}+iV)$ is convex, we must have $w\in\mathcal{I}_{\mathbf{s}}^*(\Omega^{\mathbf{s}}+iV)$, so that $\mathcal{I}_{\mathbf{s}}(w)\in\Omega^{\mathbf{s}}+iV$. Thus, (5.19) belongs to $\Omega^{\mathbf{s}}$. In particular, the coefficient of $E_j$ in (5.19) should be positive, that is, we must have

$$e^{-p} + (2s)^{-1}\big(e^{-q} - (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lk}\|^2\big)\|z_{kj}\|^2$$
$$- (2s)^{-1}(1-\delta)e^{-t_l}\|v_{lj}\|^2 - s^{-1}(1-\delta)e^{-t_l}\|z_{kj}\|^2 > 0. \tag{5.20}$$

Substitution of $e^{-t_l} = (1-\delta f)^{-1}$ shows that (5.20) is equivalent to

$$\delta\big((2s)^{-2}\|v_{lk}\|^2\|z_{kj}\|^2 + (2s)^{-1}\|v_{lj}\|^2 + s^{-1}\|z_{kj}\|^2 - (e^{-p} + (2s)^{-1}e^{-q}\|z_{kj}\|^2)f\big)$$
$$> (2s)^{-2}\|v_{lk}\|^2\|z_{kj}\|^2 + (2s)^{-1}\|v_{lj}\|^2 + s^{-1}\|z_{kj}\|^2 - \big(e^{-p} + (2s)^{-1}e^{-q}\|z_{kj}\|^2\big).$$

By (5.15) the right-hand side is equal to $-1$, and we arrive at

$$\delta\big((2s)^{-1}(f-1)\big((2s)^{-1}\|v_{lk}\|^2\|z_{kj}\|^2 + \|v_{lj}\|^2 + 2\|z_{kj}\|^2\big) + f\big) < 1. \tag{5.21}$$

Thus we must have (5.21) for any $\delta$ $(0\leq\delta\leq 1)$ satisfying (5.16).

**Claim 2.** We have $(2s)^{-1}\|v_{lj}\|^2\|v_{lk}\|^2 \leq \|z_{kj}\|^2$.

We shall prove Claim 2 by absurdity, so that we start with the assumption $(2s)^{-1}\|v_{lj}\|^2\|v_{lk}\|^2 > \|z_{kj}\|^2$. Let $x > 0$ be arbitrary (for the moment), and replace $v_{lj}, v_{lk}$ with $xv_{lj}, xv_{lk}$ respectively in the foregoing discussion. Then, by definition, $z_{kj}$ is replaced by $x^2 z_{kj}$. We put

$$F(x) := (2s)^{-1}\big(e^p\|v_{lj}\|^2 + e^q\|v_{lk}\|^2\big)x^2,$$
$$G(x) := (2s)^{-1}(F(x)-1)\big((2s)^{-1}\|v_{lk}\|^2\|z_{kj}\|^2 x^6 + 2\|z_{kj}\|^2 x^4 + \|v_{lj}\|^2 x^2\big) + F(x),$$

where we note by (5.15) that

$$e^{-p} = 1 + (2s)^{-1} \|v_{lj}\|^2 x^2 + (2s)^{-1} \|z_{kj}\|^2 x^4, \quad e^{-q} = 1 + (2s)^{-1} \|v_{lk}\|^2 x^2,$$

and the condition (5.16) is replaced by $\delta < F(x)^{-1}$, and (5.21) by $\delta \cdot G(x) < 1$. By a straightforward computation, we obtain

$$(F(x) - 1)e^{-p}e^{-q} = (2s)^{-1} \left( (2s)^{-1} \|v_{lj}\|^2 \|v_{lk}\|^2 - \|z_{kj}\|^2 \right) x^4 - 1. \quad (5.22)$$

In particular, this together with the assumption shows that $F(x) > 1$ for large $x > 0$. We put $H(x) := e^{-p}e^{-q}G(x)$. Since we see by definition of $F(x)$ that $F(x)e^{-p}e^{-q}$ is a polynomial in $x$ of degree 6, the definition of $G(x)$ and (5.22) show that $H(x)$ is a polynomial in $x$ of degree 10. The coefficient of $x^{10}$ is equal to

$$(2s)^{-3} \left( (2s)^{-1} \|v_{lj}\|^2 \|v_{lk}\|^2 - \|z_{kj}\|^2 \right) \|v_{lk}\|^2 \|z_{kj}\|^2.$$

Thus, if $z_{kj} \neq 0$, then the assumption implies $G(x) > 0$ for large $x > 0$. In case $z_{kj} = 0$, we have by a straightforward computation

$$H(x) = (2s)^{-1} \left( 1 + (2s)^{-1} \|v_{lj}\|^2 x^2 \right)^2 \|v_{lk}\|^2 x^2,$$

so that $G(x) > 0$.

Now, fixing a large $x > 0$ with $F(x) > 1$ and $G(x) > 0$, we choose $\delta > 0$ satisfying $\delta < F(x)^{-1}$. Then we have $0 < \delta < 1$ automatically, and we must have the inequality $\delta < G(x)^{-1}$ for any such $\delta$. This forces $F(x)^{-1} \leq G(x)^{-1}$, or $G(x) \leq F(x)$. But this is a contradiction, because the definition of $G(x)$ together with $F(x) > 1$ implies the reverse strict inequality $G(x) > F(x)$ (recall $v_{lj} \neq 0$). $\square$

Since $v_{lj}$ and $v_{lk}$ are non-zero, the conclusion of Claim 2 implies, in particular, that $n_{kj} \neq 0$. Let $\{e_m\}_{m=1}^{n_{kj}}$ be an orthonormal basis of $V_{kj}$. Then, the equality ${}^{\mathsf{s}}L_{v_{lj}} v_{lk} = {}^{\mathsf{s}}L_{v_{lk}} v_{lj}$ (see [4, Lemma 7.7]) implies

$$\|z_{kj}\|^2 = \sum_{m=1}^{n_{kj}} \left\langle {}^{\mathsf{s}}L_{v_{lj}} v_{lk} | e_m \right\rangle^2 = \sum_{m=1}^{n_{kj}} \left\langle {}^{\mathsf{s}}L_{v_{lk}} v_{lj} | e_m \right\rangle^2 = \sum_{m=1}^{n_{kj}} \left\langle v_{lj} | v_{lk} \triangle e_m \right\rangle^2. \quad (5.23)$$

Therefore the conclusion of Claim 2 is rewritten as

$$(2s)^{-1} \|v_{lj}\|^2 \|v_{lk}\|^2 \leq \sum_{m=1}^{n_{kj}} \left\langle v_{lj} | v_{lk} \triangle e_m \right\rangle^2.$$

We make $v_{lj}$ run over an orthonormal basis of $V_{lj}$ and sum up the resulting formulas. Then we obtain

$$(2s)^{-1} n_{lj} \|v_{lk}\|^2 \leq \sum_{m=1}^{n_{kj}} \|v_{lk} \triangle e_m\|^2 = (2s)^{-1} n_{kj} \|v_{lk}\|^2,$$

where the last equality follows from (5.2). Hence $n_{lj} \leq n_{kj}$. This completes the proof of Lemma 5.9. $\square$

Lemma 5.9 together with the statement (2) of Lemma 5.3 give

**Proposition 5.10.** If $n_{lk} \neq 0$, then one has $n_{lj} = n_{kj}$.

5.3. **Third step.** We next show that if $n_{kj} \neq 0$, then it holds that $n_{lk} = n_{lj}$. To do so, we set, for given $v_{lj} \in V_{lj}$ and $v_{kj} \in V_{kj}$,

$$U_{lk} := \tfrac{1}{2}(v_{lj} \triangle v_{kj} + v_{kj} \triangle v_{lj}).$$

We know $U_{lk} \in V_{lk}$ by (2.4), and by [4, Lemma 4.9], we have

$$\|U_{lk}\|^2 \leq (2s_k)^{-1} \|v_{lj}\|^2 \|v_{kj}\|^2. \tag{5.24}$$

**Lemma 5.11.** $\|U_{lk}\|^2 = (2s)^{-1} \|v_{lj}\|^2 \|v_{kj}\|^2.$

*Proof.* In view of (5.24), it is enough to show that the strict inequality

$$\|U_{lk}\|^2 < (2s)^{-1} \|v_{lj}\|^2 \|v_{kj}\|^2 \tag{5.25}$$

for some $v_{lj}$ and $v_{kj}$ causes a contradiction. Note that (5.25) forces that $v_{lj}$ and $v_{kj}$ are both non-zero. We put

$$
\begin{aligned}
p &:= \log\left(1 + (2s)^{-1} \|v_{kj}\|^2\right), \\
q &:= \log\left(1 + (2s)^{-1} \|v_{lj}\|^2 - (2s)^{-1}e^{-p} \|U_{lk}\|^2\right).
\end{aligned}
\tag{5.26}
$$

Observe that (5.25) guarantees that $q$ is a a real number. Consider two points

$$z_1 := \mathcal{I}_{\mathbf{s}}\left(E + i(v_{lj} + v_{kj})\right), \quad z_2 := \overline{z_1}$$

in $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$. We know by the proof of [4, Lemma 7.10] that (note that $p, q$ are written as $t_k, t_l$ respectively there and we put $z_{lk} := e^{-p}U_{lk}$ here)

$$
z_1 = \sum_{m \neq j,k,l} E_m - (f-1)E_j + \left(e^{-p} + (2s)^{-1}e^{-q} \|z_{lk}\|^2\right) E_k + e^{-q}E_l - e^{-q}z_{lk}
$$
$$
+ i\left(-e^{-p}v_{kj} + e^{-q}\left({}^{\mathbf{s}}L_{v_{lj}}z_{lk} - (2s)^{-1} \|z_{lk}\|^2 v_{kj} + {}^{\mathbf{s}}L_{v_{kj}}z_{lk} - v_{lj}\right)\right),
$$

where, for brevity, we have put

$$
\begin{aligned}
f &:= (2s)^{-1}\left(e^{-p} + (2s)^{-1}e^{-(2p+q)} \|U_{lk}\|^2\right) \|v_{kj}\|^2 \\
&\quad + (2s)^{-1}e^{-q} \|v_{lj}\|^2 - s^{-1}e^{-(p+q)} \|U_{lk}\|^2.
\end{aligned}
$$

Using (5.26), we see that

$$f = e^{-(p+q)}\left((2s)^{-1}\left(1 + (2s)^{-1} \|v_{lj}\|^2\right) \|v_{kj}\|^2 + (2s)^{-1}e^p \|v_{lj}\|^2 - s^{-1} \|U_{lk}\|^2\right).$$

Let $w := (1-u)z_1 + uz_2$ $(0 \leq u \leq 1)$, where putting $\delta := 4u(1-u)$ so that $0 \leq \delta \leq 1$, we impose that $u$ satisfies

$$1 - \delta f > 0. \tag{5.27}$$

Now, by definition, we have

$$
w = \sum_{m \neq j,k,l} E_m - (f-1)E_j + \left(e^{-p} + (2s)^{-1}e^{-q} \|z_{lk}\|^2\right) E_k + e^{-q}E_l - e^{-q}z_{lk}
$$
$$
+ i(1-2u)\left(-e^{-p}v_{kj} + e^{-q}\left({}^{\mathbf{s}}L_{v_{lj}}z_{lk} - (2s)^{-1} \|z_{lk}\|^2 v_{kj} + {}^{\mathbf{s}}L_{v_{kj}}z_{lk} - v_{lj}\right)\right).
$$

In Proposition 5.2, we set

$$t_j := -\log(1 - \delta f), \quad t_k := p, \quad t_l := q, \tag{5.28}$$
$$w_{lk} := e^{-p}U_{lk}(= z_{lk}), \quad w_{kj} := i(1-2u)v_{kj}, \quad w_{lj} := i(1-2u)v_{lj},$$

and put

$$\eta := \exp\left(L_{w_{lj}} + L_{w_{kj}}\right)\exp\left(L_{w_{lk}}\right)\exp\left(t_j H_j + t_k H_k + t_l H_l\right).$$

We remark that (5.27) guarantees that $t_j$ is a real number. Then the formula in Proposition 5.2 becomes ${}^{\mathbf{s}}\eta^{-1}E = w$ (note $S_{lk} = -(1-\delta)U_{lk}$ in the current situation). Since $\mathcal{I}_{\mathbf{s}}^*(w) = \eta\mathcal{I}_{\mathbf{s}}^*(E) = \eta E$, Proposition 5.1 yields

$$\begin{aligned}
\operatorname{Re}\mathcal{I}_{\mathbf{s}}^*(w) = \sum_{m\neq j,k,l} E_m &+ e^{t_j}E_j + \left(e^p - (2s)^{-1}(1-\delta)e^{t_j}\|v_{kj}\|^2\right)E_k \\
&+ \left(e^q + (2s)^{-1}e^{-p}\|U_{lk}\|^2 - (2s)^{-1}(1-\delta)e^{t_j}\|v_{lj}\|^2\right)E_l \\
&+ \left(1 - (1-\delta)e^{t_j}\right)U_{lk}.
\end{aligned} \tag{5.29}$$

Since $\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is convex, we must have $w \in \mathcal{I}_{\mathbf{s}}(\Omega + iV)$, so that $\mathcal{I}_{\mathbf{s}}^*(w) \in \Omega + iV$. Hence (5.29) should belong to $\Omega$. Therefore, the coefficient of $E_l$ in (5.29) should be positive, that is,

$$e^q + (2s)^{-1}e^{-p}\|U_{lk}\|^2 - (2s)^{-1}(1-\delta)e^{t_j}\|v_{lj}\|^2 > 0.$$

By a calculation using (5.26) and (5.28), we see that this is equivalent to

$$\delta\left((2s)^{-1}(f-1)\|v_{lj}\|^2 + f\right) < 1. \tag{5.30}$$

Let $x > 0$ be arbitrary and we replace $v_{lj}, v_{kj}$ with $xv_{lj}, xv_{kj}$ respectively. Then, by definition, $U_{lk}$ is replaced by $x^2 U_{lk}$. We put

$$\begin{aligned}
F(x) := e^{-(p+q)}\big((2s)^{-1}\|v_{kj}\|^2\left(1 + (2s)^{-1}\|v_{lj}\|^2 x^2\right)x^2 \\
+ (2s)^{-1}e^p\|v_{lj}\|^2 x^2 - s^{-1}\|U_{lk}\|^2 x^4\big),
\end{aligned} \tag{5.31}$$

$$G(x) := (2s)^{-1}(F(x)-1)\|v_{lj}\|^2 x^2 + F(x). \tag{5.32}$$

We note that (5.26) is replaced by

$$e^p = 1 + (2s)^{-1}\|v_{kj}\|^2 x^2, \tag{5.33}$$

$$e^q = 1 + (2s)^{-1}\|v_{lj}\|^2 x^2 - (2s)^{-1}e^{-p}\|U_{lk}\|^2 x^4. \tag{5.34}$$

The condition (5.27) is replaced by $\delta \cdot F(x) < 1$, and (5.30) by $\delta \cdot G(x) < 1$. Since

$$e^p e^q = e^p\left(1 + (2s)^{-1}\|v_{lj}\|^2 x^2\right) - (2s)^{-1}\|U_{lk}\|^2 x^4,$$

a calculation using (5.33) gives

$$\begin{aligned}
\left(F(x)-1\right)e^p e^q &= (2s)^{-1}\|v_{kj}\|^2\left(1 + (2s)^{-1}\|v_{lj}\|^2 x^2\right)x^2 - (2s)^{-1}\|U_{lk}\|^2 x^4 - e^p \\
&= (2s)^{-1}\left((2s)^{-1}\|v_{kj}\|^2\|v_{lj}\|^2 - \|U_{lk}\|^2\right)x^4 - 1. \tag{5.35}
\end{aligned}$$

Thus our assumption (5.25) says that $F(x) > 1$ for sufficiently large $x > 0$.

On the other hand, since $F(x)e^p e^q$ is a polynomial in $x$ of degree at most 4 by (5.31) and (5.33), we see from (5.32) and (5.35) that $G(x)e^p e^q$ is a polynomial in $x$ of degree 6. The coefficient of $x^6$ is

$$(2s)^{-2}\|v_{lj}\|^2\left((2s)^{-1}\|v_{lj}\|^2\|v_{kj}\|^2 - \|U_{lk}\|^2\right). \tag{5.36}$$

Therefore, our assumption (5.25) implies that $G(x) > 0$ for large $x > 0$.

Let us fix a large $x > 0$ so that we have $F(x) > 1$ and $G(x) > 0$. Suppose that $\delta > 0$ satisfies $\delta < F(x)^{-1}$, so that $0 < \delta < 1$ automatically. Then we must have $\delta < G(x)^{-1}$ for any such $\delta$. This implies $F(x)^{-1} \leqq G(x)^{-1}$, or $G(x) \leqq F(x)$. But this is impossible, because definition (5.32) and $F(x) > 1$ give the reverse strict inequality $G(x) > F(x)$ (note $v_{lj} \neq 0$). $\qquad\square$

**Proposition 5.12.** If $n_{kj} \neq 0$, then one has $n_{lk} = n_{lj}$.

*Proof.* If $n_{kj} \neq 0$, then we choose $v_{kj} \neq 0$, so that the linear map $V_{lj} \ni v_{lj} \mapsto U_{lk} \in V_{lk}$ is injective by virtue of Lemma 5.11. Therefore we have $n_{lj} \leq n_{lk}$. The reverse inequality follows from (1) of Lemma 5.3. $\qquad\square$

5.4. **Last step.** The concluding step is parallel to that of [9, Subsection 5.5].

**Lemma 5.13.** If at least two of $n_{lk}, n_{lj}, n_{kj}$ are non-zero, they are all equal.

*Proof.* In view of Propositions 5.10 and 5.12, the proof is completely similar to that of [9, Lemma 5.15]. $\qquad\square$

Now we see that the numbers $n_{kj}$ are independent of $j, k$ (see [9, Proposition 5.16] for the proof). Then the following proposition due to Vinberg tells us that $\Omega$ is a symmetric cone.

**Proposition 5.14** ([13, Proposition 3]). The irreducible homogeneous convex cone $\Omega$ is a symmetric cone if and only if the numbers $n_{kj}$ are independent of $j, k$.

Therefore $\Omega + iV$ is symmetric by [3, Theorem X.1.1], which completes the proof of Theorem 1.1. $\qquad\square$

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