In [6], symmetry relations (Ch. VI, (6.6)) and special values (Ch. VI, (6,11')) of Macdonald symmetric polynomials have been given. By a combinatorial argument similar to the one employed in their paper, we see that for any (k,r,n)-admissible partition λ, the multiplicity of the factor $(1-t^{k+1}q^{r-1})$ in r.h.s. of (6,11') is $[\frac{n}{k+1}]$. Moreover, for $\mu \in \pi_{\eta,N}$ or $\pi'_{\eta,N}$, the same results as Lemma 3.12 follow as well. Hence from symmetry relations, through the same argument as Theorem 3.13, we conclude that the Macdonald symmetric polynomial is well-defined and satisfies the wheel conditions if λ is (k,r,n)-admissible.

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monomial symmetric polynomials are linearly independent, it follows that

$$\sum_{\mu\in\pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu}e_\mu = 0$$

in $R^{(d)}_{M,k+1}/(R^{(d+1)}_{M,k+1}+\sum_p \mathbb{C}r^p_d)$. Note that $\mathcal{J}_{M,k+1} = \sum_{d=0}^{M^{k+1}}\sum_p \mathbb{C}r^p_d$. Therefore in $R_{M,k+1}/\mathcal{J}_{M,k+1}$, we have

$$\sum_{\mu\in\pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu}e_\mu = \sum_{\mu\in\pi_{k+1},|\mu|\geq d+1,\mu_1\leq M} c_\mu e_\mu.$$

For any $(k,r,k+1)$-non-admissible partition $\lambda\in\pi_{k+1}$ such that $\lambda_1\leq M$, there exists some d and m so that $\lambda\in\pi_{k+1,d}(m)$. Moreover, the set $\pi_{k+1,d}(m)$ contains at most one $(k,r,k+1)$-non-admissible partition λ, and for all $\mu\in\pi_{k+1,d}(m)$ such that $\mu\neq\lambda$, we have $\mu\succ\lambda$. Therefore e_λ can be written in $R_{M,k+1}/\mathcal{J}_{M,k+1}$ as follows:

$$e_\lambda = \sum_{\mu\succ\lambda,\mu_1\leq M} c'_\mu e_\mu.$$

Let $\lambda\in\pi_n$ be a (k,r,n)-non-admissible partition such that $\lambda_1\leq M$. Then there exists i such that $\lambda_i-\lambda_{i+k}<r$. We set $\mu:=(\lambda_i,\cdots,\lambda_{i+k})\in\pi_{k+1}$ and $\nu:=(\lambda_1,\cdots,\lambda_{i-1},\lambda_{i+k+1},\cdots,\lambda_n)$. Since μ is $(k,r,k+1)$-non-admissible, from the above argument, we can rewrite μ as a linear combination of greater monomials $\{e_{\mu'};\mu'\succ\mu\}$ in $R_{M,k+1}/\mathcal{J}_{M,k+1}$. Hence e_λ can be rewritten in $R_{M,n}/\mathcal{J}_{M,n}$ as follows:

$$\begin{aligned} e_\lambda &= e_\mu e_\nu \\ &= \left(\sum_{\mu'\succ\mu,\mu'_1\leq M} c_{\mu'}e_{\mu'}\right)e_\nu \\ &= \sum_{\lambda'\succ\lambda,\lambda'_1\leq M} c_{\lambda'}e_{\lambda'}. \end{aligned}$$

Here, in the last $=$, we set $\lambda':=\mu'\cup\nu$.

If $e_{\lambda'}$ is still (k,r,n)-non-admissible for some λ', we further rewrite $e_{\lambda'}$ as a linear combination of greater monomials. Since $\{\lambda\in\pi_n;\lambda_1\leq M\}$ is a finite set, this procedure stops in finite times. □

Corollary 4.2. $\dim J^{(k,r)}_M \leq \sharp\{\lambda\in\pi_n;\lambda$ *is* (k,r,n)*-admissible and* $\lambda_1\leq M\}$.

By Corollary 3.14 and Corollary 4.2, we complete the proof of Theorem 3.6.

5. Application to Macdonald symmetric polynomials

We can apply the method in Section 3 to a proof of Theorem 1.1.

In this space,

$$\begin{aligned}
0 &= r_d^p = \sum_{\substack{i_1+\cdots+i_{k+1}=d \\ i_j\geq 0}} \prod_{j=1}^{k+1} e_{i_j}\tau^{p_j i_j} \\
&= \sum_{\substack{\nu\in\mathbb{Z}_{\geq 0}^{k+1} \\ \nu_j\leq r-2}} \tau^{p_1\nu_1+\cdots+p_{k+1}\nu_{k+1}} \left(\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^{k+1},\sum_j \mu_j=d \\ \mu_j=\nu_j+(r-1)\kappa_j,\kappa_j\in\mathbb{Z}_{\geq 0}}} \prod_j e_{\mu_j} \right) \\
&= \sum_{\substack{\lambda\in\pi_{k+1} \\ \lambda_1\leq r-2}} \left(\sum_{\nu\in\mathfrak{S}_{k+1}\lambda} \tau^{p_1\nu_1+\cdots+p_{k+1}\nu_{k+1}} \right) \left(\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^{k+1},\sum_j \mu_j=d \\ \mu_j=\lambda_j+(r-1)\kappa_j,\kappa_j\in\mathbb{Z}_{\geq 0}}} \prod_j e_{\mu_j} \right)
\end{aligned}$$

We set $\pi_{k+1,d} := \{\lambda \in \pi_{k+1}; |\lambda| = d\}$. For a sequence of nonnegative integers $m := (m_0, \cdots, m_{r-2})$ such that $\sum m_i = k+1$, we define a subset $\pi_{k+1,d}(m)$ by

$$\pi_{k+1,d}(m) := \{\mu \in \pi_{k+1,d}\ ;\ \sharp\{i; \mu_i \equiv a\ mod\ (r-1)\} = m_a \text{ for every } 0 \leq a \leq r-2\}.$$

We denote by $m_i^{(\lambda)}$ the multiplicity of i in λ. Define $m(\lambda) := (m_0^{(\lambda)}, \cdots, m_{r-2}^{(\lambda)})$. Then,

$$\begin{aligned}
r_d^p &= \sum_{\substack{\lambda\in\pi_{k+1} \\ \lambda_1\leq r-2}} \left(\sum_{\mu\in\pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) \left(\sum_{\nu\in\mathfrak{S}_{k+1}\lambda} \tau^{p_1\nu_1+\cdots+p_{k+1}\nu_{k+1}} \right). \\
&= \sum_{\substack{\lambda\in\pi_{k+1} \\ \lambda_1\leq r-2}} \left(\sum_{\mu\in\pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) m_\lambda(\tau^{p_1}, \cdots, \tau^{p_{k+1}}).
\end{aligned}$$

Here, $c_{\lambda,\mu} = \prod_i m_i^{(\lambda)}!/\prod_i m_i^{(\mu)}!$ and m_λ is the monomial $\mathfrak{S}_{k+1}$-symmetric polynomial (not Laurent).

Since $\lambda_1 \leq r-2$, the degree of

$$\sum_{\substack{\lambda\in\pi_{k+1} \\ \lambda_1\leq r-2}} \left(\sum_{\mu\in\pi_{k+1,d}(m(\lambda))} c_{\lambda,\mu} e_\mu \right) m_\lambda(x_1, \cdots, x_{k+1}) \tag{23}$$

in each variable x_i is less than $r-2$. On the other hand, we can choose the values of x_i from $\tau^0, \tau^1, \cdots, \tau^{r-2}$ independently. Hence the expression (23) is identically zero in the quotient space $R_{M,k+1}^{(d)}/(R_{M,k+1}^{(d+1)} + \sum_p \mathbb{C}r_d^p)$. Since is the orthogonal complement of $\bar{J}_M^{(k,r)}$ with respect to the coupling $\langle , \rangle$. For $p=(p_1,\cdots,p_{k+1})\in\mathbb{Z}^{k+1}$, let r_d^p be the coefficient of z^d in

$$e(\tau^{p_1}z)\cdots e(\tau^{p_{k+1}}z)=\sum_d r_d^p z^d.$$

By the symmetry of exchanging $z\leftrightarrow z^{-1}$ in the current $e(z)$, we have $r_d^p = r_{-d}^p$. We denote by $\mathcal{J}_M$ the ideal of R_M generated by the elements r_d^p. Set $\mathcal{J}_{M,n} := \mathcal{J}_M \cap R_{M,n}$. Then the space (22) coincides with $\mathcal{J}_{M,n}$. Since $\dim R_{M,n}/\mathcal{J}_{M,n} = \dim \bar{J}_M^{(k,r)}$, the condition (20) is equivalent to the relations in the quotient space

$$r_d^p = 0 \quad \text{for all } p=(p_1,\cdots,p_{k+1})\in\mathbb{Z}^{k+1} \text{ and } d\geq 0.$$

Proposition 4.1. *The image of the set* $\{e_\lambda;\lambda\in\pi_n$ *is* (k,r,n)*-admissible,*$\lambda_1\leq M\}$ *spans the quotient space* $R_{M,n}/\mathcal{J}_{M,n}$.

Proof. We introduce a total ordering for partitions and monomials. For two partitions λ and μ such that $|\lambda|>|\mu|$, we define $\lambda\succ\mu$. For two partitions λ and μ such that $|\lambda|=|\mu|$, we define $\lambda\succ\mu$ if $\lambda_1>\mu_1$ or $\lambda_1=\mu_1,\lambda_2>\mu_2$ or $\lambda_1=\mu_1,\lambda_2=\mu_2,\lambda_3>\mu_3$ or $\cdots$. We define for the corresponding monomials e_λ and e_μ, $e_\lambda\succ e_\mu$.

Let us calculate r_d^p.

$$\begin{aligned} e(\tau^{p_1}z)\cdots e(\tau^{p_{k+1}}z) &= \prod_{j=1}^{k+1}\sum_{i_j=-M}^{M} e_{|i_j|}(\tau^{p_j}z)^{i_j} \\ &= \sum_{d\in\mathbb{Z}} z^d \left(\sum_{\substack{i_1+\cdots+i_{k+1}=|d| \\ i_j\geq 0}} \prod_{j=1}^{k+1} e_{i_j}\tau^{p_j i_j} + \sum_{\substack{\lambda\in\pi_{k+1} \\ |\lambda|>|d|}} c_{\lambda,|d|}e_\lambda \right). \end{aligned}$$

Hence, for any nonnegative integer d,

$$r_d^p = \sum_{\substack{i_1+\cdots+i_{k+1}=d \\ i_j\geq 0}} \prod_{j=1}^{k+1} e_{i_j}\tau^{p_j i_j} + \sum_{\substack{\lambda\in\pi_{k+1} \\ |\lambda|>d}} c_{\lambda,d}e_\lambda.$$

We define $R_{M,k+1}^{(d)}$ by

$$R_{M,k+1}^{(d)} := \bigoplus_{\lambda\in\pi_{k+1},|\lambda|\geq d,\lambda_1\leq M} \mathbb{C}e_\lambda,$$

and we consider a quotient space

$$R_{M,k+1}^{(d)}/(R_{M,k+1}^{(d+1)} + \sum_{p\in\mathbb{Z}^{k+1}} \mathbb{C}r_d^p).$$

We have shown $u_\mu(P_\lambda) = 0$ at the specialization (16) for all $\mu \in \pi'_{\eta,N}$. Hence from Remark 3.3 and Remark 3.10, we conclude that $\varphi(P_\lambda)$ satisfies the wheel condition (17).

□

Corollary 3.14. *The space $I^{(k,r)}$ and $I_M^{(k,r)}$ are well-defined for any positive integer M, and we have $J_M^{(k,r)} \supseteq I_M^{(k,r)}$.*

4. Estimate of $\dim J_M^{(k,r)}$

We have already constructed the polynomials satisfying the zero conditions. In this section, we show that $J_M^{(k,r)} = I_M^{(k,r)}$ by giving an upper estimate of the dimension of $J_M^{(k,r)}$.

Fix $g'_0,\ g'_1,\ g'_2,\ g'_3 \gg 1$ such that $g'_0 = g'_1 + g'_2 + g'_3$. We take the limit $t \to 1$, $q \to \tau$, $a \to \tau^{g'_0}$, $b \to -\tau^{g'_1}$, $c \to \tau^{g'_2+1/2}$, $d \to -\tau^{g'_3+1/2}$, where τ is a primitive $(r-1)$-th root of unity. In this limit the wheel condition (17) reduces to

$$f = 0 \qquad \text{if } x_i = \tau^{p_i} x_0 \qquad (1 \le i \le k+1) \tag{20}$$

for all $p_1, \cdots, p_{k+1} \in \mathbb{Z}$ and $x_0 \in \mathbb{C}$. We denote by $\bar{J}^{(k,r)} \subseteq \bar{\Lambda}_n$ the space of $(BC)_n$-symmetric polynomials satisfying (20). We define

$$\bar{J}_M^{(k,r)} = \{f \in \bar{J}^{(k,r)}; \deg_{x_1} f \le M\}.$$

Note that $\dim_{\mathbb{C}(u,b,c,d)} J_M^{(k,r)} \le \dim_{\mathbb{C}} \bar{J}_M^{(k,r)}$.

We consider the commutative ring $R_M := \mathbb{C}[e_0, e_1, e_2, \cdots, e_M]$ for indeterminates $\{e_i\}$. We count the weight of e_i as 1 and the degree of e_i as i. We set $e_\lambda := \prod_{i=1}^n e_{\lambda_i}$ for $\lambda \in \pi_n$. We denote by $R_{M,n} \subseteq R_M$ the space spanned by the monomials e_λ such that $\lambda \in \pi_n$ and $\lambda_1 \le M$.

We use the dual language (see [2]). There is a nondegenerate coupling:

$$\begin{aligned} R_{M,n} \times \bar{\Lambda}_{n,M} &\to \mathbb{C}; \\ \langle e_\lambda, \widehat{m}_\mu \rangle &= \delta_{\lambda,\mu}. \end{aligned} \tag{21}$$

We introduce an abelian current

$$e(z) \ := \ \sum_{i=1}^{M} e_i(z^i + z^{-i}) + e_0.$$

It satisfies

$$\langle e(z_1)e(z_2)\cdots e(z_n), f\rangle = f(z_1, z_2, \cdots, z_n) \quad \text{ for } f \in \bar{\Lambda}_{n,M}.$$

Then for any $f \in \bar{J}_M^{(k,r)}$, we have

$$\langle e(\tau^{p_1} z)\cdots e(\tau^{p_{k+1}} z) e(z_{k+2})\cdots e(z_n), f\rangle = 0 \quad \text{for all } (p_1, \cdots, p_{k+1}) \in \mathbb{Z}^{k+1}.$$

Hence the space

$$\begin{aligned} \operatorname{span}_{\mathbb{C}}\{ & e(\tau^{p_1} z)\cdots e(\tau^{p_{k+1}} z) e(z_{k+2})\cdots e(z_n) \\ & ;\ z, z_{k+2}, \cdots, z_n \in \mathbb{C}, p_1, \cdots, p_{k+1} \in \mathbb{Z}\} \end{aligned} \tag{22}$$

(i) for each i, $\sharp(\varphi(C_i)) \geq N$ in $\mathbb{C}(u,b,c,d)$;
(ii) for all choices of $c_i \in C_i$, $Z(g(c_1,\cdots,c_n)) > 0$ (resp. ≥ 0).

Motivated by the observation above, we define certain sets of partitions.

Definition 3.11. A partition η is called thick if $\eta_i \gg \eta_{i+1} \gg 0$ for all i. For a thick partition $\eta \in \pi_n$, a set of N^n partitions is defined by $\pi_{\eta,N} := \{\mu \in \pi_n; \mu_i = \eta_i + d_i \text{ for all } i \text{ where } 0 \leq d_i \leq N-1\}$.

For a thick partition $\eta \in \pi_{n-k}$, we define $\pi'_{\eta,N} := \{\mu \in \pi_n; \mu_1 - \mu_{k+1} < r, \mu_i = \eta_{i-k} + d_{i-k} \text{ for } k+1 \leq i \leq n \text{ where } 0 \leq d_i \leq N-1\}$.

When we use these sets $\pi_{\eta,N}$ and $\pi'_{\eta,N}$, we choose a sufficiently large N such that $N \gg M$ and any thick partition η such that $\eta_i - \eta_{i+1} \gg \max(M, 2[\frac{n}{k+1}](r-1))$, $\eta_i \gg \max(M, 2[\frac{n}{k+1}](r-1))$. We do not specify N and η in the below.

Lemma 3.12. *For $\mu \in \pi_{\eta,N}$ or $\mu \in \pi'_{\eta,N}$, P_μ has no pole at the specialization* (16)*. Moreover*

$$Z(u_0(P_\mu)) = \begin{cases} [\frac{n}{k+1}] & \text{if } \mu \in \pi_{\eta,N} \ (\eta \in \pi_n), \\ [\frac{n}{k+1}] - 1 & \text{if } \mu \in \pi'_{\eta,N} \ (\eta \in \pi_{n-k}). \end{cases}$$

Proof. If μ is an element of $\pi_{\eta,N}$ or $\pi'_{\eta,N}$, then $\mu_i \gg \mu_{i+k+1}$ for $1 \leq i \leq n-k-1$. Hence from Lemma 3.4, we see P_μ has no pole at (16).

If $\mu \in \pi_{\eta,N}$, then for each $1 \leq l \leq [\frac{n}{k+1}]$, $\mu_i \gg \mu_{i+(k+1)l-1}$ $(1 \leq i \leq n-(k+1)l+1)$ and $\mu_i \gg \mu_{i+(k+1)l}$ $(1 \leq i \leq n-(k+1)l)$. Hence from Proposition 3.8, $Z(u_0(P_\mu)) = [\frac{n}{k+1}]$.

If $\mu \in \pi'_{\eta,N}$, then $\mu_1 - \mu_{k+1} \leq r-1$. Hence from Proposition 3.8, $(i,l) = (1,1)$ is the only different situation from the case $\mu \in \pi_{\eta,N}$. Therefore $Z(u_0(P_\mu)) = [\frac{n}{k+1}] - 1$. □

Now we are ready to prove a part of Theorem 3.6.

Theorem 3.13. *For any (k,r,n)-admissible λ, Koornwinder-Macdonald polynomial P_λ has no pole at the specialization* (16) *and $\varphi(P_\lambda)$ satisfies the wheel condition* (17)*.*

Proof. Since λ is (k,r,n)-admissible, $Z(u_0(P_\lambda)) = [\frac{n}{k+1}]$ from Corollary 3.9.

Let $N \gg |\lambda|$ and let $\mu \in \pi_{\eta,N}$ where $\eta \in \pi_n$. Then from Lemma 3.12, P_μ has no pole at the specialization (16) and $Z(u_0(P_\mu)) = [\frac{n}{k+1}]$. From the duality relation (12),

$$u_\mu(P_\lambda) = \frac{u_\lambda(P_\mu)}{u_0(P_\mu)} u_0(P_\lambda).$$

Therefore, $Z(u_\mu(P_\lambda)) \geq 0$.

Since this holds for all $\mu \in \pi_{\eta,N}$, from Remark 3.10, we see that P_λ has no pole at the specialization (16).

Let $\mu \in \pi'_{\eta,N}$ $(\eta \in \pi_{n-k})$. Then from Lemma 3.12, P_μ has no pole at the specialization (16) and $Z(u_0(P_\mu)) = [\frac{n}{k+1}] - 1$. From the duality relation (12), through the same argument as the above, $Z(u_\mu(P_\lambda)) \geq 1$.

Here, we define a subspace $I^{(k,r)}$ *of* Λ'_n

$$I^{(k,r)} := \mathrm{span}_{\mathbb{C}(u,b,c,d)}\{\varphi(P_\lambda); \lambda \text{ is } (k,r,n)\text{-admissible } \},$$

and we set

$$I_M^{(k,r)} := \mathrm{span}_{\mathbb{C}(u,b,c,d)}\{\varphi(P_\lambda); \lambda \text{ is } (k,r,n)\text{-admissible and } \lambda_1 \le M\}.$$

First, we prepare some propositions and lemmas.

Definition 3.7. For $p \in \mathbb{C}(t,q,b,c,d)$, we denote by $Z(p) \in \mathbb{Z}$ the multiplicity of $(t^{(k+1)/m}q^{(r-1)/m} - \omega)$ in p. That is,

$$p = (t^{k+1}q^{r-1} - 1)^{Z(p)}p',$$

where the factor $p' \in \mathbb{C}(t,q,b,c,d)$ has neither pole nor zero at (16).

Proposition 3.8. *For any partition* $\lambda \in \pi_n$*, we have*

$$\begin{aligned} Z(u_0(P_\lambda)) &= \sharp\{(i,l) \in \mathbb{Z}^2_{>0}; \lambda_i - \lambda_{i+(k+1)l-1} \ge (r-1)l+1\} \\ &\quad -\sharp\{(i,l) \in \mathbb{Z}^2_{>0}; \lambda_i - \lambda_{i+(k+1)l} \ge (r-1)l+1\}. \end{aligned}$$

Proof. Recall Remark 2.4. The factor P_λ^{diff} has the factors of the form $(1 - t^x q^y)$ $(x, y \in \mathbb{Z}_{\ge 0})$.

If $j - i + 1 = (k+1)l$ and $\lambda_i - \lambda_j \ge (r-1)l + 1$, then $u_0(P_\lambda)$ has the factor $(1 - t^{(k+1)l}q^{(r-1)l})$ in the numerator of P_λ^{diff}. If $j - i = (k+1)l$ and $\lambda_i - \lambda_j \ge (r-1)l + 1$, then $u_0(P_\lambda)$ has the factor $(1 - t^{(k+1)l}q^{(r-1)l})$ in the denominator of P_λ^{diff}. Otherwise, there does not exist the factor $(1 - t^{(k+1)l}q^{(r-1)l})$ in P_λ^{diff}.

On the other hand, P_λ^{sum} and P_λ^{single} have neither pole nor zero at the specialization (16).

□

Corollary 3.9. *For any* (k,r,n)*-admissible* λ*, we have* $Z(u_0(P_\lambda)) = [\frac{n}{k+1}]$.

Proof. Since $\lambda_i - \lambda_{i+k} \ge r$,

$$\begin{aligned} Z(u_0(P_\lambda)) &= \sharp\{(i,l) \in \mathbb{Z}^2_{>0}; i + (k+1)l - 1 \le n\} \\ &\quad -\sharp\{(i,l) \in \mathbb{Z}^2_{>0}; i + (k+1)l \le n\} \\ &= \sum_{l \ge 1} \max\{(n - (k+1)l + 1), 1\} - \sum_{l \ge 1} \max\{(n - (k+1)l), 1\} \\ &= \left[\frac{n}{k+1}\right] \end{aligned}$$

□

Remark 3.10. For $g \in \Lambda_n$, we take an integer N such that the degree of g in each variable x_i is less than $N/2$. Then to prove that $g = 0$ (respectively, g has no pole) at the specialization (16), it is sufficient to show that there exist n subsets $C_1, \cdots, C_n \subseteq \mathbb{C}(b,c,d)[q^{\pm 1}, t^{\pm 1}]$, which satisfy the following two conditions:

Lemma 3.4. *If $\lambda \in \pi_n$ satisfies*

$$\lambda_i - \lambda_{i+k+1} > 2\left[\frac{n}{k+1}\right](r-1) \quad \text{for } 1 \leq i \leq n-k-1,$$

then P_λ has no pole at the specialization (16).

Proof. Suppose that there exists μ such that $\varphi(E_\mu(X)) = \varphi(E_\lambda(X))$, that is

$$\begin{aligned}&\{\varphi(t^{n-i}q^{\mu_i}a + t^{-n+i}q^{-\mu_i}a^{-1}); 1 \leq i \leq n\}\\ &= \{\varphi(t^{n-i}q^{\lambda_i}a + t^{-n+i}q^{-\lambda_i}a^{-1}); 1 \leq i \leq n\}.\end{aligned}$$

Since u and $a = bcdq^{-1}$ are generic, it must be satisfied that

$$\begin{aligned}&\{\varphi(t^{n-(k+1)l-i}q^{\mu_{(k+1)l+i}}) \ ; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\}\\ &= \{\varphi(t^{n-(k+1)l-i}q^{\lambda_{(k+1)l+i}}) \ ; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\}\end{aligned}$$

for $1 \leq i \leq k+1$. Hence

$$\begin{aligned}&\{(r-1)l + \mu_{(k+1)l+i} \ ; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\}\\ &= \{(r-1)l + \lambda_{(k+1)l+i} \ ; l \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq (k+1)l+i \leq n\}\end{aligned}$$

for $1 \leq i \leq k+1$.

Then for any $1 \leq i \leq k+1$, there exists $l_i \geq 0$ such that $(r-1)l_i + \mu_{(k+1)l_i+i} = \lambda_i$ and there exists $l'_i \geq 0$ such that $(r-1)l'_i + \lambda_{(k+1)l'_i+i} = \mu_i$. If $l'_i \neq 0$, then by the hypothesis,

$$\begin{aligned}\mu_i - \mu_{(k+1)l_i+i} &= (r-1)l'_i + \lambda_{(k+1)l'_i+i} - \lambda_i + (r-1)l_i\\ &< (r-1)(l'_i + l_i) - 2\left[\frac{n}{k+1}\right](r-1)l'_i\\ &\leq 2\left[\frac{n}{k+1}\right](r-1)(1-l'_i)\\ &\leq 0.\end{aligned}$$

Hence l'_i must be equal to 0, namely $\lambda_i = \mu_i$. Inductively, we have $\lambda_{(k+1)l+i} = \mu_{(k+1)l+i}$ for all $l \geq 0$. It follows that $\lambda = \mu$. Therefore from Lemma 2.1, P_μ has no pole at the specialization (16). □

We are going to construct a basis of $J_M^{(k,r)}$.

Definition 3.5. $\lambda \in \pi_n$ is called (k,r,n)-*admissible* if

$$\lambda_i - \lambda_{i+k} \geq r \qquad (1 \leq \forall i \leq n-k). \tag{19}$$

Our main result is

Theorem 3.6. *For any (k,r,n)-admissible λ, Koornwinder-Macdonald polynomial P_λ has no pole at the specialization* (16). *Moreover, for any positive integer M, we have*

$$I_M^{(k,r)} = J_M^{(k,r)}.$$

Remark 2.4. Note that in (14), there appear only factors of the form $(1-t^xq^y)$, $x,y\in\mathbb{Z}_{\geq 0}$. In (13), there appear only factors of the form $(1-t^xq^ya^2)$, $x,y\in\mathbb{Z}_{\geq 0}$. In (15), there appear only factors of the form $(1-t^xq^ya^2)$, $(1-t^xq^yab)$, $(1-t^xq^yac)$, $(1-t^xq^yad)$, $x,y\in\mathbb{Z}_{\geq 0}$.

3. The space $I_M^{(k,r)}$ and $J_M^{(k,r)}$

In this section, we describe zero conditions and construct symmetric Laurent polynomials satisfying the zero conditions.

First, we describe a specialization of the parameters. Let k,r be integers such that $1\leq k\leq n-1$ and $r\geq 2$. Let m be the greatest common divisor of $(k+1)$ and $(r-1)$. Let ω be a primitive m-th root of unity. Let $\omega_1\in\mathbb{C}$ be such that $\omega_1^{(r-1)/m}=\omega$.

Definition 3.1. For an indeterminate u, we consider the specialization of t and q:

$$t=u^{(r-1)/m}, q=\omega_1u^{-(k+1)/m}. \tag{16}$$

Then for integers $x,y\in\mathbb{Z}$, $t^xq^y=1$ if and only if $x=(k+1)l$, $y=(r-1)l$ for some $l\in\mathbb{Z}$. Moreover, the multiplicity of $(t^{(k+1)/m}q^{(r-1)/m}-\omega)$ in $(t^{(k+1)l}q^{(r-1)l}-1)$ is 1.

We define the subject of our study. We denote by Λ'_n the corresponding space $\Lambda'_n:=\bar{\Lambda}_n\otimes\mathbb{C}(u,b,c,d)$.

Definition 3.2. A sequence $(s_1,\cdots,s_{k+1})$ $(s_1,\cdots,s_{k+1}\in\mathbb{Z}_{\geq 0})$ is called a *wheel sequence* if $s_1+\cdots+s_{k+1}=r-1$. For $f\in\Lambda'_n$, we consider the following *wheel condition*:

$$\begin{array}{l} f=0,\quad \text{if } x_{i+1}=tq^{s_i}x_i \qquad (1\leq i\leq k) \\ \text{for all wheel sequences } (s_1,\cdots,s_{k+1}). \end{array} \tag{17}$$

We consider the subspace $J^{(k,r)}\subseteq\Lambda'_n$

$$J^{(k,r)}:=\{f\in\Lambda'_n; f \text{ satisfies (17)}\}. \tag{18}$$

We denote by $\Lambda'_{n,M}$ the subspace consisting of $f\in\Lambda'_n$ such that the degree of f in each x_i is less than M. We set $J_M^{(k,r)}:=J^{(k,r)}\cap\Lambda'_{n,M}$.

Remark 3.3. For any partition $\mu\in\pi_n$, $u_\mu(x_1)/u_\mu(x_{k+1})=t^kq^{\mu_1-\mu_{k+1}}$. Hence the condition $\mu_1-\mu_{k+1}\leq r-1$ corresponds to the existence of the wheel sequence: $s_{k+1}=r-1-(\mu_1-\mu_{k+1})\geq 0$. The wheel conditions for $f(x)\in\Lambda'_n$ correspond to $u_\mu(f)=0$ at the specialization (16) for any partition $\mu\in\pi_n$ such that $\mu_1-\mu_{k+1}\leq r-1$.

For $f(t,q,b,c,d)\in\mathbb{C}[t,q,b,c,d]$, we use a specialization mapping φ

$$\begin{array}{rccc} \varphi: & \mathbb{C}[t,q,b,c,d] & \longrightarrow & \mathbb{C}(u,b,c,d) \\ & f(t,q,b,c,d) & \mapsto & f(u^{(r-1)/m},\omega_1u^{-(k+1)/m},b,c,d), \end{array}$$

and we extend φ to those elements of the field $\mathbb{C}(t,q,b,c,d)$ for which the specialized denominator does not vanish.

Then the eigenvalue $E_\lambda(X)$ of the operator $D(X)$ is given by

$$\begin{aligned} D(X)P_\lambda &= E_\lambda(X)P_\lambda \\ E_\lambda(X) &:= \prod_{i=1}^{n}(X + t^{n-i}q^{\lambda_i}(abcdq^{-1})^{1/2} + t^{-n+i}q^{-\lambda_i}(abcdq^{-1})^{-1/2}). \end{aligned}$$

We use the dominance ordering $\lambda > \mu$ for partitions λ and μ. We have

Lemma 2.1. *Let* $c_{\lambda\mu}$ *be*

$$P_\lambda =: \widehat{m}_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}\widehat{m}_\mu.$$

If there does not exist $\nu < \lambda$ *such that* $E_\lambda(X) = E_\nu(X)$ *at a certain specialization of parameters, then for any* $\mu < \lambda$, $c_{\lambda\mu}$ *has no pole at the same specialization.*

Proof. It is clear from the defining equality of P_λ

$$P_\lambda := \left(\prod_{\mu<\lambda}\frac{D(X) - E_\mu(X)}{E_\lambda(X) - E_\mu(X)}\right) m_\lambda.$$

□

In the rest of paper, we always assume

$$a = bcdq^{-1}. \tag{11}$$

From Lemma 2.1, we see that P_λ has no pole at (11). The condition (11) is called the *self-duality* condition. We set $\Lambda_n := \bar{\Lambda}_n \otimes_{\mathbb{C}} \mathbb{C}(t, q, b, c, d)$.

From [1], we have the following propositions:

Proposition 2.2 (duality)**.** *For all* $\lambda, \mu \in \pi_n$, *Koornwinder-Macdonald polynomials* P_λ *and* $P_\mu \in \Lambda_n$ *satisfy the following duality relation:*

$$\frac{u_\mu(P_\lambda)}{u_0(P_\lambda)} = \frac{u_\lambda(P_\mu)}{u_0(P_\mu)}. \tag{12}$$

Here, the definition of u_μ *is the one in* (6).

Proposition 2.3.

$$u_0(P_\lambda) = P_\lambda^{sum} \times P_\lambda^{diff} \times P_\lambda^{single},$$

$$P_\lambda^{sum} := \prod_{i<j} t^{-(\lambda_i+\lambda_j)/2}\frac{(t^{2n+1-i-j}a^2;q)_{\lambda_i+\lambda_j}}{(t^{2n-i-j}a^2;q)_{\lambda_i+\lambda_j}}, \tag{13}$$

$$P_\lambda^{diff} := \prod_{i<j} t^{-(\lambda_i-\lambda_j)/2}\frac{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(t^{j-i};q)_{\lambda_i-\lambda_j}}, \tag{14}$$

$$P_\lambda^{single} := \prod_{i} a^{-\lambda_i}\frac{(t^{n-i}a^2, t^{n-i}ab, t^{n-i}ac, t^{n-i}ad;q)_{\lambda_i}}{(t^{n-i}a, -t^{n-i}a, t^{n-i}aq^{1/2}, -t^{n-i}aq^{1/2};q)_{\lambda_i}}. \tag{15}$$

Here, $(a;q)_l := \prod_{i=0}^{l-1}(1-aq^i)$ *and* $(a_1, a_2, \cdots, a_p;q)_l := (a_1;q)_l(a_2;q)_l\cdots(a_p;q)_l$.

To be precise,

$$D_r := \sum_{\substack{J\subset\{1,\cdots,n\},0\le|J|\le r\\ \epsilon_j=\pm1,j\in J}} U_{J^c,r-|J|}(x)V_{\epsilon J,J^c}(x)T_{\epsilon J,q}$$

$$\begin{aligned}
V_{\epsilon J,K}(x) &:= \prod_{j\in J} a_*\frac{1-ax_j^{\epsilon_j}}{1-x_j^{\epsilon_j}}\frac{1-bx_j^{\epsilon_j}}{1+x_j^{\epsilon_j}}\frac{1-cx_j^{\epsilon_j}}{1-q^{1/2}x_j^{\epsilon_j}}\frac{1-dx_j^{\epsilon_j}}{1+q^{1/2}x_j^{\epsilon_j}}\\
&\times \prod_{j,j'\in J,j<j'} t^{-1}\frac{1-tx_j^{\epsilon_j}x_{j'}^{\epsilon_{j'}}}{1-x_j^{\epsilon_j}x_{j'}^{\epsilon_{j'}}}\frac{1-tqx_j^{\epsilon_j}x_{j'}^{\epsilon_{j'}}}{1-qx_j^{\epsilon_j}x_{j'}^{\epsilon_{j'}}}\\
&\times \prod_{j\in J,k\in K} t^{-1}\frac{1-tx_j^{\epsilon_j}x_k}{1-x_j^{\epsilon_j}x_k}\frac{1-tx_j^{\epsilon_j}x_k^{-1}}{1-x_j^{\epsilon_j}x_k^{-1}}\\
U_{K,p}(x) &:= (-1)^p \sum_{\substack{L\subset K,|L|=p\\ \epsilon_l=\pm1,l\in L}}\prod_{l\in L} a_*\frac{1-ax_l^{\epsilon_l}}{1-x_l^{\epsilon_l}}\frac{1-bx_l^{\epsilon_l}}{1+x_l^{\epsilon_l}}\frac{1-cx_l^{\epsilon_l}}{1-q^{1/2}x_l^{\epsilon_l}}\frac{1-dx_l^{\epsilon_l}}{1+q^{1/2}x_l^{\epsilon_l}}\\
&\times \prod_{l,l'\in L,l<l'} t^{-1}\frac{1-tx_l^{\epsilon_l}x_{l'}^{\epsilon_{l'}}}{1-x_l^{\epsilon_l}x_{l'}^{\epsilon_{l'}}}\frac{1-tq^{-1}x_l^{-\epsilon_l}x_{l'}^{-\epsilon_{l'}}}{1-q^{-1}x_l^{-\epsilon_l}x_{l'}^{-\epsilon_{l'}}}\\
&\times \prod_{l\in L,k\in K\setminus L} t^{-1}\frac{1-tx_l^{\epsilon_l}x_k}{1-x_l^{\epsilon_l}x_k}\frac{1-tx_l^{\epsilon_l}x_k^{-1}}{1-x_l^{\epsilon_l}x_k^{-1}}
\end{aligned}$$

and

$$a_{r,s} := (-1)^{r-s}\sum_{r\le l_1\le\cdots\le l_{r-s}\le n}(t^{n-l_1}a_*+t^{-n+l_1}a_*^{-1})\cdots(t^{n-l_{r-s}}a_*+t^{-n+l_{r-s}}a_*^{-1}),$$

where $a_* := (abcdq^{-1})^{1/2}$, $T_{\epsilon J,q} := \prod_{j\in J} T_{\epsilon_j j,q}$, and

$$(T_{\pm j,q}f)(x_1,\cdots,x_n) := f(x_1,\cdots,x_{j-1},q^{\pm1}x_j,x_{j+1},\cdots,x_n).$$

For an indeterminate X, by taking the linear combination of $\{D_r\}$, we can define the operator $D(X)$

$$D(X) := \sum_{i=0}^n D'_i X^{n-i},$$

where $\{D'_i; 0\le i\le n\}$ are defined inductively as follows

$$\begin{aligned}
D'_0 &= 1\\
D'_i &= D_i - \sum_{j<i} a_{i,j}D'_j.
\end{aligned}$$

assuming that the former is already proved. The details of the proofs are given in the main body of the paper.

In order to study the values of P_λ on the submanifold (4), we use (7) by choosing $\mu = (\mu_1, \dots, \mu_n)$ in such a way that

$$\mu_i - \mu_{i+1} = s_i \quad \text{for } i = 1, \dots, k, \tag{8}$$

$$\mu_i - \mu_{i+1} > 2[\tfrac{n}{k+1}](r-1) \quad \text{for } i = k+1, \dots, n-1. \tag{9}$$

From the definition of the Koornwinder-Macdonald polynomial P_μ, it has no pole at the specialization (2) if (9) is valid. Without specialization (2) we have an explicit formula for $u_0(P_\lambda)$ and $u_0(P_\mu)$, and we can easily count the order of zeros (or poles) for them. Using (7), we can prove that $u_\mu(P_\lambda)$ vanishes at (2). Since there exist enough μ's satisfying the conditions (8) and (9), the Laurent polynomial P_λ itself should vanish at (4).

This much is the proof of the first half of Theorem 1.2. Let $J^{(k,r)}$ be the space of symmetric Laurent polynomials P of type $(BC)_n$ satisfying the wheel conditions, and for a positive integer M, let $J^{(k,r)}_M$ be its subspace consisting of P such that the degree of P in each variable x_i is less than M. Because of the invariance for $x_i \leftrightarrow x_i^{-1}$, the dimension of this subspace is finite. From the first half of the proof, we have a lower estimate of the dimension of $J^{(k,r)}_M$. We give an upper estimate of the dimension of the same space by considering its dual space. This is a standard technique originated in the paper by Feigin and Stoyanovsky [3]. Showing that these two estimates are equal, we finish the proof of Theorem 1.2.

2. Properties of the Koornwinder-Macdonald polynomials

Let n be the number of variables. We denote by W_n the group generated by permutations and sign flips ($W_n \cong \mathfrak{S}_n \ltimes (\mathbb{Z}_2)^n$). We consider a W_n-symmetric Laurent polynomial ring

$$\bar{\Lambda}_n = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]^{W_n}. \tag{10}$$

We denote by π_n the set of partitions of length n, $\lambda = (\lambda_1, \cdots, \lambda_n)$. We denote by $\widehat{m}_\lambda$ a monomial W_n-symmetric Laurent polynomial:

$$\widehat{m}_\lambda(x) \ := \ \sum_{\nu \in W_n \lambda} \prod_i x_i^{\nu_i}.$$

The Koornwinder-Macdonald polynomial $P_\lambda(x)$ corresponding to λ is a simultaneous eigenfunction of the difference operators $\{D_r; 1 \le r \le n\}$ (see [1]). The corresponding eigenvalues $E^{(r)}_\lambda$ are of the form

$$E^{(r)}_\lambda := \widehat{m}_{1^r}(x_\lambda) + \sum_{0 \le s < r} a_{r,s} \widehat{m}_{1^s}(x_\lambda)$$

where $x_\lambda = (t^{n-1} q^{\lambda_1} (abcdq^{-1})^{1/2}, t^{n-2} q^{\lambda_2} (abcdq^{-1})^{1/2}, \cdots, t^0 q^{\lambda_n} (abcdq^{-1})^{1/2})$ and $a_{r,s} \in \mathbb{C}[t^{\pm 1}, q^{\pm 1}, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, (abcdq^{-1})^{\pm 1/2}]$.

The condition that a polynomial vanishes on the submanifold (4) is called the wheel condition corresponding to the submanifold (4) and a partition λ satisfying the condition (3) is called a (k,r,n)-admissible partition. Note that if we set $s_{k+1} = r-1-\sum_{i=1}^{k} s_i$, it follows that $x_{k+1}/x_1 = tq^{s_{k+1}}$ from (4) and (1) .

In this paper, we obtain a similar result in the case of n-variable symmetric Laurent polynomials of type $(BC)_n$. Here we say a Laurent polynomial in the variables $x_1,\dots,x_n$ is of type $(BC)_n$ if and only if it is symmetric and invariant for the change of the variable x_1 to x_1^{-1}. The original case in [2] corresponds to A_n. We use the self-dual Koornwinder-Macdonald polynomials P_λ of type $(BC)_n$ [4, 1] in order to characterize the space of symmetric Laurent polynomials of type $(BC)_n$ satisfying the wheel conditions. The Koornwinder-Macdonald polynomials depend on six parameters t,q,a,b,c,d. The self-duality requires

$$a = q^{-1}bcd. \tag{5}$$

We set $W_n := \mathfrak{S}_n \ltimes (\mathbb{Z}_2)^n$. Our main result is

Theorem 1.2. *Let $\lambda = (\lambda_1,\dots,\lambda_n)$ be a (k,r,n)-admissible partition. Then, the self-dual Koornwinder-Macdonald polynomial $P_\lambda \in \mathbb{C}(t,q,b,c,d)[x_1^{\pm1},\dots,x_n^{\pm1}]^{W_n}$ has no pole at (2), and when it is specialized at (2), it satisfies the wheel conditions corresponding to (4). Conversely, the space of symmetric Laurent polynomials of type $(BC)_n$ in $\mathbb{C}(u,b,c,d)[x_1^{\pm1},\dots,x_n^{\pm1}]^{W_n}$ satisfying the wheel conditions is spanned by the self-dual Koornwinder-Macdonald polynomials P_λ specialized at (2) where λ are (k,r,n)-admissible partitions.*

Although the statement of Theorem 1.2 is quite analogous to that of Theorem 1.1, our proof of Theorem 1.2 is different from that of Theorem 1.1 given in [2]. In fact, our method gives an alternative proof simpler than the one given in [2] for the A_n case. We use the duality relation for the self-dual Koornwinder-Macdonald polynomials P_λ. In [5], we obtain a further result by an application of the method used in this paper.

Let us explain the duality relation and the method of our proof. Let $\mu = (\mu_1,\dots,\mu_n)$ be a partition. For $f \in \mathbb{C}(t,q,b,c,d)[x_1^{\pm1},\dots,x_n^{\pm1}]$, we define a specialization $u_\mu(f)$ of f by

$$u_\mu(f) := f(t^{n-1}q^{\mu_1}a, t^{n-2}q^{\mu_2}a, \cdots, q^{\mu_n}a). \tag{6}$$

Here, a is given by (5). In particular, we have

$$u_0(f) = f(t^{n-1}a, t^{n-2}a, \dots, a).$$

The duality relations reads as

$$\frac{u_\mu(P_\lambda)}{u_0(P_\lambda)} = \frac{u_\lambda(P_\mu)}{u_0(P_\mu)}. \tag{7}$$

To prove the two statements, (i) P_λ has no pole at (2), and (ii) P_λ specialized at (2) satisfies the wheel conditions corresponding to (4), we use the duality relation with special choices of μ. Here, we explain only the latter

ZEROS OF SYMMETRIC LAURENT POLYNOMIALS OF TYPE $(BC)_n$ AND SELF-DUAL KOORNWINDER-MACDONALD POLYNOMIALS SPECIALIZED AT $t^{k+1}q^{r-1} = 1$

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ABSTRACT. A characterization of the space of symmetric Laurent polynomials of type $(BC)_n$ which vanish on a certain set of submanifolds is given by using the self-dual Koornwinder-Macdonald polynomials. A similar characterization was given previously for symmetric polynomials of type A_n by using the Macdonald polynomials. We use a new method which exploits the duality relation. The method simplifies a part of the proof in the A_n case.

1. INTRODUCTION

Let k, r, n be positive integers. We assume that $n \geq k+1$ and $r \geq 2$. In [2], n-variable symmetric polynomials satisfying certain zero conditions are characterized by using the Macdonald polynomials [6] specialized at

$$t^{k+1}q^{r-1} = 1. \tag{1}$$

To be precise, the paper [2] works in the following setting. Denote by m the greatest common divisor of $k+1$ and $r-1$. Let ω be an m-th primitive root of unity. Then, the variety given by $t^{\frac{k+1}{m}} q^{\frac{r-1}{m}} = \omega$ is an irreducible component of (1). It is uniformized as follows. Let $\omega_1 \in \mathbb{C}$ be such that $\omega_1^{(r-1)/m} = \omega$. We consider the specialization of t, q in terms of the uniformization parameter u,

$$t = u^{(r-1)/m}, q = \omega_1 u^{-(k+1)/m}. \tag{2}$$

The following result was obtained in [2].

Theorem 1.1. *For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfying*

$$\lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq n-k), \tag{3}$$

the Macdonald polynomial $P_\lambda \in \mathbb{C}(t,q)[x_1, \dots, x_n]^{\mathfrak{S}_n}$ has no pole at (2), and when it is specialized at (2), it vanishes on the submanifold given by

$$x_i/x_{i+1} = tq^{s_i} \qquad \text{for } 1 \leq i \leq k \tag{4}$$

for each choice of non-negative integers s_i such that $\sum_{i=1}^k s_i \leq r-1$. Conversely, the space of symmetric polynomials $P \in \mathbb{C}(u)[x_1, \dots, x_n]^{\mathfrak{S}_n}$ satisfying the above condition is spanned by the Macdonald polynomials P_λ specialized at (2) where λ satisfies (3).