EXISTENCE AND QUASICONFORMAL DEFORMATIONS OF TRANSVERSELY HOLOMORPHIC FOLIATIONS

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Abstract. Existence of complex codimension-one transverse structure is studied using the complex dilatation. As an application, a version of quasiconformal surgeries of foliations are considered.

1. Introduction

In the study of foliations of codimension greater than one, it is natural to restrict oneself to foliations which admit transverse geometric structures. In the present paper, we consider transversely holomorphic foliations of complex codimension one, namely, foliations whose holonomy pseudogroups are generated by biholomorphic local diffeomorphisms of $\boldsymbol{C}$. One might consider that the situation is very restrictive, however, there are many interesting examples. For example, if a holomorphic vector field on $\boldsymbol{C}^2$ with Poincaré type singularities is given, then one can construct naturally a transversely holomorphic foliation of S^3. Although transversely holomorphic foliations of closed 3-manifolds are classified by Brunella [5], Ghys [7] and Carrière [6], it seems difficult to tell if a given flow admits transverse holomorphic structures. In addition, if the ambient manifold is of dimension greater than three, there are very complicated examples [8] so that it seems very hard to classify such foliations. Thus it is important to find good criteria for foliations admitting transverse holomorphic structures as well as to find methods to construct such foliations.

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On open manifolds, a homotopy theoretic approach can be found for example in a work of Haefliger [9]. In this paper, we introduce the notion of quasiconformal foliations, and show such foliations are in fact transversely holomorphic. Quasiconformal foliations are foliation version of quasiconformal groups. Indeed, the construction of holomorphic structure is an application of Tukia's method found in [12]. A version of quasiconformal surgeries of foliations are also considered. This is almost equivalent to the extention problem of given transverse holomorphic structures on the boundary. Under an additional but natural condition, such an extention is possible if the foliation is quasiconformal. The surgery considered here is closely related to the Julia sets for complex codimension one foliations given by Ghys, Gomez-Mont and Saludes [8], and also to characteristic classes of foliations.

This paper is organized as follows. In the section 2, relevant definitions are given. In the section 3, the main theorem is proved. As a corollary, it is shown that under a natural condition, two transversely holomorphic foliations can be glued possibly after changing the transverse structure on one piece. The relevant tools are complex dilatations and the measurable Riemann mapping theorem (see [1] for details). Finally, some examples are presented in the section 4.

2. Definitions

Let $\mathcal{F}$ be a transversely oriented, real codimension two foliation of a manifold M. If the boundary of M is nonempty, then assume that $\mathcal{F}$ is transversal to the boundary. Although we are interested in smooth foliations, we only assume that $\mathcal{F}$ is transversely quasiconformal. Roughly speaking, $\mathcal{F}$ is said to be transversely quasiconformal if the holonomy pseudogroup is generated by quasiconformal local homeomorphisms of $\boldsymbol{C}$. A more precise definition will be given soon later.

Let $\{U_i\}$ be a locally finite foliation chart so that each U_i is homeomorphic to $V_i \times D_i$, where V_i is an open set of $\boldsymbol{R}^{\dim M-2}$ and D_i is an open disc in $\boldsymbol{R}^2$ (if $\mathcal{F}$ is not smooth, we assume that such a chart exists). Let φ_{ji} be the transition function from U_i to U_j, then every φ_{ji} is of the form (ψ_{ji}, γ_{ji}), where γ_{ji} is a function defined on an open subset of D_i. Fix now for each i an identification of D_i with an open disc in $\boldsymbol{C}$, then each γ_{ji} is a local homeomorphism of $\boldsymbol{C}$. Let T be the disjoint union of D_i's, then T can be considered as an open subset of $\boldsymbol{C}$ and also as a subset of M. We call this T a complete transversal. When we need a measure class of T, we consider the restriction of the Lebesgue measure of $\boldsymbol{C}$ to T. We may assume that this measure class coincides with the natural one induced from M if every γ_{ji} preserves the Lebesgue measure class. This is indeed the case if $\mathcal{F}$ is smooth or every γ_{ji} is a quasiconformal homeomorphism.

Definition 2.1. Let $T = \amalg D_i$ be a complete transversal and consider it as an open subset of $\boldsymbol{C}$. Set $\Gamma_1 = \{\gamma_{ji}\}$, where γ_{ji} is as above. We denote by Γ the holonomy pseudogroup associated with T, namely, let Γ be the pseudogroup generated by Γ_1. If we denote by Γ_n the set of local homeomorphisms of $\boldsymbol{C}$ obtained as the composition of at most n elements of Γ_1, then $\Gamma = \bigcup \Gamma_n$. For an element γ of Γ, the domain of γ is denoted by $\operatorname{dom}\gamma$ and the range of γ is denoted by $\operatorname{range}\gamma$.

A foliation is said to be transversely quasiconformal if every γ_{ji} is a quasiconformal local homeomorphism (see [1] for the definition of quasiconformal homeomorphisms). To be more precise, recall the notion of complex dilatation.

Definition 2.2. For an orientation preserving quasiconformal local homeomorphism f of $\boldsymbol{C}$, we denote by $\mu_f(z)$ the complex dilatation (Beltrami coefficient) of f, namely, we set

$$\mu_f(z) = \frac{f_{\bar{z}}(z)}{f_z(z)},$$

where $f_{\bar{z}} = \dfrac{\partial f}{\partial \bar{z}}$ and $f_z = \dfrac{\partial f}{\partial z}$. Such an f is said to be K-quasiconformal for $K \geq 1$ if $\|\mu_f\| \leq \dfrac{1-K}{1+K}$, where $\|\mu_f\|$ denotes the essential supremum of $|\mu_f|$ on the domain of f.

It is known that the partial derivatives are well-defined almost everywhere for quasiconformal local homeomorphisms. A quasiconformal local homeomorphism f is biholomorphic if and only if it is 1-quasiconformal, or equivalently, $|\mu_f(z)| = 0$ a.e. z.

Definition 2.3. Let $\mathcal{F}$ be a real codimension two foliation of a manifold M. If $\partial M \neq \phi$, assume that $\mathcal{F}$ is transversal to ∂M. Let T be a complete transversal for $\mathcal{F}$ and let Γ be the holonomy pseudogroup associated with T. Then, $\mathcal{F}$ is said to be K-quasiconformal with respect to T if every element of Γ is K-quasiconformal.

Since the foliation is assumed to be transversely oriented, $1 > |\mu_\gamma(z)| \geq 0$ for $\gamma \in \Gamma$.

Remark 2.4. The notion of K-quasiconformality depends on the choice of complete transversals. However, provided that M is compact and $\mathcal{F}$ is smooth, if $\mathcal{F}$ is K-quasiconformal for some choice, then $\mathcal{F}$ is K'-quasiconformal for other choices with some $K' \geq 1$.

The notion of transversely quasiconformal foliations is a foliation version of quasiconformal groups. Let G be a group of quasiconformal self-homeomorphisms of an open subset U of $\boldsymbol{C}P^1$ and assume that the action is orientation preserving. The group G is said to be a quasiconformal group if there is a constant $k < 1$ such that $|\mu_g(z)| \leq k$ for any $g \in G$ and a.e. z. A theorem of Tukia [12] shows then that there is a K-quasiconformal homeomorphism $f : U \to \boldsymbol{C}P^1$ such that the action of $f \circ G \circ f^{-1}$ on $f(U)$ is holomorphic. The main theorem in this paper is a foliation version of his theorem.

Definition 2.5. Let $\mathcal{F}$ be a transversely quasiconformal foliation of a manifold M which is transversely orientable and of real codimension two. Let $\{V_i \times D_i, (\psi_{ji}, \gamma_{ji})\}$ be a foliation chart as above. Set $T = \amalg D_i$ and consider T as an open subset of $\boldsymbol{C}$, and let $\varGamma$ be the holonomy pseudogroup associated with T. Let f be a K-quasiconformal homeomorphism from T to its image, then one can form a new foliation $\mathcal{F}'$ whose foliation chart is given by $\{V_i \times f(D_i), (\psi_{ji}, f \circ \gamma_{ji} \circ f^{-1})\}$. The foliation $\mathcal{F}'$ is called a K-quasiconformal conjugate of $\mathcal{F}$. The complex dilatation μ_f of f is called the transverse complex dilatation of the conjugacy.

Definition 2.5 is reduced to a more natural form for transversely holomorphic foliations.

Definition 2.6. Let $\mathcal{F}$ be a transversely holomorphic foliation of M, of complex codimension one. The transverse complex dilatation $\mu_f^{\pitchfork}$ of a foliation preserving diffeomorphism f of M into itself is defined to be the complex dilatation of f in the transverse direction with respect to the transverse holomorphic structure of $\mathcal{F}$. If $||\mu_f^{\pitchfork}||_\infty \leq \dfrac{K-1}{K+1}$, f is said to be transversely K-quasiconformal.

Finally, we introduce the complex dilatation for germs of elements of $\varGamma$.

Definition 2.7. Let $\varGamma$ be a topological groupoid acting on an open subset T of $\boldsymbol{C}$. Suppose that $\varGamma$ is generated by orientation preserving quasiconformal local homeomorphisms of T. We denote by $[\gamma]_x$ the germ of element γ of $\varGamma$ at x if $x \in \operatorname{dom}\gamma$, and set $\varGamma_x = \{[\gamma]_x \,|\, \gamma \in \varGamma, x \in \operatorname{dom}\gamma\}$. For an element $[\gamma]_x$ in $\varGamma_x$, we choose its representative γ' and set $\mu_{[\gamma]_x}(x) = \mu_{\gamma'}(x)$. By abuse of notation, $\mu_{[\gamma]_x}(x)$ is denoted again by $\mu_\gamma(x)$.

Note that $\mu_\gamma(x)$ is well-defined for a.e. x and any γ. An important property of $\varGamma_x$ is that $\varGamma_x$ acts on $\varGamma_{\gamma x}$ from the left, indeed, $\varGamma_{\gamma x}[\gamma]_x = \{[\gamma'\gamma]_x \,|\, \gamma' \in \varGamma_{\gamma x}\} = \varGamma_x$ if $\gamma \in \varGamma_x$. In what follows, we denote $[\gamma]_x$ simply by γ, because only the germ of elements of $\varGamma$ is relevant.

3. Proof of Theorem

The proof presented here is almost identical to Tukia's one in [12] for group actions. Here we make adaptations for foliations and formulate a boundary relative version, which is needed for surgeries. Before proving the main theorem, we introduce the following:

Definition 3.1. Let $\mathbb{D}$ be the Poincaré disc and let X be a subset of $\mathbb{D}$ bounded in the Poincaré metric. Let $P(X)$ be the center of the unique hyperbolic ball $D(x,r)$ with the properties that 1) $D(x,r) \supset X$ and 2) if $D(y,r') \supset X$ and $y \neq x$, then $r' > r$, where $D(x,r)$ denotes the hyperbolic ball centered at x and of radius r. We call $P(X)$ the hyperbolic mean of X.

The existence of such a $D(x,r)$ is shown in [12].

We need one more definition before proving the main theorem.

Definition 3.2. Let $\mathcal{F}$ be a transversely orientable, real codimension two foliation of a manifold M. Let W be a codimension zero submanifold of M and suppose that ∂W is transversal to $\mathcal{F}$. Denote by T_W a complete transversal for $\mathcal{F}|_W$, and denote by Γ_W the holonomy pseudogroup of $\mathcal{F}|_W$ associated with T_W. Choose a complete transversal T of $\mathcal{F}$ such that T contains T_W and that $T \setminus T_W$ is a complete transversal for $\mathcal{F}|_{M \setminus W}$. Let Γ be the holonomy pseudogroup associated with T. Set then $\widetilde{\Gamma}_W = \{\gamma \in \Gamma \,|\, \operatorname{dom}\gamma \subset T_W, \operatorname{range}\gamma \subset T_W\}$.

Elements of $\widetilde{\Gamma}_W$ represent holonomies along leaf paths connecting points of W but not necessarily contained in W.

The main theorem is as follows.

Theorem 3.3.

1) *Let $\mathcal{F}$ be a real codimension two foliation of a manifold M. If $\partial M \neq \phi$, then assume that $\mathcal{F}$ is transversal to ∂M. If $\mathcal{F}$ is K-quasiconformal with respect to a complete transversal T, then $\mathcal{F}$ admits a transverse holomorphic structure after taking a transverse K-quasiconformal conjugate of $\mathcal{F}$.*
2) *Let W be a codimension zero submanifold of M and assume that ∂W is transversal to $\mathcal{F}$. Assume that $\mathcal{F}$ is K-quasiconformal and that a transverse holomorphic structure is given to $\mathcal{F}|_W$. Define T_W, T, Γ_W and $\widetilde{\Gamma}_W$ as in Definition 3.2. Suppose now that $\widetilde{\Gamma}_W$ is an extention of Γ_W by biholomorphic local diffeomorphisms of T_W, then the transverse holomorphic structure of $\mathcal{F}|_W$ extends to a transverse holomorphic structure of $\mathcal{F}$ on M*

after taking a transverse K-quasiconformal conjugate of $\mathcal{F}$ which is transversely holomorphic on W.

Before giving a proof, we explain the condition assumed in 2). Suppose that the given transverse holomorphic structure of $\mathcal{F}|_W$ extends to the whole manifold by modifying T by a quasiconformal homeomorphism f which is biholomorphic on T_W. Then, $f \circ \Gamma \circ f^{-1}$ is generated by biholomorphic local diffeomorphisms. Let γ be an element of Γ which corresponds to a leaf path connecting points of W but not necessarily contained in W. Then $f \circ \gamma \circ f^{-1}$ is biholomorphic. Since f is biholomorphic when restricted to W, the mapping γ itself should be biholomorphic. Hence the assumption in 2) is indispensable. See Example 4.8 for an easy counterexample when this compatibility condition is dropped. The easiest case where the compatibility condition is satisfied is that $\mathcal{F}$ is in fact a flow and that each orbit meets the boundary at most once.

Proof. First we show 1). This part is essentially due to Tukia. We repeat his proof with necessary adaptations, basically following the notations in [12].

Denote by $\mathbb{D} \subset \boldsymbol{C}$ the Poincaré disc. For $x \in T$ and $\gamma \in \Gamma_x$, define a Möbius transformation of $\mathbb{D}$ by the formula

$$T_\gamma(x)(z) = \frac{\gamma_{\bar{z}}(x) + \overline{\gamma_z(x)}z}{\gamma_z(x) + \overline{\gamma_{\bar{z}}(x)}z},$$

where $z \in \mathbb{D}$. If $\gamma' \in \Gamma_{\gamma x}$ is a quasiconformal homeomorphism with the complex dilatation $\mu_{\gamma'}$, then $T_\gamma(x)(\mu_{\gamma'}(\gamma x)) = \mu_{\gamma'\gamma}(x)$, where $\gamma \in \Gamma_x$. For $x \in T$, set $M_x = \{\mu_\gamma(x) \,|\, \gamma \in \Gamma_x\}$, then we have

$$\begin{aligned} T_\gamma(x)(M_{\gamma x}) &= \{T_\gamma(x)(\mu_{\gamma'}(\gamma x)) \,|\, \gamma' \in \Gamma_{\gamma x}\} \\ &= \{\mu_{\gamma'\gamma}(x) \,|\, \gamma' \in \Gamma_{\gamma x}\} \\ &= \{\mu_{\gamma'\gamma}(x) \,|\, (\gamma'\gamma) \in \Gamma_x\} \\ &= M_x. \end{aligned}$$

For $x \in T$, we set $\mu(x) = P(M_x)$ if M_x is bounded, where $P(M_x)$ is the hyperbolic mean of M_x, and $\mu(x) = 0$ either M_x is unbounded or $x \notin T$. Although foliations are considered, μ is still measurable. To see this, recall that Γ is generated by Γ_1, which is countable. We give an order to elements of Γ_1 and denote by γ_i the i-th element. Set $G_i = \{\gamma_1, \gamma_2, \dots, \gamma_i\}$ and let Γ'_n be the subset of Γ_n which consists of the composition of elements of G_n. Then clearly Γ'_n is countable and $\bigcup \Gamma'_n = \Gamma$.

Let M_x' be the subset of $\mathbb{D}$ obtained by collecting $\mu_{\gamma'}(x)$, where $\gamma' \in \Gamma_n' \cap \Gamma_x$. We set $\mu_n(x) = P(M_x')$, where $P(\phi)$ is set to be 0. An elementary argument shows that the sequence $\{\mu_n(x)\}$ converges to $\mu(x)$ if M_x is bounded. As $\mu_n(x)$ is the unique point determined in a measurable way as in Definition 3.1, μ_n is a measurable function. Hence μ is also a measurable function.

Finally let f be the quasiconformal mapping with $\mu_f(x) = \mu(x)$ a.e. x given by the measurable Riemann mapping theorem, then

$$\begin{aligned}\mu_f(x) &= P(M_x)\\ &= P(T_\gamma(x)(M_{\gamma x}))\\ &= T_\gamma(x)(P(M_{\gamma x}))\\ &= T_\gamma(x)(\mu_f(\gamma x))\\ &= \mu_{f\gamma}(x)\end{aligned}$$

for a.e. $x \in T$ and every $\gamma \in \Gamma_x$. This implies that $f \circ \Gamma \circ f^{-1}$ acts as holomorphic transformations on $f(T)$. The dilatation of f can be estimated exactly as in [12].

The second part is shown as follows. Let $\widetilde{W}$ be the saturation of W in M and let $T_{\widetilde{W}}$ be the corresponding subset of T, then $T_{\widetilde{W}}$ is open subset of T invariant under the action of Γ. We define a measurable function μ' on T instead of μ as follows. For $x \in T_{\widetilde{W}}$, set $M_x' = \{\mu_\gamma(x) \,|\, \gamma \in \Gamma_x, \gamma x \in T_W\}$. If $\gamma \in \Gamma_x$, then we have

$$\begin{aligned}T_\gamma(x)(M_{\gamma x}') &= \{T_\gamma(x)(\mu_{\gamma'}(\gamma x)) \,|\, \gamma' \in \Gamma_{\gamma x}, \gamma'\gamma x \in T_W\}\\ &= \{\mu_{\gamma'\gamma}(x) \,|\, \gamma' \in \Gamma_{\gamma x}, \gamma'\gamma x \in T_W\}\\ &= \{\mu_{\gamma'\gamma}(x) \,|\, (\gamma'\gamma) \in \Gamma_x, \gamma'\gamma x \in T_W\}\\ &= M_x'.\end{aligned}$$

The compatibility condition on Γ_W and $\widetilde{\Gamma}_W$ implies that $M_x' = \{0\}$ if $x \in T_W$. Set now

$$\mu'(x) = \begin{cases} P(M_x') & \text{if } x \in T_{\widetilde{W}} \text{ and } M_x' \text{ is bounded,}\\ P(M_x) & \text{if } x \notin T_{\widetilde{W}} \text{ and } M_x \text{ is bounded.}\end{cases}$$

As in the case 1), $\mu'(x)$ is essentially bounded, measurable and invariant under the action of Γ. Since $\mu'(x) = 0$ for $x \in T_W$, the conjugacy is transversely holomorphic on W. This completes the proof. □

Remark 3.4. Sullivan also made a similar construction in [11] involving the barycenter of M_x instead of the hyperbolic mean.

Remark 3.5. Even if the foliation is not transversely orientable, one can find an invariant transverse conformal structure under the same condition. After conjugation, the holonomy pseudogroup is generated by biholomorphic and bi-antiholomorphic local differomorphisms of $\boldsymbol{C}$.

A version of quasiconformal surgery is formulated as follows. Consider the following situation: let M_1 and M_2 be manifolds with boundaries ∂M_1 and ∂M_2. Let $\mathcal{F}_i$ be a transversely holomorphic foliation of M_i transvesal to the boundary ($i = 1, 2$). Let N_1 and N_2 be the union of several components of ∂M_1 and ∂M_2, repectively. Assume that there is a foliation preserving, transversely quasiconformal homeomorphism φ from $(N_1, \mathcal{F}_1|_{N_1})$ to $(N_2, \mathcal{F}_2|_{N_2})$. If one tries to glue M_1 and M_2 by φ, a situation as in the part 2) of Theorem 3.3 occurs. Pulling back the structure by φ, $\mathcal{F}_1$ is given a transverse holomorphic structure on a collar neighborhood W of N_1, because $\mathcal{F}_1$ is transversal to the boundary. The problem is if this structure can be extended to the whole M_1. Let T_W, Γ_W and $\widetilde{\Gamma}_W$ as in Definition 3.2. The latter should be an extention of Γ_W by biholomorphic local diffeomorphisms of T_W. This is sufficient if one more condition is fulfilled.

Corollary 3.6. *Suppose that $\widetilde{\Gamma}_W$ is an extention of Γ_W by biholomorphic local diffeomorphisms of T_W and that φ is transversely K-quasiconformal. Then $M_1 \cup_\varphi M_2$ admits a transversely holomorphic foliation which is the same as $\mathcal{F}_2$ on M_2 and which is transversely K^2-quasiconformal conjugate to $\mathcal{F}_1$ on M_1.*

Proof. Let ℓ be a leaf path, then we may assume that ℓ is transversal to ∂W. If ℓ comes into a component of W and goes out of W, then by pushing ℓ slightly into the interior $\mathrm{int}(M \setminus W)$ of $M \setminus W$, ℓ can be modified so that ℓ stays in $\mathrm{int}(M \setminus W)$ because W is a collar. Hence we may assume that ℓ meets ∂W at most twice. Since the foliation is transversely holomorphic when restricted respectively to W and to $\mathrm{int}(M \setminus W)$, and since the transverse complex dilatation of φ is bounded, the complex dilatation along ℓ is bounded. Hence 2) of Theorem 3.3 can be applied so that one can find a transverse complex structure on M_1. The estimate of distortion follows from the fact that the composition of a K_1-quasiconformal map and a K_2-quasiconformal map is K_1K_2-quasiconformal. □

Remark 3.7.

1) The gluing map φ is a priori transversely K-quasiconformal for some K if N_1 is compact and φ is smooth.
2) If $\mathcal{F}$ is a flow, the transverse complex dilatation of φ is just the complex dilation of the mapping $\varphi|_{N_1}$.

Remark 3.8. As examples in the next section show, this surgery need not produce a new foliation.

Remark 3.9. This kind of surgeries of transversely holomorphic foliations are considered in [8] when the gluing mappings are transversely holomorphic. Corollary 3.6 shows that these mappings need not be transversely holomorphic if one is allowed to modify the transverse holomorphic structure on one piece. We will study such surgeries in detail for one-dimensional foliations in the next section.

4. Examples

First we introduce a simple example where the main theorem is related with quasiconformal deformations of foliations.

Example 4.1. Let $\alpha \in \boldsymbol{C}$ and define a mapping $f : \boldsymbol{C} \to \boldsymbol{C}$ by setting $f_\alpha(z) = e^{2\pi\sqrt{-1}\alpha} z$. Note that α and $\alpha+1$ give the same mapping. Assume that $\alpha, \beta \in \boldsymbol{C} \backslash \boldsymbol{R}$, and define a homeomorphism φ of $\boldsymbol{C}$ to itself by setting

$$\varphi(z) = z\,|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}.$$

We also assume that $\dfrac{\operatorname{Im}\beta}{\operatorname{Im}\alpha} > 0$ so that φ is orientation preserving. The homeomorphism φ is in fact a quasiconformal homeomorphism and we have $\varphi \circ f_\alpha = f_\beta \circ \varphi$, indeed,

$$\begin{aligned}
&\varphi \circ f_\alpha(z)\\
=&\varphi(e^{2\pi\sqrt{-1}\alpha} z) = e^{2\pi\sqrt{-1}\alpha} z (e^{-2\pi \operatorname{Im}\alpha})^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}\,|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}\\
=&e^{2\pi\sqrt{-1}\beta} z\,|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}\\
=&f_\beta \circ \varphi(z).
\end{aligned}$$

The complex dilatation of φ is given by

$$\begin{aligned}
\frac{\varphi_{\bar z}}{\varphi_z}(z) &= \frac{\left(-\sqrt{-1}\frac{\beta-\alpha}{2\operatorname{Im}\alpha}\right)|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}\,\frac{z}{\bar z}}{\left(1-\sqrt{-1}\frac{\beta-\alpha}{2\operatorname{Im}\alpha}\right)|z|^{-\sqrt{-1}\frac{\beta-\alpha}{\operatorname{Im}\alpha}}}\\
&= \frac{\alpha-\beta}{\bar\alpha-\beta}\frac{z}{\bar z}.
\end{aligned}$$

Note that $\bar\alpha - \beta \neq 0$ because $\dfrac{\operatorname{Im}\beta}{\operatorname{Im}\alpha} > 0$.

Let $R = \left\{(t,z) \in \boldsymbol{R} \times \boldsymbol{C} \,\middle|\, |z| \leq e^{2\pi \operatorname{Im}\alpha t}\right\}$ and set $H_\alpha = R/(t+1,z) \sim (t, f_\alpha(z))$. The foliation of $\boldsymbol{R} \times \boldsymbol{C}$ by the lines $\boldsymbol{R} \times \{z\}$ naturally induces a transversely holomorphic foliation of H_α and the orientation of the leaves. Since $\alpha \in \boldsymbol{C} \setminus \boldsymbol{R}$, this foliation is transversal to the boundary. Hence ∂H_α is naturally a complex torus.

According to Corollary 3.6, a new transversely holomorphic structure will be defined if we modify the complex structure of ∂H_α. For example, if we construct ∂H_β in a parallel way and replace the complex structure of ∂H_α with that of ∂H_β, the mapping $\widetilde{\varphi} : H_\alpha \to H_\beta$ defined by $\widetilde{\varphi}(t, z) = (t, \varphi(z))$ describes the deformation.

In order to proceed further, fix the longitude ℓ and the meridian m of ∂H_α by setting $\ell = \{(t, e^{-2\pi\sqrt{-1}\alpha t})\}_{t\in\boldsymbol{R}}$ and $m = \{(0, e^{2\pi\sqrt{-1}t})\}_{t\in\boldsymbol{R}}$. These ℓ and m inherit the orientation of $\boldsymbol{R}$.

If $\operatorname{Im}\alpha < 0$, it is natural to consider that the complex structure of ∂H_α is given from the inside of H_α, namely, the complex structure is given so that its natural orientation is opposite to the orientation as the boundary of H_α. In this case, ∂H_α is isomorphic to $\boldsymbol{C}/\Gamma$ with $\Gamma = \boldsymbol{Z}\ell + \boldsymbol{Z}m$, where ℓ is considered as 1 and m is considered as $-\alpha$. The modulus of ∂H_α is thus equal to $-\alpha$ modulo $\mathrm{PSL}(2;\boldsymbol{Z})$. On the other hand, if $\operatorname{Im}\alpha > 0$, the complex structure is natural if it is chosen so that its natural orientation coincides with the orientation as the boundary. In this case, ∂H_α is again isomorphic to $\boldsymbol{C}/\Gamma$ with $\Gamma = \boldsymbol{Z}\ell + \boldsymbol{Z}m$, where ℓ is again considered as 1 and m is considered as $-\alpha$. However, taking the orientation into account, the modulus of ∂H_α is equal to $-1/\alpha$ modulo $\mathrm{PSL}(2;\boldsymbol{Z})$, in other words, under the usual normalizing conditions, m corresponds to 1 and ℓ corresponds to $-1/\alpha$.

As firstly remarked, α and $\alpha+1$ give the same mapping f_α and thus everything would be the same even if α is replaced with $\alpha+1$. This is indeed the case; adding 1 to α corresponds to the Dehn twist of H_α along m in the clockwise direction, and hence the flow remains isomorphic as a transversely holomorphic flow. In particular, the modulus of the boundary remains unchanged. The value α modulo $\boldsymbol{Z}$ can be detected by the residue of the Bott class [2] (see also [10]). Roughly speaking, it is the linear holonomy along the leaves. Indeed, the residue of the Bott class for H_α is equal to α modulo $\boldsymbol{Z}$.

The above example is closely related with the characteristic classes and also with the Julia sets in the sense of Ghys, Gomez-Mont and Saludes [8]. It is clearly seen in the following example of Bott.

Example 4.2 (Bott)**.** Consider S^3 as the unit sphere in $\boldsymbol{C}^2$ and let (z, w) be the standard coordinate of $\boldsymbol{C}^2$. Let X be the holomorphic vector field on $\boldsymbol{C}^2$ defined by $X = \lambda z\dfrac{\partial}{\partial z} + \mu w\dfrac{\partial}{\partial w}$. Then X determines a holomorphic foliation, denoted by $\widetilde{\mathcal{F}}_{\lambda,\mu}$ of $\boldsymbol{C}^2 \setminus \{0\}$. We assume that $\lambda/\mu \notin \boldsymbol{R}$, then X is transversal to S^3 and to the tori $T_r = \{|z|^2 = r^2,\ |w|^2 = 1 - r^2\}$ contained in S^3. Thus X naturally determines a transversely holomorphic flow, denoted by $\mathcal{F}_{\lambda,\mu}$ on S^3 and a complex structure of T_r.

Consider now the Heegaard decomposition of S^3 by cutting along $T_{1/2}$. Namely, set $\tau_1 = \{(z,w) \in S^3 \,|\, |w| \leq 1/2\}$, $\tau_2 = \{(z,w) \in S^3 \,|\, |z| \leq 1/2\}$ and consider S^3 as $\tau_1 \cup \tau_2$. Then one can find a diffeomorphism from τ_1 to $H_{\mu/\lambda}$ which is transversely holomorphic, and also a transversely holomorphic diffeomorphism from τ_2 to $H_{\lambda/\mu}$. In particular, the moduli of $\partial\tau_1$ and $\partial\tau_2$ are equal to λ/μ and μ/λ (modulo $\mathrm{PSL}(2;\boldsymbol{Z})$), respectively. Conversely speaking, by gluing τ_1 and τ_2 in the standard way, we can reconstruct the original foliation of S^3. The gluing will be discussed in Example 4.5.

We can deform the foliation by varying λ and μ. By using Example 4.1 and the above decomposition, this deformation can be also viewed as a transverse quasiconformal deformation. Quasiconformal deformations are closely related with the Julia set defined in [8] defined as follows. First find a continuous vector field X of the form $X = g(z,\bar{z})\dfrac{\partial}{\partial z}$ of certain regularity such that X is leafwise constant and that $\dfrac{\partial g}{\partial \bar{z}}\dfrac{\partial}{\partial z} \otimes d\bar{z} = \mu\dfrac{\partial}{\partial z} \otimes d\bar{z}$, where μ is the transverse complex dilatation. It can be shown that it is always possible, and then the Julia set is given by the intersection of the zeroes of all such vector fields. The Julia set of $\mathcal{F}_{\lambda,\mu}$ is precisely two closed orbits placed in the core of τ_1 and τ_2. Indeed, such an X which corresponds to the deformation of $\mathcal{F}_{\lambda,\mu}$ to $\mathcal{F}_{\lambda',\mu'}$ is given on $\tau_1 \cong H_\alpha$ by $X = \left(\frac{\alpha-\alpha'}{\bar{\alpha}-\alpha'}\, z\log|z|^2 + h(z)\right)\dfrac{\partial}{\partial z}$ with h being holomorphic, where $\alpha = \mu/\lambda$ and $\alpha' = \mu'/\lambda'$. An easy additional argument shows that $h(z)$ should be of the form $h(z) = kz$, where $k \in \boldsymbol{C}$ if X extends to the whole S^3 and if $\lambda/\mu \in \boldsymbol{C} \setminus \boldsymbol{R}$.

Before discussing the gluings, we introduce another example on S^3. It is probably known although we could not find literature.

Example 4.3. Let X be the 2-plane field on $\boldsymbol{C}^2 \setminus \{0\}$ defined by $X = \lambda z\dfrac{\partial}{\partial z} + \mu\bar{w}\dfrac{\partial}{\partial \bar{w}}$. More precisely, X is the 2-plane field spanned by the vector fields Y and Z defined as follows. First write $z = x_1 + \sqrt{-1}y_1$, $w = x_2 + \sqrt{-1}y_2$, $\lambda = a + \sqrt{-1}b$ and $\mu = p + \sqrt{-1}q$. Set $R_i = x_i\dfrac{\partial}{\partial x_i} + y_i\dfrac{\partial}{\partial y_i}$ and $N_i = -y_i\dfrac{\partial}{\partial x_i} + x_i\dfrac{\partial}{\partial y_i}$ for $i = 1, 2$, and define Y and Z by

$$\begin{aligned} Y &= aR_1 + bN_1 + pR_2 - qN_2, \\ Z &= aN_1 - bR_1 - pN_2 - qR_2. \end{aligned}$$

It is easy to see that X is integrable, in fact, the leaves are $\{(z_0e^{\lambda t}, w_0e^{\overline{\mu t}})\}_{t\in\boldsymbol{C}}$. We denote this foliation of $\boldsymbol{C}^2 \setminus \{0\}$ by $\widetilde{\mathcal{G}}_{\lambda,\mu}$, then $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is transversely holomorphic because it is the pull-back of the foliation $\widetilde{\mathcal{F}}_{\lambda,\mu}$ by the mapping $(z,w) \mapsto (z,\bar{w})$.

A concrete foliation chart for $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is given as follows. Let φ_1 be the mapping from $\boldsymbol{C} \times \boldsymbol{C}$ to $\boldsymbol{C}^\times \times \boldsymbol{C}$ defined by

$$\varphi_1(t,z) = (e^{2\pi\sqrt{-1}t}, \bar{z}\overline{e^{2\pi\sqrt{-1}(\mu/\lambda)t}}).$$

Set now $\widetilde{H}_{\mu/\lambda} = \boldsymbol{C} \times \boldsymbol{C}/(t+1,z) \sim (t, ze^{2\pi\sqrt{-1}(\mu/\lambda)})$ and denote the equivalence class of (t,z) by $[t,z]$. The mapping φ_1 induces a diffeomorphism, denoted again by φ_1, of $\widetilde{H}_{\mu/\lambda}$ to $\boldsymbol{C}^\times \times \boldsymbol{C}$. Indeed, let Log be a fixed branch of the logarithmic function, then $\varphi_1^{-1}(u,v) = \left[\dfrac{1}{2\pi\sqrt{-1}}\operatorname{Log} u, \bar{v}e^{-(\mu/\lambda)\operatorname{Log} u}\right]$. When pulled back to $\widetilde{H}_{\mu/\lambda}$, the foliation $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is nothing but the suspension of the mapping $f_{\mu/\lambda}$ defined in Example 4.1 so that $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is transversely holomorphic on $\widetilde{H}_{\mu/\lambda}$. Similarly, set $\widetilde{H}_{\lambda/\mu} = \boldsymbol{C} \times \boldsymbol{C}/(s+1,w) \sim (s, we^{2\pi\sqrt{-1}(\lambda/\mu)})$ and define a diffeomorphism from $\widetilde{H}_{\lambda/\mu}$ to $\boldsymbol{C} \times \boldsymbol{C}^\times$ by $\varphi_2([s,w]) = (we^{2\pi\sqrt{-1}(\lambda/\mu)s}, \overline{e^{2\pi\sqrt{-1}s}})$. The foliation $\widetilde{\mathcal{G}}_{\lambda,\mu}$ pulled back to $\widetilde{H}_{\lambda/\mu}$ is again transversely holomorphic. Note that $\varphi_2^{-1}(u,v) = \left[\dfrac{1}{2\pi\sqrt{-1}}\overline{\operatorname{Log} v}, ue^{-(\lambda/\mu)\overline{\operatorname{Log} v}}\right]$.

Thus it suffices to see that the transition from $\widetilde{H}_{\mu/\lambda}$ to $\widetilde{H}_{\lambda/\mu}$ is transversely holomorphic. Simply by calculating, we see that

$$\begin{aligned}
\varphi_2^{-1} \circ \varphi_1([t,z]) &= \varphi_2^{-1}(e^{2\pi\sqrt{-1}t}, \bar{z}\overline{e^{2\pi\sqrt{-1}(\mu/\lambda)t}}) \\
&= \left[\frac{1}{2\pi\sqrt{-1}}\overline{\operatorname{Log}\left(\bar{z}\overline{e^{2\pi\sqrt{-1}(\mu/\lambda)t}}\right)}, e^{2\pi\sqrt{-1}t}e^{-(\lambda/\mu)\overline{\operatorname{Log}\bar{z}\overline{e^{2\pi\sqrt{-1}(\mu/\lambda)t}}}}\right] \\
&= \left[\frac{1}{2\pi\sqrt{-1}}\overline{\operatorname{Log}\left(\bar{z}\overline{e^{2\pi\sqrt{-1}(\mu/\lambda)t}}\right)}, e^{-(\lambda/\mu)\overline{\operatorname{Log}\bar{z}}}\right].
\end{aligned}$$

The second component is indeed *holomorphic* in z, moreover, for an appropriate choice of the branch, the function $\overline{\operatorname{Log}\bar{z}}$ can be replaced by $\operatorname{Log} z$.

Set now $\mathcal{N} = x_1\dfrac{\partial}{\partial x_1} + y_1\dfrac{\partial}{\partial y_1} + x_2\dfrac{\partial}{\partial x_2} + y_2\dfrac{\partial}{\partial y_2}$, then $Y \cdot \mathcal{N} = a(x_1^2+y_1^2) + p(x_2^2+y_2^2)$ and $Z \cdot \mathcal{N} = -b(x_1^2+y_1^2) - q(x_2^2+y_2^2)$. By considering $\dfrac{1}{\lambda}X$ instead of X, we see that $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is transversal to S^3 if and only if $\mu/\lambda \notin \boldsymbol{R}^-$. If we assume moreover that $\mu/\lambda \notin \boldsymbol{R}$, then $\widetilde{\mathcal{G}}_{\lambda,\mu}$ is transversal to the torus T_r as in Example 4.2. Assume now $\mu/\lambda \in \boldsymbol{C}/\boldsymbol{R}^-$ and denote by $\mathcal{G}_{\lambda,\mu}$ the induced flow on S^3.

There are no extensions of $\mathcal{G}_{\lambda,\mu}$ to holomorphic vector fields with a Poincaré type singularity at the origin. Indeed, $\mathcal{G}_{\lambda,\mu}$ has two closed orbits $C_1 = \{(e^{2\pi\sqrt{-1}t}, 0)\}_{t\in\boldsymbol{R}}$ and $C_2 = \{(0, e^{2\pi\sqrt{-1}s})\}_{s\in\boldsymbol{R}}$ on S^3. If $\mathcal{F}$ is induced by a holomorphic vector field, the both C_1 and C_2 are positively linked. Note that the flow $\mathcal{G}_{\lambda,\mu}$ is given by the vector field $\left(b(x_1^2+y_1^2)+q(x_2^2+y_2^2)\right)Y+\left(a(x_1^2+y_1^2)+p(x_2^2+y_2^2)\right)Z$ if we assume that C_1 is oriented in the standard way. Indeed, this vector field is equal to $(a^2+b^2)R_1$ if $w=0$ and to $-(p^2+q^2)R_2$ if $z=0$. It follows that C_1 and C_2 are negatively linked. Thus there are no such extensions. This example shows that the extendability of a transversely holomorphic foliation of S^3 to a holomorphic vector field on $\boldsymbol{C}^2$ depends on its realization, namely, the embedding of S^3 into $\boldsymbol{C}^2 \setminus \{0\}$.

Remark 4.4. Assume that $\lambda/\mu \in \boldsymbol{C} \setminus \boldsymbol{R}$, then $\mathcal{G}_{\lambda,\mu}$ is obtained as follows: first consider the foliation $\mathcal{F}_{\lambda,\mu}$ of S^3 as in Example 4.2. Decompose then S^3 as $H_\alpha \cup H_{1/\alpha}$, where $\alpha=\lambda/\mu$. Then glue H_α and $H_{1/\alpha}$ again after turn $H_{1/\alpha}$ over. One obtains again S^3 but now the flow inside $H_{1/\alpha}$ is modified.

In general, transversely holomorphic flows on the Lens spaces can be constructed as follows (S^3 and $S^2 \times S^1$ are included).

Example 4.5. We retain the notations in the previous examples. Assume that $\operatorname{Im}\alpha < 0 < \operatorname{Im}\beta$ and set $\tau_1 = H_\alpha$ and $\tau_2 = H_\beta$, then the flow on H_α is repelling while the flow on H_β is attracting. One can glue τ_1 and τ_2 if the complex structures of their boundaries agree, and will obtain a transversely holomorphic flow on a Lens space. Let ℓ_i and m_i be the longitude and the meridian of τ_i chosen as in Example 4.1. Then, the topological type of the manifold is determined by the image of m_2 on $\partial\tau_1$. We denote by $L(p,q)$ if m_2 is mapped to $p\ell_1+qm_1$. Assume that ℓ_2 is mapped to $r\ell_1+sm_1$. Taking the orientation into account, we may assume that the attaching map from $\partial\tau_2$ to $\partial\tau_1$ is represented by an element $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ of $\mathrm{SL}(2;\boldsymbol{Z})$. Noticing that $\operatorname{Im}\beta > 0 > \operatorname{Im}\alpha$, we see that the complex structures on $\partial\tau_2$ and on $\partial\tau_1$ coincide via the attaching map if and only if $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot (-\alpha) = -\dfrac{1}{\beta}$, where $\mathrm{PSL}(2;\boldsymbol{Z})$ acts on $\boldsymbol{C}P^1$ by linear fractional transformations. In other words, one can always attach H_β to H_α by Corollary 3.6 provided that $\operatorname{Im}\alpha < 0 < \operatorname{Im}\beta$, but once α is fixed, H_β should be deformed into $H_{\beta'}$ with $\beta' = -\dfrac{r\alpha - s}{p\alpha - q}$. The Bott class of the foliation is given by $\alpha+\beta = \alpha - \dfrac{r\alpha - s}{p\alpha - q}$ modulo $\boldsymbol{Z}$ (more precisely, it is an element of $H^3(L(p,q);\boldsymbol{C}/\boldsymbol{Z})$, see [2]).

It is easy to see that $L(\pm p, \pm q)$ are diffeomorphic regardless of the choice of the sign. It is also known that if $p, p' \geq 0$ then $L(p,q) \cong L(p',q')$ if and only if 1) $p=p'$ and $q \equiv \pm q' \mod p$, or 2) $p=p'$ and $qq' \equiv \pm 1 \mod p$ [3], [4]. Thus we will have several distinct flows and its Bott class on a Lens space.

1) $L(\pm 1, q) \cong S^3$. If $p=\pm 1$, then $L(p,q)$ is diffeomorphic to S^3 and $s = \pm(1+qr)$.

Hence $\alpha+\beta = \alpha \mp r + \dfrac{1}{\alpha \mp q} \equiv \alpha + \dfrac{1}{\alpha \mp q} \mod \boldsymbol{Z}$. One can easily verify that the number q and r reflect the Dehn surgeries, namely, twisting along the meridian of H_α and H_β when they are attached. These foliations are obtained from linear holomorphic vector fields, namely, if $p=1$, then in the coordinates as in Example 4.2, the foliation is given by $z\dfrac{\partial}{\partial z} + (\alpha - q)w\dfrac{\partial}{\partial w}$. If $p=-1$, the foliation is given by $z\dfrac{\partial}{\partial z} + (\alpha - q)\bar{w}\dfrac{\partial}{\partial \bar{w}}$.

2) $L(0, \pm 1) \cong S^2 \times S^1$. In this case $r = \mp 1$ and $\beta = -\alpha \mp s$. Hence $\alpha + \beta = \mp s \equiv 0$ modulo $\boldsymbol{Z}$. If $q=-1$, then the foliation is the suspension of the automorphism of $\boldsymbol{C}P^1$ defined by $f_\alpha(z) = e^{2\pi\sqrt{-1}\alpha}z$ for $z \in \boldsymbol{C} \subset \boldsymbol{C}P^1$. If $q=1$, then the foliation is obtained as follows. First consider the foliation of $\boldsymbol{R}^3$ by real lines $\{\{(x,y)\} \times \boldsymbol{R}\}$, where $(x,y) \in \boldsymbol{R}^2$. Remove the origin, then the mapping φ_α : $\boldsymbol{R}^3 \setminus \{0\} \to \boldsymbol{R}^3 \setminus \{0\}$ defined by $\varphi_\alpha(p) = e^{2\pi\sqrt{-1}\alpha}p$, $p \in \boldsymbol{R}^3$, preserves the foliation. Thus there is an induced foliation of $(\boldsymbol{R}^3 \setminus \{0\})/(p \sim \varphi_\alpha(p))$. It is easy to see that the foliation is transversely holomorphic under the natural identification of this space with $\boldsymbol{C}P^1 \times S^1$. This is the foliation corresponding to the case where $q=1$.

3) $L(2, \pm 1) \cong \boldsymbol{R}P^3$. In this case $r = \pm(2s-1)$ and

$$\alpha + \beta = \alpha \mp \left(s - \frac{1}{2}\right) + \frac{1}{2(2\alpha \mp 1)} \equiv \frac{1}{2}(2\alpha \mp 1) \pm \frac{1}{2(2\alpha \mp 1)} \mod \boldsymbol{Z}.$$

Indeed, taking the double cover, the foliation of S^3 induced by $z\dfrac{\partial}{\partial z} + (2\alpha \mp 1)w\dfrac{\partial}{\partial w}$ is obtained. If we replace $p=2$ by -2, then again we are in the situation as in Example 4.3.

4) General case ($p \neq 0$): one has the following equation, namely,

$$\alpha + \beta = \alpha - \frac{r\alpha - s}{p\alpha - q} = \frac{1}{p}(p\alpha - r) + \frac{1}{p(p\alpha - q)}.$$

After taking a $|p|$-fold covering, one obtains S^3 equipped with the foliation as in the case 1). The foliation is defined by a vector field $\dfrac{\partial}{\partial z} + (p\alpha - q)w\dfrac{\partial}{\partial w}$ or $\dfrac{\partial}{\partial z} + (p\alpha - q)\bar{w}\dfrac{\partial}{\partial \bar{w}}$.

Remark 4.6. Let M be a closed 3-manifold and let $\mathcal{F}$ be a nonsingular Morse-Smale flow on M. Assume that the monodromy of closed orbits are either repelling or attracting. Then one can show that there are only a single repelling orbit and a single attracting orbit. Hence M is a Lens space. Example 4.5 can be considered as an example of this kind.

There is a simple example where the set M_x appeared in Section 3 is bounded but there is a sequence of monodromies associated with loops passing through x which is not convergent.

Example 4.7. Define $f : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by setting $f(x,y) = (-2y, x/2)$ and consider the suspension of f. Clearly f is not holomorphic if we consider $\boldsymbol{R}^2$ as $\boldsymbol{C}$ in the standard way. Set now $\psi(x,y) = (x/2, y)$, then $\psi \circ f \circ \psi^{-1}(x,y) = (-y, x)$. Thus by identifying $\boldsymbol{R}^2$ with $\boldsymbol{C}$ via ψ, f is holomorphic. As $f(z) = \frac{5}{4}\sqrt{-1}z - \frac{3}{4}\sqrt{-1}\bar{z}$ and $f^2(z) = -z$,

$$\mu_{f^n} = \begin{cases} -\dfrac{3}{5} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Form now the suspension of f, then one can find a sequence of the holonomy pseudogroup whose complex dilatations do not converge although the foliation clearly admits a transverse holomorphic structure. The conjugacy ψ can be found by Theorem 3.3. Indeed, some calculation shows that the function μ in the proof of Theorem 3.3 is equal to $-1/3$, which is in tern the complex dilatation of the mapping ψ.

Finally, there is an easy example which shows the necessity of the compatibility condition in Theorem 3.3.

Example 4.8. Let $\mathcal{F}$ be the foliation of $T^2 \times [0,1]$ by the intervals $\{p\} \times [0,1]$, $p \in T^2$. Give $T^2 \times \{0\}$ and $T^2 \times \{1\}$ two distinct complex structure, and extend them trivially to $W_0 = T^2 \times [0,\epsilon)$ and $W_1 = T^2 \times (1-\epsilon, 1]$, respectively, where ϵ is a small positive real number. It is then obvious that the transverse holomorphic structure on $W_0 \cup W_1$ cannot be extended to any transverse holomorphic structure of $\mathcal{F}$ on the whole $T^2 \times [0,1]$.

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