Reconstructions of distances by energy forms*†‡

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Abstract

We prove that, if a metric measure space admits a stratification so that each stratum satisfies the strong doubling condition, then the intrinsic distance induced from the Cheeger-type energy form coincides with the original distance. In other words, we can reconstruct the distance function by the Cheeger-type energy form. We also observe that this reconstruction does not work for the Korevaar-Schoen-type energy form.

1 Introduction

The theory of Sobolev spaces for functions on an arbitrary metric measure space is making remarkable progress in recent years (see [C], [HK], [He], [KoSc], etc.). There the Sobolev space is defined as a space of functions with finite energies, and there are several definitions of energy forms on a metric measure space. Among them, in this article, we shall consider Cheeger's and Korevaar and Schoen's ones ([C], [KoSc]), and intend to reveal the difference between them from the geometric point of view.

Our main theorem (Theorem 5.2) asserts that, if a metric measure space admits a stratification so that each stratum satisfies the strong doubling condition (in the sense of Ranjbar-Motlagh [R2]), then the intrinsic distance defined by using the Cheeger-type energy form coincides with the original distance. Here the strong doubling condition (Definition 5.1) can be regarded as a generalization of Measure Contraction Property in [S] as well as the weak measure contraction property of Bishop-Gromov type in [KuSh], and the intrinsic distance (Definition 4.1) is defined as in [BM]. The coincidence between the original distance and the intrinsic distance induced from the canonical Dirichlet form is known for Riemannian manifolds and, more generally, for Alexandrov spaces with lower curvature bounds ([KMS, Theorem 7.1]).

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*Partially supported by the Grant-in-Aid for Scientific Research for Young Scientists (B) 16740034 from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

†Mathematics Subject Classification (2000): 58C05, 53C60.

‡Keywords: intrinsic distance, Cheeger-type energy form, Korevaar-Schoen-type energy form, strong doubling condition, measure contraction property.

One aspect of the theorem is that we can reconstruct the distance function by using the energy form, and another one is that we can distinguish metric spaces by comparing the energy forms on them (Corollary 6.1). We will observe that the analogue is not true for the Korevaar-Schoen-type energy form in the case of Banach spaces (§6). Thus it seems that the Korevaar-Schoen-type energy form is suitable when we consider *Riemannian* spaces such as Alexandrov spaces, rather than *Finsler* spaces.

Acknowledgements. I would like to express my gratitude to Professor Takashi Shioya for his valuable comments.

2 Preliminaries for Cheeger-type energy form

This section is devoted to recalling the definition and some fundamental properties of the Cheeger-type energy form. See [C] for detail. Throughout this article, let (X, d_X) be a metric space and μ be a Borel regular measure on X such that $0 < \mu(B(x,r)) < \infty$ holds for all $x \in X$ and $r > 0$. Here $B(x,r)$ denotes the open ball with center x and radius r. For real numbers $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. We will use some terminologies on Dirichlet forms (with quotation marks '$\cdots$'), consult [FOT] for them.

A Borel measurable function $g : X \longrightarrow [0,\infty]$ is called an *upper gradient* for a function $f : X \longrightarrow \mathbb{R}$ if, for any unit speed curve $\gamma : [0,l] \longrightarrow X$, we have

$$\big|f\big(\gamma(0)\big) - f\big(\gamma(l)\big)\big| \le \int_0^l g\big(\gamma(t)\big)\, dt.$$

Definition 2.1 (Cheeger-type energy form) For $p \in (1,\infty)$ and $f \in L^p(X)$, we define the *Cheeger-type p-energy* $E_p^C(f)$ of f by

$$E_p^C(f) := \inf_{\{(f_i,g_i)\}_{i=1}^\infty} \liminf_{i\to\infty} |g_i|_{L^p}^p,$$

where the infimum is taken over all sequences $\{(f_i, g_i)\}_{i=1}^\infty$ satisfying that $f_i \to f$ in $L^p(X)$ as $i \to \infty$ and that g_i is an upper gradient for f_i for each i. The *Cheeger-type $(1,p)$-Sobolev space* $H^{1,p}(X)$ is defined as a space

$$H^{1,p}(X) := \{f \in L^p(X) \,|\, E_p^C(f) < \infty\}$$

equipped with a norm $|f|_{H^{1,p}} := |f|_{L^p} + E_p^C(f)^{1/p}$.

It is clear by definition that E_p^C is 'Markovian'. Indeed, for any $f \in H^{1,p}(X)$, we have $E_p^C(0 \vee f \wedge 1) \le E_p^C(f)$. It is also known that E_p^C is 'closed' in the sense that $(H^{1,p}(X), |\cdot|_{H^{1,p}})$ is complete ([C, Theorem 2.7]).

Remark 2.2 In general, the Cheeger-type 2-energy form E_2^C is *not bilinear* in the sense that the symmetric form $\mathcal{E} : H^{1,2}(X) \times H^{1,2}(X) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{E}(f_1, f_2) := \frac{1}{4}\{E_2^C(f_1 + f_2) - E_2^C(f_1 - f_2)\}$$

is not bilinear. In particular, $\mathcal{E}$ is not actually a Dirichlet form. However, it is a heart of our reconstruction. Compare this with the Korevaar-Schoen-type energy form which will be defined in §6.

A function $g \in L^p(X)$ is called a *generalized upper gradient* for $f \in H^{1,p}(X)$ if there exist sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ in $L^p(X)$ such that $f_i \to f$ and $g_i \to g$ in $L^p(X)$ as $i \to \infty$, respectively, and that g_i is an upper gradient for f_i for each i. By definition, it clearly holds that $E_p^C(f) \le |g|_{L^p}^p$. A generalized upper gradient $g \in L^p(X)$ for $f \in H^{1,p}(X)$ is said to be *minimal* if it satisfies $|g|_{L^p}^p = E_p^C(f)$.

Theorem 2.3 ([C, Theorem 2.10]) *For any $f \in H^{1,p}(X)$, there exists a unique minimal generalized upper gradient $g \in L^p(X)$ for f.*

We denote by $g_{f,p} \in L^p(X)$ the unique minimal generalized upper gradient for $f \in H^{1,p}(X)$.

Proposition 2.4 ('Strong locality', [C, Corollary 2.25]) *For functions $f_1, f_2 \in H^{1,p}(X)$ and a real number $a \in \mathbb{R}$, if $f_1 = f_2 + a$ holds a.e. on a measurable set $A \subset X$, then we have $g_{f_1,p} = g_{f_2,p}$ a.e. on A.*

We also state two more known properties for the later use.

Lemma 2.5 ([C, Lemma 1.7]) *Let g_1 and g_2 be upper gradients for functions f_1 and f_2, respectively. Then, for any $\varepsilon > 0$, the function $g_1(|f_2| + \varepsilon) + (|f_1| + \varepsilon)g_2$ is an upper gradient for $f_1 f_2$.*

Theorem 2.6 ([C, Theorem 2.5]) *Let $f \in H^{1,p}(X)$ and $\{f_i\}_{i=1}^{\infty} \subset H^{1,p}(X)$ be a sequence such that $f_i \to f$ in $L^p(X)$ as i tends to ∞. Then we have $E_p^C(f) \le \liminf_{i\to\infty} E_p^C(f_i)$.*

3 Regularity

In this section, we show the 'regularity' of E_p^C under some appropriate assumptions on X.

Definition 3.1 (Doubling condition) A metric measure space (X, d_X, μ) is said to satisfy the (*local*) *doubling condition* if there exist constants $C_D = C_D(X) \ge 1$ and $R_D = R_D(X) > 0$ such that

$$\mu\big(B(x, 2r)\big) \le C_D \mu\big(B(x, r)\big)$$

holds for every $x \in X$ and $r \in (0, R_D]$.

Definition 3.2 (Poincaré inequality) A metric measure space (X, d_X, μ) is said to satisfy the (*local*) *weak Poincaré inequality of type* $(1, p)$ if there exist constants $C_P = C_P(X) \ge 1$, $R_P = R_P(X) > 0$, and $\Lambda = \Lambda(X) \ge 1$ such that we have

$$⨍_{B(x,r)} \left| f - ⨍_{B(x,r)} f \, d\mu \right| d\mu \le C_P r \left(⨍_{B(x,\Lambda r)} (g_{f,p})^p \, d\mu \right)^{1/p}$$

for all $x \in X$, $r \in (0, R_P]$, and for all $f \in H^{1,p}(B(x, \Lambda r))$.

As usual, for a measurable set $A \subset X$, we define $⨍_A f\, d\mu := \mu(A)^{-1} \int_A f\, d\mu$. A metric space (X, d_X) is said to be *geodesic* if any two points $x, y \in X$ can be connected by a minimal geodesic between them, i.e., a rectifiable, constant speed curve $\gamma : [0, l] \longrightarrow X$ satisfying $\gamma(0) = x$, $\gamma(l) = y$, and $\mathrm{length}(\gamma) = d_X(x, y)$. A subset $V \subset X$ is said to be *convex* if every two points in V is joined by a minimal geodesic contained in V. Henceforth, let (X, d_X, μ) be a complete, geodesic metric measure space and assume the following.

Assumption 3.3 *There exists an (at most) countable family of open sets in X, say $\{U_n\}_{n=1}^{\infty}$, which satisfies the following:*

(1) $\overline{U_n} \subset U_{n+1}$ *for all* $n \geq 1$;

(2) $X = \bigcup_{n=1}^{\infty} U_n$;

(3) *Denote the connected components of* $V_n := \overline{U_n} \setminus U_{n-1}$ $(= (U_n \setminus \overline{U_{n-1}})^-)$ *by* $\{V_{n,\alpha}\}_{\alpha=1}^{N_n}$, $1 \leq N_n \leq \infty$, *where we put* $U_0 := \emptyset$. *Then each* $V_{n,\alpha}$ *is convex and, for any* $x \in X$, $R > 0$, *and any* $n \geq 1$, *only finitely many* $V_{n,\alpha}$*'s intersect with* $B(x, R)$;

(4) *For any* $x \in X$, $R > 0$, *and any* $n \geq 1$, *we have*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-p} \mu\big((B(U_n, \varepsilon) \setminus U_n) \cap B(x, R)\big) < \infty$$

for a common $p \in (1, \infty)$;

(5) *Each* $(V_{n,\alpha}, d_X, \mu)$ *satisfies the doubling condition and the weak Poincaré inequality of type* $(1, p)$ *for* p *in* (4).

In Assumption 3.3(5), we need to treat not only balls contained in $V_{n,\alpha}$, but also the intersections of balls and $V_{n,\alpha}$, so that it requires the *smoothness* of the boundary of $V_{n,\alpha}$. We also remark that, by the doubling condition in (5) and the completeness, (X, d_X) is proper. So that μ is a Radon measure and (2) implies that, for any $x \in X$ and $R > 0$, we have $B(x, R) \subset U_n$ for some n. (4) means that, roughly speaking, ∂U_n has a codimension at least p in $U_{n+1} \setminus U_n$. In particular, we have $\mu(\partial U_n) = 0$. Therefore X may have various dimensions.

Example 3.4 Let $X = \mathbb{R}^m \cup \mathbb{R}^n / \sim$, $m \leq n$, and $n \geq 2$, where $0_{\mathbb{R}^m} \sim 0_{\mathbb{R}^n}$. We consider the induced length metric and $\mu := \mathcal{L}^m|_{\mathbb{R}^m} + \mathcal{L}^n|_{\mathbb{R}^n}$ on X, where $\mathcal{L}^k$ denotes the k-dimensional Lebesgue measure. Then X satisfies Assumption 3.3 by putting $U_1 = \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}$, $U_2 = X$, and $p = 2$.

For $f \in L^p(X)$ and $x \in V_{n,\alpha}$, we set

$$M(f)(x) := \sup_{0 < r \leq R_D(V_{n,\alpha})/5} ⨍_{B(x,r) \cap V_{n,\alpha}} |f|\, d\mu.$$

The following two lemmas are proved in the standard ways (see [HK] and [He]).

Lemma 3.5 (Maximal function theorem) *Assume Assumption* 3.3.

(i) *For $f \in L^1(X)$ and $t > 0$, we have*

$$\mu(\{x \in V_{n,\alpha} \mid M(f)(x) > t\}) \leq 2C_D(V_{n,\alpha})t^{-1}|f|_{L^1(\{|f| \geq t/2\} \cap V_{n,\alpha})}.$$

(ii) *For $f \in L^p(X)$, we have $|M(f)|_{L^p(V_{n,\alpha})} \leq C_1\big(p, C_D(V_{n,\alpha})\big)|f|_{L^p(V_{n,\alpha})}$.*

Lemma 3.6 *Assume Assumption* 3.3 *and let $f \in H^{1,p}(X)$. For Lebesgue points $x, y \in V_{n,\alpha}$ of f with $d_X(x, y) \leq \min\{R_P(V_{n,\alpha})/2, R_D(V_{n,\alpha})/10\Lambda(V_{n,\alpha})\}$, we have*

$$|f(x) - f(y)| \leq C_2(C_D, C_P)d_X(x, y)\big\{M(g_f^p)(x)^{1/p} + M(g_f^p)(y)^{1/p}\big\}.$$

We define, as a 'core',

$$\mathcal{C} := \{f \in H^{1,p}(X) \cap C_0(X) \mid f \text{ is locally Lipschitz on } U_n \setminus \overline{U_{n-1}} \text{ for all } n \geq 1\}, \quad (3.1)$$

where $C_0(X)$ denotes the set of continuous functions on X with compact supports.

For a continuous function $f : X \longrightarrow \mathbb{R}$ and a point $x \in X$, we define

$$\operatorname{Lip} f(x) := \lim_{r \to 0} \sup_{y \in B(x,r) \setminus \{x\}} \frac{|f(x) - f(y)|}{d_X(x, y)}.$$

Note that $\operatorname{Lip} f$ is Borel measurable and, if f is Lipschitz continuous, then it does not exceed the Lipschitz constant of f. It is not difficult to show that, for a locally Lipschitz function f, $\operatorname{Lip} f$ is an upper gradient for f ([C, Proposition 1.11]).

Theorem 3.7 ('Regularity', cf. [C, Theorem 4.24]) *Assume Assumption* 3.3. *Then the set $\mathcal{C}$ is dense in both $(H^{1,p}(X), |\cdot|_{H^{1,p}})$ and $(C_0(X), |\cdot|_\infty)$.*

Proof. The density in $(C_0(X), |\cdot|_\infty)$ is well-known (see [He, Theorem 6.8]), so that it suffices to show that every function $f \in H^{1,p}(X)$ is approximated by a sequence of functions in $\mathcal{C}$ with respect to $|\cdot|_{H^{1,p}}$. Note that, without loss of generality, we can suppose that $|f| \leq M$ for some $M > 0$ and that $\operatorname{supp} f \subset B(x_0, R)$ for some $x_0 \in X$ and $R > 0$. Furthermore, by Assumption 3.3(2), (3), and (4), we know $\operatorname{supp} f \subset U_N$ for some $N \geq 1$, $\operatorname{supp} f \cap V_n \subset \bigcup_{\alpha=1}^{\alpha_n} V_{n,\alpha}$ for some $\alpha_n < \infty$, and

$$C(n) := \limsup_{\varepsilon \to 0} \varepsilon^{-p}\mu\big((B(U_n, \varepsilon) \setminus U_n) \cap B(x_0, R)\big) < \infty$$

for all $n \geq 1$. Put

$$C_D(V_n) := \max_{1 \leq \alpha \leq \alpha_n} C_D(V_{n,\alpha}), \quad R_D(V_n) := \min_{1 \leq \alpha \leq \alpha_n} R_D(V_{n,\alpha}),$$

and define $C_P(V_n)$, $R_P(V_n)$, and $\Lambda(V_n)$ in the same manner. In the remainder of this proof, we will omit '$\cap B(x_0, R)$' for brevity.

We fix $n \geq 1$ for a while and consider an approximation of $f|_{V_n}$. For $l \geq 1$, set

$$A_l := \{x \in V_n \mid \text{ Lebesgue point of } f, M(g_f^p)(x) \leq l^p\}.$$

Then, by Lemma 3.5(i), we find

$$\mu(V_n \setminus A_l) = \mu(\{M(g_f^p) > l^p\} \cap V_n) \leq 2C_D(V_n) l^{-p} |g_f|^p_{L^p(\{g_f^p \geq l^p/2\} \cap V_n)},$$

and hence $\lim_{l\to\infty} l^p \mu(V_n \setminus A_l) = 0$. It follows from Lemma 3.6 that, for $x, y \in A_l \cap V_{n,\alpha}$ with $\alpha \leq \alpha_n$ and $d_X(x,y) \leq \min\{R_P(V_n)/2, R_D(V_n)/10\Lambda(V_n)\}$, we have

$$|f(x) - f(y)| \leq 2C_2(C_D, C_P) l d_X(x,y).$$

Since $V_{n,\alpha} \cap V_{n,\beta} = \emptyset$ if $\alpha \neq \beta$, we can choose a positive $\delta \leq \min\{R_P/2, R_D/10\Lambda\}/4$ for which $B(V_{n,\alpha}, 2\delta) \cap B(V_{n,\beta}, 2\delta) = \emptyset$ holds if $\alpha \neq \beta$ and $\alpha, \beta \leq \alpha_n$. Take l large enough to satisfy $l^{-2} < \delta$ and

$$\mu(V_n \setminus A_l) < \inf_{x \in V_n \cap B(x_0,R)} \mu\big(B(x,\delta) \cap V_n\big).$$

We remark that the right hand side is positive by the doubling condition together with the convexity of $V_{n,\alpha}$. We can extend $f|_{A_l}$ to $W_{n,l} := B(V_n, l^{-2}) \setminus \overline{U_{n-1}}$ by a local version of MacShane's lemma, more precisely,

$$f_{n,l}(x) := \inf\{f(y) + 2C_2 l d_X(x,y) \,|\, y \in A_l \cap B(x, 2\delta)\}.$$

Note that, for any $x \in W_{n,l}$, we have $\mu(A_l \cap B(x,2\delta)) > 0$ by our construction and that, for any $y_1, y_2 \in A_l \cap B(x, 2\delta)$, we have $|f(y_1) - f(y_2)| \leq 2C_2 l d_X(y_1, y_2)$. Hence $f_{n,l} = f$ on A_l and $f_{n,l}$ is locally Lipschitz on $W_{n,l}$ (with a Lipschitz constant $2C_2 l$). It follows from Assumption 3.3(4) that

$$\lim_{l\to\infty} l^p \mu(W_{n,l} \setminus A_l) = \lim_{l\to\infty} l^p \big\{\mu\big(B(U_n, l^{-2}) \setminus U_n\big) + \mu(V_n \setminus A_l)\big\} = 0,$$

and hence, by Proposition 2.4,

$$\begin{aligned}
&|f - f_{n,l}|_{H^{1,p}(W_{n,l})} \\
&\leq |M + (M + 4C_2\delta l)|_{L^p(W_{n,l}\setminus A_l)} + |g_{f - f_{n,l}}|_{L^p(W_{n,l}\setminus A_l)} \\
&\leq (2M + 4C_2\delta l)\mu(W_{n,l} \setminus A_l)^{1/p} + |g_f|_{L^p(W_{n,l}\setminus A_l)} + 2C_2 l \mu(W_{n,l} \setminus A_l)^{1/p} \\
&\to 0
\end{aligned}$$

as l tends to the infinity.

Fix $m \geq 1$ and define a partition of unity $\{\varphi_n\}_{n=1}^\infty$ by $\varphi_0 \equiv 0$ and

$$\varphi_n(x) := \begin{cases} 1 - \varphi_{n-1}(x) & \text{if } x \in \overline{U_n}, \\ 1 - \big(\operatorname{dist}(x, U_n) l^2\big)^{1/m} & \text{if } 0 < \operatorname{dist}(x, U_n) \leq l^{-2}, \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 1$, inductively. Note that, by Assumption 3.3(1), $\sum_{n=1}^\infty \varphi_n = 1$ holds on $B(x_0, R)$ if l is sufficiently large. Set $f_l := \sum_{n=1}^\infty \varphi_n f_{n,l}$. It is a finite sum since $\operatorname{supp} f$ is bounded, so that $f_l \in C_0(X)$ and f_l is locally Lipschitz on each $U_n \setminus \overline{U_{n-1}}$. We have

$$|f - f_l|_{H^{1,p}} \leq \sum_{n=1}^\infty |\varphi_n (f - f_{n,l})|_{H^{1,p}(W_{n,l}\setminus A_l)},$$

and

$$|\varphi_n(f-f_{n,l})|_{L^p(W_{n,l}\setminus A_l)} \le \big(2M+4C_2(V_n)\delta l\big)\mu(W_{n,l}\setminus A_l)^{1/p} \to 0$$

as l tends to the infinity.

We next estimate $|g_{\varphi_n(f-f_{n,l})}|_{L^p(W_{n,l}\setminus A_l)}$. Let $\{(f_i,g_i)\}_{i=1}^\infty$ be a sequence such that $f_i \to f-f_{n,l}$ and $g_i \to g_{f-f_{n,l}}$ in $L^p(W_{n,l})$ as $i\to\infty$, respectively, and that g_i is an upper gradient for f_i. Clearly we may assume $|f_i| \le 2M+4C_2\delta l$. Since φ_n is bounded, $\varphi_n f_i$ tends to $\varphi_n(f-f_{n,l})$ in $L^p(W_{n,l})$ as $i\to\infty$. Hence it follows from Theorem 2.6 and Lemma 2.5 that

$$\begin{aligned}
&|g_{\varphi_n(f-f_{n,l})}|_{L^p(W_{n,l}\setminus A_l)}\\
&\le \liminf_{i\to\infty}\big\{|\operatorname{Lip}\varphi_n\cdot(|f_i|+i^{-1})|_{L^p(W_{n,l})}+|(\varphi_n+i^{-1})g_i|_{L^p(W_{n,l})}\big\}\\
&\le (2M+4C_2\delta l)|\operatorname{Lip}\varphi_n|_{L^p(W_{n,l})}+|g_{f-f_{n,l}}|_{L^p(W_{n,l}\setminus A_l)}.
\end{aligned}$$

On one hand, in the first part of this proof, we already observe that

$$\lim_{l\to\infty}|g_{f-f_{n,l}}|_{L^p(W_{n,l}\setminus A_l)}=0.$$

On the other hand, we have

$$\begin{aligned}
|\operatorname{Lip}\varphi_n|^p_{L^p(W_{n,l})} &= \int_0^\infty \mu(\{(\operatorname{Lip}\varphi_n)^p>t\}\cap W_{n,l})\,dt\\
&= \int_0^\infty pt^{p-1}\mu(\{\operatorname{Lip}\varphi_n>t\}\cap W_{n,l})\,dt.
\end{aligned}$$

It follows from $|(d/ds)\big(1-(sl^2)^{1/m}\big)| = (1/m)l^{2/m}s^{(1-m)/m}$ together with Assumption 3.3(4) that

$$\begin{aligned}
&|\operatorname{Lip}\varphi_n|^p_{L^p(W_{n,l})}/p\\
&\le \int_{l^2/m}^\infty t^{p-1}\big\{\mu\big(B(U_n,(mtl^{-2/m})^{-m/(m-1)})\setminus U_n\big)\\
&\qquad+\mu\big(B(U_{n-1},(mtl^{-2/m})^{-m/(m-1)})\setminus U_{n-1}\big)\big\}\,dt\\
&\le \{C(n-1)+C(n)\}\int_{l^2/m}^\infty t^{p-1}(mtl^{-2/m})^{-mp/(m-1)}\,dt\\
&= \{C(n-1)+C(n)\}m^{-mp/(m-1)}l^{2p/(m-1)}\Big[-\frac{m-1}{p}t^{-p/(m-1)}\Big]_{l^2/m}^\infty\\
&= \{C(n-1)+C(n)\}\frac{m-1}{p}m^{-p} = \frac{C(n-1)+C(n)}{p}\frac{m-1}{m}m^{1-p}\\
&\to 0
\end{aligned}$$

as m tends to the infinity. Therefore we obtain $|f-f_l|_{H^{1,p}}\to 0$ as $m\to\infty$ and then $l\to\infty$. This completes the proof. □

4 Intrinsic distance

Theorem 3.7 allows us to adopt $\mathcal{C}$ as a set of test functions for defining the intrinsic distance according to Biroli and Mosco ([BM]).

Definition 4.1 (Intrinsic distance) For $p \in (1,\infty)$ and $x, y \in X$, define the *p-intrinsic distance* between x and y by

$$d_p(x,y) := \sup\{f(x) - f(y) \mid f \in \mathcal{C},\ g_{f,p} \le 1 \text{ a.e. on } X\}.$$

We first recall Cheeger's theorem on the minimality of $\operatorname{Lip} f$ for a locally Lipschitz function f.

Theorem 4.2 ([C, Theorem 6.1]) *Let (X, d_X, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1,p)$ for some $p \in (1,\infty)$. Then, for any locally Lipschitz function $f \in H^{1,p}(X)$, we have $g_{f,p} = \operatorname{Lip} f$ a.e. on X.*

The following is an immediate generalization of Theorem 4.2 through Proposition 2.4.

Lemma 4.3 *Let (X, d_X, μ) be a complete metric measure space satisfying Assumption 3.3 for some $p \in (1,\infty)$. Then, for any function $f \in H^{1,p}(X)$ which is locally Lipschitz on each $U_n \setminus \overline{U_{n-1}}$, we have $g_{f,p} = \operatorname{Lip} f$ a.e. on X.*

Proposition 4.4 *Let (X, d_X, μ) be a complete, geodesic metric measure space satisfying Assumption 3.3 for some $p \in (1,\infty)$. Then we have*

$$d_X \le d_p \le 2C_2\big(C_D(V_{n,\alpha}), C_P(V_{n,\alpha})\big) d_X$$

on $V_{n,\alpha}$ for each $n, \alpha \ge 1$. Here C_2 is a constant in Lemma 3.6. In particular, d_p gives the same topology on X as d_X.

Proof. Fix two points $x, y \in V_{n,\alpha}$ and a function $f \in \mathcal{C}$ with $g_{f,p} \le 1$ a.e. on X. For any $\varepsilon > 0$, we can find Lebesgue points $x', y' \in V_{n,\alpha}$ of f satisfying $d_X(x,x') \le \varepsilon$, $d_X(y,y') \le \varepsilon$, $|f(x) - f(x')| \le \varepsilon$, and $|f(y) - f(y')| \le \varepsilon$. It follows from Lemma 3.6 that

$$\begin{aligned}|f(x) - f(y)| &\le |f(x') - f(y')| + 2\varepsilon \le 2C_2 d_X(x', y') + 2\varepsilon \\ &\le 2C_2 d_X(x,y) + 4C_2\varepsilon + 2\varepsilon.\end{aligned}$$

Since both $f \in \mathcal{C}$ and $\varepsilon > 0$ are arbitrary, we obtain $d_p(x,y) \le 2C_2 d_X(x,y)$.

Put $f(z) := \max\{d_X(x,y) - d_X(x,z), 0\}$ for $z \in X$. Then f is 1-Lipschitz, $f \in \mathcal{C}$, and clearly $f(x) - f(y) = d_X(x,y)$. Therefore we obtain $d_X(x,y) \le d_p(x,y)$. $\square$

5 Strong doubling condition and main theorem

To improve the estimate $d_X \le d_p \le 2C_2 d_X$ in Proposition 4.4 to the equality $d_p = d_X$, we need a kind of measure contraction property. A measurable map $\Phi : X \times X \times [0,1] \longrightarrow X$ is called a *geodesic bicombing* if, for each $x, y \in X$, a map $[0,1] \ni t \longmapsto \Phi(x,y,t) \in X$ gives a minimal geodesic between x and y.

Definition 5.1 ([R2]) A geodesic metric measure space (X, d_X, μ) is said to satisfy the (*local*) *strong doubling condition* along a geodesic bicombing $\Phi : X \times X \times [0,1] \longrightarrow X$ if there exist positive numbers $a = a(X) > 0$ and $R = R(X) > 0$ such that we have

$$\mu\big(\Phi(x, A, t)\big) \geq a\mu(A)$$

for any $x \in X$, $r \in (0, R]$, $t \in [1/2, 1]$, and any measurable subset $A \subset B(x, r)$.

Finite dimensional Alexandrov spaces with lower curvature bounds as well as Riemannian manifolds with lower Ricci curvature bounds satisfy the strong doubling condition (see [KuSh], [R2]). Clearly the strong doubling condition implies the doubling condition. Furthermore, it is shown in [R2] that, if (X, d_X, μ) satisfies the local strong doubling condition, then it satisfies the weak Poincaré inequality of type $(1, 1)$ for the Cheeger-type energy form (see also [R1]).

Theorem 5.2 *Let* (X, d_X, μ) *be a complete, geodesic metric measure space satisfying Assumption* 3.3(1), (2), (3), *and* (4) *for some* $p \in (1, \infty)$. *If, in addition, each* $(V_{n,\alpha}, d_X, \mu)$ *satisfies the strong doubling condition* (*along a geodesic bicombing* $\Phi? = \Phi_{n,\alpha}$), *then we have* $d_X = d_p$ *on* X.

Proof. Note that the strong doubling condition of $V_{n,\alpha}$ implies Assumption 3.3(5). We already know $d_X \leq d_p$ by Proposition 4.4, so that we need only to show $d_p \leq d_X$. We first show this inequality on $V_{n,\alpha}$. To do this, it is sufficient to prove that every $f \in \mathcal{C}$ with $g_{f,p} \leq 1$ a.e. on X is 1-Lipschitz on $V_{n,\alpha}$. Suppose that there exist two distinct points $x, y \in V_{n,\alpha}$ and a function $f \in \mathcal{C}$ with $g_{f,p} \leq 1$ a.e. on X such that we have $|f(x) - f(y)| \geq (1 + 2\varepsilon)d_X(x, y)$ for some $\varepsilon > 0$. Since $V_{n,\alpha}$ is convex, without loss of generality, we may assume $d_X(x, y) \leq R(V_{n,\alpha})/2$. We remark that $f|_{V_{n,\alpha}}$ is locally Lipschitz by Lemma 3.6. Since f is continuous, we can find a sufficiently small $r > 0$ such that $|f(w) - f(z)| \geq (1+\varepsilon)d_X(w, z)$ holds for all $w \in B(x, r) \cap V_{n,\alpha}$ and $z \in B(y, r) \cap V_{n,\alpha}$. We define $A := \{z \in V_{n,\alpha} \,|\, \mathrm{Lip}\, f(z) \geq 1 + \varepsilon\}$, denote by χ_A the characteristic function of A, and set k as a smallest integer not smaller than $-\log_2 r$. We put $\Phi_0 := \Phi$ and $\Phi_i(x, z, t) := \Phi(x, \Phi_{i-1}(x, z, 1/2), t)$ for $i = 1, 2, \ldots, k-1$, inductively. Then we have, for every $z \in B(y, r) \cap V_{n,\alpha}$,

$$\sum_{i=0}^{k-1} \int_{1/2}^{1} \chi_A\big(\Phi_i(x, z, t)\big)\, dt > 0$$

since $|f(\Phi_{k-1}(x, z, 1/2)) - f(z)| \geq (1+\varepsilon)d_X(\Phi_{k-1}(x, z, 1/2), z)$. Note also that $\mu(B(y, r) \cap$

$V_{n,\alpha}) > 0$. By the strong doubling condition, we obtain

$$\begin{aligned}0 &< \sum_{i=0}^{k-1} \int_{B(y,r)\cap V_{n,\alpha}} \int_{1/2}^{1} \chi_A\big(\Phi_i(x,z,t)\big)\,dt\,d\mu(z)\\ &= \sum_{i=0}^{k-1} \int_{1/2}^{1} \int_{B(y,r)\cap V_{n,\alpha}} \chi_A\big(\Phi_i(x,z,t)\big)\,d\mu(z)\,dt\\ &= \sum_{i=0}^{k-1} \int_{1/2}^{1} \int_{\Phi_i(x,B(y,r)\cap V_{n,\alpha},t)} \chi_A(z)\,\big(\Phi_i(x,\cdot,t)_*(d\mu)\big)(z)\,dt\\ &\le \sum_{i=0}^{k-1} a(V_{n,\alpha})^{-(i+1)} \int_{1/2}^{1} \int_{\Phi_i(x,B(y,r)\cap V_{n,\alpha},t)} \chi_A(z)\,d\mu(z)\,dt.\end{aligned}$$

Therefore we have $\mu(A) > 0$, but it is a contradiction. Thus every $f \in \mathcal{C}$ with $g_{f,p} \le 1$ a.e. on X is 1-Lipschitz on $V_{n,\alpha}$, so that we obtain $d_p = d_X$ on $V_{n,\alpha}$.

For general $x \in V_{n,\alpha}$ and $y \in V_{m,\beta}$, let $\gamma : [0, d_X(x,y)] \longrightarrow X$ be a minimal geodesic between them. By Assumption 3.3(2) and (3), we have

$$\gamma([0, d_X(x,y)]) \subset \bigcup_{k=1}^{N}\bigcup_{\sigma=1}^{N} V_{k,\sigma}$$

for some $N \ge 1$. Fix $\varepsilon > 0$ and set $t_0 := 0$ and

$$t_1 := \sup\{t \in [0, d_X(x,y)] \,|\, \gamma(t) \in V_{n,\alpha}\}.$$

If $t_1 > d_X(x,y) - \varepsilon/N^2$, then we put $t_2 := d_X(x,y)$. If not, then we put $t_2 := \sup\{t \in [0, d_X(x,y)] \,|\, \gamma(t) \in V_{k,\sigma}\}$, where $k, \sigma \le N$ are such that $\gamma(t_1 + \varepsilon/N^2) \in V_{k,\sigma}$. Note that, by the definition of t_1, we have $(n', \alpha') \ne (n, \alpha)$. We iterate this construction and obtain a sequence $0 = t_0 \le t_1 < t_1 + \varepsilon/N^2 \le t_2 < \cdots \le t_M = d_X(x,y)$. By our construction, we observe $M \le N^2$. Since the proof is common, we assume $t_M > t_{M-1} + \varepsilon/N^2$ in the following. By the first part of this proof, we know $d_p(\gamma(t_0), \gamma(t_1)) = d_X(\gamma(t_0), \gamma(t_1))$ and $d_p(\gamma(t_l + \varepsilon/N^2), \gamma(t_{l+1})) = d_X(\gamma(t_l + \varepsilon/N^2), \gamma(t_{l+1}))$ for $l = 1, 2, \ldots, M-1$, and hence we conclude that

$$\begin{aligned}&d_p(x,y)\\ &\le d_p\big(\gamma(t_0),\gamma(t_1)\big) + \sum_{l=1}^{M-1}\big\{d_p\big(\gamma(t_l),\gamma(t_l+\varepsilon/N^2)\big) + d_p\big(\gamma(t_l+\varepsilon/N^2),\gamma(t_{l+1})\big)\big\}\\ &\le d_X\big(\gamma(t_0),\gamma(t_1)\big) + 2(M-1)C_2\frac{\varepsilon}{N^2} + \sum_{l=1}^{M-1} d_X\big(\gamma(t_l+\varepsilon/N^2),\gamma(t_{l+1})\big)\\ &\le d_X(x,y) + 2C_2\varepsilon,\end{aligned}$$

where we set $C_2 := \max_{1\le k,\sigma\le N} C_2(C_D(V_{k,\sigma}), C_P(V_{k,\sigma}))$. Since $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

In a quite general setting, the condition $\mathrm{Lip}\, f \le 1$ a.e. does not imply the 1-Lipschitz continuity of a Lipschitz function f. At least, the Poincaré inequality is necessary to ensure that, if f has zero energy, then it is constant.

6 Distinguish metric spaces by energy forms

By Theorem 5.2 with $X = V_{1,1}$, we find the following.

Corollary 6.1 *Let d and d' be two equivalent distance functions on a measure space (X, μ) such that both (X, d) and (X, d') are complete and geodesic and that both (X, d, μ) and (X, d', μ) satisfy the strong doubling condition. If $g^d_{f,p} = g^{d'}_{f,p}$ holds a.e. on X for some $p \in (1, \infty)$ and every $f \in \mathcal{C}$, then we have $d = d'$.*

Here we denote by $g^d_{f,p}$ and $g^{d'}_{f,p}$ the minimal generalized upper gradients for f with respect to d and d', respectively. We remark that the core $\mathcal{C}$ defined as (3.1) is common to d and d' since they are equivalent.

Corollary 6.1 means that we can distinguish two (equivalent) distance functions by comparing energy measures of functions in the core $\mathcal{C}$. In the reminder of this article, we shall observe that, if we consider the Korevaar-Schoen-type energy form in place of the Cheeger-type one, then we can not distinguish some metric spaces. In particular, we can not reconstruct the distance function by using the Korevaar-Schoen-type energy form. In the following, we treat only the case of $p = 2$.

Definition 6.2 (Korevaar-Schoen-type energy form) For $f \in L^2(X)$, define the *Korevaar-Schoen-type 2-energy* $E_2^{KS}(f)$ of f by

$$E_2^{KS}(f) := \limsup_{r \to 0} \int_X \left\{ \int_{B(x,r)\setminus\{x\}} \frac{|f(x) - f(y)|^2}{r^2} \frac{d\mu(y)}{\mu(B(y,r))^{1/2}} \right\} \frac{d\mu(x)}{\mu(B(x,r))^{1/2}}.$$

This definition is due to [S] and is slightly different from that in [KoSc]. By [S, Theorem 5.6], E_2^{KS} gives a Dirichlet form if (X, d_X, μ) satisfies the strong Measure Contraction Property.

Example 6.3 Let $n \geq 2$ and $(X, d_X, \mu) = (\mathbb{R}^n, |\cdot|, dx)$ be an n-dimensional Banach space with the standard measure. Then there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that we have $E_2^{KS}(f) = \int_{\mathbb{R}^n} \langle df, df \rangle \, dx$ for every $f \in C_0^\infty(\mathbb{R}^n)$. On the other hand, we know $E_2^C(f) = \int_{\mathbb{R}^n} |df|_*^2 \, dx$, where we denote by $(\mathbb{R}^n, |\cdot|_*)$ the dual space of $(\mathbb{R}^n, |\cdot|)$.

Note that the moduli space consisting of inner products on $\mathbb{R}^n$, that is, the space of positive definite symmetric $n \times n$ matrices, is obviously finite dimensional. However, the moduli space of norms on $\mathbb{R}^n$ is infinite dimensional, so that many norms give the same Korevaar-Schoen-type energy form. For example, for all l_p-norm $|\cdot|_p$ with $p \in [1, \infty]$, the associated inner product is a constant mutilple of the standard Euclidean one (because they are symmetric enough). If we denote by c_p that constant, then the Korevaar-Schoen-type energy form with respect to the norm $c_p^{1/2}|\cdot|_p$ coincides with the standard Dirichlet form on $\mathbb{R}^n$. Therefore the Korevaar-Schoen-type energy form can not distinguish these (uncountably many) norms.

References

[BM] M. Biroli and U. Mosco, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. **169** (1995), 125–181.

[C] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428–517.

[FOT] M. Fukushima, Y. Oshima, and M. Takeda, Dirichlet forms and symmetric Markov processes, de Gruyter Studies in Mathematics 19, Walter de Gruyter & Co., Berlin, 1994.

[HK] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.

[He] J. Heinonen, Lectures on analysis on metric spaces, Universitext, Springer-Verlag, New York, 2001.

[KoSc] N. J. Korevaar and R. M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), 561–659.

[KMS] K. Kuwae, Y. Machigashira, and T. Shioya, *Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces*, Math. Z. **238** (2001), 269–316.

[KuSh] K. Kuwae and T. Shioya, *On generalized measure contraction property and energy functionals over Lipschitz maps*, Potential Anal. **15** (2001), 105–121.

[R1] A. Ranjbar-Motlagh, *A note on the Poincaré inequality*, Studia Math. **154** (2003), 1–11.

[R2] A. Ranjbar-Motlagh, *On the Poincaré inequality for abstract spaces*, preprint, 2002.

[S] K.-T. Sturm, *Diffusion processes and heat kernels on metric spaces*, Ann. Probab. **26** (1998), 1–55.