APPLICATION OF FLOER HOMOLOGY OF LANGRANGIAN SUBMANIFOLDS TO SYMPLECTIC TOPOLOGY

KENJI FUKAYA

Contents

1. Introduction

In this article, we review theory of Floer homology of Lagrangian submanifolds developped in [13] jointly with Oh, Ohta, Ono. We include some of new results whose detail will appear later in [12] and also some results which will be included in a revised version of [13]. In this article, we emphasise its application to symplectic topology of Lagrangian submanifold. In other surveys [10, 11, 26], more emphasise is put on its application to Mirror symmetry.

The author would like to thank the organizers of the school "Morse theoretic methods in non-linear analysis and symplectic topology" to give him an opportunity to write this article. He would also like to thank his collaborators Y-G. Oh, H. Ohta, K. Ono and acknowledge that many of the results discussed in this article are our joint work.

2. Lagrangian submanifold of $\mathbb{C}^n$

In this section we extract some arguments of the Floer theory and use it to study Lagrangian submanifolds of $\mathbb{C}^n$. We do not introduce Floer homology in this section yet. Let $x_i + \sqrt{-1}y_i$, $i = 1, \cdots, n$ be the canonical coordinate of $\mathbb{C}^n$. We put the standard symplectic structure $\omega = \sum dx_i \wedge dy_i$ to $\mathbb{C}^n$.

Definition 2.1. An n dimensional submanifold L of $\mathbb{C}^n$ is said to be a *Lagrangian submanifold* if $\omega|_L = 0$.

In this article we always assume that our Lagrangian submanifold is compact.

For a Lagrangian submanifold $L \subset \mathbb{C}^n$ we define an *energy function* $E : \pi_1(L) \to \mathbb{R}$ by

$$E_L(\gamma) = \sum_i \int_{S^1} \gamma^* x_i dy_i = \int_{D^2} u^*\omega. \tag{2.1}$$

Here $u : D^2 \to \mathbb{C}^n$ is a map with $u(\partial D^2) \subseteq L$ and $[u(\partial D^2)] = \gamma$.

Remark 2.1. For a general Lagrangian submanifold $L \subset M$, we can define $E_L : H_2(M, L) \to \mathbb{R}$ in the same way.

The following result due to Gromov is one of the most basic results of symplectic topology.

Theorem 2.1 (Gromov [18])**.** *$E_L \neq 0$ for any embedded Lagrangian submanifold $L \subset \mathbb{C}^n$.*

Theorem 2.1 in particular implies $H_1(L;\mathbb{Q}) \neq 0$ for any Lagrangian submanifold L of $\mathbb{C}^n$. To state the next result we need to define Maslov index of Lagrangian submanifold.

Definition 2.2. The set of all real n dimensional $\mathbb{R}$ linear subspaces V of $\mathbb{C}^n$ with $\omega|_V = 0$ is called *Lagrangian Grassmannian.* We write it as $GLag_n$.

Let Gr_n be the set of all real n dimensional $\mathbb{R}$ linear subspaces V of $\mathbb{C}^n$, that is the Grassmanian manifold.

Lemma 2.2. *$\pi_1(GLag_n) = \mathbb{Z}$. The map $GLag_n \to Gr_n$ induces a surjective homomorphism $\mathbb{Z} = \pi_1(GLag_n) \to \pi_1(Gr_n) = \mathbb{Z}_2$.*

See for example [1] for its proof.

In case $L \subset \mathbb{C}^n$ is a Lagrangian submanifold, we define a map $Ti_L : L \to GLag_n$ by

$$Ti_L(p) = T_pL \in GLag_n. \tag{2.2}$$

Definition 2.3. The *Malsov index* $\eta_L : \pi_1(L) \to \mathbb{Z} = \pi_1(GLag_n)$ is a homomorphism induced by Ti_L.

Remark 2.2. In case of general Lagrangian submanifold $L \subset M$, we can define $\eta_L : \pi_2(M, L) \to \mathbb{Z}$ in a similar way. The composition of $\pi_2(M) \to \pi_2(M, L)$ and η_L coincides with $\beta \mapsto 2\beta \cap c^1(M)$, where $c^1(M)$ is the first chern class.

Lemm 2.2 imnplies that, if L is oriented, then image of η_L is in $2\mathbb{Z}$.

Theorem 2.3 ([13])**.** *Let $L \subseteq \mathbb{C}^n$ be a Lagrangian submanifold. If $H^2(L;\mathbb{Q}) = 0$ and if L is spin, then $\eta_L \neq 0$.*

Remark 2.3. We actually do not need to assum L is spin. See [13].

Let us consider the case $L = S^1 \times S^n$ for example. The assumption of Theorem 2.3 is satisfied if $n > 2$.

For $n = 1$, $L = S^1 \times S^1$, it was proved by Viterbo [34] and Polterovich [28], that there exists $\gamma \in \pi_1(L)$ such that $E(\gamma) > 0$ and $\eta_L(\gamma) = 2$. For general n we can prove the following.

Proposition 2.4 ([25, 13]). *Let $L = S^1 \times S^n \subset \mathbb{C}^n$ be a Lagrangian submanifold with $n \geq 2$. We choose the generator $\gamma \in \pi_1(L) = \mathbb{Z}$, such that $E(\gamma) > 0$.*

If n is odd then there exists a positive integer ℓ such that $\eta_L(\ell\gamma) = n + 1$.

If n is even then either $\eta_L(\gamma) = 2$ or exists a positive integer ℓ such that $\eta_L(\ell\gamma) = 2 - n$.

Remark 2.4. In case $\eta_L(\gamma) > 0$, Proposition 2.4 follows from [25]. Proposition 2.4 follows from the results of [13] directly as we will see in §5, though it was not written in its 2000 version explicitely.

Actually, in the case n is even, we can show $\eta_L(\gamma) = 2$. Namely we can exclude the second case. See §13.

Before proving the results stated above we mention some of the constructions of Lagrangian submanifolds. See [2] for detail.

We say $i : L \to \mathbb{C}^n$ is a *Lagrangian immersion* if i is an immersion, $\dim L = n$ and if $i^*\omega = 0$. The map $Ti_L : L \to GLag_n$ is defined in the same way.

Two Lagrangian immersions i_0, i_1 are said to be regular homotopic to each other if there exists a faimily of Lagrangian immersions i_s $s \in [0,1]$ joining them.

Theorem 2.5 (Gromov[19]-Lees[23]). (1) *There exists a Lagrangian immersion $i : L \to \mathbb{C}^n$ if and only if $TL \otimes \mathbb{C}$ is a trivial bundle.*

(2) *We assume that $TL \otimes \mathbb{C}$ is trivial. Then $i \mapsto [i_L]$ induces a bijection from the set of regular homotopy classes of Lagrangian immersions $i : L \to \mathbb{C}^n$ to the set of the homotopy classes of maps $L \to GLag_n$.*

The proof is based on Gromov's h-principle. See [9].

Theorem 2.5 implies that every element of $\pi_n(GLag_n)$ is realized by a Lagrangian immersion $i : S^n \to GLag_n$. (We remark that Theorem 2.1 implies that none of them is realized by a Lagrangian *embedding*.)

Theorem 2.6 (Audin-Lalonde-Polterovich[2]). *If $i_L : L \to \mathbb{C}^n$ is a Lagrangian immersion and $i_{L'} : L' \to \mathbb{C}^m$ is a Lagrangian embedding then there exists a Lagrangian embedding $i : L \times L' \to \mathbb{C}^{n+m}$ such that $Ti : L \times L' \to GLag_{n+m}$ is homoltopic to $sum \circ (Ti_L \otimes Ti_{L'})$. Where $sum : GLag_n \times GLag_m \to GLag_{n+m}$ is given by $(V, W) \mapsto V \oplus W$.*

Proof: Let us take $f : L \to \mathbb{R}^m \subset \mathbb{C}^m$ such that $I = (i_L, f) : L \to \mathbb{C}^{n+m}$ is an embedding. Clearly $I^*\omega = 0$. Then Weinstein's theorem (see [1, 9]) implies that there exists a neighborhood U of $I(L)$ in $\mathbb{C}^{n+m}$ and a symplectic embedding $I' : U \subset T^*L \times \mathbb{C}^m$ such that $I' \circ I$ is an embedding $q \mapsto ((q, 0), 0)$. Then, for small ϵ, the map $I_+ : L \times L' \to T^*L \times \mathbb{C}^m$, $I_+(q, Q) = ((q, 0), \epsilon i_{L'}(Q))$ sends $L \times L'$ to $I'(U)$. We put $i_{L,L'} = I' \circ I_+ : L \times L' \to \mathbb{C}^{n+m}$. □

We take $L = S^n$ and $L' = S^1$. We remark that there is an obvious Lagrangian embedding $S^1 \to \mathbb{C}$. Maslov index of it is given by $\eta_{S^1}[S^1] = 2$. Hence we have an embedding $S^1 \times S^n \to \mathbb{C}^{n+1}$ such that $\eta_{S^1 \times S^n}([S^1]) = 2$. Note regular homotopy class of Lagrangian immersion of $S^1 \times S^n$ is identified with

$\pi_1(GLag_n) \times \pi_n(GLag_n)$. The construction above realize any elements of the form $(2, \alpha) \in \pi_1(GLag_n) \times \pi_n(GLag_n)$ as an Lagrangian embedding.

In [28] Polterovich constructed a Lagrangian embedding $S^1 \times S^{2n-1} \subset \mathbb{C}^{2n}$ such that $\eta_L(\gamma) = 2n$, as follows[1]. ($\gamma \in \pi_1(L)$ is a generator with $E(\gamma) > 0$.) Let $S^{2n-1} = \{\vec{x} \in \mathbb{R}^{2n} | |\vec{x}| = 1\}$. We then put $L = \{\theta\vec{x} \in \mathbb{C}^{2n} | \vec{x} \in S^{2n-1}, \theta \in \mathbb{C}, |\theta| = 1\}$. We can show that L is diffeomorphic to $S^1 \times S^{2n-1}$ and is a Lagrangian submanifold. We can also prove $\eta_L(\gamma) = 2n$.

In case $k = 2$, $L = S^1 \times S^3$, then $\eta_L(\gamma)$ is 2 or 4 according to Proposition 2.4. They both actually occur. In case $k = 3$, $L = S^1 \times S^5$, Proposition 2.4 implies $\eta_L(\gamma)$ is 2 or 6. They again both occur. In case $k = 4$, $L = S^1 \times S^7$, Proposition 2.4 implies $\eta_L(\gamma)$ is either 2, 4 or 8. 2 and 8 occur. However example with $\eta_L(\gamma) = 4$ does not seem to be known.

Problem 2.1. Are there any Lagrangian submanifold $L \subset \mathbb{C}^{2n}$ diffeomorphic to $S^1 \times S^{2n-1}$ such that $\eta_L(\gamma) \neq 2, 2n$.

As for the the other factor $\pi_{2n-1}(GLag_{2n})$, there is no restriction in the case $\eta_{S^1 \times S^{2n-1}}(\gamma) = 2$. However, in other case, especially in the case $\eta_{S^1 \times S^{2n-1}}(\gamma) = 2n$ the following seems to be open.

Problem 2.2. For which homotopy class $\alpha \in \pi_{2n-1}(GLag_{2n})$, the class $(2n, \alpha) \in \pi_1(GLag_{2n}) \times \pi_{2n-1}(GLag_{2n})$ is realized by a Lagrangian embedding ?

The first case to study seems to be the case of $S^1 \times S^3$. We remark Theorem 13.1 determines when the regular homotopy class of Lagrangian immersion $S^1 \times S^{2m} \to \mathbb{C}^{2m+1}$ is realized by a Lagrangian embedding. We will discuss more results on Lagrangian submanifolds in $\mathbb{C}^n$ in §§11,12,13.

3. Perturbing Cauchy-Riemann equation

Let us start the proof of Theorems 2.1,2.3. We use the moduli space of holomorphic disks which bound L. We define it below. Let $\beta \in \pi_2(\mathbb{C}^n, L) \cong \pi_1(L)$.

Definition 3.1. We consider a map $\varphi : D^2 \to \mathbb{C}^n$ with the following properties.

(1) φ is holomorphic.
(2) $\varphi(\partial D^2) \subset L$.
(3) The homotopy type of φ is β.

We denote by $\operatorname{Int} \widetilde{\mathcal{M}}(L; \beta)$ the set of all such maps. We can define a topology on it (see [13]).

We consider the group $PSL(2; \mathbb{R})$. It can be identified with the group of biholomorphic maps $D^2 \to D^2$. We denote $\{g \in PSL(2; \mathbb{R}) | g(1) = 1\}$ by $Aut(D^2, 1)$. (Here we identify $D^2 = \{z \in \mathbb{C} | |z| < 1\}$, $PSL(2; \mathbb{R}) = Aut(D^2)$. Aut denotes the group of biholomorphic automorphisms.) Then we put

$$\operatorname{Int} \widehat{\mathcal{M}}(L; \beta) = \frac{\operatorname{Int} \widetilde{\mathcal{M}}(L; \beta)}{Aut(D^2, 1)}, \qquad \operatorname{Int} \mathcal{M}(L; \beta) = \frac{\operatorname{Int} \widetilde{\mathcal{M}}(L; \beta)}{PSL(2; \mathbb{R})}.$$

[1] The author thanks Polterovich who pointed out his example and corrected an error author made during author's lecture at Montreal.

The moduli spaces $\operatorname{Int}\widehat{\mathcal{M}}(L;\beta)$, $\operatorname{Int}\mathcal{M}(L;\beta)$ can be compactified by including stable maps (from open Riemann surface of genus 0). (See [13] for its precise definition.) We denote them by $\widehat{\mathcal{M}}(L;\beta)$, $\mathcal{M}(L;\beta)$ respectively. These moduli spaces are used to define and study Floer homology of Lagrangian submanifold. We will discuss it in §5. In this section we use it more directly to prove Theorems 2.1,2.3.

For this purpose, we use a similar but a different moduli space using perturbation of the Cauchy-Riemann equation (which is the equation that φ is holomorphic) by Hamiltonian vector field. We need some notations to describe it.

Definition 3.2. For each positive R we consider a smooth function $\chi_R : \mathbb{R} \to [0,1]$ such that

(1) $\chi_R(t) = 1$, if $|t| > R$,
(2) $\chi_R(t) = 0$, if $|t| < R - 1$,
(3) The C^k norm of χ_R is bounded uniformly on R.

We next take a smooth function $H : \mathbb{C}^n \to \mathbb{R}$ of compact support and let

$$X_H = J \operatorname{grad} H \tag{3.1}$$

be the Hamiltonian vector field generated by it.

Definition 3.3. We consider a map $\varphi = \varphi(\tau, t) : \mathbb{R} \times [0,1] \to \mathbb{C}$ with the following properties.

$$\frac{\partial\varphi}{\partial\tau}(\tau,t) = J\left(\frac{\partial\varphi}{\partial t}(\tau,t) - \chi_R(\tau)X_H(\varphi(\tau,t))\right), \tag{3.2a}$$

$$\varphi(\tau,0), \varphi(\tau,1) \in L, \tag{3.2b}$$

$$\int_{\mathbb{R}\times[0,1]} \varphi^*\omega < \infty. \tag{3.2c}$$

We denote by $\mathcal{N}(R)$ the set of all such φ.

Note (3.2a) implies that φ is holomorphic outside $[-R,R] \times [0,1]$. Therefore (3.2c) and removable singularity theorem of Gromov [18] and Oh [24] imply the following.

Lemma 3.1. *There exists* $p_{+\infty}, p_{-\infty} \in L$ *such that* $\lim_{\tau\to\pm\infty}\varphi(\tau,t) = p_{\pm\infty}$.

We remark that $D^2 \setminus \{1,-1\}$ is biholomorphic to $\mathbb{R} \times [0,1]$. Hence element φ of $\mathcal{N}(R)$ may be regarded as a map $\overline{\varphi} : (D^2, \partial D^2) \to (\mathbb{C}^n, L)$.

Definition 3.4. Let $\mathcal{N}(R;\beta)$ be the set of all $\varphi \in \mathcal{N}(R)$ such that the homotopy class of $\overline{\varphi}$ is β.

For a fixed $p_0 \in L$, let $\mathcal{N}(R;\beta;p_0)$ be the set of all element of $\varphi \in \mathcal{N}(R)$ such that $\lim_{\tau\to+\infty}\varphi(\tau,t) = p_0$. We define $\mathcal{N}(R;p_0)$ in the same way.

Let $\exp_t^{X_H} : \mathbb{C}^n \to \mathbb{C}^n$ be a one parameter group of transformations associated to X_H.

Assumption 3.1. $\exp_1^{X_H}(L) \cap L = \emptyset$.

Since L is compact we can always find H satisfying Assumption 3.1. Now the following three propositions are basic points of the proof of Theorems 2.1,2.3.

Proposition 3.2. *There exists C depending only on H and L (and independent of R) such that if $\mathcal{N}(R;\beta) \ne \emptyset$ then*

$$\beta \cap \omega > -C. \tag{3.3}$$

The proof is based on well established argument and is omitted. We remark that Definition 3.2 (3) is essential for the proof of Proposition 3.2.

Proposition 3.3. *If Assumption* 3.1 *is satisfied then, there exists R_0 depending only on H and L such that $\mathcal{N}(R) = \emptyset$ for $R > R_0$.*

Proof. (*Sketch*) If not there exists $\varphi_i \in \mathcal{N}(R_i)$ with $R_i \to \infty$. We can use Proopsition 3.2 and elliptic regurality to show the existence of $\tau_i \in [-R_i+2, R_i-2]$, $\delta > 0$ such that the C^∞ norm of φ_i on $[\tau_i - \delta, \tau_i + \delta] \times [0,1]$ is bounded and that

$$\lim_{i\to\infty} \int_{[\tau_i-\delta,\tau_i+\delta]\times[0,1]} \varphi^*\omega = 0. \tag{3.4}$$

(3.4), (3.2a) and elliptic regurality implies that

$$\lim_{i\to\infty} \sup_{t\in[0,1]} \left\| \frac{\partial \varphi}{\partial t}(\tau_i, t) - V_H \right\| = 0.$$

This implies

$$\lim_{i\to\infty} dist\left(\exp_1^{X_H}(\varphi(\tau_i,0)), \varphi(\tau_i,1)\right) = 0.$$

In view of (3.2b) this contradicts to Assumption 3.1. $\square$

For the next proposition we need an assumption.

Assumption 3.2. If $E(\beta) > 0$ then $\eta_L(\beta) \le 0$.

Let $\beta_0 = 0 \in \pi_2(\mathbb{C}^n, L)$.

Proposition 3.4. *Under Assumption* 3.2, *we may choose p_0 such that $\mathcal{N}(0;\beta;p_0) = \emptyset$, for any $\beta \ne \beta_0$.*

Proof. (*Sketch*) We have

$$\dim \mathcal{M}(L;\beta) = n + \eta_L(\beta) - 3, \tag{3.5}$$

(see [13]), for $\beta \ne \beta_0$. Hence, by Assumption 3.2 we have

$$\dim \bigcup_{\varphi\in\mathcal{M}(L;\beta)} \varphi(S^1) \le n + \eta_L(\beta) - 3 + 1 \le n-2.$$

Therefore we may take p_0 which is not contained in the union of $\bigcup_{\varphi\in\mathcal{M}(L;\beta)} \varphi(S^1)$ for various β. We remark that the equation (3.2a) is the Cauchy-Riemann equation when R is zero. Proposition 3.4 holds. $\square$

Lemma 3.5. *$\mathcal{N}(0;\beta_0;p_0)$ is one point.*

The lemma follows immedately from the fact that each element of $\mathcal{N}(0;\beta_0)$ is a constant map. We put

$$\mathcal{N}(\beta;p_0) = \bigcup_{R\ge 0} \mathcal{N}(R;\beta;p_0) \times \{R\}. \tag{3.6}$$

We can perturb our moduli spaces so that $\mathcal{N}(\beta;p_0)$ is a manifold with boundary.

Now we are in the position to prove Theorem 2.1. The main point is as follows.

Lemma 3.6. *If $E : \pi_2(\mathbb{C}^n, L) \to \mathbb{R}$ is zero then $\mathcal{N}(\beta; p_0)$ is compact.*

Proof. (*Sketch*) We prove by contradiction. Let $\varphi_i \in \mathcal{N}(R_i; \beta_0; p_0)$ be a divergent sequence. We may assume by Proposition 3.3 that R_i converges to R. Then there exists $p_i = (\tau_i, t_i) \in \mathbb{R} \times [0,1]$ such that

$$|d_{p_i}\varphi_i| = \sup\{|d_x\varphi_i||x \in \mathbb{R} \times [0,1]\}. \tag{3.7}$$

We put $\epsilon_i = 1/D_i$. We consider the following three cases separately.
Case 1 : $|d_{p_i}\varphi_i| = D_i$ diverges. $D_i dist(p_i, \partial(\mathbb{R} \times [0,1])) = C_i \to \infty$.
Case 2 : $|d_{p_i}\varphi_i| = D_i$ diverges. $D_i dist(p_i, \partial(\mathbb{R} \times [0,1]))$ is bounded.
Case 3 : $|d_{p_i}\varphi_i| = D_i$ is bounded. $|\tau_i|$ diverges.

In Case 1, we define $h_i : D^2(C_i) \to \mathbb{C}^n$ by $h_i(z) = \varphi_i(\epsilon_i z + p_i) - \varphi_i(p_i)$. (3.7) implies that dh_i is uniformly bounded. Moreover $h_i(0) = 0$ and $|d_0 h_i| = 1$. Hence, by elliptic regularity, h_i converges to $h : \mathbb{C} \to \mathbb{C}^n$. Since $|d_0 h| = 1$, h is not a constant. On the other hand, by (3.2a),

$$\sup\left\{\left|\frac{\partial h_i}{\partial \tau} - J\frac{\partial h_i}{\partial t}(z)\right| \,\middle|\, z \in D^2(C_i)\right\} < \epsilon_i C$$

where C is a constant independent of i. Therefore h is holomorphic. Moreover by (3.2c) the integral $\int_{D^2(R_i)} h_i^*\omega$ is uniformly bounded. If follows that $\int_{D^2(R_i)} h^*\omega$ is finite. Hence the holomorphicity of h implies that h is a constant map. This is a contradiction.

In Case 2, we consider a similar limit and obtain a map $h : \mathfrak{h} = \{z \in \mathbb{C}|\mathrm{Im} z \geq 0\} \to \mathbb{C}^n$ which is nonconstant and holomorphic. Moreover $h(\mathbb{R}) \subset L$ and $\int_{\mathfrak{h}} h^*\omega$ is finite. Therefore by removable singularity theorem of Gromov [18] and Oh [24], h can be extended to $h^+ : D^2 \to \mathbb{C}^n$ with $h^+(\partial D^2) \subset L$. (Namely when we identify $D^2 \setminus \{1\} = \mathfrak{h}$, $h^+ = h$ on $\mathfrak{h}$.) Since $E = 0$ by assumption, it follows that

$$\int_{\mathfrak{h}} h^*\omega = \int_{D^2} h^{+*}\omega = 0.$$

Therefore, since h is holomorphic, h is a constant map. This is a contradiction.

We next consider Case 3. We can show that D_i is bounded away from zero in the same way as the Case 1. We define $h_i : \mathbb{R} \times [0,1] \to \mathbb{C}^n$ by $h_i(z) = \varphi_i(z - (\tau_i, 0))$. Since $|\tau_i|$ diverges it follows that h_i, after taking a subsequence if necessary, converges to a holomomorphic map $h : \mathbb{R} \times [0,1] \to \mathbb{C}^n$ which is nonconstant. Moreover $h(\mathbb{R} \times \{0,1\}) \subset L$. Furthermore :

$$\int_{\mathbb{R}\times[0,1]} h^*\omega \leq \limsup \int_{\mathbb{R}\times[0,1]} \varphi_i^*\omega < \infty.$$

By Lemma 3.1 and $E = 0$, we find that $\int_{\mathbb{R}\times[0,1]} h^*\omega = 0$. Since h is holomorphic, this implies that h is a constant map. This is a contradiction.

The proof of Lemma 3.6 is now complete. □

Now the rest of the proof of Theorem 2.1 is as follows. We remark that

$$\dim \mathcal{N}(R; \beta) = n + \eta_L(\beta) \tag{3.8}$$

can be proved in the same way as (3.5). Then

$$\dim \mathcal{N}(R; \beta; p_0) = \eta_L(\beta) \tag{3.9}$$

and hence

$$\dim \mathcal{N}(\beta; p_0) = \eta_L(\beta) + 1. \tag{3.10}$$

In particular $\partial\mathcal{CN}(\beta_0; p_0)$ is a one dimensional manifold. It is compact by Lemma 3.6. Its boundary is one point by Proposition 3.4, Lemma 3.5. This is a contradiction.

Thus we proved Theorem 2.1. □

Remark 3.1. To make the above argument rigorous we need to find a perturbation by which various moduli spaces have fundamental chain. We omit the arguments on transversality in various places of this article. We explain one novel argument on transversality in §6, however. We do not discuss sign (orientation) either. Those points are discussed in detail in [13]. (The detail of the new argument on transversality in §6 will be in [12].)

4. Maslov index of Lagrangian submanifold with vanishing second Betti number.

In this section, we continue the argument of the previous section and sketch the proof of Theorem 2.3. We here assume $\eta_L : \pi_2(\mathbb{C}^n, L) \to \mathbb{Z}$ is zero. It follows from (3.9) and (3.10) that $\dim \mathcal{N}(\beta; p_0) = 1$, $\dim \mathcal{N}(R; \beta; p_0) = 0$, $\dim \mathcal{M}(L; \beta) = n - 3$.

We next need the following result :

Theorem 4.1 ([13, 31])**.** *A spin structure of L determines an orientation of $\mathcal{M}(L; \beta)$, $\mathcal{N}(\beta; p_0)$ etc.*

Hereafter we assume that L is spin and use the orientation obtained by Theorem 4.1. Since we do not assume $E = 0$, Lemma 3.6 does not imply compactness of $\mathcal{N}(\beta; p_0)$. We are going to study its compactification. We use our assumption $\eta_L = 0$ in the next lemma.

Lemma 4.2. *If $\eta_L = 0$, we may choose p_0 such that, for any $\varphi \in \mathcal{M}(L; \beta)$, we have $p_0 \notin \varphi(S^1)$.*

Proof. Since $\dim \mathcal{M}(L; \beta) = n - 3$, it follows that the union of all $\varphi(S^1)$ for $\varphi \in \mathcal{M}(L; \beta)$ is $n - 2$ dimensional. The lemma follows. □

Using Lemma 4.2, we can describe compactification of $\mathcal{N}(\beta; p_0)$ and of $\mathcal{N}(R; \beta; p_0)$. We put $A = (\mathbb{R} \times \{0, 1\}) \cup \{-\infty\}$ in this section. We define

$$ev : \mathcal{N}(R; \beta; p_0) \times A \to L$$

by

$$ev(\varphi, (\tau, \pm 1)) = \varphi(\tau, \pm 1), \qquad ev(\varphi, -\infty) = \lim_{\tau \to -\infty} \varphi(\tau, t). \tag{4.1}$$

We also define $\widehat{\mathcal{M}}(L; \beta) \to L$ by

$$ev([\varphi]) = \lim_{\tau \to +\infty} \varphi(\tau, t). \tag{4.2}$$

Now we have

Proposition 4.3. *We assume $\eta_L = 0$. Then $\mathcal{N}(R;\beta;p_0)$, $\mathcal{N}(R;\beta;p_0)$ has compactifications $\mathcal{CN}(R;\beta;p_0)$, $\mathcal{CN}(R;\beta;p_0)$ which has Kuranishi structure with boundary. The boundary is identified as*

$$\partial\mathcal{CN}(R;\beta;p_0) = \bigcup_{\beta_1+\beta_2=\beta} \widehat{\mathcal{M}}(L;\beta_1) \times_L (\mathcal{N}(R;\beta_2;p_0)\times A) \quad \beta \neq \beta_0$$
$$\partial\mathcal{CN}(\beta;p_0) \setminus \mathcal{N}(0;\beta;p_0) = \bigcup_{\beta_1+\beta_2=\beta} \widehat{\mathcal{M}}(L;\beta_1) \times_L (\mathcal{N}(\beta_2;p_0)\times A) \quad \beta_0 = 0.$$

Remark 4.1. We do not explain the definition of "Kuranishi structure with boundary" here. See [14]. If the reader is not familier with it, he may read the statement as "there is a perturbation so that the virtual fundamental chain satisfies the equality ... ".

Proof. (*Sketch*) The proof of Proposition 4.3 goes in a similar way as the proof of Lemma 3.6. We consider a divergent sequence $\varphi_i \in \mathcal{N}(R_i;\beta_0;p_0)$. R_i is bounded by Proposition 3.3 and hence we may assume $R_i \to R$. We choose $p_i = (\tau_i, t_i) \in \mathbb{R}\times[0,1]$ satisfying (3.7). Then we consider Cases 1,2 and 3 in the proof of Lemma 3.6. Case 1 does not occur by the same reason as the proof of Lemma 3.6. So we need to consider Cases 2 and 3 only.

If $\lim \tau_i = \tau \in \mathbb{R}$ and $\lim t_i = t_\infty \in \{0,1\}$, the limit correspond to an element

$$([h^+],(\varphi_\infty,(\tau,t_\infty))) \in \widehat{\mathcal{M}}(L;\beta_1)\times_L (\mathcal{N}(R;\beta_2;p_0)\times A).$$

Here $h^+ : D^2 \to \mathbb{C}^n$ is a bubble at (τ, t_∞).

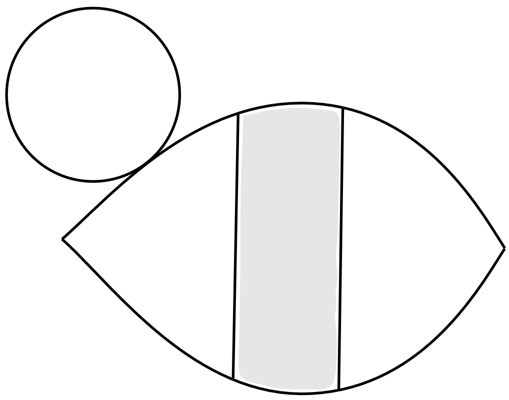

Figure 1

If $\tau_i \to -\infty$ we have an element

$$([h^+],(\varphi_\infty,(-\infty))) \in \widehat{\mathcal{M}}(L;\beta_1)\times_L (\mathcal{N}(R;\beta_2;p_0)\times A).$$

Using Lemma 4.2 we may assume $\tau_i \to +\infty$ does not occur. (See Figure 2.)

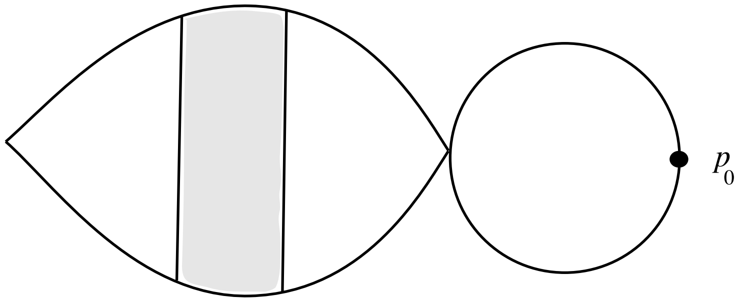

Figure 2

We next consider Case 3. $\tau_i \to +\infty$ does not occur by the same reason. In the case $\tau_i \to -\infty$, we again get an element

$$([h^+], (\varphi_\infty, (-\infty))) \in \widehat{\mathcal{M}}(L; \beta_1) \times_L (\mathcal{N}(R; \beta_2; p_0) \times A).$$

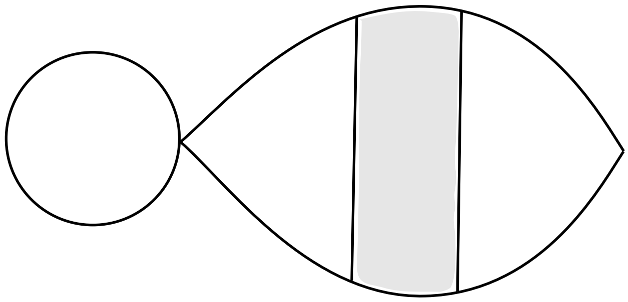

Figure 3

This implies Proposition 4.3. □

The next lemma is a consequence of Proposition 3.2 and Gromov compactness.

Lemma 4.4. *There exists only a finitely many $\beta \in \pi_1(\mathbb{C}^n, L)$ such that $\mathcal{N}(\beta; p_0) \neq \emptyset$ and $E(\beta) < 0$.*

First we consider the case when the following is satisfied in addition.

Assumption 4.1. There exists a unique $\beta \in \pi_1(\mathbb{C}^n, L)$ such that $\mathcal{N}(\beta; p_0) \neq \emptyset$ and $E(\beta) < 0$.

Moreover if $\beta' \in \pi_1(\mathbb{C}^n, L)$ with $E(\beta') > 0$, $-E(\beta) > E(\beta') > 0$, then $\widehat{\mathcal{M}}(L; \beta') = \emptyset$.

This assumption is rather restrictive. We will explain the argument to remove it later in the next section.

Now we assume $\eta_L = 0$, L is spin and Assumption 4.1. We consider the boundary $\partial \mathcal{CN}(\beta_0; p_0)$ by using Lemma 4.3. We have

$$\partial \mathcal{CN}(\beta_0; p_0) \setminus \mathcal{N}(0; \beta_0; p_0) = \bigcup_{\beta_1+\beta_2=\beta_0} \widehat{\mathcal{M}}(L; \beta_1) \times_L (\mathcal{N}(\beta_2; p_0) \times A). \tag{4.3}$$

Since $E(\beta_1) > 0$ if $\widehat{\mathcal{M}}(L; \beta_1) \neq \emptyset$ and since $E(\beta_0) = 0$ it follows from (4.3) and Assumption 4.1 that

$$\partial \mathcal{CN}(\beta_0; p_0) \setminus \mathcal{N}(0; \beta_0; p_0) = \widehat{\mathcal{M}}(L; -\beta) \times_L (\mathcal{N}(\beta; p_0) \times A). \tag{4.4}$$

Note that $\mathcal{CN}(\beta_0;p_0)$ is a one dimensional oriented chain and $\mathcal{N}(0;\beta_0;p_0)$ is a one point. Hence the right hand side of (4.4) is an oriented zero dimensional manifold whose order counted with sign is 1.

By the second half of Assumption 4.1, we can show that $\widehat{\mathcal{M}}(L;-\beta) = \operatorname{Int}\widehat{\mathcal{M}}(L;-\beta)$ and defines a cycle. Its dimension is $n-2$. Hence

$$ev_*([\widehat{\mathcal{M}}(L;-\beta)]) \in H_{n-2}(L;\mathbb{Z}).$$

Similarly we find

$$ev_*([\mathcal{N}(0;\beta;p_0)]) \in H_2(L;\mathbb{Z}).$$

We now have

$$ev_*([\widehat{\mathcal{M}}(L;-\beta)]) \cdot ev_*([\mathcal{N}(0;\beta;p_0)]) = 1,$$

since it is equal to the right hand side of (4.4). (Here $\cdot$ is the intersection pairing.) This implies $H^2(L;\mathbb{Q}) \neq 0$. Theorem 2.3 is thus proved under additional hypothesis Assumption 4.1.

5. Floer homology and a spectre sequence.

We now introduce Floer cohomology of Lagrangian submanifold and explain how it can be used to study Lagrangian submanifold of, say $\mathbb{C}^n$.

Let L be a compact Lagrangian submanifold of a symplectic manifold M. (In case M is noncompact we assume that M is convex at infinity. See [8].) Let us define a universal Novikov ring Λ by

$$\Lambda = \left\{ \sum a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{R}, \lambda_i \to +\infty, \lambda_i \in \mathbb{R} \right\}. \tag{5.1}$$

Actually Λ is a field.

Let $\eta_L : \pi_2(M,L) \to \mathbb{Z}$ be the Maslov index and $E : \pi_2(M,L) \to \mathbb{R}$ is defined by integrating the symplectic form. We say that L is relatively spin if there exists $st \in H^2(M;\mathbb{Z}_2)$ which is sent to the second Stiefel-Whiteney class of L. (If L is spin then it is relatively spin.)

Theorem 5.1 ([13]). *We assume L is relatively spin. Then there exists a series of elements $\beta_i \in \pi_2(M,L)$ with $E(\beta_i) > 0$, $E(\beta_i) \le E(\beta_{i+1})$ and cohomology classes*

$$o_{\beta_i} \in H^{2-\eta_L(\beta_i)}(L;\mathbb{Q}),$$

such that $o_\beta = 0$ if $2-\eta_L(\beta) = 0$ or n, and $o_\beta = 0$ if $E(\beta) \le 0$. It has the following properties.

If $o_{\beta_i} = 0$ for all β_i then there exists a Λ module $HF(L,L)$ and a spectral sequence E_* such that

(1) $E_2 \cong H(L;\Lambda)$.

(2) $E_\infty \cong HF(L,L)$.

(3) *The differential $d^i = T^{E(\beta_i)} d^{\beta_i}$ is a Λ module homomorphism induced by $d^{\beta_i} : H^k(L;\mathbb{Q}) \to H^{k+1-\eta_L(\beta_i)}(L;\mathbb{Q})$.*

(4) *$d_\beta([1]) = 0$. Here $[1] \in H^0(L;\mathbb{Q})$ is a generator.*

(5) *The fundamental cocycle $[L] \in H^n(L;\mathbb{Q})$ is not contained in the image of d_β.*

(6) *If $\Phi : M \to M$ is a Hamiltonian diffeomorphism and L is transversal to $\Phi(L)$, then*

$$\dim_\Lambda HF(L,L) \le \sharp(L \cap \Phi(L)).$$

Remark 5.1. Floer cohomology $HF(L,L)$ is not an invariant of symplectic diffeomorphism type of (M,L) but depends on an element of a moduli space $\mathcal{M}(L)$ of bounding chains b. We introduced the notion of bounding chain and denote its moduli space by $\mathcal{M}(L)$ in [13]. We do no discuss bounding chain here, since we use only the properties stated above in this article. (Namely any choice of bounding chain is good for the applications which appear in this article.)

For other applications, especially for applications to Mirror symmetry, the space $\mathcal{M}(L)$ itself plays a crucial role.

We first show

Lemma 5.2. *Theorem* 5.1 *implies Theorem* 2.3.

Proof. Let $L \subset \mathbb{C}^n$ be a spin Lagrangian submanifold. We assume $H^2(L;\mathbb{Q}) = 0$ and $\eta_L = 0$ and deduce a contradiction.

Since $o_\beta \in H^{2-\eta(\beta)}(L;\mathbb{Q}) = H^2(L;\mathbb{Q})$ it follows that we can define $HF(L,L)$. By (6), we have $HF(L,L) = 0$. Hence by (1)(2), $[L] \in H^n(L)$ does not survive until E_∞. Since $[L]$ is not in the image of d_β by (5), it follows that $d_\beta([L]) \neq 0$ for some β. However this is impossible since $d_\beta([L]) \in H^{n+1+\eta_L(\beta)}(L;\mathbb{Q}) = 0$. □

We next apply Theorem 5.1 to the case of $(M,L) = (\mathbb{C}^{n+1}, S^1 \times S^n)$ and prove Proposition 2.4. Namely we prove the following two lemmas.

Lemma 5.3 (Oh [25])**.** *Let $L = S^1 \times S^n$ be a Lagrangian submanifold of $\mathbb{C}^n$ and n is odd. We choose the generator $\beta \in \pi_2(\mathbb{C}^{n+1}, L)$ so that $E(\beta) > 0$. Then $\eta_L(\beta)$ is positive and divides $n+1$.*

Proof. We first remark that $E \neq 0$ by Theorem 2.1. Hence there is a unique choice of β as in the statement of Lemma 5.3.

Since $\eta(\beta)$ is even, it follows that $o_\beta \in H^{even}(L;\mathbb{Q})$. Since n if odd, the cohomology group $H^{even}(L;\mathbb{Q})$ is nonzero for $H^0(L)$ and $H^{n+1}(L)$ only. In that case $o_\beta = 0$ by Theorem 5.1. Therefore $HF(L,L)$ is well defined. Since $L \subseteq \mathbb{C}^{n+1}$, it follows that $HF(L,L) \cong 0$. By (4) and (5) there exists k, k' such that $d_{k\beta}(1) \neq 0$, and $[L] = d_{k'\beta}u$ for some $u \in H(L)$. Since $d_{k\beta}$ is odd degree either $d_{k\beta}(1) = c[S^1]$ or $d_{k\beta} = c[S^n]$. ($c \in \mathbb{Q}$.) (Since $E(\beta) > 0$, it follows that $k > 0$.) In the case $d_{k\beta}(1) = [S^1]$, we have $\eta_L(k\beta) = 2$. Hence $k = 1$ and $\eta_L(\beta) = 2$.

In the case $d_{k\beta}(1) = [S^n]$, we have $\eta_L(k\beta) = n+1$. Hence $\eta_L(\beta)$ divides $n+1$. □

Lemma 5.4 ([13, 25])**.** *Let $L = S^1 \times S^n$ be a Lagrangian submanifold of $\mathbb{C}^{n+1}$ and n is even. We choose the generator $\beta \in \pi_2(\mathbb{C}^{n+1}, L)$ so that $E(\beta) > 0$. Then either $\eta_L(\beta) = 2$ or it is nonpositive and divides $2-n$.*

Proof. If $o_{k\beta} \neq 0$ then, since $\deg o_{k\beta}$ is nonzero and even, it follows that $o_{k\beta} \in H^n(L;\mathbb{Q})$. It implies $2 - \eta_L(k\beta) = n$. Hence $\eta_L(\beta)$ is negative and divides $2-n$.

If $o_{k\beta}$ are all zero, then Floer cohomology $HF(L,L)$ is well defined and is zero. Hence $d_{k\beta}(1) \neq 0$ for some k. Since $[L]$ is not in the image of $d_{k\beta}$, it follows that $d_{k\beta}(1) = c[S^1]$. Hence $\eta_L(k\beta) = 2$. Herefore $k = 1$ and $\eta_L(\beta) = 2$. □

Remark 5.2. The second possibility $\eta_L(\beta) < 0$ in Lemma 5.4 will be eliminated in §13. (Theorem 13.1.)

6. Homology of loop space and Chas-Sullivan bracket.

In §§6,7,8, we will explain a construction of a filtered A_∞ structure on the cohomology group $H(L)$ of Lagrangian submanifold L. We take a bit different way from one in [13] and uses De-Rham cohomology.

Remark 6.1. In [13] a variant of singular chain complex was used. The way taken in [13] has an advantage that we can work over $\mathbb{Z}$ coefficient at least in case L is semipositive[2]. The way taken here has an advantage that we can then keep more symmetry. Especially, cyclic symmetry is established in §9.

Let L be a compact smooth manifold. We denote its free loop space by $\mathcal{L}(L)$. Namely

$$\mathcal{L}(L) = \{\ell : S^1 \to L \mid \ell \text{ is piecewise smooth.}\} \tag{6.1}$$

M.Chas and D.Sullivan [5] introduced a structure of graded Lie algebra on the homology of $\mathcal{L}(L)$. Let us recall it here. We identifty $S^1 = \mathbb{R}/\mathbb{Z}$. Let $f_i : P_i \to \mathcal{L}(L)$ be cycles. (We write $\hat{P}_i = (P_i, f_i)$.) We put

$$P_1 * P_2 = \{(x, y, t) \in P_1 \times P_2 \times S^1 \mid (f_1(x))(0) = (f_2(y))(t)\} \tag{6.2}$$

We define $f_1 * f_2 : P_1 * P_2 \to \mathcal{L}(L)$ by the following formula.

$$((f_1 * f_2)(x, y, t))(s) = \begin{cases} (f_2(y))(2s) & 2s \le t, \\ (f_1(x))(2s - t) & t \le 2s \le t + 1, \\ (f_2(x))(2s - 1) & t + 1 \le 2s. \end{cases} \tag{6.3}$$

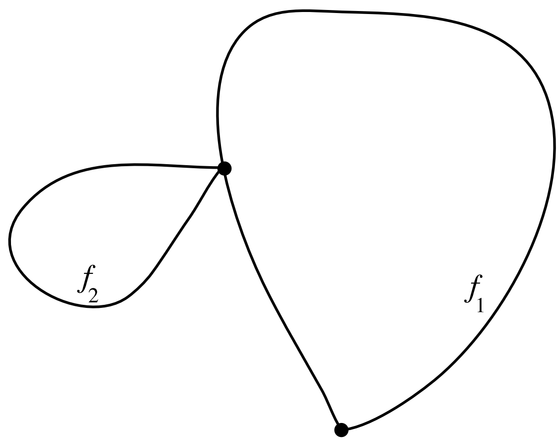

Figure 4

We then put

$$\hat{P}_1 * \hat{P}_2 = (P_1 * P_2, f_1 * f_2). \tag{6.4}$$

If $\hat{P}_1$, $\hat{P}_2$ are cycles, of dimension k_1,k_2 respectively, then, under an appropriate transversality condition, $\hat{P}_1 * \hat{P}_2$ is a cycle of dimension $k_1 + k_2 - n + 1$. Here $n = \dim L$. Therefore $*$ defines a map

$$* : H_{k_1}(\mathcal{L}(L)) \otimes H_{k_2}(\mathcal{L}(L)) \to H_{k_1+k_2-n+1}(\mathcal{L}(L)). \tag{6.5}$$

[2] Actually we can do it in general using the method of [15].

Definition 6.1 ([5])**.** We define *loop bracket* $\{\cdot,\cdot\}$ by

$$\{[\hat{P}_1],[\hat{P}_2]\} = [\hat{P}_1 * \hat{P}_2] + (-1)^{(\deg P_1+1)(\deg P_2+1)}[\hat{P}_2 * \hat{P}_1].$$

Theorem 6.1 ([5])**.** *Loop bracket satisfies Jacobi identity. Namely it defines a structure of graded Lie algebra on $H_*(\mathcal{L}(L))$.*

Actually we can work in chain level and construct an L_∞ algebra, (that is a homotopy version of graded Lie algebra).

There are various ways to work out transversality problem to justify Chas-Sullivan's construction. Here we use the following one which works best with our construction of virtual fundamental chain of the moduli space of pseudoholomorphic disks.

Definition 6.2. (P,f,ω) is said to be an *approximate De-Rham chain* of $\mathcal{L}(L)$ if the following holds.

(1) P is an oriented smooth manifold of finite dimension (with or withour boundary). ω is a smooth diferential form on P of compact support.
(2) $f : P \to \mathcal{L}(L)$ is a smooth map.
(3) The map $ev : P \to L$ defined by $ev(x) = (f(x))(0)$ is a submersion. In case ∂P is nonempty we assume that $ev : \partial P \to L$ is also a submersion.

We say (P,f,ω) is an *approximate De-Rham cycle* if P does not have a boundary and $\omega = 0$.

We define the degree of approximate De-Rham chain by

$$\deg(P,f,\omega) = \dim P - \deg\omega.$$

We put

$$\partial(P,f,\omega) = (\partial P,f,\omega) + (-1)^{\deg P}(P,f,d\omega) \tag{6.6}$$

An approximate De-Rham cycle (P,f,ω) of degree k determines an element $H_k(\mathcal{L}(L);\mathbb{R})$ as follows. Let $H_c^*(P;\mathbb{R})$ be the De Rham cohomology group of compact support and $PD : H_c^{\dim P-k}(P;\mathbb{R}) \to H_k(P;\mathbb{R})$ be the Poincaré duality. Then we associate $f_*(PD([\omega])) \in H_k(\mathcal{L}(L);\mathbb{R})$ to (P,f,ω). It is easy to see that any element of $H_*(\mathcal{L}(L);\mathbb{R})$ can be realized by an approximate De-Rham cycle.

An approximate De Rham chain (P,f,ω) determines a smooth differential form $ev^L(P,f,\omega)$ of L of codimension $k = \deg(P,f,\omega)$ as follows.

$$ev^L(P,f,\omega) = ev!(\omega) \in \Omega^{n-k}(L). \tag{6.7}$$

Here $ev!$ is integration along fiber. It is well defined since ω is of compact support and $ev : P \to L$ is a submersion.

We can easily check

$$PD([ev^L(P,f,\omega)]) = f_*(PD([\omega]))$$

if (P,f,ω) is an approximate De Rham cycle.

The following lemma follows easily from Stokes' theorem.

Lemma 6.2. $d(ev^L(P,f,\omega)) = ev^L(\partial(P,f,\omega))$

Now we go back to the loop bracket. Let (P_i, f_i, ω_i) be approximate De-Rham chains of degree k_i for $i = 1, 2$.

Lemma 6.3. $(P_1 * P_2, f_1 * f_2, \omega_1 \times \omega_2)$ *is an approximate De-Rham chain of degree* $k_1 + k_2 - n + 1$.

Proof. We remark that the map

$$P_1 \times P_2 \times S^1 \to L \times L, (x, y, t) \mapsto ((f_1(x))(0), (f_2(y))(t))$$

is transversal to diagonal by Condition (3) of Definition 6.2. The rest of the proof is straightforward. $\square$

Thus we can define $*$ in the chain level using approximate De-Rham chains. Hence we can define loop bracket also in the chain level. By the argument of [5], loop bracket satisfies graded Jacobi identity in the chain level modulo parametrization of the loop. Hence we can prove easily that it induces L_∞ structure. (The argument we need to do so is the same as the argument to show that the (based) loop space is an A_∞ space. See [32, 33]).

Now we want to use moduli space of pseudoholomorphic disks to construct a chain (approximate De Rham chain) in the loop space. Let M be a symplectic manifold and L be its Lagrangian submanifold. We assume that L is compact and relatively spin and M is convex at infinity (in case M is noncompact). For each $\beta \in \pi_2(M; L)$ we define the spaces $\widetilde{\mathcal{M}}(L; \beta)$, $\widehat{\mathcal{M}}(L; \beta)$, $\mathcal{M}(L; \beta)$ in the same way as Definition 3.1. Actually since $M \neq \mathbb{C}^n$ in general, we need to consider sphere buble to compactify them. But the argument to handle sphere bubble of $\widehat{\mathcal{M}}(L; \beta)$ is the same as the case of the moduli space of pseudoholomorphic map from Riemann surface without boundary (see [14]) since the sphere bubble is a codimension 2 phenomenon. So we do not discuss it. As for the compactification $\mathcal{M}(L; \beta)$ there exists one new point to discuss which we mention briefly in §14 (see Theorem 14.2 and its proof) and will be discussed in detail in the reviced version of [13].

Now we consider $\widehat{\mathcal{M}}(L; \beta)$. Recall

$$\widehat{\mathcal{M}}(L; \beta) = \frac{\widetilde{\mathcal{M}}(L; \beta)}{Aut(D^2, 1)}$$

and the group $Aut(D^2, 1)$ is contractible. Hence we take a lift

$$\mathrm{Lift}_\beta : \widehat{\mathcal{M}}(L; \beta) \to \widetilde{\mathcal{M}}(L; \beta)$$

and fix it. We define a map $ev : \widetilde{\mathcal{M}}(L; \beta) \to \mathcal{L}(L)$. by $ev(\varphi) = \varphi|_{\partial D^2}$, and consider a map $ev \circ \mathrm{Lift}_\beta : \widehat{\mathcal{M}}(L; \beta) \to \mathcal{L}(L)$. We want to use the chain $(\widehat{\mathcal{M}}(L; \beta), ev \circ \mathrm{Lift}_\beta)$ to construct Floer cohomology etc.

For the purpose of transversality, we want to replace $(\widehat{\mathcal{M}}(L; \beta), ev \circ \mathrm{Lift}_\beta)$ by an appropriate approximate De-Rham chain. To describe it precisely we need to use the notion of Kuranishi structure more systematically. We will do it in [12]. Here we sketch the argument by simplifying the situation. Let us consider a Banach manifold B together with a Banach bundle $E \to B$ and a section $s : B \to E$ which is not necessary transversal to zero. We assume that the differential of s is Fredholm. Let $\tilde{f} : B \to \mathcal{L}(L)$ be a smooth map. (In our application $B = Map((D^2, \partial D^2); (M, L))$, and $s = 0$ gives the equation $\varphi \in B$ to be pseudoholomorphic.) We assume $s^{-1}(0)$ is compact. Then we can find a finite dimensional space W and a family of perturbations s_w of s parametrized by $w \in W$ which has the following properties.

(1) We put $P = \{(x,w) \in B \times W | s_w(x) = 0\}$. Then P is a smooth manifold.
(2) The projection $\pi_W : P \to W$ is smooth and proper.
(3) If we put $f(x,w) = \tilde f(x)$ then $f : P \to \mathcal{L}(L)$ is a smooth map.
(4) The composition $ev \circ f : P \to L$ is a submersion.

(4) becomes possible by taking W of very big dimension.)

Now let ω_W be a smooth form on W of top dimension. We assume that it is of compact support and $\int_W \omega_W = 1$. We put $\omega = \pi_W^* \omega_W$. Then (P, f, ω) is an approximate De Rham cycle.

In our actual application, we have locally an orbibundle $E_\alpha \to U_\alpha$ on an orbifold U_α together with its sections s_α such that $\cup_\alpha s_\alpha^{-1}(0) = \widehat{\mathcal{M}}(L;\beta)$. Moreover $(E_\alpha, U_\alpha, s_\alpha)$ are glued in an appropriate sense. More precisely we have a Kuranishi structure on $\widehat{\mathcal{M}}(L;\beta)$. We can use a smooth family of multisections (see [14]) in a similar way as above to obtain an approximate De Rham chain for each $\widehat{\mathcal{M}}(L;\beta)$. We write it by the same symbol $\widehat{\mathcal{M}}(L;\beta)$ for simplicity.

Theorem 6.4 ([12])**.** *We can choose* Lift_β *and approximate De Rham chain* $\widehat{\mathcal{M}}(L;\beta)$ *such that*

$$\partial \widehat{\mathcal{M}}(L;\beta) + \frac{1}{2} \sum_{\beta = \beta_1 + \beta_2} \{\widehat{\mathcal{M}}(L;\beta_1), \widehat{\mathcal{M}}(L;\beta_2)\} = 0. \tag{6.8}$$

The proof is similar to the proof of Proposition 4.3 and will be given in detail in [12].

Remark 6.2. The method to realize virtual fundamental chain using approximate De Rham chain is somewhat similar to Ruan's approach [30].

7. Iterated integral and Gerstenhaber bracket.

In §6, we studyed homology of the loop space $\mathcal{L}(L)$. The relation of homology of L and of $\mathcal{L}(L)$ is classical. Especially there is a construction by Chen [6], which we review here.

Let L be a smooth manifold and $(\Omega(L), d, \wedge)$ be its De Rham complex. We put

$$\begin{aligned} \Omega(L)[1]^k &= \Omega^{k+1}(L), \\ B_k(\Omega(L)[1]) &= \underbrace{\Omega(L)[1] \times \cdots \times \Omega(L)[1]}_{k\text{copy}} \\ B(\Omega(L)[1]) &= \bigoplus_k B_k(\Omega(L)[1]) \end{aligned}$$

and define

$$\overline{\mathfrak{m}}_k : \Omega(L)[1]^{k\otimes} \to \Omega(L)[1]$$

of degree $+1$ by

$$\overline{\mathfrak{m}}_k(u) = (-1)^{\deg u} du, \quad \overline{\mathfrak{m}}_2(u,v) = (-1)^{\deg u(\deg v + 1)} u \wedge v \tag{7.1}$$

and $\overline{\mathfrak{m}}_k = 0$ for $k \neq 1, 2$. It will define a structure of A_∞ algebra (which will be defined later). We now define

$$\hat d : B(\Omega(L)[1]) \to B(\Omega(L)[1]),$$

by

$$\begin{aligned}\hat{d}(u_1 \otimes \cdots \otimes u_k) = &\sum_i (-1)^{*_i} u_1 \otimes \cdots \otimes \overline{\mathfrak{m}}_1(u_i) \otimes \cdots \otimes u_k + \\ &\sum_i (-1)^{*_i} u_1 \otimes \cdots \otimes \overline{\mathfrak{m}}_2(u_i, u_{i+1}) \otimes \cdots \otimes u_k\end{aligned} \tag{7.2}$$

where $*_i = \deg u_1 + \cdots + \deg u_{i-1} + i - 1$, here deg is the degree as a differential form. It is easy to see that $\hat{d}^2 = 0$ and hence $(B(\Omega(L)[1]), \hat{d})$ is a cochain complex.

To define iterated integral we modify it a bit. We fix a base point p_0 of L and define $\Omega_0(L)$ as follows.

$$\Omega_0^k(L) = \Omega^k(L), \quad k \neq 0,$$
$$\Omega_0^k(L) = \{f \in C^\infty(L) | f(p_0) = 0.\}$$

We define $B(\Omega_0(L)[1])$ in a similar way.

It is easy to see that $\hat{d}$ preserves $B(\Omega_0(L)[1])$ and hence $(B(\Omega_0(L)[1]), \hat{d})$ is a cochain complex also.

We denote by $\mathcal{L}_0(L)$ the based loop space. Namely :

$$\mathcal{L}_0(L) = \{\ell : S^1 \to L | \ell(0) = p_0\}.$$

Theorem 7.1 (Chen [6]). *There exists a cochain homomorphism*

$$\mathrm{Ich} : (B(\Omega_0(L)[1]), \hat{d}) \to \Omega(\mathcal{L}_0(L), d)$$

where $(\Omega(\mathcal{L}_0(L)), d)$ is the De-Rham complex of the based loop space.

Proof. Since $\mathcal{L}_0(L)$ is infinite dimensional we need to be careful to define De Rham complex $\Omega(\mathcal{L}_0(L), d)$. We do not discuss this point. See [6]. Instead we take a smooth chain (P, f) of $\mathcal{L}_0(L)$ and define an integration of $\mathrm{Ich}(u_1 \otimes \cdots \otimes u_k)$ over (P, f) as follows.

We put

$$C_k = \{(t_1, \cdots, t_k) \in [0,1]^k | 0 \le t_1 \le \cdots \le t_k \le 1\}. \tag{7.3}$$

We define a map $ev : \mathcal{L}_0(L) \times C_k \to L^k$, by

$$ev(\ell, (t_1, \cdots, t_k)) = (\ell(t_1), \cdots, \ell(t_k)). \tag{7.4}$$

We now define

$$\int_P f^*(\mathrm{Ich}(u_1 \otimes \cdots \otimes u_k)) = \int_{P \times C_k} (ev \circ (f \times id))^*(u_1 \wedge \cdots \wedge u_k). \tag{7.5}$$

We can prove that

$$\int_{\partial P} f^*(\mathrm{Ich}(u_1 \otimes \cdots \otimes u_k)) = \int_P f^*(\mathrm{Ich}(\hat{d}(u_1 \otimes \cdots \otimes u_k))). \tag{7.6}$$

by studying the boundarry of $P \times C_k$. (We omit the detail.) This implies that Ich is a cochain homomorphism. □

We define a free loop space version of the homomorphism of Theorem 7.1. We consider

$$Hom(B(\Omega(L)[1]), \Omega(L)[1]) = \prod_k Hom(B_k(\Omega(L)[1]), \Omega(L)[1]),$$

and define a boundary operator δ on it as follows. Let $\varphi \in Hom(B(\Omega(L)[1]),\Omega(L)[1])$ then

$$\begin{aligned}(\delta\varphi)(a_1\otimes\cdots\otimes a_k) = d(&\varphi(a_1\otimes\cdots\otimes a_k))\\ &- (-1)^{\deg\varphi}(\varphi\circ\hat{d})(a_1\otimes\cdots\otimes a_k)\\ &+ (-1)^{(\deg u+1)\deg\varphi}\overline{\mathfrak{m}}_2(a_1,\varphi(a_2\otimes\cdots\otimes a_k))\\ &+ \overline{\mathfrak{m}}_2(\varphi(a_1\otimes\cdots\otimes a_{k-1}),a_k)\end{aligned}$$

It is easy to check that $\delta^2 = 0$.

We denote by $S^D(\mathcal{L}(L))$ the set of all approximate De Rham chains on $\mathcal{L}(L)$. It is a chain compex.

Proposition 7.2. *There exists a chain homomorphism*

$$\mathrm{Ich}^* : S^D(\mathcal{L}(L)) \to Hom(B(\Omega(L)[1]),\Omega(L)[1])$$

Proof. We define a map

$$ev_+ = (ev, ev_0) : \mathcal{L}(L)\times C_k \to L^{k+1}$$

by

$$ev_+(\ell,(t_1,\cdots,t_k))) = (\ell(t_1),\cdots,\ell(t_k),\ell(0)).$$

Now we put

$$\begin{aligned}&\mathrm{Ich}^*((P,f,\omega))(u_1\otimes\cdots\otimes u_k))\\ &\qquad = (ev_0\circ\pi)!\,(\omega\wedge(ev\circ(f\times id))^*(u_1\wedge\cdots\wedge u_k))\,.\end{aligned}\tag{7.7}$$

Here $ev_0\circ\pi : P\times C_k \to L$ is the composition of $P\times C_k\to P$, $f : P\to\mathcal{L}(L)$ and $ev_0 : \mathcal{L}(L)\to L$. $(ev_0\circ\pi)!$ is the integration along fiber.

It is straightforward to check that Ich^* is a chain map. □

We next recall Gerstenhaber bracket, which was in troduced by [16] to study deformation theory of associative algebra.

We restrict ourselves to the case of $\Omega(L)$. We define a structure of differential graded Lie algebra on $Hom(B(\Omega(L)[1]),\Omega(L)[1])$ as follows. Let

$$\varphi_i \in Hom(B_{k_i}(\Omega(L)[1]),\Omega(L)[1]).$$

We put

$$\begin{aligned}&(\varphi_1\circ\varphi_2)(u_1\otimes\cdots\otimes u_{k_1+k_2-1})\\ &\quad=\sum_i(-1)^{*_i}\varphi_1(u_1\otimes\cdots\varphi_2(u_i\otimes\cdots u_{i+k_2-1})\cdots\otimes u_{k_1+k_2-1}).\end{aligned}\tag{7.8}$$

where $*_i = (\deg\varphi_2)(\deg u_1+\cdots+\deg u_{i-1}+i-1)$. We then define :

$$\{\varphi_1,\varphi_2\} = \varphi_1\circ\varphi_2 - (-1)^{\deg\varphi_1\deg\varphi_2}\varphi_2\circ\varphi_1.\tag{7.9}$$

We call $\{\cdot,\cdot\}$ the *Gerstenhaber bracket.*

Theorem 7.3 ([16])**.** $(Hom(B(\Omega(L)[1]),\Omega(L)[1]),\delta,\{\cdot,\cdot\})$ *is a differential graded Lie algebra.*

Now we have :

Proposition 7.4. *We have* $\delta(\mathrm{Ich}^*((P,f,\omega))) = \mathrm{Ich}^*(\partial(P,f,\omega))$ *and*

$$\mathrm{Ich}^*(\{(P_1,f_1,\omega_1),(P_2,f_2,\omega_2)\}) = \{\mathrm{Ich}^*((P_1,f_1,\omega_1)),\mathrm{Ich}^*((P_2,f_2,\omega_2))\}$$

where $\{\cdot,\cdot\}$ *in the left hand side is Loop bracket and* $\{\cdot,\cdot\}$ *in the right hand side is Gerstenhaber bracket.*

The proof is straight forward calculation and is omitted. The proposition implies that Ich^* is a homomrphism of differential graded Lie algebra.

Remark 7.1. Chas-Sullivan in [5] already mentioned that their construction is an analogy of Gerstenhaber bracket.

Remark 7.2. Precisely speaking the loop bracket defines an L_∞ structure since the Jacobi identity holds only modulo homotopy. However Jacobi identity on $S^D(\mathfrak{L}(L))$ fails only because of parametrization. The difference of parametrization is killed by Ich^*.

8. A_∞ deformation of De Rham complex.

We now use the result of §§6 7 to define an A_∞ deformation of the De Rham complex. We first recall the definition of filtered A_∞ algebra. Let $C_{\mathbb{R}}$ be a graded $\mathbb{R}$ vector space. Let

$$\Lambda_0 = \left\{\sum a_i T^{\lambda_i} \in \Lambda \,\middle|\, \lambda_i \ge 0\right\}, \quad \Lambda_+ = \left\{\sum a_i T^{\lambda_i} \in \Lambda \,\middle|\, \lambda_i > 0\right\}.$$

Λ_0 is a local ring and Λ_+ is its maximal ideal. Λ_0 has a filtration $F^\lambda\Lambda_0 = T^\lambda\Lambda_0$ which defines a (non Archimedian) norm on it. We put $C^k \cong C_{\mathbb{R}}\hat\otimes\Lambda_0$ here and hereafter $\hat\otimes$ means that we are taking an approariate completion with respect to the (non Archimedian) norm on Λ_0. (We omit the detail refer [13].) $C[1]^k = C^{k+1}$. We put

$$B_k(C[1]) = \underbrace{C[1]\hat\otimes_{\Lambda_0}\cdots\hat\otimes_{\Lambda_0}C[1]}_{k}, \quad B(C[1]) = \bigoplus_k B_k(C[1])$$

and consider a series of Λ_0 module homomorphisms

$$\mathfrak{m}_k : B_k(C[1]) \to C[1]$$

of odd degree. We assume that it is written as

$$\mathfrak{m}_k = \sum_i T^{\lambda_i}\mathfrak{m}_{k,i}$$

where $\lambda_i \ge 0$, $\lambda_i \to +\infty$ and $\mathfrak{m}_{k,i}$ is induced by $\mathbb{R}$ linear map

$$\underbrace{C_{\mathbb{R}} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} C_{\mathbb{R}}}_{k} \to C_{\mathbb{R}},$$

of degree d_i+1. (Here d_i are even numbers. We assume that λ_i and d_i is independent of k.) We assume also $\lambda_0 = 0$ and $\lambda_i > 0$ for $i > 0$.

Definition 8.1. We say that $(C,\mathfrak{m}_k)$, $k = 0,1,\cdots$ is a filtered A_∞ algebra if

$$\sum_{k+\ell=n+1}\sum_i (-1)^*\mathfrak{m}_k(x_1\otimes\cdots\mathfrak{m}_\ell(x_i\otimes\cdots\otimes x_{i+\ell-1})\cdots\otimes x_n) = 0$$

where $* = \deg x_1 + \cdots + \deg x_{i-1} + i - 1$ (deg is a degree before shift) and if $\mathfrak{m}_0 \equiv 0$ mod Λ_+.

We take $C_{\mathbb{R}}^k = \Omega^k(L)$ the De-Rham complex. We are going to define $\mathfrak{m}_k$. We take $\mathfrak{m}_{k,0} = \overline{\mathfrak{m}}_k$. We recall that $\overline{\mathfrak{m}}_1 = \pm d$, $\overline{\mathfrak{m}}_2 = \pm\wedge$ and other $\overline{\mathfrak{m}}_k$ are zero. We put

$$\varphi_k = \sum_{i>0} \mathfrak{m}_{k,i} \in Hom(B_kC[1], C[1]), \quad \varphi = (\varphi_0, \varphi_1, \cdots). \tag{8.1}$$

The next proposition is in principle due to Gerstenhaber.

Proposition 8.1. $\mathfrak{m}_k$ *is an filtered* A_∞ *algebra if and only if*

$$\delta\varphi + \frac{1}{2}\{\varphi, \varphi\} = 0.$$

We remark we can generalize Gersenhaber blacket $\{\cdot, \cdot\}$ in an obvious way to $\prod_k Hom(B_kC[1], C[1])$.

Now let L be a Lagrangian submanifold of M. We assume that it is relativelly spin and is compact.

Definition 8.2.

$$\alpha(L) = \sum_i T^{E(\beta)} \widehat{\mathcal{M}}(L; \beta_i).$$

$\alpha(L)$ is a Λ_0 valued approximate De Rham chain of $\mathcal{L}(L)$.

Here $\beta_i \in \pi_2(M, L)$ such that $0 = E(\beta_0) < E(\beta_1) \le \cdots$. We define

$$\varphi = (\varphi_0, \varphi_1, \cdots) = \mathrm{Ich}^*(\alpha), \tag{8.2}$$

where Ich^* is as in (7.7) We use φ_k to define $\mathfrak{m}_k$ by (8.1).

Theorem 8.2. *The operator* $\mathfrak{m}_k$ *above defines a structure of filtered* A_∞ *algebra on* $\Omega(L)\hat{\otimes}\Lambda_0$.

Proof. Theorem 8.2 follows immediately from Theorem 6.4, Propositions 7.4, 8.1. □

To define a structure of filtered A_∞ algebra on the cohomology group $H^*(L; \mathbb{R})$ we use the following theorem. Let $(C, \mathfrak{m}_k)$ be a filtered A_∞ algebra. We remark that $\mathfrak{m}_{1,0}$ is induced from an $\mathbb{R}$ linear map $C_{\mathbb{R}}^k \to C_{\mathbb{R}}^{k+1}$, which we write $\overline{\mathfrak{m}}_1$. Using $\mathfrak{m}_{0,0} = 0$, (which follows from $\mathfrak{m}_0 \equiv 0 \mod \Lambda_+$.), we can prove $\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_1 = 0$. Let $H^*(C_{\mathbb{R}})$ be the cohomology group $H^*(C; \overline{\mathfrak{m}}_1)$ that is nothing but the De Rham cohomology group in our main example. We consider $H^*(C; \Lambda_0) = H^*(C_{\mathbb{R}}) \otimes_{\mathbb{R}} \Lambda_0$.

Theorem 8.3 ([13])**.** *There exists a structure of filtered* A_∞ *algebra on* $H^*(C; \Lambda_0)$ *such that it is homotopy equivalent to* $(C, \mathfrak{m})$.

Theorem 8.3 is a filtered version of a classical result of homotopical algebra (see Kadeishvili [21] etc.) and is proved in the reviced version of [13] (see also [11]). We also refer [13] (and [11]) for the definition of homotopy equivalence of filtered A_∞ algebra.

Theorems 8.2 and 8.3 imply that we have a structure of filtered A_∞ algebra on $H^*(L; \Lambda_0)$ for a compact relatively spin Lagrangian submanifold L. We now show how the structures of Theorem 5.1 is deduced from it. First we consider

$$\mathfrak{m}_0 : B_0H(L; \Lambda_0) = \Lambda_0 \to H(L; \Lambda_0).$$

We put

$$o_{\beta_i} = \mathfrak{m}_{0,i}(1) \in H^{2+d_i}(L; \mathbb{R}).$$

By Definition 8.2 we can show that $d_i = -\eta_L(\beta_i)$. Hence $o_{\beta_i} \in H^{2-\eta_L(\beta_i)}(L;\mathbb{R})$ as required.

We next assume that o_β are all zero. Then

$$(\mathfrak{m}_1 \circ \mathfrak{m}_1)(x) = \pm \mathfrak{m}_2(\mathfrak{m}_0(1), x) \pm \mathfrak{m}_2(x, \mathfrak{m}_0(1)) = 0.$$

We define

Definition 8.3.

$$HF(L, L; \Lambda_0) = \frac{\mathrm{Ker}\mathfrak{m}_1}{\mathrm{Im}\mathfrak{m}_1}.$$

We remark that $H^*(L;\Lambda_0)$ has a filtration induced by the filtration $F^\lambda\Lambda_0 = T^\lambda\Lambda_0$. We can use it to construct a spectral sequence in Theorem 5.1. The formula

$$\mathfrak{m}_1 = \sum_{i \geq 0} T^{\lambda_i}\mathfrak{m}_{1,i}$$

implies that the differential of the spectral sequence is induced by $\mathfrak{m}_{1,i}$. We put $d_{\beta_i} = \mathfrak{m}_{1,i}$.

Remark 8.1. Actually the filtration $F^\lambda\Lambda_0 = T^\lambda\Lambda_0$ is parametrized by $\lambda_i \in \mathbb{R}_{\geq 0}$ and is rather unusual. (Usually spectral sequence is induced by a filtration parametrized by integer.) Moreover the ring Λ_0 is not Noetherian. This causes serious trouble to construct spectral sequence and prove its convergence. This problem is resolved in [13].

Remark 8.2. In the above argument we use Λ_0 in place of Λ. Off course Floer homology over Λ_0 induces Floer homology over Λ. The reason we need to work over Λ is that (6) of Theorem 5.1 (or more generally the invariance of Floer homology under Hamiltonian deformation of Lagrangian submanifold) is not true over Λ_0 and can be proved only over Λ coefficient. See [13].

We need more argument to establish the properties asserted in Theorem 5.1. (Especially (4)(5)(6) of it.) We do not explain it here and refer [13].

We mention one application of the construction of this article.

Theorem 8.4. *Let L be a compact relatively spin Lagrangian submanifold of M. We assume that L admits a metric of negative sectional curvature and* $\dim L$ *is even. Then $o_\beta = 0$. Moreover $HF(L;L) \cong H(L;\Lambda)$.*

Proof. (*Sketch*) We remark that $\deg \widehat{\mathcal{M}}(L;\beta) = n-2+\eta_L(\beta)$ is even, since n is even and $\eta_L(\beta)$ is even. Let us decompose $\mathcal{L}(L)$ into connected components $\cup\mathcal{L}_{[\gamma]}(L)$, where $\gamma \in \pi_1(L)$ and $[\gamma]$ be its conjugacy class. Since L has negative curvature $H_i(\mathcal{L}_{[\gamma]}(L)) \cong H_i(S^1)$ for $\gamma \neq 1$. We remark that if $\widehat{\mathcal{M}}(L;\beta)$ is nonempty then it is at least one dimensional. (In fact if $\widehat{\mathcal{M}}(L;\beta)$ is nonempty then $\mathcal{M}(L;\beta)$ is nonempty.) This implies that $\widehat{\mathcal{M}}(L;\beta)$ is homologous to zero if $\partial\beta \neq 1$. Hence $\mathfrak{m}_{k,i}$ is nonzero only for β_i with $\partial\beta_i = 1$. However since L is aspherical $\mathcal{L}_{[1]}(L)$ is homotopy equivalent to L. We can use it to show that Ich^* is trivial on $\mathcal{L}_{[1]}(L)$. Thus the A_∞ algebra $(\Omega(L)\hat{\otimes}\Lambda_0, \mathfrak{m}_k)$ is homotopy equivalent to the De Rham complex. Theorem 8.4 follows. □

Remark 8.3. Actually there is one point where the argument above is not sufficient. Namely since $\widehat{\mathcal{M}}(L;\beta)$ is not a cycle in general (see Theorem 6.4), it is not clear how to use vanishing of cohomology of $\mathcal{L}_{[\partial\beta]}(L)$ to modify $\widehat{\mathcal{M}}(L;\beta)$ to zero, without changing the homotopy type of filtered A_∞ algebra induced by it on $\Omega(L)\hat{\otimes}\Lambda_0$.

We can overcome this point in the following way.

First there is a notion of gauge equivelence between elements satisfing Maurer-Cartan equation (that is the conclusion of Theorem 6.4), such that gauge equivalent $\widehat{\mathcal{M}}(L;\beta)$ induces a homotopy equivalent A_∞ structure on $\Omega(L)\hat{\otimes}\Lambda_0$. See [11].

Second we find that the set of homotopy equivalence class of elements satisfing Maurer-Cartan equation, is independt of the homotopy type of differential graded Lie algebra (or more generally of L_∞ algebra).

Third we can show that and differential graded Lie algebra C is homotoyp equivalent to an L_∞ algebra defined on cohomology group of C. This fact is an L_∞ analogue of Theorem 8.3 and is proved by various people including [21].

Since our solution $\widehat{\mathcal{M}}(L;\beta)$ of Maurer-Cartan equation has degree where cohomology group vanish, it follows that it is gauge equivalent to zero. Summing up we find that the A_∞ structure induced by $\widehat{\mathcal{M}}(L;\beta)$ is homotopy equivalent to one induced by zero. Theorem 8.4 follows.

Remark 8.4. Theorem 8.4 implies that negatively curved spin manifold of even dimension is not embeded to $\mathbb{C}\times M$ (for any symplectic manifold M which is convex at infinity) as a Lagrangian submanifold. This fact was established in stronger form by Viterbo. Namely negatively curved manifold is not embeded as a Lagrangian submanifold to $\mathbb{C}\times M$ or $\mathbb{C}P^n$. (He does not need to assume neigher L is spin nor that L is of even dimension.) We will discuss related problems in §14.

Remark 8.5. The idea to use Chan-Sullivan's String topology to study open string theory is also applied by [4]. Their interest is in its application to Physics. We here emphasise its application to sympletic topology. The application of our approach to mirror symmetry will be discussed elsewhere.

The idea to use homology of loop space in Floer theory is independently proposed in [3], where Barraud and Cornea applied it in the case when there exist no pseudoholomorphic disk and Floer homology is isomorphic to the usual homology[3]. F.Lalonde informed the author that together with authors of [3] he is now trying to apply it in more general situation.

9. S^1 equivariant homology of Loop space and cyclic A_∞ algebra.

The loop space $\mathcal{L}(L)$ has a canonical S^1 action defined by $(s\cdot\ell)(t) = \ell(t+s)$. Chas-Sullivan [5] also defined a blacket (which they call string blacket) on the S^1 equivariant homology $H_*^{S^1}(\mathcal{L}(L))$. Namley they define

$$\{\cdot,\cdot\} : H_k^{S^1}(\mathcal{L}(L)) \otimes H_\ell^{S^1}(\mathcal{L}(L)) \to H_{k+\ell-n+2}^{S^1}(\mathcal{L}(L)), \tag{9.1}$$

by

$$\{x,y\} = I_*\{I^*(x), I^*(y)\} \tag{9.2}$$

where $I_* : H_k(\mathcal{L}(L)) \to H_k^{S^1}(\mathcal{L}(L))$ and $I^* : H_k^{S^1}(\mathcal{L}(L)) \to H_{k+1}(\mathcal{L}(L))$ are obvious maps and $\{\}$ in the right hand side is the loop bracket. In a similar way, we can construct it in the chain level using S^1 equivariant approximation De Rham chain of $\mathcal{L}(L)$. Here S^1 equivariant approximation De Rham chain of $\mathcal{L}(L)$ is an

[3] Our main application in §§11,12,13,14 is in the case when Floer homology may not be well defined in the sense of [13].

approximate De Rham chain (P, f, ω) such that S^1 acts on P and that f,ω are S^1 equvariant.

We next consider $\mathcal{M}(L;\beta)$. We may regard it as an S^1 equivariant approximate De Rham chain of $\mathcal{L}(L)$ of degree $n-3+\eta_L(\beta)$. Then we have

$$\partial\mathcal{M}(L;\beta) + \frac{1}{2}\sum_{\beta=\beta_1+\beta_2}\{\mathcal{M}(L;\beta_1), \mathcal{M}(L;\beta_2)\} = 0. \tag{9.3}$$

We next define a cyclic Bar complex $B_k^{cyc}(C[1])$ deviding $B_k(C[1])$ by the equivalence relation generated by

$$x_1 \otimes \cdots \otimes x_k \sim (-1)^{(\deg x_k+1)(\deg x_1+\cdots \deg x_{k-1}+k-1)} x_k \otimes x_1 \otimes \cdots \otimes x_{k-1}.$$

We can define a Gerstenhaver bracket on $\prod_{k\ge 1} Hom(B_k^{cyc}(C[1]), \mathbb{R})$ in a similar way. We have homomorphism

$$\mathrm{Ich} : \bigoplus_k B_k^{cyc}(C[1]) \to \Omega^*_{S^1}(\mathcal{L}(L))$$

here the right hand side is the set of S^1 equivarent forms. (See [17].) Its dual Ich^* is a chain homomorphism and sends string blacket to Gerstenhaber bracket.

Now we put

$$\overline{\alpha}(L) = \sum_i T^{E(\beta)}\mathcal{M}(L;\beta_i).$$

which is an S^1 equivalent approximate De Rham chain of $\mathcal{L}(L)$. Hence pulling it back by Ich^*, we have an element $\mathrm{Ich}^*(\overline{\alpha}(L))$ of $\prod_{k\ge 1} Hom(B_k^{cyc}(\Omega(L)[1]), \Lambda_+)$ Then we have

$$\delta(\mathrm{Ich}^*(\overline{\alpha}(L))) + \frac{1}{2}\{\mathrm{Ich}^*(\overline{\alpha}(L)), \mathrm{Ich}^*(\overline{\alpha}(L))\} = 0.$$

It defines a family of operations

$$\mathfrak{m}_k^+ : B_k^{cyc}(\Omega(L)[1]) \to \Lambda_0.$$

It is related to the operations $\mathfrak{m}_k$ in the last section by the fomula

$$\langle \mathfrak{m}_k(u_1, \cdots, u_k), u_{k+1}\rangle = \mathfrak{m}_{k+1}^+(u_1, \cdots, u_k, u_{k+1}) \tag{9.4}$$

if we take perturbation appropriately. Here $\langle\cdot,\cdot\rangle : \Omega^k(L) \otimes \Omega^{n-k}(L) \to \mathbb{R}$ is defined by

$$\langle u, v\rangle = \int_L u \wedge v.$$

(9.4) implies that $\mathfrak{m}_k$ satisfies the following cyclic symmetry.

$$\langle \mathfrak{m}_k(u_1, \cdots, u_k), u_{k+1}\rangle = (-1)^* \langle \mathfrak{m}_k(u_{k+1}, u_1 \cdots, u_{k-1}), u_k\rangle \tag{9.5}$$

where $* = (\deg x_{k+1} + 1)(\deg x_1 + \cdots \deg x_k + k)$. We will discuss the contents of this section in more detail in [12].

Remark 9.1. We constructed filtered cyclic A_∞ algebra in this section using De Rham cohomology. Hence it is defined over $\mathbb{R}$. The author does not know how to do it keeping cyclic symmetry over $\mathbb{Z}$ even in semipositive case. (Actually he does not know how to do it for classical cohomology either.) Compare Remark 6.1.

10. L_∞ structure on $H(S^1 \times S^n; \mathbb{Q})$.

In this section we consider the case of $S^1 \times S^n$ in $\mathbb{C}^{n+1}$ and calculate the leading term of the symmetrization of A_∞ structure of it.

We first discuss symmetrization of (filtered) A_∞ algebra briefly. Let $B_k(C[1])$ is as in §8. We divide it by the equivalence relation generated by

$$\begin{aligned}(10.1) \qquad & x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \\ & \sim (-1)^{(\deg x_i + 1)(\deg x_{i+1}+1)} x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_k,\end{aligned}$$

and denote it by $E_k(C[1])$. $\mathfrak{m}_k$ induces $\mathfrak{l}_k : E_k(C[1]) \to C[1]$ by

$$\mathfrak{l}_k([x_1 \otimes \cdots \otimes x_k]) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{*_\sigma} \mathfrak{m}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) \tag{10.2}$$

here $*_\sigma = \sum_{i<j;\sigma(i)>\sigma(j)} (\deg x_i + 1)(\deg x_j + 1)$. In case $\mathfrak{m}_k = \overline{\mathfrak{m}}_k$ is induced by the structure of (graded commutative) differential graded algebra by (7.1) then we can check by an easy calculation that the induced operations $\mathfrak{l}_k$ becomes zero. Hene the part of $\mathfrak{l}_k$ on $E_k(\Omega(L)[1])$ induced by wedge product of differential forms vanishes.

$\mathfrak{l}_k$ defines an L_∞ structure. Here we define :

Definition 10.1. $(C, \mathfrak{l}_k)$ is said to be a filtered L_∞ algebra if the following holds :

$$\sum_{k+\ell=n} \sum_{I,J} (-1)^* \mathfrak{l}_{\ell+1}(\mathfrak{l}_k(x_{i_1} \otimes \cdots x_{i_k}) \otimes x_{j_1} \otimes \cdots \otimes x_{j_\ell}) = 0$$

where the second sum is taken over all $I = \{i_1, \cdots, i_k\}, J = \{j_1, \cdots, j_\ell\}$ with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_\ell$, $I \cap J = \emptyset$, $I \cup J = \{1, \cdots, n\}$ and

$$* = \sum_{a,b; i_a > j_b} (\deg x_{i_a} + 1)(\deg x_{j_b} + 1).$$

We thus obtained a filtered L_∞ algebra $(\Omega(L), \mathfrak{l}_k)$ for a relatively spin compact Lagrangian submanifold $L \subseteq M$. We consider the case $L = S^1 \times S^n \subset \mathbb{C}^{n+1}$ and calculate the leading term of $\mathfrak{l}_k$.

We put

$$\langle \mathfrak{l}_k(u_1, \cdots, u_k), u_{k+1} \rangle = \mathfrak{l}^+_{k+1}(u_1, \cdots, u_k, u_{k+1}). \tag{10.3}$$

Let us chooose a generator $\gamma \in \pi_1(S^1 \times S^n)$ such that $E(\gamma) = \lambda_1 > 0$. (Such γ exists by Theorem 2.1.) We consider the case $\eta_L(\gamma) = 2, n+1$. They are the only cases we have an examples. Let $a, b, c, e \in H^*(S^1 \times S^n; \mathbb{Z})$ be generators of degree $1, n, n+1, 0$ respectively.

Theorem 10.1. *If $\eta_L(\gamma) = 2$ then*

$$\mathfrak{l}^+_{k+1}(a, \cdots, a, c) \equiv \pm T^{\lambda_1} \frac{1}{k!} \mod T^{2\lambda_1}.$$

All other operations $\mathfrak{l}^+$ among generators vanish.

Theorem 10.2. *If $\eta_L(\gamma) = n+1$ (n is odd) then*

$$\mathfrak{l}^+_{k+2}(a, \cdots, a, b, c) \equiv \pm T^{\lambda_1} \frac{1}{(k+1)!} \mod T^{2\lambda_1}.$$

All other operations $\mathfrak{l}^+$ among generators vanish.

Remark 10.1. We can show that the left hand side of Theorems 10.1, 10.2 is well defined modulo $T^{2\lambda_1}$. Namely it is independent of the perturbation etc. This is because then are leading terms. (In other words it is because the cup product $\overline{\mathfrak{m}}_2$ will cancel out after symmetrization.) On the other hand, $\mathfrak{m}^+_{k+1}(a, \cdots, a, c)$ etc. depends on perturbation etc. even modulo $T^{2\lambda_1}$. This is because cup (or wedge) product is the leading term. $\mathfrak{m}^+_{k+1}$ is well defined only up to homotopy equivalent. We will discuss this point more in the reviced version of [13].

To prove Theorems 10.1,10.2, we first remark that Theorem 6.4 implies that $\widehat{\mathcal{M}}(S^1 \times S^n; \gamma)$ is a cycle in $S^D(\mathcal{L}(S^1 \times S^n))$. (This is because there is no $\gamma' \in \pi_1(S^1 \times S^n)$ with $0 < E(\gamma') < E(\gamma)$.)

Actually we can prove more. Namely, since our Lagrangian submanifold is monotone we can use a result of [13] to find an appropriate (single valued) perturbation so that the fundamental chain of $\widehat{\mathcal{M}}(S^1 \times S^n; \gamma)$ is a cycle over $\mathbb{Z}$.

Hence in case of Theorems 10.1, we have $[\widehat{\mathcal{M}}(S^1 \times S^n; \gamma)] \in H_{n+1}(\mathcal{L}(S^1 \times S^n); \mathbb{Z})$, and in case of Theorems 10.2 we have $[\widehat{\mathcal{M}}(S^1 \times S^n; \gamma)] \in H_{2n}(\mathcal{L}(S^1 \times S^n); \mathbb{Z})$. We are going to calculate them below. We first use the following :

Lemma 10.3. *$[\widehat{\mathcal{M}}(S^1 \times S^n; \gamma)]$ is in the image of $I^* : H_k^{S^1}(\mathcal{L}(S^1 \times S^n); \mathbb{Q}) \to H_{k+1}(\mathcal{L}(S^1 \times S^n); \mathbb{Q})$.*

This is a consequence of the §9.

We next remark that it is easy to show

$$H_*(\mathcal{L}(X \times Y)) = H_*(\mathcal{L}(X)) \otimes H_*(\mathcal{L}(Y)). \tag{10.4}$$

Let $\gamma_0 \in \pi_1(S^1)$ is a generator and $\mathcal{L}_{\gamma_0}(S^1)$ be the component containing γ_0. It is easy to see that $H_*(\mathcal{L}_{\gamma_0}(S^1)) \cong H_*(S^1)$, $H_*^{S^1}(\mathcal{L}_{\gamma_0}(S^1)) \cong H_*^{S^1}(S^1) \cong \mathbb{Z}$.

Now we consider the component of $\mathcal{L}_\gamma(S^1 \times S^n)$. Then using a commutative diagram

$$\begin{tikzcd}
\mathcal{L}(S^n) \arrow[d, "\subset"'] & \\
\mathcal{L}_\gamma(S^1 \times S^n) \arrow[r, "/S^1"] \arrow[d, "\pi"'] & \mathcal{L}_\gamma(S^1 \times S^n)/S^1 \arrow[d, "\pi"'] \\
\mathcal{L}_{\gamma_0}(S^1) \arrow[r, "/S^1"] & \text{one point}
\end{tikzcd}$$

Diagram 1

where the left vertical maps are fibration, we find that

$$H_k^{S^1}(\mathcal{L}_\gamma(S^1 \times S^n)) \cong H_k(\mathcal{L}(S^n)) \tag{10.5}$$

and the map $I^* : H_k^{S^1}(\mathcal{L}_\gamma(S^1 \times S^n)) \to H_{k+1}(\mathcal{L}_\gamma(S^1 \times S^n))$ is identifided with $x \mapsto [S^1] \otimes x$ where we use the identification (10.4) and $H_*(\mathcal{L}_{\gamma_0}(S^1)) \cong H_*(S^1)$.

We now recall the calculation of homology group of loop space of S^n. (See [29] for detail.)

Let $E(a, b, \cdots, c)$ be the free graded commutative graded algebra generated by $a, b, \cdots, c$. (Namely if all of $a, b, \cdots, c$ are of even degree then $E(a, b, \cdots, c)$ are polynomial algebra and if all of them are of odd degree then $E(a, b, \cdots, c)$ is an exterior algebra.) We recall the following classical result. Let $\mathcal{L}_0(X)$ be the based loop space.

Lemma 10.4. *If n is odd then $H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \cong E(x)$ with $\deg x = n-1$. If n is even then $H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \cong E(x,y)$ with $\deg x = n-1$, $\deg y = 2n-2$.*

To caclucate the cohomology of free loop space $\mathcal{L}(S^n)$ we use the Leray-Serre spectral sequence associated to the fiberation $\mathcal{L}_0(S^n) \to \mathcal{L}(S^n) \to S^n$. Let $[S^n]$ be the fundametal cohomology class of S^n. Then the E^2 term of the spectral sequence is $(\mathbb{Q}[S^n] \otimes \mathbb{Q}[pt]) \otimes H^*(\mathcal{L}_0(S^n);\mathbb{Q})$.

Lemma 10.5. *The boundary of the spectral sequence $(\mathbb{Q}[S^n]\otimes\mathbb{Q}[pt])\otimes H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \Rightarrow H^*(\mathcal{L}(S^n);\mathbb{Q})$ is zero if n is odd and is given by*

$$d(x \otimes [pt]) = 0, \qquad d(y \otimes [pt]) = 2x \otimes [S^n].$$

if n is even.

Now we consider first the case when $\eta_L(\gamma) = 2$. Then $[\widehat{\mathcal{M}}(S^1 \times S^n;\gamma)] \in H_{n+1}(\mathcal{L}(S^1 \times S^n);\mathbb{Z})$ corresponds to an element of $H_n(\mathcal{L}(S^n);\mathbb{Q})$. We can use Lemmas 10.4,10.5 to see that it corresponds to $\ell\ [pt] \otimes [S^n]$ for some $\ell \in \mathbb{Z}$.

Lemma 10.6. $\ell = \pm 1$.

Postponing the proof of Lemma 10.6 for a while, let us complete the proof of Theorem 10.1. We remark that the operations $\mathfrak{l}_k$ depends only on the homology class of $\widehat{\mathcal{M}}(S^1 \times S^n;\gamma)$. Hence we may assume that $\widehat{\mathcal{M}}(S^1 \times S^n;\gamma) = S^1 \times S^n$ and elements $(s,x) \in S^1 \times S^n$ corresponds to the curve $t \mapsto (s+t,x)$ in $S^1 \times S^n$. Let dt be the one form on S^1 and Ω be the volume form of S^n. Then a,b,c are the De Rham cohomology classes represented by dt, Ω and $dt \wedge \Omega$ respectively. Let us write $\mathfrak{m}_k = \sum_{i=0}^{\infty} T^{i\lambda_1}\mathfrak{m}_{k,i\gamma}$. Then, by definition, we have

$$\begin{aligned}
&\langle \mathfrak{m}_{k,\gamma}(a,\ \dots\ ,a), c\rangle \\
&= \pm \int_{x\in S^n}\int_{t_1=0}^{1}\int_{t_2=0}^{t_1}\ \dots\ \int_{t_k=0}^{t_{k-1}} d(s+t_1)\wedge\ \dots\ \wedge d(s+t_k)\wedge ds\wedge\Omega \\
&= \pm 1/k!.
\end{aligned}$$

This implies Theorem 10.1. □

Let us prove Lemma 10.6. By the same argument as above we can show (without using Lemma 10.6) that $\langle \mathfrak{m}_{k,\gamma}(a,\ \dots\ ,a), c\rangle = \pm\ell/k!$. In particular $\mathfrak{m}_{1,\gamma}(a) = \pm\ell e$. Since e should be on the image of $\mathfrak{m}_1$ by Theorem 5.1, it follows that $\ell = \pm 1$ as required. (We remark that since our Lagrangian submanifold is monotone we can work over integer. (See [13].) Hence we can prove not only $\ell \neq 0$ but also $\ell = \pm 1$.) □

We next consider the situation of Theorem 10.2. Then $[\widehat{\mathcal{M}}(S^1 \times S^n;\gamma)] \in H_{2n}(\mathcal{L}(S^1 \times S^n);\mathbb{Z})$ corresponds to an element of $H_{2n-1}(\mathcal{L}(S^n);\mathbb{Z})$. We can use Lemmas 10.4,10.5 to see that it corresponds to $\ell\ x \otimes [S^n]$ for some $\ell \in \mathbb{Z}$.

Lemma 10.7. $\ell = \pm 1$.

Postponing the proof of Lemma 10.7 for a while let us complete the proof of Theorem 10.2.

Let $P \to \mathcal{L}(S^n)$ be the cycle representing x. For $z \in P$, let $\mu_z : S^1 \to S^n$ be the curve represented by it. We consider the map $ev : P \times S^1 \to S^n \times S^n$ defined by $ev(z,t) = (\mu_z(0), \mu_z(t))$. By the definition of the class x, we find easily that $ev : P \times S^1 \to S^n \times S^n$ is of degree one.

Now by Lemma 10.7 $[\widehat{\mathcal{M}}(S^1 \times S^n;\gamma)]$ is homologous to the cycle represented by $f : P \times S^1 \to \mathcal{L}(S^1 \times S^n)$ given by $f(z,s)(t) = (s+t, \mu_z(t))$

Then, by definition, we have

$$\begin{aligned}
&\sum_{i=0}^{k} \langle \mathfrak{m}_{k+1,\gamma}(\overbrace{a, \ \dots \ , a}^{i}, b, \overbrace{a, \ \dots \ , a}^{k-i}), c\rangle \\
&= \pm \int_{(x,s)\in P\times S^1} \int_{t=0}^{1} \int_{t_1=0}^{1} \int_{t_2=0}^{t_1} \ \dots \ \int_{t_k=0}^{t_{k-1}} \\
&\qquad\qquad d(s+t_1) \wedge \ \dots \ \wedge d(s+t_k) \wedge dt \wedge ev^*(\Omega \wedge \Omega) \\
&= \pm 1/k!.
\end{aligned}$$

Theorem 10.2 follows. □

To prove Lemma 10.7 we use the above argument to show $\mathfrak{m}_{1,\gamma}(b) = \pm \ell e$. The required equality $\ell = \pm 1$ then follows from (the $\mathbb{Z}$ version of) Theorem 5.1. □

Remark 10.2. We remark that we still did not yet elliminate the possibility $\eta_L(\gamma) = 2-n$ when n is even. (We will elliminate this possibility in §14.) If this happens then $[\widehat{\mathcal{M}}(S^1\times S^n;\gamma)] \in H_1(\mathcal{L}_\gamma(S^1\times S^n)) \cong \mathbb{Z}$. Let us suppose for example that it is a generator. This means that, to calculate $\mathfrak{l}_k$ we may assume that $[\widehat{\mathcal{M}}(S^1\times S^n;\gamma)] \cong S^1$ such that $s \in S^1$ corresponds to the loop $t \mapsto (s+t, x_0)$ for some fixed x_0. Then we can calculate as above and obtain

$$\mathfrak{l}^+_{k+1}(a, \cdots, a, a) \equiv \pm T^{\lambda_1} \frac{1}{k!} \mod T^{2\lambda_1}.$$

and all other operations are zero modulo $T^{2\lambda_1}$. Let us suppose $\mathfrak{l}^+_{k+1}(a,\cdots,a,a) = T^{\lambda_1}/k!$ for simplicity. Then $\mathfrak{l}_k(a,\cdots,a) = T^{\lambda_1} b/k!$.

We remark that $\mathfrak{l}_k$ $(k = 0,1,\cdots)$ induces a coderivation $\hat{d}$ on $E(H(S^1\times S^m)) = E(a,b,c,e)$ in the same way as $\mathfrak{m}_k$ induces a coderivation on $B(H(L))$. We then have, in our case,

$$\hat{d} = \sum T^{\lambda_1} \frac{1}{k!} b \frac{\partial^k}{\partial a^k} \tag{10.6}$$

In other words, $\hat{d}$ is $f(a,b,c,e) \mapsto T^{\lambda_1} b f(a+1,b,c,e)$. (Here we identify element of $E(a,b,c,e)$ as a (Λ_{nov} coefficient) polynomial of a,c tensored with an element of $\mathbb{Q}[1,b]\otimes\mathbb{Q}[1,e] \cong E[b,e]$. Note that a,c has even degree after shifted and b,e has odd degree after shifted.) From this calculation, we find easily that the homology of $\hat{d}$ vanishes. This is actually consistent. In fact the $\hat{d}$ cohomology of Lagrangian submanifold of $\mathbb{C}^n$ should vanish. (This fact can be proved in a way similar to (6) of Theorem 5.1.) The argument here shows that the calculation of $\hat{d}$ cohomolgy of $E(H(S^1\times S^m))$ is not enough to eliminate the possibility $\eta_L(\gamma) = 2-n$. Actually we are going to use $H(\mathcal{L}(S^1\times S^m))$ (which is closely related to $B(H(S^1\times S^m))$) to eliminate the possibility $\eta_L(\gamma) = 2-n$.

We remark also that though (10.6) does not actually occur for a Lagrangian $S^1\times S^n$ in $\mathbb{C}^{n+1}$, it can occur for Lagrangian $S^1\times S^n$ in other symplectic manifolds. For example, if for some special Lagrangian submanifold L in a Calabi-Yau 3 fold M with $L \cong S^1\times S^2$, the equality (10.6) seems to hold modulo $T^{2\lambda_1}$.

11. Lagrangian submanifold of $\mathbb{C}^3$.

We first state our main result on Lagrangian submanifold of $\mathbb{C}^3$. We recal that a 3 dimensional manifold L is said to be *prime* if $L \cong L_1 \sharp L_2$ implies $L_1 \cong S^3$ or $L_2 \cong S^3$. Here $\sharp$ stands for connected sum and $\cong$ means diffeomorphism. Two Lagrangian immersions $i_0 : L \to M$ and $i_1 : L \to M$ are said to be *Legendrian regular homotopic* to each other if there exists a smooth family of Lagrangian immersions $i_t : L \to M$ connecting them such that $E : \pi_2(M, i_t(L)) \to \mathbb{R}$ is independent of t. (Here $\pi_2(M, i_t(L)) \cong \pi_1(L)$ is the set of homotopy class of pair of maps (f, g), $f : S^1 \to L$, $g : D^2 \to M$ such that $i_t \circ f = g|_{\partial D^2}$.)

Theorem 11.1 ([12])**.** *An oriented and connected prime* 3 *dimensional manifold* L *can be embeded to* $\mathbb{C}^3$ *as a Lagranaian submanifold if and only if* L *is diffeomorphic to* $S^1 \times \Sigma_g$ *where* Σ_g *is an oriented* 2 *dimensional manifold.*

Moreover a Lagrangian immersion $i : L = S^1 \times \Sigma_g \to \mathbb{C}^3$ *is Legendrian regular homotopic to an Lagrangian embedding if and only if there exists* $\gamma \in \pi_1(L)$ *such that* $E(\gamma) > 0$, $\eta(\gamma) = 2$.

Remark 11.1. For any oriented 3 manifold L, we have $TL \otimes \mathbb{C} \cong \mathbb{C}^3$. Hence Theorem 2.5 implies that L has a Lagrangian immersion $i : L \to \mathbb{C}^3$. We may assume that L is transversal to itself. Hence applying Lagrangian surgery (Lalonde-Sikorav [22], Polterovich [27]), we can prove that there exists k such that $L \sharp k(S^1 \times S^2)$ can be embeded as a Lagrangian submanifold of $\mathbb{C}^3$. (Here $k(S^1 \times S^2)$ is a conneted sum of k copies of $S^1 \times S^2$.)

The following seems to be open yet.

Problem 11.1. Let L_i be prime oriented 3 manifolds which are not diffeomorphic to $\mathbb{Q}$ homology sphere or $S^1 \times S^2$. Let $L = L_1 \sharp \cdots \sharp L_k$, $k \geq 2$. Can any such L be embeded to $\mathbb{C}^3$ as a Lagrangian submanifold?

The fact that $S^1 \times \Sigma_g$ can be embeded to $\mathbb{C}^3$ as a Lagrangian submanifold follows from Theorem 2.6. We can also prove that if $E(S^1) > 0$ and $\eta(S^1) = 2$ for an Lagrangian immersion $i : S^1 \times \Sigma_g \to \mathbb{C}^3$ then it is Legendrian regular homotopic to an embedding, by carefully examining the proof of Theorem 2.6. So the main new part of the proof of Theorem 11.1 is the proof of “only if” part.

Let $L \subset \mathbb{C}^3$ be an embeded Lagrangian submanifold. We assume that L is oriented and prime. By Theorem 2.1 $H_1(L; \mathbb{Q}) \neq 0$. Hence by a well known result of 3 manifold topology (see for example [20]), L is diffeomorphic either to $S^1 \times S^2$ or an aspherical manifold. Here a manifold L is said to be *aspherical* if $\pi_k(L) = 0$ for $k \geq 2$. We can generalize Theorem 11.1 in both cases to higher dimensions. We will discuss it in next two sections together with a sketch of their proofs.

12. Aspherical Lagrangian submanifold.

In this section we consider an aspherical Lagrangian submanifold L of a symplectic manifold M of arbitrary dimension. We assume M is convex at infinity in case M is noncompact. To prove Theorem 11.1 and its generalization in the case of aspherical Lagrangian submanifold L we are going to use the moduli space $\mathcal{N}(R, \beta)$ introduced in §3. To study this moduli space we need to use an assumption similar to Assumption 3.1 in §3. We state it below. Let $H : M \times [0, 1] \to \mathbb{R}$ be a smooth function of compact support. We put $H_t(x) = H(x, t)$, then $H_t : M \to \mathbb{R}$ is a smooth function of compact support. We denote by X_{H_t} the Hamiltonian vector field generated by H_t. Namely $dH_t = i_{X_{H_t}}(\omega)$. Let $\exp_t^{X_H} : M \to M$ be a one parameter group of transformations associated with X_H. Namely

$$\frac{\partial}{\partial t} \exp_t^{X_H}(x)\bigg|_{t=t_0} = X_{H_{t_0}}(\exp_{t_0}^{X_H}(x)).$$

$\exp_t^{X_H}$ is a symplectic diffeomorphism for each t.

Assumption 12.1. $\exp_1^{X_H}(L) \cap L = \emptyset$.

Theorem 12.1. *Let $L \subset M$ be a Lagrangian submanifold. We assume that L is relatively spin and aspherical. We also assume Assumption* 12.1. *Then there exists $\beta \in \pi_2(M, L)$ with the following properties.*

(1) $E(\beta) > 0$.
(2) $\eta_L(\beta) = 2$.
(3) $\partial\beta \in \pi_1(L)$ *is nonzero. Its centralizer $Z_{\partial\beta} = \{\gamma \in \pi_1(L) | \gamma\,(\partial\beta) = (\partial\beta)\,\gamma\}$ is of finite index in $\pi_1(L)$.*

Remark 12.1. In case $L = T^n \subset \mathbb{C}P^n$ existence of $\beta \in \pi_2(\mathbb{C}P^n, L)$ with $\eta_L(\beta) = 2$ was conjectured by M.Audin and is independently proved by Y. Eliashberg and by K. Cielieback also.

Remark 12.2. (A) Let us assume that $c^1\cap : \pi_1(M) \to \mathbb{Z}$ is zero in Theorem 12.1. Then η_L induces a homomorphism $\eta_L : \pi_1(L) \to \mathbb{Z}$. We now have an exact sequence

$$0 \to (\operatorname{Ker}\eta_L) \cap Z_{\partial\beta} \;\to\; Z_{\partial\beta} \overset{\eta_L/2}{\longrightarrow} \mathbb{Z} \to 0. \tag{12.1}$$

(We remark that the image of η_L is even since L is orientable.) Therefore $Z_{\partial\beta} \cong \mathbb{Z} \times ((\operatorname{Ker}\eta_L) \cap Z_{\partial\beta})$ by (2). It follows that the finite covering space $\hat{L}$ of L with $\pi_1(\hat{L}) = Z_{\partial\beta}$ is homotopy equivalent to $S^1 \times L'$ for a $K(Z_{\partial\beta}/\mathbb{Z}, 1)$ space L'.

(B) Under the assumption of Theorem 12.1, the finite covering space $\hat{L}$ of L with $\pi_1(\hat{L}) = Z_{\partial\beta}$ is homotopy equivalent to an S^1 bundle over L' for a $K(Z_{\partial\beta}/\mathbb{Z}, 1)$ space L'. If the image of $c^1\cap : \pi_1(M) \to \mathbb{Z}$ is $m\mathbb{Z}$, we can show that the Euler class of this S^1 bundle is divisible by m in a similar way.

Let us consider the situation of Theorem 11.1. As we remarked before either $L \cong S^1 \times S^2$ or L is aspherical. We discuss the first case in the next section. So we may assume that L satisfies the assumption of Theorem 12.1. (We remark that any oriented 3 manifold is spin.) Moreover $c^1(\mathbb{C}^3) = 0$. Hence we have $Z_{\partial\beta} \cong \mathbb{Z} \times ((\operatorname{Ker}\eta_L) \cap Z_{\partial\beta})$ as in Remark 12.2 (A). Using a standard result of 3 dimensional topology, (we remark that L is sufficiently large since $H_1(L; \mathbb{Q}) \neq 0$), we can prove that $\hat{L}$ is *diffeomorphic* to $\Sigma_g \times S^1$. Let $G = \pi_1(L)/\pi_1(\hat{L})$. It acts freely on $\hat{L}$ and $\hat{L}/G = L$. Since $G = \operatorname{Ker}\eta_L/((\operatorname{Ker}\eta_L) \cap Z_{\partial\beta})$ it follows that G acts freely on Σ_g and trivially on the S^1 factor. Hence Σ_g/G is again a Rieman surface. We can use this fact and $Aut(\pi_1(\Sigma_g)) \subset PSL(2; \mathbb{R})$ for $g \geq 2$ to show that actually $Z_{\partial\beta} = \pi_1(L)$. Namely L is diffeomorphic to $S^1 \times \Sigma_g$. This proves Theorem 11.1 except the case $L = S^1 \times S^2$. $\square$

We now sketch the proof of Theorem 12.1. We consider a map $\varphi = \varphi(\tau, t) : \mathbb{R} \times [0,1] \to M$ with the following properties. Here χ is as in Definition 3.2.

$$\frac{\partial \varphi}{\partial \tau}(\tau,t) = J\left(\frac{\partial \varphi}{\partial t}(\tau,t) - \chi_R(\tau) X_{H_t}(\varphi(\tau,t))\right), \tag{12.2a}$$

$$\varphi(\tau,0), \varphi(\tau,1) \in L, \tag{12.2b}$$

$$\int_{\mathbb{R}\times[0,1]} \varphi^*\omega < \infty. \tag{12.2c}$$

(Compare Definition 3.6.) We denote by $\mathcal{N}(R)$ the set of all such φ. In the same way as §3, we can extend φ to a map $\overline{\varphi} : (D^2, \partial D^2) \to (M, L)$. Let $\mathcal{N}(R;\beta)$ be the set of all $\varphi \in \mathcal{N}(R)$ such that the homotopy class of $\overline{\varphi}$ is β. Element of $\mathcal{N}(R;\beta)$ may be regarded as a map $(D^2, \partial D^2) \to (M, L)$. We define a map $ev : \mathcal{N}(R;\beta) \to \mathcal{L}(L)$ by $ev(\varphi) = \varphi|_{\partial D^2}$. We put

$$\mathcal{N}(\beta) = \bigcup_{R\in[0,\infty)} \mathcal{N}(R;\beta) \times \{R\}$$

and define $ev : \mathcal{N}(\beta) \to \mathcal{L}(L)$ in an obvious way. We remark

$$\dim \mathcal{N}(\beta) = n + 1 + \eta_L(\beta). \tag{12.3}$$

Definition 12.1. We define $\mathfrak{B}(L,H)$ by

$$\mathfrak{B}(L,H) = \sum_\beta T^{\beta\cap\omega} ev_*[\mathcal{N}(\beta)].$$

Here we may regard $ev_*[\mathcal{N}(\beta)]$ as an approximate De Rham chain of $\mathcal{L}(L)$ in a similar way to §6. We can prove an analogue of Lemma 4.4. Together with Gromov compactness it implies the following.

Lemma 12.2. *There exists C such that if $\beta \cap \omega < -C$ then $\mathcal{N}(\beta)$ is empty.*

Moreove, for any C, there exists only a finite number of β such that $\beta \cap \omega < C$ and $\mathcal{N}(\beta) \neq \emptyset$.

Lemma 12.2 implies that

$$\mathfrak{B}(L,H) \in S^D(L) \hat{\otimes} \Lambda,$$

where $\hat{\otimes}$ means the completion of the algebraic tensor product. Now the main point of the proof of Theorem 12.1 is the following equality.

Theorem 12.3. *Let $L \subset M$ be a Lagrangian submanifold. We assume that L is relatively spin and Assumption* 12.1. *Then we have*

$$\partial \mathfrak{B}(L,H) + \{\alpha(L), \mathfrak{B}(L,H)\} = [L]. \tag{12.4}$$

Here we embed $L \to \mathcal{L}(L)$ as the set of constant maps. Hence $[L] \in S^D_n(\mathcal{L}(L))$. $\alpha(L)$ is defined in Definition 8.2.

Remark 12.3. We remark that we do *not* assume that L is aspherical in Theorem 12.3.

Proof. (Sketch) We are going to study the boundary of $\mathfrak{B}(L,H)$. Let $\varphi_i \in \mathcal{N}(R_i;\beta)$ be a divergent sequence. In a way similar to the proof of Proposition 3.3, we can show that R_i is bounded. (This is the place we use Assumption 12.1.) There exists $p_i = (\tau_i, t_i) \in \mathbb{R} \times [0,1]$ such that

$$|d_{p_i}\varphi_i| = \sup\{|d_x\varphi_i| | x \in \mathbb{R} \times [0,1]\}. \tag{12.5}$$

We then consider three cases separately. (Compare the proof of Lemma 3.6.)

Case 1: $|d_{p_i}\varphi_i| = D_i$ diverges. $D_i dist(p_i, \partial(\mathbb{R} \times [0,1])) = C_i \to \infty$.
Case 2 : $|d_{p_i}\varphi_i| = D_i$ diverges. $D_i dist(p_i, \partial(\mathbb{R} \times [0,1]))$ is bounded.
Case 3 : $|d_{p_i}\varphi_i| = D_i$ is bounded. $|\tau_i|$ diverges.
Case 4 : $R_i \to 0$.

Case 1 is a sphere bubble. Hence it happens in codimension two. So it does not contribute to $\partial\mathfrak{B}(L,H)$. (We can make this argument rigorous in the same way as [14].)

Cases 2 and 3 correspond to a bubble at the boundary ∂D^2. We can show that this gives the term $\{\alpha(L), \mathfrak{B}(L,H)\}$.

Let us consider Case 4. We remark that the equation (12.2a) becomes pseudoholomorphicity for $R = 0$. Therefore, the limit $\lim_{i\to 0}\varphi_i$ will give a pseudoholomorphic map $\varphi : (D^2, \partial D^2) \to (M, L)$. The moduli space of such maps has an extra symmetry $\{g \in PSL(2;\mathbb{R}) | g(1) = 1\} = Aut(D^1, 1)$. The action of this group is nontrivial if $\beta \neq 0$. So in that case the contribution of Case 4 is zero as a chain. In case $\beta = 0$ we have $[L]$.

We thus obtain the fomula (12.4). The detail will be in [12]. □

To apply Theorem 12.3 to the proof of Theorem 12.1, we use a series of lemmas of purely topological nature. For $\gamma \in \pi_1(L)$, let $\mathcal{L}_{[\gamma]}(L) \subset \mathcal{L}(L)$ be the set of all loops in the free homotopy class of γ. (Here $[\gamma]$ stands for the free homotopy class of γ.) Let $Z_\gamma \subset \pi_1(L)$ be the centralizer. Let $\hat{L}_\gamma$ be the covering space of L corresponding to Z_γ.

Lemma 12.4. *If $\gamma \in \pi_1(L)$ is nonzero, then the natural projection induces a homeomorphism : $\pi_* : \mathcal{L}_{[\gamma]}(\hat{L}_\gamma) \to \mathcal{L}_{[\gamma]}(L)$*

Proof. Let $\ell : S^1 \to L$ be a loop in $\mathcal{L}_{[\gamma]}(L)$. Let $p_0 \in L$ be the base point. We can choose a path $m : [0,1] \to L$ joining p_0 to $\ell(1)$ such that $m^{-1} \circ \ell \circ m$ is homotopic to γ. It then lifts to a loop $\tilde{m}^{-1} \circ \tilde{\ell} \circ \tilde{m}$ in $\mathcal{L}_{[\gamma]}(\hat{L}_\gamma)$. It is easy to see that $\pi_*(\tilde{\ell}) = \ell$. Hence π_* is surjective.

We next assume $\pi_*(\tilde{\ell}_1) = \pi_*(\tilde{\ell}_2) = \ell$ and $\tilde{\ell}_1, \tilde{\ell}_2 \in \mathcal{L}_{[\gamma]}(\hat{L}_\gamma)$. There exists $g \in G = \pi_1(L)/Z_\gamma$ such that $g \cdot \tilde{\ell}_1 = \tilde{\ell}_2$. We remark that both $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are in the free homotopy class of γ. Since γ is in the center of $\pi_1(L_{[\gamma]})$ it follows that $g^{-1}\gamma g = \gamma$. Since $g \in G = \pi_1(L)/Z_\gamma$, it follows $g = 1$. Namely π_* is injective. □

Lemma 12.5. *Let L is a $K(\pi,1)$ space and $\gamma \in \pi_1(L) = \pi$, then $ev : \mathcal{L}_{[\gamma]}(\hat{L}_\gamma) \to \hat{L}_\gamma$ is a homotopy equivalence.*

Proof. If $\gamma = 1$, then $\hat{L}_\gamma = \hat{L}_1 = L$. Let $\tilde{L}$ be the universal covering space of L. Then $\mathcal{L}_{[1]}(L) = \mathcal{L}_{[1]}(\tilde{L})/\pi$. Since $\tilde{L}$ is contractible it follows that $\mathcal{L}_{[1]}(\tilde{L})$ is contractible. Hence $ev : \mathcal{L}_{[1]}(L) \to L$ is a homotopy equivalence.

Next we assume $\gamma \neq 1$. We put $g_\gamma = \gamma \in \pi$, which acts on $\tilde{L}$. We put $X = \{\ell : \mathbb{R} \to \tilde{L} \mid \ell(t+1) = g\ell(t)\}$. Z_γ acts on X freely and $X/Z_\gamma = \mathcal{L}_{[\gamma]}(\hat{L}_\gamma)$. Hence it suffices to show that X is contractible.

We consider a map $ev_0 : X \to \tilde{L}$, $\ell \mapsto \ell(0)$. It is a fibration. Hence X is homotopy equivalent to the fiber of ev_0. The fiber can be identified to the space of path $\ell' : [0,1] \to \tilde{L}$ joining $\tilde{p}_0$ with $g\tilde{p}_0$ and hence is contractible. □

Lemma 12.6. *Let L is an n dimensional aspherical manifold and $\gamma \in \pi_1(L)$. Then $H_k(\mathcal{L}_{[\gamma]}(L);\mathbb{Z}) = 0$ for $k \notin \{0, \cdots, n\}$. Moreover, if $H_n(\mathcal{L}_{[\gamma]}(L);\mathbb{Z}) \neq 0$ then Z_γ is of finite index in $\pi_1(L)$.*

Proof. $H_k(\mathcal{L}_{[\gamma]}(L);\mathbb{Z}) \cong H_k(\hat{L}_\gamma;\mathbb{Z})$ by Lemmas 12.4,12.5. Since $\hat{L}_\gamma$ is a covering space of n dimensional manifold L it follows that $H_k(\hat{L}_\gamma;\mathbb{Z}) = 0$ for $k \notin \{0, \cdots, n\}$. If $H_n(\hat{L}_\gamma;\mathbb{Z}) \neq 0$, it follows that $\hat{L}_\gamma$ is compact. Hence Z_γ is of finite index in $\pi_1(L)$. □

Lemma 12.7. *Let L be an n dimensional compact oriented aspherical manifold and $[L] \in H_n(\mathcal{L}_0(L);\mathbb{Q})$. Then for any $[P] \in H_*(\mathcal{L}(L);\mathbb{Q})$ we have $\{[L],[P]\} = 0$.*

Proof. For $p \in L$ let ℓ_p be the constant loop at p. For $x \in P$ let ℓ_x be the loop corresponding to it. Now $[P] * [L]$ is supported at

$$P * L = \{(p,x,t) \in L \times P \times S^1 | \ell_p(t) = \ell_x(0)\}.$$

If $(p,x,t) \in P * L$ then $(p,x,t') \in P * L$ for any t'. Moreover $* : P * L \to \mathcal{L}(L)$ sends (p,x,t) to the same loop as (p,x,t'). This implies $[P] * [L] = 0$.

On the other hand, $[L] * [P]$ is supported at

$$L * P = \{(x,p,t) \in L \times P \times S^1 | \ell_x(t) = \ell_p(0)\}.$$

If $(p,x,t) \in L * P$ then $(p,x,t') \in L * P$ for any t'. Moreover $*(x,p,t)$ is different from $*(x,p,t')$ only by the parametrization. This implies $[L] * [P] = 0$. □

Now we go back to the (sketch of the) proof of Theorem 12.1. By (12.4) we have $\beta \in \pi_2(M)$ such that

$$\{\mathcal{M}(L;\beta), \mathcal{N}(-\beta)\} \neq 0.$$

We remark that $\dim \mathcal{M}(L;\beta) = n - 2 + \eta_L(\beta)$, $\mathcal{N}(-\beta) = n + 1 - \eta_L(\beta)$. Hence, by Lemma 12.6, $n - 2 + \eta_L(\beta),\ n + 1 - \eta_L(\beta) \in \{0, \cdots, n\}$. Since $\eta_L(\beta)$ is even, this implies $\eta_L(\beta) = 2$. Then since $\dim \mathcal{M}(L;\beta) = n$ and is nonzero, it follows from Lemma 12.6 that the centralizer Z_γ of $\gamma = \partial\beta \in \pi_1(L)$ is of finite index in $\pi_1(L)$. This implies Theorem 12.1. □

Remark 12.4. Actually there is one point we need a clearification in the proof above. Namely the chain $\mathcal{N}(-\beta)$ is not necessary a cycle. So we need to work in the chain level. So the way to apply Lemma 12.6 is not so clear. We can overcome this trouble in a way similar to Remark 8.3 by using a theorem that any L_∞ algebra is homotopy equivalent to an L_∞ algebra defined on its homology group. (We use also Lemma 12.7.) The detail will appear in [12].

Remark 12.5. Loop space homology was used in [35] in a related context. To find a relation of Floer homology to [35] was one of the motivations of the author to modify the construction of [13] to ones described in §§6,7,8,9. In [35] and in [34], Viterbo used closed geodesic. Closed geodesic appears also in the study of Lagrangian submanifold using contact homology and in the approach by Eliashberg and Cielieback mentioned in Remark 12.1 using [7]. Closed geodesic is closely related to the homology of loop space. In a sense, our approach is more topological than one using closed geodesic.

It seems possible to describe relation of Floer homology of Lagrangian submanifold to [35] and to [7], using the ideas developped in §§6,7,8,9. We will discuss it elsewhere.

13. Lagrangian submanifold homotopy equivalent to $S^1 \times S^{2m}$.

Theorem 13.1. *Let $L \subset M$ be a Lagrangian submanifold. We assume that L is homotopy equivalent to $S^1 \times S^{2m}$. We also assume Assumption 12.1. Then there exists $\beta \in \pi_2(M, L)$ such that $E(\beta) > 0$, $\eta_L(\beta) = 2$, $\partial\beta \in \pi_1(L)$ is a generator*[4].

Proof. (Sketch) (We need to apply the same remark as Remark 12.4, to make the argument below precise.)

We put $n = 2m$. Lemma 5.4 (and its proof) implies that, if the theorem is false, then there exists $\beta \in \pi_2(M, L)$ with $\mathcal{M}(L;\beta) \neq 0$, $\eta_L(\beta) = 2 - n$. Moreover by Theorem 12.3, we may assume

$$\{\mathcal{M}(L;\beta), \mathcal{N}(-\beta)\} = [L]. \tag{13.1}$$

Then $\dim \mathcal{N}(-\beta) = 2n$. Let $\gamma = \partial\beta$. We remark that $[\mathcal{N}(-\beta)] \in H_n(\mathcal{L}_\gamma(S^1 \times S^n))$. Hence $[\mathcal{N}(-\beta)]$ is either of the form $[pt] \otimes a$ or $[S^1] \times a'$ where $H(\mathcal{L}_\gamma(S^1 \times S^n)) \cong H(\mathcal{L}_\gamma(S^1)) \otimes H(\mathcal{L}(S^{2n})) \cong H(S^1) \otimes H(\mathcal{L}(S^n))$.

By Lemma 10.5, $x \otimes [S^n]$ in the E^2 term of the spectral sequence does not survive. (Note $\deg(x \otimes [S^n]) = \deg a'$.) Using this fact, we can prove that $[\mathcal{N}(-\beta)]$ lies in the image of $H(S^1) \otimes H(\mathcal{L}_0(S^n))$. Here $\mathcal{L}_0(S^n)$ denotes the based loop space.

We define $ev_0 : \mathcal{L}_\gamma(S^1 \times S^n) \to S^1 \times S^n$ by $ev_0(\ell) = \ell(0)$. Then by definition $ev_0(\{P, Q\}) \subseteq ev_0(P) \cup ev_0(Q)$. Moreover the image of ev_0 of elements of $H(S^1) \otimes H(\mathcal{L}_0(S^n))$ is on $S^1 \times \{p_0\}$ where $p_0 \in S^n$ is the base point. Furthermore $\dim \mathcal{M}(L;\beta) = 0$.

Hence the support of $ev(\{\mathcal{M}(L;\beta), \mathcal{N}(-\beta)\})$ is contained in a one dimensional space. On the other hand, $ev_{0*}[L] = [L]$. This contradicts to (13.1). □

14. Lagrangian submanifold of $\mathbb{C}P^n$.

As we mentioned in Remark 8.4 Viterbo proved that if L admits a metric of negative curvature it can not be embeded to $\mathbb{C}P^n$ as a Larangian submanifold. Theorem 12.1 does *not* imply this result even in spin case since Assumption 12.1 may not be satisfied. We however have an alternative argument which implies Viterbo's result in spin case.

Theorem 14.1. *Let L be a Lagrangian submanifold of $\mathbb{C}P^n$. We assume that L is asperical and spin. Then there exists $\beta \in \pi_2(\mathbb{C}P^n, L)$ with the following properties.*

(1) $E(\beta) > 0$.
(2) $\eta_L(\beta) = 2$.
(3) *$\partial\beta \in \pi_1(L)$ is nonzero. Its centralizer $Z_{\partial\beta} = \{\gamma \in \pi_1(L) | \gamma\,(\partial\beta) = (\partial\beta)\,\gamma\}$ is of finite index in $\pi_1(L)$.*

[4]The author thanks to Prof. A. Kono who provides informations on the homology of free loop space useful to prove this Theorem.

Proof. (Sketch) We are going to construct an S^1 equivariant chain $\overline{\mathfrak{B}}(L) \in S^D(\mathcal{L}(L))$ which has a similar property as (12.4). We fix $p_0 \in \mathbb{C}P^n \setminus L$. For each $\beta \in \pi_2(\mathbb{C}P^n, L)$, we consider the moduli space of maps $\varphi : D^2 \to \mathbb{C}P^n$ with the following properties.

(1) φ is holomorphic.
(2) $\varphi(\partial D^2) \subset L$.
(3) The homotopy type of φ is β.
(4) $\varphi(0) = p_0$.

(Compare Definition 3.1.) Let $\mathcal{N}'(L;\beta)$ be the space of all such φ. We consider $Aut(D^2, 0) = \{g \in PSL(2;\mathbb{R}) | g(0) = 0\} \cong S^1$. It acts on $\mathcal{N}'(L;\beta)$ in an obvious way. $\varphi \mapsto \varphi|_{S^1}$ defines an S^1 equivariant map $ev : \mathcal{N}'(L;\beta) \to \mathcal{L}(L)$.

Hereafter we denote by $S^D_{S^1}(\mathcal{L}(L))$ the set of all S^1 equivariant De Rham chains in $\mathcal{L}(L)$.

We can use an argument similar to §6 and may regard $ev_*[\mathcal{N}'(L;\beta)] \in S^D_{S^1}(\mathcal{L}(L))$.

Definition 14.1. We define $\overline{\mathfrak{B}}(L) \in S^D_{S^1}(\mathcal{L}(L))\hat{\otimes}\Lambda$ by

$$\overline{\mathfrak{B}}(L) = \sum_\beta T^{\beta\cap\omega} ev_*[\mathcal{N}'(\beta)].$$

In §9 we regarded $\mathcal{M}(L;\beta)$ as an S^1 equivariant approximate De Rham chain of $\mathcal{L}(L)$ of degree $n - 3 + \eta_L(\beta)$ and used it to define $\overline{\alpha}(L) = \sum_i T^{E(\beta)}\mathcal{M}(L;\beta_i) \in S^D_{S^1}(\mathcal{L}(L))$.

Theorem 14.2. *We normalize our symplectic form ω so that $\omega \cap S^2 = 1$ for the generator $[S^2] \in H_2(\mathbb{C}P^n;\mathbb{Z})$. We then have*

$$\partial\overline{\mathfrak{B}}(L) + \{\overline{\alpha}(L), \overline{\mathfrak{B}}(L)\} \equiv [L] \mod T^2.$$

Proof. (Sketch) We consider a divergent series of elements φ_i of $\mathcal{N}'(L;\beta)$. Then, in the limit, one of the following occurs.

(1) A bubble occurs at the boundary.
(2) A bubble occurs at interior.

(1) gives the term $\{\overline{\alpha}(L), \overline{\mathfrak{B}}(L)\}$.

In general (2) is a phenomenon of codimension 2 and do not contribute to our formula. However there is an exception. Namely φ_i may converge to a union of trivial disk $\varphi_0 : D^2 \to \mathbb{C}^2$ that is a constant map to $p \in L$, and a pseudoholomorphic sphere $\varphi : S^2 \to \mathbb{C}P^n$ such that $\varphi(\infty) = p_0$ and $\varphi(0) = p = \varphi_0(D^2)$.

The reason that this is codimension one phenomenon is that the map $\varphi_0 \vee \varphi : D^2 \vee S^2 \to \mathbb{C}P^n$ is *not* stable, since its group of automorphism is S^1. We can analyse the neighborhood of this map in the moduli space and can show that they correspond to the boundary point of the moduli space $\mathcal{N}'(L;\beta)$. (We remark that in order such phenomenon to occur, $\partial\beta$ must be zero.) We are interested in the case when $\varphi \cap \omega = 1$. It is easy to see that, for each p, there exists exactly one such φ. (This is because there exists exactly one rational curve of degree one containing p and ∞.) Hence this gives $[L]$. Theorem 14.2 follows. (More detail of the proof will be in the reviced version of [13].) ∎

Now we can use Theorem 14.2 in place of Theorem 12.3 and prove Theorem 14.1 in the same way as Theorem 12.1. We also replace Lemma 12.6 by the following :

Lemma 14.3. *Let L is an n dimensional aspherical manifold and $\gamma \in \pi_1(L)$. If $x \in H_k^{S^1}(\mathcal{L}_{[\gamma]}(L);\mathbb{Z})$ for $k \notin \{0,\cdots,n-1\}$, then $\{x,y\} = 0$ for any y. Moreover, if $x \in H_{n-1}^{S^1}(\mathcal{L}_{[\gamma]}(L);\mathbb{Z})$ and $\{x,y\} \neq 0$ for some y, then Z_γ is of finite index in $\pi_1(L)$.*

Lemma 14.3 follows immediately from Lemma 12.6 and (9.2). □

Remark 14.1. We remark that we do not assume L to be aspherical in Theorem 14.2. So it can be used to study Lagrangian submanifold L of $\mathbb{C}P^n$ for more general L. For example the case $L = S^1 \times S^n$ can be studyed in a way similar to §13. The case when L is rational homology sphere is also of interest since Gromov's theorem 2.1 is not generalized directly to a Lagrangian submanifold of $\mathbb{C}P^n$.

One may also study Lagrangian submanifold of more general symplectic manifold M than $\mathbb{C}P^n$. For example the case when M is uniruled may be handled in a similar way.

The author is planning to explore these points elsewhere.

References

[1] V.I. Arnol'd, and A.V. Givental, *Symplectic Geometry.* in "Dynamical Systems IV" Encyclopaedia of Mathematical Sciences, ed. by V.I. Arnold and S.P. Novikov, Springer-Verlag, 1985.
[2] M.Audin, F.Lalonde, and L.Polterovich, *Symplectic rigidity: Lagrangian submanifolds.* in "Holomorphic curves in symplectic geometry", 271–321, Progr. Math., 117, Birkhäser, Basel, 1994.
[3] J.F. Barraud and O.Cornea *Lagrangian Intersections and the Serre Spectral Sequence.* math.DG/0401094
[4] A.S. Cattaneo J. Fröhlich, and P.B. Jürg. *Topological field theory interpretation of string topology.* Comm. Math. Phys. 240 (2003), no. 3, 397–421.
[5] M.Chas and D.Sullivan *String Topology.* preprint, math.GT/9911159.
[6] K.-T.Chen *Iterated integrals of differential forms and loop space homology.* Ann. of Math. (2) 97 (1973), 217–246.
[7] Y.Eliashberg. A.Givental, and H.Hofer. *Introduction to symplectic field theory.* GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673
[8] Y.Eliashberg, and M.Gromov. *Convex symplectic manifolds.* in "Several complex variables and complex geometry, Part 2," 135–162, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.
[9] Y.Eliashberg, and N.Mishachev. *Introduction to the h-principle.* Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.
[10] K. Fukaya. *Floer homology and mirror symmetry I.* in "Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds," 15–43, AMS/IP Stud. Adv. Math., 23, Amer. Math. Soc., Providence, RI, 2001.
[11] K. Fukaya. *Deformation theory, homological algebra and mirror symmetry.* in "Geometry and physics of branes," Ser. High Energy Phys. Cosmol. Gravit., IOP, Bristol, 2003.
[12] K.Fukaya. in preparation.
[13] K.Fukaya, Y.G.Oh, H.Ohta, and K.Ono. *Langrangian intersection Floer theory -anomaly and obstruction-.* preprint, http://www.kusm.kyoto-u.ac.jp/∼fukaya/ fukaya.html, 2000.
[14] K.Fukaya and K.Ono. *Arnold conjecture and Gromov-Witten invariants.* Topology, 38:933 – 1048, 1999.
[15] K.Fukaya and K.Ono. *Floer homology and Gromov-Witten invariant over integer of general symplectic manifolds—summary.* Taniguchi Conference on Mathematics Nara '98, 75–91, Adv. Stud. Pure Math., 31, Math. Soc. Japan, Tokyo, 2001.

[16] M.Gerstenhaber. *The cohomology structure of an associative ring.* Ann. of Math. (2) 78 1963 267–288.

[17] T.G.Goodwillie. *Cyclic homology, derivations, and the free loopspace.* Topology 24 (1985), no. 2, 187–215.

[18] M.Gromov. *Pseudoholomorphic curves in symplectic manifolds.* Invent. Math., 82(2):307–347, 1985.

[19] M.Gromov. *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 9. Springer-Verlag, Berlin, 1986. x+363 pp.

[20] J.Hempel 3-*Manifolds.* Ann. of Math. Studies, No. 86. Princeton University Press, Princeton, N. J., 1976

[21] T.V.Kadeishvili. *The algebraic structure in the homology of an $A(\infty)$-algebra.* Soobshch. Akad. Nauk Gruzin. SSR, 108(2):249–252 (1983), 1982.

[22] F.Lalonde, and J.-C.Sikorav. *Sous-varits lagrangiennes et lagrangiennes exactes des fibrs cotangents.* Comment. Math. Helv. 66 (1991), no. 1, 18–33.

[23] J.A.Lees. *On the classification of Lagrange immersions.* Duke Math. J. 43 (1976), no. 2, 217–224.

[24] Y-G.Oh. *Removal of boundary singularities of pseudo-holomorphic curves with Lagrangian boundary conditions.* Comm. Pure Appl. Math. 45 (1992), no. 1, 121–139.

[25] Y-G.Oh. *Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings.* Internat. Math. Res. Notices 1996, no. 7, 305–346.

[26] H.Ohta. *Obstraction to and deformation of Lagrangian intersection Floer cohomology.* In "Symplectic Geometry and Mirror Symmetry". World Sci. Press, 2001.

[27] L.Polterovich. *The surgery of Lagrange submanifolds.* Geom. Funct. Anal. 1 (1991), no. 2, 198–210.

[28] L.Polterovich. *Monotone Lagrange submanifolds of linear spaces and the Maslov class in cotangent bundles.* Math. Z. 207 (1991), no. 2, 217–222.

[29] M.Vigué-Poirrier and D.Sullivan *The homology theory of the closed geodesic problem.* J. Differential Geometry 11 (1976), no. 4, 633–644.

[30] Y.Ruan, *Virtual neighborhoods and pseudo-holomorphic curves.* Proceedings of 6th Gkova Geometry-Topology Conference. Turkish J. Math. 23 (1999), no. 1, 161–231.

[31] V.D.Silva *Products on symplectic Floer homology.* Thesis, Oxford Uinv. 1997.

[32] J.Stasheff. *Homotopy associativity of H-spaces. I, II.* Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. 108 1963 293–312.

[33] M.Sugawara. *A condition that a space is group-like.* Math. J. Okayama Univ. 7 1957 123–149.

[34] C.Viterbo. *A new obstruction to embedding Lagrangian tori.* Invent. Math. 100 (1990), no. 2, 301–320.

[35] C.Viterbo. *Exact Lagrange submanifolds, periodic orbits and the cohomology of free loop spaces.* J. Differential Geom. 47 (1997), 420–468.

Department of Math. Facult of Science, Kyoto University
E-mail address: `fukaya@math.kyoto-u.ac.jp`