METRIC RIEMANNIAN GEOMETRY

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Contents

1. Introduction

This artible is a survey of (a part of) Riemannian geometry. Riemannian geometry is a huge area which occupies, I believe, at least 1/3 of whole differential geometry. So obviously we need to restrict attention to some part of it to write an article in this handbook. (M. Berger's books [18, 19] deal with wider topics.) Let me mention first what is *not* included in this article but should have been included in the survey of Riemannian geometry.

(1) We do not include elementary or introductory part of Riemannian geometry. For example topics covered in [104] Section II,III or [97] are not in this article. We assume the reader to have some knowledge about it.
(2) We focus our attention to global results, and results of local nature are rarely discussed.
(3) One powerful tool to study global Riemannian geometry is partial differential equation, especially nonlinear one. We do not discuss it[1]. The theory of geodesic (which is a theory of nonlinear *ordinary* differential equation) is one of the main tool used in this article. Linear partial differential equation, especially Laplacian, is mentioned only when it is closely related to the other topics included in this article.
(4) We do not discuss manifolds of nonpositive curvature.
(5) We do not discuss scaler curvature.

After removing so many important and interesting topics there are still many things missing in this article. For example results such as filling volume ([74]) is not discussed. Study of closed geodesic is not included either.

So what is included in this article ?

We focus the part of Riemannian geometry which describes relations of curvature (sectional or Ricci curvature) to topology of underlining manifold. Since we do not discuss nonpositively curved manifold, the main target is manifold of (almost) nonnegative curvature and more generally the class of manifolds with curvature bounded from below. The study of such Riemannian manifolds is started with sphere theorems in 50's where comparison theorems are introduced by Rauch as an important tool of study.

At the begining of 70's Cheeger (and Weinstein) proved finiteness theorems which provides another kinds of statements to be established other than sphere theorems. Soon after that M. Gromov intorduced many new ideas, results and tools, such as Gromov-Hausdorff convergence, almost flat manifold theorem, Betti number estimate, etc., and gave tremendous influence to the area. These present the first turning point of the development of metric Riemannian geometry.

[1]So for example famous result by Hamilton on the 3 manifold of positive Ricci curvature is not discussed.

In 1980's global Riemannian geometry was a very rapidly developping area. Especially the class of Riemannian manifolds with sectional curvature bounds from below and above are studied extensively. One of the important progress on 1980's is the theory of collapsing Riemannian manifolds.

Those topics are discussed in §2 ∼ §13. After a brief review of sphere theorem in §2, we describe finiteness theorem in §3. In §4, while explaining a rough sketch of the proof of a sphere theorem we review several basic facts on global Riemannian geometry, such as Rauch's comparison theorem, cut point, conjugate point, injectivity radius etc. One of the main tools of global Riemannian geometry is Gromov-Hausdorff distance, which we define in §5 and will prove Gromov's precompactness theorem. The proof of finiteness theorems is discussed in §6 ∼ §9. We try to sketch various (different) techniques used to prove finiteness theorem etc. there, rather than to concentrate on one method and to give its full detail. Collapsing Riemannian manifolds (under the bound of absolute value of sectional curvature) is discussed in §10 ∼ §13.

In §14 ∼ §18, we discuss the class of Riemannian manifolds under sectional curvature bound from below (but not above). The basic tool to study it is Morse theory of distance function, which was initiated by Grove-Shiohama. We discuss it and its application to sphere theorem in §14. We explain application of the same method to finiteness theorem in §15. The thema of §16 is noncompact manifolds of nonnegative curvature. Besides its own interest, it is used in many places to study compact Riemannian manifold. Our focus in this article is on compact case, so we restrict our discussion on noncompact manifolds to ones which have a direct application to compact manifolds.

New turning point of development of metric Riemannian geometry came at some point in 1990's when several mathematicians belonging to new generation (such as Perelman and Colding) began to work in this field. In §17 and §18 we discuss Aleandrov space. It is a metric space which has curvature $> -\infty$ in some generalized sense. The notion of curvature on a metric space which is not a manifold was introduced by Alexandrov long ago. Recently various applications of it to Riemannian geometry (study of *smooth* Riemannian manifolds) were discovered. It makes this topic more popular among Riemannian geometers. An important structure theorem of Alexandrov space is obtained by Perelman and his collaborators, which we review in §17 and §18.

§19 ∼ §23 we discuss the class of Riemannian manifolds with Ricci curvature bounded from below. First betti number and fundmental group are the topics studied extensively under this curvature assumption. We review some of such study in §19. The thema of §20 is (mainly) a special case, that is the case of Einstein manifold. Our discussion of Einstein manifold is restricted to those related to the other parts of this article. We discuss Einstein manifold here since it provides rich examples of new phenomenon which appears when we replace the assumption sectional curvature $\geq$ const, by Ricci curvature $\geq$ const. Also it is an area where results we discuss in §§21,22,23 provide (and will provide) powerfull tool. §§21,22,23 are review of results obtained recently by Colding and Cheeger-Colding on the class of manifolds whose Ricci curvature is bounded from below. Here we emphasise geometric part of the story and omits most of the analytic parts of the proof, though analytic parts are as important as geometric parts.

It is of course impossible to write full detail of the proof in this article. However, rather than stating as many results as possible without proof, the author tried to survey as many ideas, tools, techniques, methods of proofs etc. as possible. In that sense, the emphasis of this article is on methods of proofs and not of their outcome. (Off course important applications of various techniques are explained.) Since this is a survey article there is no new results in it.

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Notations used in this article.

$T_pM =$ the tangent space, $\quad \mathrm{Exp}_p : T_pM \to M,$ the exponential map.
$B_p(R,X) = \{x \in X | d(x,p) < R\},$ for a metric space (X,d) and $p \in X$.
$K_M =$ the sectional curvature of M, $\quad \mathrm{Vol}(M) =$ the volume of M,
$Ricci_M =$ the Ricci curvature of M, $\quad Diam(M) =$ the Diameter of M.
$i_M(p) =$ the injectivity radius of M at p (Definition 4.1),
$\overline{xy} =$ a minima geodesic joining x and y,
$\angle xyz =$ the angle between $\overline{xy}$ and $\overline{yz}$ at y.
$\mathfrak{S}_n(D) = \{M \mid Ricci_M \ge -(n-1), \dim = n, Diam(M) \le D\}$,
$\mathfrak{S}_n(D,v) = \{M \in \mathfrak{S}_n(D) \mid \mathrm{Vol}(M) \ge v\}$,
$\mathfrak{S}_n(D, i > \rho) = \{M \in \mathfrak{S}_n(D) \mid \forall p \;\, i_M(p) \ge \rho\}$,
$\mathfrak{M}_n(D) = \{M || K_M| \le 1, \dim = n, Diam(M) \le D\}$.
$\mathfrak{M}_n(D,v) = \{M \in \mathfrak{M}_n(D) \mid \mathrm{Vol}(M) \ge v\}$.
$\mathfrak{M}'_n(D,v) = \{M \mid K_M \ge -1, Diam(M) \le D, \mathrm{Vol}(M) \ge v\}$.
$d_{GH}(X_1,X_2) =$ the Gromov-Hausdorff distance (Definition 3.2).
$\mathbb{S}^n(\kappa) =$ simply connected Riemannian manifold with $K_M \equiv \kappa$
$A_p(a,b;M) = \{x \in M | a \le d(p,x) \le b\}$,
$S_p(a;M) = \{x \in M | d(p,x) = a\}$.
$\lim^{GH}_{i\to\infty} X_i = X$ means $\lim_{i\to\infty} d_{GH}(X_i,X) = 0$.

The symbol $\doteq$ means almost equal. The argument using this symbol is note rigorous. We use it only when we sketch the proof.

The symbol $\tau(\epsilon_1,\cdots,\epsilon_k|a_1,\cdots,a_m)$ stand for the positive number depending only on $\epsilon_1,\cdots,\epsilon_k,a_1,\cdots,a_m$ and satisfying

$$\lim_{\epsilon_1,\cdots,\epsilon_k\to 0} \tau(\epsilon_1,\cdots,\epsilon_k|a_1,\cdots,a_m) = 0,$$

for each fixed $a_1,\cdots,a_m$. In other words

$$f(\epsilon_1,\cdots,\epsilon_k|a_1,\cdots,a_m) < \tau(\epsilon_1,\cdots,\epsilon_k|a_1,\cdots,a_m)$$

is equivalent to the following statement.

For each $\delta, a_1,\cdots,a_m$ there exists ϵ such that if $\epsilon_1 < \epsilon,\cdots,\epsilon_k < \epsilon$ then

$$f(\epsilon_1,\cdots,\epsilon_k|a_1,\cdots,a_m) < \delta.$$

2. Sphere theorems

There are several pioneering works in metric Riemannian geometry (such as Myers' theorem (Theorem 5.4), Hadamard-Cartan's theorem (Theorem 4.6), study of convex surface in $\mathbb{R}^3$ etc.). But let me set the begining of metric Riemannian geometry at the time when the following theorem was proved. From now on, we denote by K_M the sectional curvature of a Riemannian manifold M. We assume all Riemannian manifolds are complete unless otherwise stated.

Theorem 2.1 (Rauch's sphere theorem [129])**.** *There exists a positive constant ϵ_n depending only on the dimension n such that, if a simply connected Riemannian manifold M satisfies $1 \geq K_M \geq 1-\epsilon_n$, then M is homeomorphic to a sphere.*

This theorem is a first of the theorems which are called "sphere theorem". In this section, we mention some of the most important sphere theorems[2].

Theorem 2.2 (Klingenberg [94], Berger, [17])**.** *If a simply connected Riamannian manifold M satisfies $1 \geq K_M > 1/4$, then it is homeomorphic to a sphere.*

If M satisfies $1 \geq K_M \geq 1/4$, then M is either homeomorphic to a sphere or is isometric to a symmetric space of compact type[3].

Theorem 2.2 is a generalization of Rauch's theorem, and is optimal results among those characterizing spheres under assumption of the sectional curvature from above and below[4]. (We remark that the sectional curvature of complex, or quaternionic projective space, or Caylay plane is between 1 and 1/4.)

Theorem 2.3 (Bochner [151])**.** *If the curvature tensor R of a simply connected Riemannian manifold M satisfies*

$$\frac{C}{2} \leq \frac{-R_{ijk\ell}\xi^{ij}\xi^{k\ell}}{\Vert\xi\Vert} \leq C$$

for any antisymmetric 2 tensor ξ (where C is a positive constant), then the homology group over $\mathbb{R}$ of M is isomorphic to the homology group of the sphere.

The assumption of Theorem 2.3 is on curvature operator and is more restrictive than one on sectional curvature. Hence Theorem 2.3 follows from Theorem 2.2. (Theorem 2.3 was proved earier.) We mention Theorem 2.3 since the idea of its proof is quite different from one of Theorem 2.2. We mention them later in §19.

[2]In this article we mention only a part of many sphere theorems. The reader may find more in [139].

[3]more precisely, one of complex or quaternionic projective space or Caylay plane

[4]Several results which relax the condition of Theorem 2.2 to $1 \geq K_M \geq 1/4 - \epsilon$ are known. See [3].

Theorem 2.4. *If M is simply connected and if $1 \geq K_M \geq 1-\epsilon$, then M is diffeomorphic to a sphere.*

The difference between Theorems 2.4 and 2.2 is that the conclusion of Theorem 2.4 is one on diffeomorphism type and is sharper. The constant $1-\epsilon$ in Theorems 2.4 was $1-\epsilon_n$ where ϵ_n is a positive number depending only on dimension n and was not explicite, at the time when it was first proved by Gromoll and Shikata in [65, 136]. Later it was improved to a constant $1-\epsilon$ which is independent of the dimension. It was further improved and an explicite bound ($1-\epsilon = 0.87$) was found [143]. The explicite bound is improved several times[5] . The possibility that "$1 \geq K_M > 0.25$ and $\pi_1(M) = \{1\}$ implies that M is diffeomorphic to S^n" was not yet eliminated. The best constant is not yet found.

Remark 2.1. Hitchin [85] proved that there are some exotic spheres which does not admit metric of positive scaler curvature, by using KO index theorem of Dirac operator. Gromoll-Myer [66] (and Grove-Ziller [83]) found examples of exotic sphere which has metric of nonnegative curvature. So far example of exotic sphere which has metric of (strictly) positive sectional curvature is not found.

Theorem 2.5 (Berger [17], Grove-Shiohama [82])**.** *If $K_M \geq 1/4$ and if the diameter of M is greater than π, then M is homeomorphic to a sphere.*

Berger proved that M is homotopy equivalent to a sphere under the assumption of Theorem 2.5 and Grove-Shiohama proved that M is homeomorphic to a sphere. By generalized Poincaré conjecture (proved by Smale and Freedman) the later follows from the former (in case dimension is not 3). But the proof by Grove-Shiohama (which is different from Berger's) uses Morse theory of function which is not differentiable. This technique turns out to be very useful to study Riemannian manifold under lower (but not upper) curvature bounds. (See §14.)

The next theorem is a final form of series of results due to Shiohama [137], Otsu-Shiohama-Yamaguchi [111], Perelman [118]. We will discuss it in §22.

Theorem 2.6 (Cheeger-Colding [29])**.** *There exists $\epsilon_n > 0$ such that if M satisfies $Ricci_M \geq (n-1)$, $\mathrm{Vol}(M) \geq \mathrm{Vol}(S^n) - \epsilon_n$ then M is diffeomorphic to a sphere.*

Sphere theorem is a characterization of a sphere, that is the most basic example of Riemannian manifolds.

[5]The best estimate known at the time of writing this article is about $1-\epsilon = 0.68$ ([86, 144]).

Let us recall the classification of surface (two manifolds). There it was first proved that "simply connected compact 2 dimensional manifold is a sphere", then the classification in the general case was performed by simplifying general surface by, say, surgery.

In a similar sense, sphere theorem plays an important role in metric Riemannian geometry. Especially the techniques used to prove the sphere theorems we mentioned above play an important role to study more general Riemannian manifolds.

3. Finiteness theorems and Gromov-Hausdorff distance

Another type of important results in metric Riemannian geometry are finiteness theorems. First of that kind are ones by Cheeger and by Weinstein, which appeared at the begining of 1970's. Cheeger's finiteness theorem is as follows.

Theorem 3.1 (Cheeger [25]). *For each positive numbers D,v,n, the number of diffeomorphism classes of Riemannian manifolds M with $Diam(M) \le D$, $\mathrm{Vol}(M) \ge v$, and $|K_M| \le 1$ is finite.*

The method of proof of Theorem 3.1 is closely related to the proofs of Rauch's sphere theorem and of Theorems 2.2, 2.4. We will explain it later.

Theorems 2.4 and 3.1 (and their proof) use an idea that if two Riemannian manifolds are "close" to each other then they are diffeomorphic to each other.

One way to formulate precisely what we mean by two Riemannian manifolds to be close, is by using the notion Gromov-Hausdorff distance[6]. Let us first review the definition of (usual or classical) Hausdorff distance. Let (X, d) be a metric space and Y_1, Y_2 be subspaces. We put :

$$N_\epsilon Y = \{x \in X | d(x, Y) < \epsilon\},$$

where $d(x, Y) = \inf\{d(x, y) | y \in Y\}$.

Definition 3.1. The *Hausdorff distance* $d_X(Y_1, Y_2)$ between Y_1 and Y_2 is the infinimum of $\epsilon > 0$ such that $Y_2 \subset N_\epsilon Y_1$, $Y_1 \subset N_\epsilon Y_2$.

Hausdorff distance defines a complete metric on the set of all compact subsets of a fixed complete metric space (X, d).

Gromov-Hausdorff distance is an "absolute analogue" of Hausdorff distance. Namely it defines a distance between two metric spaces (for which we do not assume to be embeded somewhere a priori).

Definition 3.2. The *Gromov-Hausdorff distance* $d_{GH}((X_1, d), (X_2, d))$ between two metric spaces (X_1, d) and (X_2, d) is an infinimum of the Haudsorff distance $d_Z(X_1, X_2)$, where Z is a metric space such that X_1, X_2 are embeded to Z by isometries.

Hereafter we write $\lim^{GH}_{i\to\infty} X_i = X$ if $\lim_{i\to\infty} d_{GH}(X_i, X) = 0$.

Gromov-Hausdorff distance defines a complete metric on the set of all the isometry classes of compact metric spaces.

The following version is sometimes convenient.

Definition 3.3 ([52]). A map $\varphi : X_1 \to X_2$ is called an *ϵ-Hausdorff approximation*, if $|d_{X_1}(\varphi(x), \varphi(y)) - d_{X_2}(x, y)| \le \epsilon$ for all $x, y \in X_1$ and if the ϵ neighborhood of the image $\varphi(X_1)$ is X_2.

[6]See [70, 76, 57] for more detailed acount on it.

If $d_{GH}(X_1, X_2) \le \epsilon$ then there exists a 3ϵ-Hausdorff approximation $X_1 \to X_2$. If there exists an ϵ-Hausdorff approximation $X_1 \to X_2$ then $d_{GH}(X_1, X_2) \le 3\epsilon$.

There are two types of important results on Gromov-Hausdorff distance which are applied to finiteness theorems. In this section, we explain results which was developped mainly in 1980's.

We first state Gromov's precompactness theorem on manifolds with Ricci curvature bound. Let n,D be a positive integer and a positive number. We denote by $\mathfrak{S}_n(D)$ the set of all isometry classes of Riemannian manifolds M such that Ricci $\ge -(n-1)$ and diameter $\le D$. Here and hereafter the diameter $Diam(X)$ of a metric space (X, d) is the supremum of $d(x, y)$ where $x, y \in X$.

Theorem 3.2 (Gromov [70])**.** *$(\mathfrak{S}_n(D), d_{GH})$ is relatively compact in the space of all isometry classes of compact metric spaces.*

The method of proof of Theorem 3.2 is related to the proofs of Rauch's sphere theorem and of Theorem 2.2. We will explain it in §5.

We next mention rigidity theorem. Gromov's precompactness theorem assumes bounds from below of Ricci curvature, which is rather a weak assumption. We need stronger assumption for rigidity theorem. We first discuss the case when Gromov studied in [70]. For n,D,v, we denote by $\mathfrak{M}_n(D, v)$ the set of all isometry classes of n dimensional Riemannian manifolds M such that $|K_M| \le 1$, $Diam(M) \le D$, and $\mathrm{Vol}(M) \ge v$.

Theorem 3.3 ([70], [93])**.** *There exists $\epsilon_n(D, v) > 0$ such that if $M_1, M_2 \in \mathfrak{M}_n(D, v)$ and if $d_{GH}(M_1, M_2) \le \epsilon_n(D, v)$, then M_1 is diffeomorphic to M_2.*

Attempts to prove a similar conclusion as Theorem 3.3 under an assumption milder than $M_1, M_2 \in \mathfrak{M}_n(D, v)$, played a very important role in the development of metric Riemannian geometry. Perelman proved that M_1 is homeomorphic to M_2 if $d_{GH}(M_1, M_2) \le \epsilon_n(D, v)$ under the assumption $K_M \ge -1$, which replace $|K_M| \le 1$ in the definition of $\mathfrak{M}_n(D, v)$. (Theorem 18.2.) Further study is done in the case when we assume Ricci curvature bounds. (See Theorem 22.3).

Theorem 3.1 follows from Theorems 3.2 and 3.3. (We leave its proof as an exercise to the reader.)

Theorem 3.2 asserts relatively compactness. Namely it implies that, for any sequence M_i of elements of $\mathfrak{S}_n(D)$, there exists a converging subsequence. Its limit M_∞ may be regarded as a "weak solution" of various problems of metric Riemannian geometry, (when we regard it as an analogy of functional analysis). Then it is natural and important to study the "regurality" of M_∞. It is closely related to the proof of Theorem 3.3. The next result is related to "regurality" question.

Theorem 3.4 ([70],[64, 116]). *Each element of $\mathfrak{M}_n(D,v)$ is a Riemannian manifold of $C^{1,\alpha}$ class*[7].

Here α is any positive number with $\alpha < 1$ and a Riemannian manifold of $C^{1,\alpha}$ class is a manifold with metric tensor g whose first derivative is C^α Hölder continuous.

The assumption of Theorem 3.4 is rather strong. There are two kinds of study to relax this condition $X \in \mathfrak{M}_n(D,v)$.

One is to remove the assumption volume $\ge v$. It means that we study limit of a sequence of Riemannian manifolds which will become degenerate. This is called the study of collapsing Riemannian manifolds. We discuss it in §10 $\sim$ 13. (See also [57].)

The other direction is to relax the assumption $|K_M| \le 1$. Theorem 3.1 is generalized as follows toward this direction.

For n,D,v, we denote by $\mathfrak{M}'_n(D,v)$ the set of all isometry classes of n dimensional Riemannian manifolds M such that $K_M \ge -1$, $Diam(M) \le D$, $\mathrm{Vol}(M) \ge v$.

Theorem 3.5 (Grove-Petersen-Wu [78, 81]). *For each n, D, v, the number of homeomorphism classes of elements of $\mathfrak{M}'_n(D,v)$ is finite*[8].

We explain the proof of Theorem 3.5 in §15.

The limit of a sequence of manifolds M satisfying $K_M \ge -1$ is an Alexandrov space. We will discuss it in §17 and §18.

Remark 3.1. (1) If M_i is a sequence of Riemannian manifolds such that $N = \lim^{GH}_{i\to\infty} M_i$ and N is a Riemannian manifold. Then $K_{M_i} \ge \kappa$ implies $K_N \ge \kappa$. Moreover we have $\dim N \le \dim M_i$.

(2) On the other hand, in case when $\Lambda \ge K_{M_i} \ge \kappa$. $\Lambda \ge K_N$ is, in general, false for $\lim^{GH}_{i\to\infty} M_i = N$. A counter example can be constructed as follows. Let Rot_θ be the rotation by angle θ of $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$ around z axis. We consider the quotient of $S^2 \times \mathbb{R}$ by the $\mathbb{Z}$ action generated by $(p,t) \to (\mathrm{Rot}_{\alpha\epsilon}(p), t+\epsilon)$. Let $M_{\epsilon,\alpha}$ be the quotient space with quotient metric. ($M_{\epsilon,\alpha}$ is diffeomorphic to $S^2 \times S^1$.) We have $1 \ge K_{M_{\epsilon,\alpha}} \ge 0$ since $M_{\epsilon,\alpha}$ is locally isometric to $S^2 \times \mathbb{R}$. The limit of $M_{\epsilon,\alpha}$ as $\epsilon \to 0$ is S^2 with some Riemannian metric g_α. $1 \ge (S^2, g_\alpha)$ does not hold unless $\alpha = 0$.

4. Geodesic coordinate, injectivity radius, comparison theorems and sphere theorem

The following theorem in differential topology is used in the proof of Theorem 2.2.

[7]The proof of this theorem is completed in [64, 116] based on the idea of Gromov[70]. There seems to be various independent research in Russia. (See for exapmle [107, 108, 16]).

[8]In case dimension is 3, [78, 81] proved only finiteness of homotopy types. Now, Perelman's stability theorem (Theorem 18.2) implies the finiteness of homeomorphism classes in general.

Theorem 4.1. *If a compact n dimensional manifold M is a union of two open sets both of which are diffeomorphic to $\mathbb{R}^n$, then M is homeomorphic to a sphere.*

In order to apply Theorem 4.1 to the proofs of sphere theorems, we want to cover M by two coordinate neighborhoods. Estimate of the size of the coordinate charts plays an important role for the study of other problems also. Let us begin with the following.

Proposition 4.2. *For each compact Riemannian manifold M, there exists a positive number ϵ_M with the following properties. If the distance between $p, q \in M$ is smaller than ϵ_M, then, there exists a unique geodesic of length $< \epsilon_M$ joining p, q.*

The proof of Proposition 4.2 is in many standard text books of Riemannian geometry. (For example in [97, 33].)

The uniquenss of such geodesic is essential for our purpose. Let us explain this point. Let M be a complete Riemannian manifold. For each $p \in M$ we define the exponential map, $\mathrm{Exp}_p : T_pM \to M$ as follows. Let $V \in T_p(M)$. There exists a geodesic $\ell : \mathbb{R} \to M$, such that $\frac{d\ell}{dt}(0) = V$. We then put $\ell(1) = \mathrm{Exp}_p V$.

Proposition 4.2 implies that $\mathrm{Exp}_p : T_pM \to M$ is a diffeomorphism on the ball of radius ϵ_M.

Definition 4.1. The *injectivity radius* of a Riemannian manifold M is a function $i_M : M \to \mathbb{R}$ which associate to $p \in M$ the positive number :

$$i_M(p) = \sup \left\{ \epsilon | \mathrm{Exp}_p : T_pM \to M \text{ is injective on } \{V \in T_pM | \|V\| < \epsilon\} \right\}.$$

Proposition 4.2 implies $i_M \geq \epsilon_M$ for compact Riemannian manifold M. (It is easy to see that i_M is continuous. Hence $i_M \geq \epsilon_M > 0$ follows easily from implicite function theorem. Proposition 4.2 is a bit more involved.)

If $R < i_M(p)$, then the restriction of the exponential map $\mathrm{Exp}_p : T_pM \to M$ to the mertric ball of radius R centered at origin, defines a coordinate of a neighborhood of p. We call it the *geodesic coordinate*.

To prove Theorem 2.2, it is important to estimate the injectivity radius i_M from below. The next result[9] provides such an estimate.

Theorem 4.3. *Suppose that* $\dim M$ *is even. If $K_M > 0$, then $i_M \geq \pi$ and M is simply connected*[10].

Suppose $\dim M$ *is odd. If $1 \geq K_M \geq 1/4$ and if M is simply connected then, $i_M \geq \pi$.*

In particuler, if M satisfies the assumption of Theorem 2.2, *then we have $i_M \geq \pi$.*

[9]This theorem is due to [17] in even dimension, and to [95, 37] in odd dimension.
[10]The second assertion is a classical result due to Sygne.

(There are several results in the non simply connected case. We omit it.) Another results we use is the following :

Proposition 4.4 (Berger)**.** *Let us assume that $K_M \ge 1/4$ and $Diam(M) \ge \pi$. We take $p, q \in M$ such that $d(p,q) = Diam(M)$. Then we have*

$$\operatorname{Int} B_p(\pi, M) \cup \operatorname{Int} B_q(\pi, M) = M.$$

(Here Int *denotes the interior.)*

The proof is in §14.

Using Theorem 4.3 and Proposition 4.4, the proof of Theorem 2.2 goes roughly as follows. By Theorem 4.3, the injectivity radius of M is not smaller than π. Especially the diameter of M is not smaller than π.

Let us first assume $1 \ge K_M > 1/4$. We replace the metric g_M of M by $(1+\delta)g_M$, where δ is a positive number sufficiently close to 0. The assumption $1 \ge K_M > 1/4$ is still satisfied. Hence M satisfies the assumption of Proposition 4.4. Hence $\operatorname{Int} B_p(\pi, M) \cup \operatorname{Int} B_q(\pi, M) = M$. Moreover $\operatorname{Int} B_p(\pi, M)$ and $\operatorname{Int} B_q(\pi, M)$ are diffeomorphic to the ball by Theorem 4.3. Therefore, by Theorem 4.1, M is homeomorphic to a sphere.

We next consider the case when $1 \ge K_M \ge 1/4$. If the diameter of M is strictly greater than π, then again Proposition 4.4 and Theorems 4.1 and 4.3 imply that M is homeomorphic to a sphere.

Finally we consider the case when the diameter of M is π. In this case, we consider the restriction of the exponential map $\operatorname{Exp}_p : T_pM \to M$ to the ball $D^n(\pi)$ of radius π. Then it is a diffeomorphism at the interior. So M is obtained from $D^n(\pi)$ by identifiying boundary points only. We examine this situation carefully and conclude that M is a symmetric space of compact type. We omit the detail. (See for example [33] Chapter 7.) □

We explain the outline of the proof of Theorem 4.3 later in this section. We first explain some basic facts. Let us begin with the following theorem. Let κ be a constant. We put

$$s_\kappa(t) = \begin{cases} \frac{\sin t\sqrt{\kappa}}{\sqrt{\kappa}} & \kappa > 0 \\ t & \kappa = 0 \\ \frac{\sinh t\sqrt{-\kappa}}{\sqrt{-\kappa}} & \kappa < 0 \end{cases} \tag{4.1}$$

Theorem 4.5 (Rauch)**.** *If $K_M \le \kappa$, then the derivative $d_x \operatorname{Exp}_p$ of the exponential map Exp_p satisfies*

$$\|d_x \operatorname{Exp}_p(V)\| \ge \|V\| s_\kappa(r).$$

Here $x \in T_p(M)$, $\|x\| = r$, $V \in T_xT_p(M) \cong T_p(M)$ and we assume $r \le \pi/\sqrt{\kappa}$ in case $\kappa > 0$.

Let $K_M \ge \kappa$. In case $\kappa > 0$ we assume $d_{tx}\operatorname{Exp}_p$ is invertible for $t \in [0,1]$. Then we have

$$\|d_x \operatorname{Exp}_p(V)\| \le \|V\| s_\kappa(r).$$

Theorem 4.5 implies that if $K_M \le 1$ then the restriction of $\operatorname{Exp}_p : T_pM \to M$ to the ball of radius π is an immersion. (Namely its Jacobi matrix is invertible.)

We remark that the equality in Theorem 4.5 holds in the case when M is of constant curvature κ.

Theorem 4.5 is used by Rauch to prove his sphere theorem. We use Jacobi field in the proof of Theorem 4.5 as follows. Let x, V be as in Theorem 4.5, and define a geodesic ℓ_s by

$$\ell_s(t) = \operatorname{Exp}_p(t(x + sV))$$

For each s, ℓ_s is a geodesic. Its derivative

$$J(t) = \left.\frac{\partial \ell_s(t)}{\partial s}\right|_{s=0} \in T_{\ell_0(t)}M$$

with respect to s, by definition, is a Jacobi field. Note $d_x \operatorname{Exp}_p(V) = J(1)$. Therefore, to prove Theorem 4.5, it suffices to estimate Jacobi field. We use the following equation (which Jacobi field satisfy) for this purpose.

$$\frac{D^2}{dt^2}J(t) + R\left(\frac{d\ell_0}{dt}(t), J(t)\right)\frac{d\ell_0}{dt}(t) = 0 \tag{4.2}$$

Here $\frac{D}{dt}$ is a covariant derivative with respect to the tangent vector $\frac{d\ell_0}{dt}(t)$ and R is a curvature tensor.

If e_1, e_2 is an orthonormal frame of a plain π in the tangent space, then $g(R(e_1, e_2)e_2, e_1)$ is the sectional curvature of the plane π. (Here g is the metric tensor.) Therefore, the second term of the equation (4.2) can be written in terms of the sectional curvature. Using it we can compare the equation (4.2) to one in case our manifold is of constant curvature. Namely if $K_M \equiv \kappa$ then (4.2) will be

$$\frac{D^2}{dt^2}J(t) + \kappa J(t) = 0. \tag{4.3}$$

Its solution is $J(t) = s_\kappa(t)V(t)$ where $\nabla_{\dot\ell(t)}V = 0$. Namely $\|J(t)\| = s_\kappa(t)$ if $K_M \equiv \kappa$. This implies Theorem 4.5. $\square$

Theorem 4.5 implies the following :

Theorem 4.6 (Hadamard-Cartan)**.** *If a complete Riemannian manifold M satisfies $K_M \le 0$, the $\operatorname{Exp}_p : T_pM \to M$ is a covering map. In particular the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*

In fact, Theorem 4.5 implies that the Jacobi matrix of $\mathrm{Exp}_p : T_pM \to M$ is of maximal rank everywhere. To prove that it is a covering map we need a bit more. We use completeness of metric for this last step. We omit it. □

By integrating the conclusion of Theorem 4.5, we can compare the distance between two points $\mathrm{Exp}_p(x)$, $\mathrm{Exp}_p(y)$, (which are close to p) to the corresponding distance in the space with constant curvature. Actually we can do it more globally and obtain Toponogov comparison theorem.

To state it we need some notation. Let $\mathbb{S}^n(\kappa)$ be the complete simply connected Riemannian manifold with constant curvature κ. Let $x', y', z' \in \mathbb{S}^n(\kappa)$. We denote by $\overline{x'y'}$ etc. the minimal geodesic joining x' and y' etc. Let $\theta = \angle y'x'z'$ be the angle between $\overline{x'y'}$ and $\overline{x'z'}$ at x'. We put $a = d(x', y')$, $b = d(x', z')$. It is easy to see that $d(y', z')$ depends only on a, b, θ, κ. We define

$$s(a, b, \theta, \kappa) = d(y', z'). \tag{4.4}$$

We remark that in case $\kappa > 0$, the number $s(a, b, \theta, \kappa)$ is definded only for $a, b < \pi/\sqrt{\kappa}$.

Let M be a Riemannian manifold and $x, y, z \in M$. We denote by $\overline{xy}$ a minimal geodesic joining x and y. (In case there are several minimal geodesic we take any of them.) Let $\angle yxz$ be the angle between $\overline{xy}$ and $\overline{xz}$ at x.

Theorem 4.7 (Alexandrov-Toponogov). *If $K_M \geq \kappa$ then we have*

$$d(y, z) \leq s(d(x, y), d(x, z), \angle yxz, \kappa)$$

If $K_M \leq \kappa$ and if $d(x, y), d(x, z) \leq i_M(x)$ then

$$d(y, z) \geq s(d(x, y), d(x, z), \angle yxz, \kappa).$$

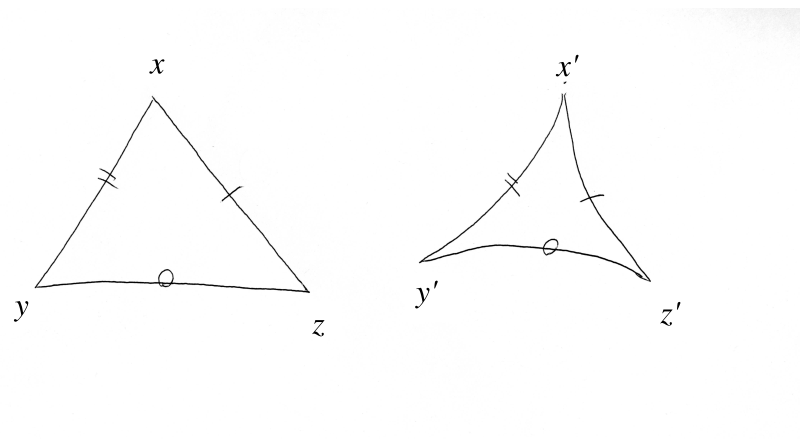

Figure 4.1

We remark that in the first inequality we do not need to assume that the triangle x, y, z is small. Actually we only need to assume one of the geodesics joining x to y and to z are minimal and the other may be any geodesic of length $\le \pi/\sqrt{\kappa}$. Theorem 4.7 is proved in many text book (See for example [33]).

As we already mentioned, Theorem 4.5 implies that, if $K_M \le 1$, then the exponential map is an immersion on the metric ball of radius π. Especially it is locally an injection there. To prove Theorem 4.1 we need global injectivity. We here introduce several terminology.

Definition 4.2. $q \in M$ is said to be a *conjugate point* of $p \in M$ if there exists x such that $q = \mathrm{Exp}_p(x)$ and that $d_x \mathrm{Exp}_p$ is not of maximal rank.

q is said to be a *cut point* of $p \in M$ if there exists $x \ne y \in T_p(M)$ such that $\mathrm{Exp}_p x = \mathrm{Exp}_p y = q$.

Example 4.1. We consider sphere S^2 of constant curvature 1. Every geodesic which start north pole np meets again at south pole sp. Hence south pole is a conjugate point of north pole.

We next divide S^2 by the involution and obtain the real projective space $\mathbb{R}P^2$. Then np and sp determine the same point $x = [np] = [sp] \in \mathbb{R}P^2$. If $c \in S^2$ is on the equator then there are minimal geodesics ℓ_1, ℓ_2 joining c to np, sp respectively. ℓ_1,ℓ_2 induce two minimal geodesics $\overline{\ell}_1$, $\overline{\ell}_2$ in $\mathbb{R}P^2$ joining x to $y = [c]$. Thus y is a cut poin of x.

Note that $i_M(p) > r$ holds if there exists neither a cut point nor a conjugate point q of p such that $d(p,q) \le r$. We can use Theorem 4.5 to estimate the distance to the conjugate point. However the problem to estimate the distance to the cut point is more global one.

We remark the following fact.

Lemma 4.8. *If $\ell : [a,b] \to M$ is the minimal geodesic, then for $t \in (a,b)$, $q = \ell(t)$ is neither a cut point nor a conjugate point of $p = \ell(0)$.*

In fact if q is a cut point then there is a geodesic ℓ' joining p to q with $|\ell'| = |\ell_{[a,t]}|$. Then the union $\ell' \cup \ell|_{[t,b]}$ of two geodesics is not smooth and has the same length as the minimal geodesic ℓ. This is a contradiction. If q is a conjugate point then by Morse index theorem (see [104, 97, 33]), $\ell_{[a,t+\epsilon]}$ is not minimal. This contradicts to the assumption.

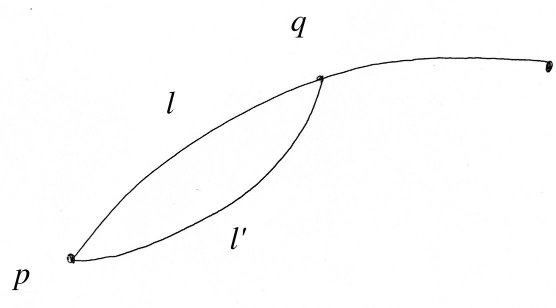

Figure 4.2

Here we state the following basic result about cut point. (See for example [33] p96 for its proof.)

Theorem 4.9 (Klingenberg). *Let M be a Riemannian manifold. We assume that q is not a conjugate point of p, for each $p, q \in M$ with $d(p, q) < r$. If there exists $p \in M$ with $i_M(p) < r$ then there exists a closed geodesic of length $< 2r$ in M.*

In view of Theorems 4.5 and 4.9, to prove Theorem 4.3, it suffices to show that the length of nontrivial closed geodesic of M is greater than 2π. We explain the brief outline of its proof below. (See [33] p100 for its detail.)

We first consider the case $\dim M$ is even. Let M be a simply connected Riemannian manifold with $1 \geq K_M > 0$. Let $\ell : S^1 \to M$ be a nontrivial geodesic of minimal length. We regard $S^1 \cong \mathbb{R}/\mathbb{Z}$. Put $p = \ell(0)$. By the parallel transport along ℓ we have a holonomy homomorphism $\mathrm{hol}_\ell : T_pM \to T_pM$. The tangent vector $\frac{d\ell}{dt}(0)$ is invariant of the holonomy. Since hol_ℓ is orthogonal transformation, and $\dim M$ is even, it follows that there exists a nonzero vector $V \in T_pM$ orthogonal to $\frac{d\ell}{dt}(0)$ such that $\mathrm{hol}_\ell(V) = V$. The parallel transport of V defines a vector field $V(t) \in T_{\ell(t)}M$, which is a parallel vector field. We put :

$$\ell_s(t) = \mathrm{Exp}_{\ell(t)}(sV(t))$$

Using $\nabla V(t) = 0$ and first variation formula (see for example [33] §1, [97] Vol II Theorem 5.1, [104] Theorem 12.2), we find that $\frac{d\ell_s}{ds}(0) = 0$. Using moreover the second variation formula (see for example [33] §6, [97] Vol II Theorem 5.4, [104] Theorem 13.1) and the positivity of curvature, we find $\frac{d^2\ell_s}{ds^2}(0) < 0$, which contradicts to the minimality of the length of ℓ.

The proof of odd dimensional case is more involved. We remark that the quotient of S^3 by a cyclic group $\mathbb{Z}/p\mathbb{Z}$ has constant positive curvature one (and is not simply connected). Its injectivity radius converges to 0 as $p \to \infty$. This shows that, to prove Theorem 4.3 in odd dimensional case, we need to use the assumption that M is simply connected.

The proof of odd dimensional case is roughly as follows. We assume that there exists a closed geodesic ℓ of length $< 2\pi$. Since M is simply connected, ℓ is null homotopic. Let ℓ_s be a homotopy such that $\ell_0 = \ell$, $\ell_1 = \mathrm{const}$. We may assume that the length of ℓ is minimal among all nontrivial closed geodesics. By using the assumption that $K_M > 1/4$ we can prove that the length of ℓ_s is always smaller than 2π. (This is the essential point of the proof. To prove this we use the fact that the Morse index (with respect to the length) of the closed geodesic of length $> 2\pi$ is not smaller than 2[11].)

[11] Let us consider the round sphere of radius 2 (that is the round sphere of curvature 1/4). The geodesic segment of length 2π, that is the geodesic segment

Now we conside the exponential map Exp_p at the tangent space of $p = \ell(0)$. Exp_p is a submersion on the ball of radius π. Hence it has a similar property to the covering map up to radius π. Especially it has homotopy lifting property there. Since the length of ℓ_s is not greater than 2π its image is of distance $\le \pi$ from p. Therefore we can lift ℓ_s to T_pM. (Note we can lift ℓ_1 since it is a constant map.) Hence we obtain a lift $\tilde\ell_0 : S^1 \to T_pM$. But this is a contradiction since $\ell_0 = \ell$ is a geodesic[12]. □

5. Packing and precompactness theorem

A similar argument as the last section is used in the proof of finiteness theorem(Theorem 3.1) and of Theorem 3.2. We explain this point here. We first discuss Theorem 3.2. The basic fact we use for its proof is the following.

Proposition 5.1. *Let $D > 0$ and $N : (0,1) \to \mathbb{N}$. We denote by $\mathfrak{Met}(D,N)$, the set of all isometry classes of complete metric spaces satisfying* (1), (2) *below. Then $\mathfrak{Met}(D,N)$ is compact with respect to the Gromov-Hausdorff distance.*

(1) *The diameter of $M \le D$.*
(2) *For each $\epsilon \in (0,1)$ there exists a finite subset Z of M with the following properties.*
 (2.a) $\sharp Z \le N(\epsilon)$.
 (2.b) *For each $x \in M$, there exists $x_0 \in Z$ satisfying $d(x,x_0) < \epsilon$.*

The proof of Proposition 5.1 is for example in [57] §2.
Here we introduce a notation.

Definition 5.1. We call the subset Z an *ϵ-net* if it satisfies (2.b) .

To deduce Theorem 3.2 from Proposition 5.1, we use the following Theorem 5.2. Let $\mathbb{S}^n(\kappa)$ be the complete simply connected Riemannian manifold with constant curvature κ. Let $B_p(R,M)$ be the metric ball in M of radius R centered at p.

Theorem 5.2 (Bishop-Gromov). *If* Ricci $\ge (n-1)\kappa$ *then the volume* $\mathrm{Vol}(B_p(R,M))$ *of the metric ball satisfies the following inequality for $r < R$.*

$$\frac{\mathrm{Vol}(B_p(R,M))}{\mathrm{Vol}(B_p(r,M))} \le \frac{\mathrm{Vol}(B_{p_0}(R,\mathbb{S}^n(\kappa)))}{\mathrm{Vol}(B_{p_0}(r,\mathbb{S}^n(\kappa)))}. \tag{5.1}$$

joining north pole with south pole, has Morse index $n-1$. (Here we consider the set of all arcs joining north pole with south pole and consider the length as a Morse funciton on it. $n-1$ is the Morse index with respect to this Morse funcition.) We compare our closed geodesic with this geodesic segment to obtain the conclusion about Morse index.

[12]This argument is not enough to handle the case $1 \ge K_M \ge 1/4$ of Theorem 2.2, (since then we can only show that π is a submersion at the *interior* of the ball of radius π.) In that case we need additional argument. We omit it.

(5.1) is called Bishop-Gromov inequality. It plays a key role to study the class of Riemannian manifolds with Ricci curvature bounded from below. The equality holds if M is of constant curvature κ.

Let us sketch a proof of Proposition5.2. We put

$$A(t) = \frac{\mathrm{Vol}(B_p(t, M))}{\mathrm{Vol}(B_{p_0}(t, \mathbb{S}^n(\kappa)))}. \tag{5.2}$$

It suffices to show that A is nonincreasing. (In case $\kappa > 0$ Theorem 5.4 implies that we need to consider $t \le \pi$ only.)

Let $\ell : [0, a) \to M$ be a minimal geodesic with $\ell(0) = p$ parametrized by arc length. Let $v = (d\ell/dt)(0)$.

We take a vector $v_* \in T_{p_0}\mathbb{S}^n(\kappa))$ with unit length We put

$$a(v, t) = \frac{\det d_{tv} \operatorname{Exp}_p}{\det d_{tv_*} \operatorname{Exp}_{p_*}} \tag{5.3}$$

Here $\det d_{tv} \operatorname{Exp}_p$ is the determinant of derivative of the exponential map. We first prove that $a(v, t)$ is a nonincreasing function of t for each fixed v.

We can prove it in a way similar to the proof of Theorem 4.5. One difference however is that our assumption in Theorem 5.2 is only on Ricci curvature while in Theorem 4.5 the assumption is on sectional curvature. However since we only need to estimate determinant of the Jacobi matrix of the exponential map, the assumption on Ricci curvature, which is a trace of curvature tensor, is enouth. This is half of the idea of the proof of Proposition 5.2. Let us fix p and move $q \in M$, and consider the set

$$V = \left\{ \frac{d\ell_{p,q}}{dt}(0) \in T_pM \,\middle|\, q \in M \right\} \tag{5.4}$$

where $\ell_{p,q}$ is the minimal geodesics joining p and q. (If there are several we take all of them.) (We take parametrization of $\ell_{p,q}$ so that the length of $\frac{d\ell_{p,q}}{dt}(0)$ is $d(p, q)$.)

We have

$$\mathrm{Vol}(B_p(R, M)) = \int_{V \cap B_0(R, R_pM)} \| \det d_x \operatorname{Exp}_p \| dx \tag{5.5}$$

(Here $\det d_x \operatorname{Exp}$ is the determinant of Jacobi matrix.) (5.5) and the fact that $a(v, t)$ is a nonincreasing function of t implies (5.1) for R, r smaller than injectivity radius.

To prove Theorem 5.2 beyond injectivity radius, we proceed as follows. We remark that V is star shaped (that is if $x \in V$ $t \in [0, 1]$ then $tx \in V$). We then modify a to a' so that $a'(t, v) = a(t, v)$ if $tv \in V$ and $a'(t, v) = 0$ if $tv \notin V$. Then a' is a nonincreasing function of t. Theorem 5.2 follows. □

Corollary 5.3. *If $Ricci_M \ge \kappa$ and $p \in M$ then*
$$\mathrm{Vol}(B_p(R, M)) \le \mathrm{Vol}(B_{p_0}(R, \mathbb{S}^n(\kappa))).$$

This corollary follows from the fact that the function A in (5.2) is nonincreasing and $\lim_{t\to 0} A(t) = 1$.

Theorem 5.2 and 5.1 imply Theorem 3.2 as follows. Let us assume that M satisfies the assumption of Theorem 3.2. It suffices to show that M satisfies the assumption of Proposition 5.1. Let $\epsilon > 0$. We take $Z \subset M$ which is maximal (with respect to inclusion) among the subsets of M satisfying "$z_1, z_2 \in Z, z_1 \neq z_2$ implies $d(z_1, z_2) > \epsilon$". The maximality implies (2.6). On the other hand, since $B_z(\epsilon/2, M)$, $z \in Z$ are disjoint to each other, it follows that :
$$\sum_{z\in Z} \mathrm{Vol}(B_z(\epsilon/2, M)) < \mathrm{Vol}\, M.$$
Since $B_z(D, M) = M$, Proposition 5.1 implies
$$\sharp Z \le \frac{\mathrm{Vol}(M)}{\sup \mathrm{Vol}(B_p(\epsilon/2, M))} \le \frac{\mathrm{Vol}(B_{p_0}(D, \mathbb{S}^n(\kappa)))}{\mathrm{Vol}(B_{p_0}(\epsilon/2, \mathbb{S}^n(\kappa)))}.$$
If we let $N(\epsilon)$ be the right hand side, then the assumption of Proposition 5.1 is satisfied. Theorem 3.2 follows. □

We remark that the following classical result is actually proved during the proof of Theorem 5.2.

Theorem 5.4 (Myers). *If M is an n dimensional complete Riemannian manifold with* Ricci $\ge (n-1)\kappa > 0$*, then M is compact and its diameter is not greater than $\pi/\sqrt{\kappa}$.*

In fact during the proof of Theorem 5.2 we proved the following under the assumption $p \in M$, $Ricci_M \ge \kappa$.

"If $t \mapsto \mathrm{Exp}_p(tv)$ is a minimal geodesic for $t \in [0, 1]$, then $\det d_v \mathrm{Exp}_p$ is not greater than $\det d_{v_0} \mathrm{Exp}_{p_0}$ where $p_0 \in \mathbb{S}^n(\kappa)$, $v_0 \in T_{p_0}\mathbb{S}^n(\kappa)$ and $|v_0| = |v|$. "

we remark that $\det d_{v_0} \mathrm{Exp}_{p_0} = 0$ if $\|v_0\| = \pi/\sqrt{\kappa}$. Therefore there exists no minimal geodesic of length $> \pi/\sqrt{\kappa}$ if $Ricci_M \ge \kappa$. Theorem 5.4 follows immediately. □

In the above argument, $B_z(\epsilon, M)$, $z \in Z$ covers M. Namely we estimate the number of metric balls (geodesic coordinate) to show Theorem 3.2. If ϵ is smaller than the injectivity radius of M, then $B_z(\epsilon, M)$ is diffeomorphic to D^n. The proof of Theorem 3.2 is related to the proof of spher theorems in this way. Theorem 4.1 deals with the case when two balls cover M and conclude that M is a sphere. If we can replace Theorem 4.1 by a statement such as "if M is covered by the balls whose number is estimated by C, then the number of diffeomorphism classes of such M is estimated by C" then finiteness theorem would follow.

Unfortunately the statement in the parenthesis above does not hold. So we need to include information how the balls are glued. Theorem 3.1 can be proved in that way. (See §6 $\sim$ 8.) Here we prove a weaker version (Weinstein [150]).

Proposition 5.5. *For each D, ϵ the number of homotopy equivalence classes of n dimensional Riemannian manifolds satisfying* (1), (2) *below is finite.* (1) $M \in \mathfrak{M}_n(D)$, (2) *The injectivity radius of M is greater than ϵ.*

To prove Proposition 5.5 we use the set Z above. We then obtain an open covering $B_z(\epsilon, M)$, $z \in Z$ of M. It is a simple covering. Namely for each $z_1, \cdots, z_k \in Z$ the intersection $\cap_{i=1}^k B_{z_i}(\epsilon, M)$ is either empty or contractible. It implies that the simplicial complex $K(Z)$ defined below is homotopy equivalent to M.

(1) The vertex of $K(Z)$ corresponds to an element of Z.
(2) $z_0, \cdots, z_k \in Z$ is the set of vertex of a k simplex of $K(Z)$ if and only if $\cap_{i=0}^k B_{z_i}(\epsilon, M) \neq \emptyset$.

Since the order of Z is estimated by the number depending only on D and ϵ, it follows that there exists only finitely many possibility for the homotopy type of $K(Z)$. Proposition 5.5 follows. □

In Theorem 3.1, there is no assumption on injectivity radius but only a bound of volume from below is assumed. Assumption on volume is more natural and geometric than one on injectivity radius. However in case absolute value of the sectional curvature is bounded, these two assumptions are equivalent.

Proposition 5.6 (Cheeger [25]). *There exists a positive number $c(n, D, v)$ depending only on n, D, v such that if $M \in \mathfrak{M}_n(D, v)$ then $i_M \geq c(n, D, v)$.*

The proof of Proposition 5.6 is closely related to the proof of Theorem 3.5. We will explain it in §15.

6. Construction of homeomorphism by isotopy theory

In §5, we discussed an estimate of the number of open sets which cover M and which are diffeomorphic to D^n, and we showed how it is used to estimate the number of homotopy types (Proposition 5.5). However as we mentioned there, we need more argument to estimate the number of diffeomorphism classes (or homeomorphism classes). We will explain some of them in the 4 sections begining from this section.

We again begin with a sphere theorem, the differentiable sphere therem (Theorem 2.4) this time.

Let M satisfy the assumptions of Theorem 2.4. Namely we assume that M is simply connected and $1 \geq K_M \geq 1 - \epsilon$. Then by Proposition 4.4 and Theorem 4.3, M is a union of two balls V_1,V_2 such that $V_i \cong D^n$.

We may assume $\partial V_i \cong S^{n-1}$. Moreover we may assume $V_1 \cap V_2 = \partial V_1 = \partial V_2$. So we obtain a diffeomorphism :

$$I : S^{n-1} \cong \partial V_1 \to \partial V_2 \cong S^{n-1}. \tag{6.1}$$

It is easy to see that if I is diffeotopic to the identity map (namely if there exists a smooth family I_t of differomorphisms such that $I_0 = I$, $I_1 = id$), then $M = V_1 \cup V_2$ is *diffeomorphic* to S^n.

Now we use the following :

Proposition 6.1. *For each compact Riemannian manifold N there exists $\epsilon_N > 0$ such that if the C^1 distance between $F : N \to N$ and the identity is smaller than ϵ_N, then F is diffeotopic to the identity.*

Here we recall

Definition 6.1. A two diffeomorphisms $F_1, F_2 : N \to N'$ are said to be *diffeotopic* to each other if there exists a smooth map $F : [1, 2] \times N \to N'$ such that $F(1, x) = F_1(x), F(2, x) = F_2(x)$ and that $x \mapsto F(t, x)$ is a diffeomorphism for each t.

The proof is elementary. To apply Proposition 6.1 to the proof of Theorem 2.5, we use the following lemma.

Lemma 6.2. *For each $\epsilon > 0$ there exists $\delta_n(\epsilon) > 0$ with the following properties. Let M be an n dimensional simply connected Riemannian manifold with $1 > K > 1 - \delta_n(\epsilon)$. Then we may choose the gluing map* (6.1) *so that its C^1 distance from identity is smaller than ϵ.*

We omit the proof. See for example [33] Chapter 7.

We are going to explain how we use the idea above to the proof of Theorems 3.3 and 3.1. Cheeger's original proof of Theorem 3.1 ([25]) is similar to the idea explained in this section.

Let M, N be Riemannian manifolds. We assume that they are covered by the same number of metric balls. Namely we assume $M = \cup_{i=1}^k B_{p_i}(\epsilon, M)$, $N = \cup_{i=1}^k B_{q_i}(\epsilon, N)$. We assume also that 10ϵ is smaller than injectivity radius of M and of N. (We put 10 by a technical reason.) We assume also that intersection pattern of the balls are the same. Namely, for each i, j, $B_{p_i}(\epsilon, M) \cap B_{p_j}(\epsilon, M) \neq \emptyset$ if and only if $B_{q_i}(\epsilon, N) \cap B_{q_j}(\epsilon, N) \neq \emptyset$.

We want to find a sufficient condition for M to be diffeomorphic to N. For this purpose we compare the chart $\cup_{i=1}^k B_{p_i}(\epsilon, M)$ of M, with the chart $N = \cup_{i=1}^k B_{q_i}(\epsilon, N)$ of N. To compare, we want to take the same domain for coordinate transformations. For this purpose we proceed as follows. Let $B_{p_i}(\epsilon, M) \cap B_{p_j}(\epsilon, M) \neq \emptyset$ then $B_{p_i}(\epsilon, M) \subset B_{p_j}(10\epsilon, M)$. For each p_i,q_j, we fix a linear isometry $T_{p_i}M \cong \mathbb{R}^n$, $T_{q_j}N \cong \mathbb{R}^n$ and use it to identify tangent spaces with $\mathbb{R}^n$. (There are various choices of identification. We take one and fix it.)

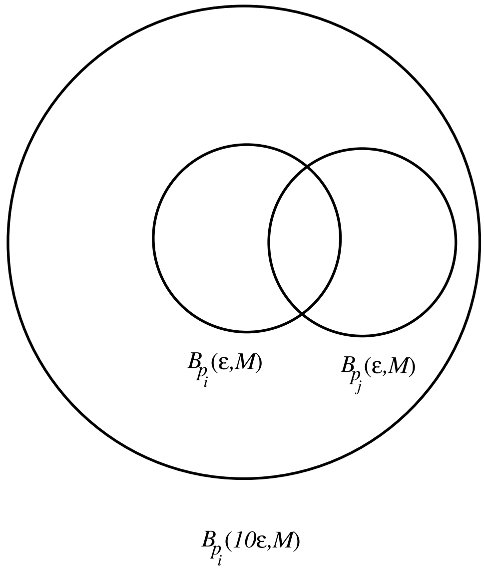

Figure 6.1

We consider the composition ;

$$\varphi_{ji}^M = \mathrm{Exp}_{p_j}^{-1} \circ \mathrm{Exp}_{p_i} : B^n(\epsilon) \to B^n(10\epsilon).$$

Here $B^n(\epsilon)$ is a metric ball of radius ϵ in $\mathbb{R}^n$ centered at origin, and $\mathrm{Exp}_{p_j}^{-1}$ is an inverse of the exponential map $\mathrm{Exp}_{p_j} : B^n(10\epsilon) \to N$. We define φ_{ji}^N in a similar way.

In the next proposition we *assume* that the C^2 norm (or $C^{1,\alpha}$ norm) of φ_{ji}^M, φ_{ji}^N is smaller than a constant C.

Proposition 6.3. *There exists $\epsilon_{n,k}(C) > 0$ such that if the C^1 distance between φ_{ji}^M and φ_{ji}^N is smaller than $\epsilon_{n,k}(C)$, then M is diffeomorphic to N.*

Cheeger proved Proposition 6.3 in the following way. First we use Proposition 6.1 to prove that the coordinate transformation φ_{ji}^M is diffeotopic to φ_{ji}^N. We then use it to construct a diffeomorphism $\cup_{i=1}^K U_i \to N$ (to its image) by induction in K. For detail see [25]. We prove Proposition 6.3 in a slightly different way in §7.

Proposition 6.3 is used to prove Theorem 3.1. For this purpose, we first observe that there is a constant C such that a Riemannian manifold satisfying the assumption of Theorem 3.1 is covered by metric balls whose number is not greater than C. Since the number of metric balls is bounded, the number of possible intersection patterns among them is also bounded. Let us fix intersection pattern of the metric balls we use. We use Proposition 6.3 and find that, if the coordinate transformations φ_{ji}^M are C^1 close to φ_{ji}^N, then M is diffeomorphic to N. If coordinate transformations φ_{ji}^M are uniformly bounded in C^2 norm then Ascoli-Alzera's theorem implies that they are precompact in C^1 topology. Theorem 3.1 will follow.

We need however to estimate second derivative of the coordinate transformaiton uniformly. Our assumption in Theorem 3.1 is on curvature, which is a second derivative of metric tensor. So one may imagine that it implies the estimate of the second derivative of coordinate transformation. However when we use geodesic coordinate, the assumption of (sectional) curvature is not enough to do so. (Cheeger [25] proved it under the additional assumption that a covariant derivative of the curvature tensor is also bounded.) To go around this trouble Cheeger in [25] proceed as follows. In place of using a statement such as "two diffeomorphism is diffeotopic to each other if they are C^1 close to each other" we can use a statement such as "two homeomorphism are isotopic to each other if they are C^0 close to each other" ([48]). And we can use isotopy extension theorem[13] to construct homemorphism $\cup_{i=1}^k U_i \to N$ by induction on k. This argument implies finiteness of homeomorphism classes and is not enough to prove Theorem 3.1 in four dimension[14]. (In higher dimension, one can use surgery etc. to deduce finiteness of diffeomorphism classses from finiteness of homeomorphism classes by purely topological argument.)

We can use harmonic coordinate (which we discuss in the next section) to find a coordinate chart such that the $C^{2,\alpha}$ norm of its coordinate transformation can be estimated uniformly.

7. Harmonic coordinate and its appliaction

As we mentioned in the last section, in order to obtain an estimate of the Hölder norm of the coordinate transformation, taking geodesic coordinate does not give an optimal result. Harmonic coordinate is the best choice for this purpose[15]. There are various other applications of harmonic coordinate[16]. It also plays an important role to prove the limit metric in Theorem 3.4 is of $C^{1,\alpha}$ class.

[13]which is much less elementary than Proposition 6.1 and is based on highly nontrivial results such as Kirby-Siebenman's result on Hauptvergmutung. See [48].

[14][115] added some technical argument and proved Theorem 3.1 in four dimension as well.

[15]In mathematical study of gauge theory, we need to take representative of gauge equivalence class in order to kill freedom of gauge transformation. This is an important point to study moduli space of connections. Here we are studying "gravity" and coordinate transformation plays a role of gauge transformation. The process to find a good coordinate is called gauge fixing in Physics. Harmonic coordinate is used in Riemannian geometry around the same time when Uhlenbeck etc. used Coulomb gauge in the study of moduli space of connections. The proof of Theorem 3.4 we present in this section is very similar to the proof by Uhlenbeck etc. of the compactification of the moduli space of self dual connections on 4 manifolds.

[16]We can use it to study Gromov-Hausdorff convergence under weaker assumption also. See §20.

Let M be a Riemannian manifold. We assume that the injectivity radius of M is much greater than r. Let $p \in M$ and $e_i(p)$, $i = 1, \cdots, n$ be an orthonormal frame of T_pM. We put $v_i(p) = \mathrm{Exp}_p(re_i(p))$, $w_i(p) = \mathrm{Exp}_p(-re_i(p))$ and define :

$$h_{p,i}(x) = \frac{d(x, w_i(p))^2 - d(x, v_i(p))^2}{4r^2} \tag{7.1}$$

We call $h_{p,i}$ an *almost linear function.* (We remark that $h_{p,i}$ is a linear function if $M = \mathbb{R}^n$.)

$h_p = (h_{p,1}, \cdots, h_{p,n})$ defines a coordinate system in a neighborhood of p. However since h_p is in principle a distance function, this coordinate does not provide optimal result for the estimate of the Hölder norm of coordinate transformation. We will replace it by a harmonic function. We consider a boundary value problem of the Laplace equation $\Delta\varphi = 0$ as follows. Let us take δ such that $r \ll \delta \ll i_M(p)$, and consider $\varphi_{p,i} : B_p(\delta, M) \to \mathbb{R}$ with the following properties.

(1) $\Delta\varphi_{p,i} = 0$.
(2) If $q \in S_p(\delta, M)$, then $\varphi_{p,i}(q) = h_{p,i}(q)$.

Definition 7.1. We call $\varphi_p = (\varphi_{p,1}, \cdots, \varphi_{p,n})$ a *harmonic coordinate.*

Using the fact that φ_i^p is C^1 close to h_i^p we can prove that φ_p defines a coordinate in a neighborhood of p.

Now we can prove an estimate of $C^{2,\alpha}$ norm of the coordinate transformation of the harmonic coordinate as follows. We put $D^n(\epsilon) = \{x \in \mathbb{R}^n | \|x\| < \epsilon\}$. We take ϵ with $10\epsilon < r$. Let $p, q \in M$ with $d(p, q) < \epsilon$. We consider the inverse φ_p^{-1} of φ_p. Then the image of $\varphi_p^{-1} : D^n(\epsilon) \to M$ is contained in the domain of $\varphi_q : B_q(r, M) \to \mathbb{R}^n$. Therefore we can define :

$$\varphi_{q,p}^M = \varphi_q \circ \varphi_p^{-1} : D^n(\epsilon) \to \mathbb{R}^n. \tag{7.2}$$

Theorem 7.1. *There exists a positive constant $C(r, \epsilon, \alpha, n)$ depending only on r, ϵ, α and the dimension n, such that the $C^{2,\alpha}$ norm of $\varphi_{q,p}^M$ is not greater than $C(r, \epsilon, \alpha, n)$.*

Also the $C^{1,\alpha}$ norm of the metric tensor in harmonic coordinate is estimated by $C(r, \epsilon, \alpha, n)$.

The proof is based on a priori estimate of harmonic functions. See [87, 88, 64], where the second half is proved. The first half follows easily from the second half. Theorem 7.1 is generalized to Theorem 20.7.

Let us prove Theorem 3.4 as a typical application of Theorem 7.1[17]. Let us take a sequence M_k of elements of $\mathfrak{M}_n(D, v)$. We denote its limit in Gromov-Haudsorff distance by X. By Theorem 4.3, the injetivity radius of M_k is greater than r, a number indepdent of k. We take ϵ

[17] The author follows the argument of [90] here.

such that $10\epsilon < r$. In the same way as §2, we can take a finite subset $\{p_{i,k} | i = 1, \cdots, I_k\} \in M_k$ with the following properties.

(1) I_k is smaller than a number independent of k.
(2) $\bigcup_i \varphi^{-1}_{p_{i,k}}(D^n(\epsilon)) = M_k$.

By (1) we may assume that I_k is independent of k by taking a subsequence if necessary. Set $I = I_k$. Then, the intersection pattern of the coordinates $\varphi^{-1}_{p_{i,k}}(D^n(\epsilon))$ has only a finite number of possibilities. Hence by taking a subsequence we may assume that the intersection pattern is independent of k. Namely we may assume that for each $i, j \le I$

$$\varphi^{-1}_{p_{i,k}}(D^n(\epsilon)) \cap \varphi^{-1}_{p_{j,k}}(D^n(\epsilon)) \tag{7.3}$$

is empty or not does not depdend on k.

Now for any i, j such that (7.3) is not empty, we consider $\varphi^{M_k}_{p_{j,k},p_{i,k}}$ defined by (7.2). We fix $\alpha < 1$, and apply Theorem 7.1 to α' with $1 > \alpha' > \alpha$. We then find that the $C^{2,\alpha'}$ norm of $\varphi^{M_k}_{p_{j,k},p_{i,k}}$ is estimated by a number independent of k. Hence we may take a subsequence and assume that that $\varphi^{M_k}_{p_{j,k},p_{i,k}}$ converges in $C^{2,\alpha}$ topology. Let us denotes its limit by

$$\varphi_{p_{j,\infty},p_{i,\infty}} : D^n(\epsilon) \to \mathbb{R}^n.$$

We use them as a coordinate transformation to obtain a smooth manifold M_∞ of $C^{2,\alpha}$ class. Moreover by the uniform $C^{1,\alpha'}$ boundedness of metric tensor, we find a Riemannian metric g_∞ on M_∞ of $C^{1,\alpha}$ class which is a limit of metrics on M_k. We can prove that M_k converges to (M_∞, g_∞) in Gromov-Hausdorff distance. Hence (M_∞, g_∞) is isometric to X. Theorem 3.4 follows. □

We next prove Theorem 3.3. We assume that the theorem is false. Then there exist $M_{1,k}, M_{2,k} \in \mathfrak{M}_n(D, v)$ such that $d_H(M_{1,k}, M_{2,k}) < 1/k$ but $M_{1,k}$ is not diffeomorphic to $M_{2,k}$. We use Theorem 3.3 to show that, after taking a subsequence, $M_{1,k}, M_{2,k}$ converges to X_1, X_2 respectively. By Theorem 3.4, X_1, X_2 are Riemannian manifolds of $C^{1,\alpha}$ class. By using the center of mass technique we will explain in the next section, we can prove that $M_{1,k}$ is diffeomorphic to X_1 and $M_{2,k}$ is diffeomorphic to X_2 for large k. On the other hand, since the Gromov-Hausdorff distance between X_1 and X_2 is zero, it follows that X_1 is isometric to X_2. Hence X_1 is diffeomorphic to X_2. This is a contradiction. □

8. Center of mass technique

In §6 we explained how isotopy extension theorem can be used to construct a homeomorphism. In fact isotopy extension theorem is very difficult to prove. We can use a method called center of mass technique which simplify those points. Center of mass technique can be used to various other problems for example to group action. In this section we explain it.

Let us start the explanation of center of mass technique by begining a proof of (a modified version of) Proposition 6.3.

In Proposition 6.3, the assumption is about exponential map Exp_p or coordinate transformation of geodesic coordinate. We actually use the case of harmonic coordinate. So we consider the following situation.

(a) $M = \cup_i \varphi_{p_i}(D^n(\epsilon))$, $N = \cup_i \psi_{q_i}(D^n(\epsilon))$ are open coverings.
(b) The intersection pattern of coordinate neighborhoods coincide to each other. Namelly $\varphi_{p_i}(D^n(\epsilon)) \cap \varphi_{p_j}(D^n(\epsilon)) \neq \emptyset$ if and only if $\psi_{q_i}(D^n(\epsilon)) \cap \psi_{q_j}(D^n(\epsilon)) \neq \emptyset$.
(c) If $\varphi_{p_i}(D^n(\epsilon)) \cap \varphi_{p_j}(D^n(\epsilon)) \neq \emptyset$, then $\varphi_{p_i}(D^n(\epsilon)) \subseteq \varphi_{p_j}(D^n(r))$.
(d) The $C^{2,\alpha}$ norm of the coordinate transformation

$$\Phi_{ij} = \varphi_{p_i}^{-1} \circ \varphi_{p_i} : D^n(\epsilon) \to \mathbb{R}^n$$

is bounded uniformly above by C. The same holds for

$$\Psi_{ij} = \psi_{q_i}^{-1} \circ \psi_{q_i} : D^n(\epsilon) \to \mathbb{R}^n.$$

(e) Φ_{ij} is close to Ψ_{ij} in C^1 norm.

Our purpose is to construct a diffeomorphism $F : M \to N$ under these assumptions.

For each $x \in \varphi_{p_i}(D^n(\epsilon))$, we put :

$$F_i(x) = \psi_{q_i} \circ \varphi_{p_i}^{-1}(x) \in N \tag{8.1}$$

This corresponds that we defined F on each coordinate chart $\varphi_{p_i}(D^n(\epsilon))$. The main point is whether we can glue them to obtain F globally. Namely in case $x \in \varphi_{p_i}(D^n(\epsilon)) \cap \varphi_{p_j}(D^n(\epsilon))$ we need to know whether

$$\psi_{q_i} \circ \varphi_{p_i}^{-1}(x) \stackrel{?}{=} \psi_{q_j} \circ \varphi_{p_j}^{-1}(x) \tag{8.2}$$

or not. It is easy to see that (8.2) does not hold. What follows from our assumption (assumption of Proposition 6.3 or the assumption (e) above) is :

$$d(\psi_{q_i} \circ \varphi_{p_i}^{-1}(x), \psi_{q_j} \circ \varphi_{p_j}^{-1}(x)) < \epsilon \tag{8.3}$$

(where ϵ is a sufficiently small positive number.) (More precisely (8.3) is on C^0 norm, but assumption (e) is on C^1 norm.)

The basic idea of center of mass technique is to take average of $F_i(x)$ over i with $x \in \varphi_{p_i}(D^n(\epsilon))$. Before we continue the proof of Proposition 6.3, we explain center of mass technique in general here.

Let $\mathfrak{m}$ a Borel probability measure on M, (namely a measure on M with $\mathfrak{m}(M) = 1$). Let us denote the support of $\mathfrak{m}$ by $Supp(\mathfrak{m})$. We define a function $d_{\mathfrak{m}}$ on M by

$$d_{\mathfrak{m}}(x) = \int d(x,p)\ d\mathfrak{m}(p). \tag{8.4}$$

Proposition 8.1. *We assume the injectivity radius of M is larger than 10ϵ. We also assume $K_M \le \kappa$ and $20\epsilon < \pi/\sqrt{\kappa}$*[18]*.*

If the diameter of $Supp(\mathfrak{m})$ is smaller than ϵ, then on

$$B_{3\epsilon}(Supp(\mathfrak{m}), M) = \{x \in M | d(x, Supp(\mathfrak{m})) < 3\epsilon\},$$

the function $d_{\mathfrak{m}}$ is convex.

Here a function on a Riemanniam manifold said to be convex if its restriction to each geodesic is convex.

We can prove Proposition 8.1 by using the convexity of the distance function d_p on $B_p(\pi/\sqrt{\kappa}, M)$[19].

Now we assume that the diameter of $Supp(\mathfrak{m})$ is smaller than ϵ. Then outside $B_{3\epsilon}(Supp(\mathfrak{m}), M)$ the value of the function $d_{\mathfrak{m}}$ is greater than 3ϵ, and on $Supp(\mathfrak{m})$ the value of the function $d_{\mathfrak{m}}$ is smaller than ϵ. Therefore $Supp(\mathfrak{m})$ attains its minimum on the interior of $B_{3\epsilon}(Supp(\mathfrak{m}), M)$. Since $d_{\mathfrak{m}}$ is convex there, the minimum is attained at unique point.

Definition 8.1. The *center of mass* is the point where $d_{\mathfrak{m}}$ attains its minimum. We write center of mass by $\mathfrak{CM}(\mathfrak{m})$.

We remark that if $M = \mathbb{R}^n$, then

$$\mathfrak{CM}(\mathfrak{m}) = \int_{\mathbb{R}^n} x \; d\mathfrak{m}(x).$$

We go back to the proof of Proposition 6.3. We take a partition of unity χ_i associated to the covering $M = \cup_i B_{p_i}(\epsilon, M)$. We define a measure $\mathfrak{F}(x)$ on N by

$$\mathfrak{F}(x) = \sum_i \chi_i(x)\delta_{F_i(x)}.$$

Here $\delta_{F_i(x)}$ is the delta measure supported at $F_i(x)$ and the summation is taken over all i with $x \in B_{p_i}(\epsilon, M)$.

By (8.3) we have $Diam(supp(\mathfrak{F}(x))) < \epsilon$. Let $F(x)$ be the center of mass of $\mathfrak{F}(x)$. Namely :

$$F(x) = \mathfrak{CM}(\mathfrak{F}(x)) = \mathfrak{CM}\left(\sum_i \chi_i(x)\delta_{F_i(x)}\right). \tag{8.5}$$

It is easy to see that $F(x)$ is a continuous function of x. Actually it is smooth. (We can prove it by using implicite function theorem.) We can prove that it is a diffeomorphism by using the following lemma.

Lemma 8.2. *If F_i $i = 1, 2, \cdots$ are C^1 close to each other then F determined by* (8.5) *is C^1 close to F_i.*

[18]In case $\kappa \le 0$ the second condition is void.

[19]This fact is a consequence of Toponogov's comparison Theorem 4.7.

The proof is elementary.

Then, to prove Proposition 6.3, we only need to show that F is injective. Suppse $F(x) = F(y)$, $x \neq y$. By using the fact that the Jacobi matrix of F is invertible, we can show that x can not be close to y. On the other hand, since F is close of F_i and since F_i is injective, we can prove that x can not be far from y. This is a cotradiction. This is an outline of the proof of Proposition 6.3. □

There are various other applications of center of mass technique. Let us mention another applicatin of it, that is an application to group action. Let M be a Riemannian manifold on which G acts. For simplicity we assume G is a finite group. We assume G has two different action on M and write them as $\psi_1 : G \to \mathrm{Diff}(M)$ and $\psi_2 : G \to \mathrm{Diff}(M)$. We assume that there exists C such that for each element $g \in G$, the C^2 norm of $\psi_1(g), \psi_2(g)$ are smaller than C.

Proposition 8.3 (Grove-Karcher)**.** *There exists a constant ϵ depending only on C, the dimension n, the injectivity radius of M, and the maximum of the absolute value of the sectional curvature of M, with the following property.*

If $d(\psi_1(g)(x), \psi_2(g)(x)) < \epsilon$ for each $g \in G, x \in M$, then there exists a diffeomorphism $\phi : M \to M$ such that $\phi(\psi_1(g)(x)) = \psi_2(g)(\phi((x))$.

See [77] for its proof. ([77] is the paper where center of mass technique first appeared).

Proposition 8.3 is applied to study Riemannian manifold whose sectional curvature is closed to 1 but whose fundamental group is not necessary trivial.

9. Embedding Riemannian manifolds by distance function.

In the last section we explained center of mass technique which we can use to construct a diffeomorphism. In §6 we mentioned another way that is to use isotopy theory. In this section, we discuss the third method which was introduced and used by Gromov [69, 70]. In [53] the author remarked that this method can be used to construct a smooth map (projection of a fiber bundle) in collapsing situation (Theorem 11.2). It was further generalized by Yamaguchi [154] (Theorem 11.3) to the case when we assumed a bound of sectional curvature from below (but not above).

We here explain an alternative proof of Theorem 3.3. This proof is completed by Katsuda [93] based on an idea of Gromov [70]. We assume $M, N \in \mathfrak{M}_n(D, v)$, $d_H(M, N) < \epsilon(n, v, D)$. (We choose $\epsilon = \epsilon(n, v, D) > 0$ later.) Let $\psi : M \to N$ be a 3ϵ Hausdorff approximation. We take a 20ϵ-net X of M. We can take X such that

(*) $$\text{If } x, x' \in X,\ x \neq x', \text{ then } d(x, x') > 10\epsilon,$$

in addition. It is easy to see that $\psi(X)$ is an 30ϵ-net of N. It is also easy to see that

$$(**) \qquad \text{if } x, x' \in X,\ x \neq x' \text{ then } d(\psi(x), \psi(x')) > \epsilon.$$

We denote by $[0,1]^X$ the set of all maps $X \to [0,1]$. It is a finite dimensional Euclidean space. The idea is to embed M(resp. N) in $[0,1]^X$ using distance function from X (resp. $\psi(X)$). In order to go around the trouble that distance function is not differentiable, we proceed as follows. We take ϵ later so that it is much smaller that the injectivity radius of M and N. We next take $\chi : \mathbb{R}_{>0} \to [0,1]$ such that

$$\chi(t) = \begin{cases} 0 & t < C\epsilon, \\ t & t \in [C^2\epsilon, C^3\epsilon], \\ \text{const} & t > C^4\epsilon. \end{cases}$$

Here C is a sufficiently large positive number which will be determined later. We may assume that $C^5\epsilon$ is smaller than the injectivity radius of M and N. (Precisely we first choose C and then choose ϵ so that this condition is satisfied.) Then we define $I_M : M \to [0,1]^X$ by $I_M(p)(x) = \chi(d(p,x))$ and $I_N : N \to \mathbb{R}^X$ by $I_N(p)(x) = \chi(d(p,\psi(x)))$. Note $\chi(t)$ is a constant where t is larger than injectivity radius. Hence I_M, I_N are smooth. We can prove the following :

Lemma 9.1. (1) *I_M, I_N are smooth embeddings.* (2) *$I_M(M)$ is contained in a tubular neighborhood $U(N)$ of $I_N(N)$.* (3) *We identify $U(N)$ with a normal bundle and let $\pi : U(N) \to N$ be the projection of the normal bundle. Then the restriction of π to $I_M(M)$ is a diffeomorphism.*

We omit the detail of the proof, (see [93]), but explain its brief idea. The reason that (1) holds is that, for each p, there are sufficiently many points $q \in X$ with $d(q,p) \in [C^2\epsilon, C^3\epsilon]$. Namely using the distance function from such q we can show the Jacobi matrix of I_M,I_N are invertible in a neighborhood of p.

To prove (2) we observe that, if $x \in X \subset M$, then the distance between $I_M(x)$ and $I_N(\psi(x))$ is small. (Namely it is something like $\text{const}\, d_H(M,N) = \text{const}\,\epsilon$.) Moreover, since X,$\psi(X)$ are enough dense in M,N, it follows that $I_M(M)$ are sufficiently close to $I_N(N)$. We next need an estimate of the size of the tubular neighborhods of $I_M(M)$, $I_N(N)$. This follows from the estimate of the second derivative of I_M and I_N, which turn out to be a consequence of the assumption on curvature of M, N. To carry out actual proof we need to estimate the size of tublar neighborhood and the distance between $I_M(x)$ and $I_N(\psi(x))$ more precisely.

To prove (3) we need to see that the Jacobi matrix of the restriction of $\pi : U(N) \to N$ to $I_M(M)$ is invertible. This follows from the fact that $I_M(M)$ is C^1 close to $I_N(N)$, namely they are close to each other together with their tangent spaces. Since the derivative of the distance function is written in terms of the angle between edges of geodesic triangles, we can prove this fact by using comparison theorems. □

Theorem 3.3 follows immediately from Lemma 9.1.

Remark 9.1. (A) We used distance function in the discussion above. We can use eigenfunction of Laplace opeartor (or green kernel) instead. Then the estimate about the derivatives of the diffeomorphism we get becomes better. (See for example [21, 56, 91]). This approach is closely related to harmonic coordinate.

(B) We took net and embed Rimennian manifolds to a finite dimensional Euclidean space in the above argument. We can use distance function from all the points and can embed Riemannian manifolds to a Hilbert or Banach space. This argument is useful for a generalization of Theorem 3.3 to an equivariant version. (Namely in the situation when a Lie group acts on M, N.) If we use eigenfunction of Laplace opeartor as we mentioned in (A), embedding to finite dimensional Euclidean space is good enough to show equivariant version also.

10. Almost flat manifold.

In this section we start discussing the case when the injectivity radius goes to zero. In the earier sections, we begin with sphere theorems and applied the method appeared there to finiteness theorems etc. In sphere theorems, we study manifolds of positive curvature. We here consider another typical Riemannian manifold that is a flat manifold. We first recall the following famous :

Theorem 10.1 (Bieberbach)**.** *If M is a compact Riemannian manifold with $K_M \equiv 0$, then there exists a finite covering $\tilde M$ of M such that $\tilde M$ is isometric to a flat torus.*

We want to study a Riemannian manifold (M, g) whose curvature is close to zero. To obtain a nontrivial result, we need some normalization. (In fact, the curvature of (M, kg) tends to 0 as $k \to \infty$ for any (M, g).) To normalize volume is not good enough eigher. (For example, $M \times S^1$ for any M carries a metric with volume 1 and curvature arbitrary small.) So let us normalize diameter to 1. In other words, we assume $|K_M|Diam(M)^2$ is small. We call such manifold *almost flat manifold*. However the assumption $|K_M|Diam(M)^2$ small does *not* imply that M is diffeomorphic to a flat manifold.

Example 10.1. We consider the group N of all 3×3 matrix of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. We consider a left invariant metric g_ϵ on N such that $g_\epsilon = \epsilon^2 dx^2 + \epsilon^2 dy^2 + \epsilon^4 dz^2$ at the unit matrix I. Let E_1,E_2,E_3 be left invariant vectors such that $E_1 = \partial/\partial x$, $E_2 = \partial/\partial y$, $E_3 = \partial/\partial z$ at I. It is well known that the curvature of Lie group with left invariant metric is calculated as follows. If E, F are left invariant orthonormal vector then the sectional curvature of the plain spanned by them is not greater $6\|[E,F]\|$. (See [24].) Hence the sectional curvature $K_{(N,g_\epsilon)}$ is bounded as $\epsilon \to 0$. On the other hand, we consider the subgroup $N_{\mathbb Z}$ consisting of matrix in N such that $x, y, z \in \mathbb Z$. $N_{\mathbb Z}$ is a dicrete subgroup of N and the quotient space $M = N_{\mathbb Z}\backslash N$ is known to be compact. We consider the metric on M induced by g_ϵ and denote it by $\overline{g}_\epsilon$. It is easy to see that the diameter $Diam(M, \overline{g}_\epsilon)$ goes to zero. Hence $Diam(M, \overline{g}_\epsilon)^2 K_{g_\epsilon}$ goes to zero. However no finite cover of M is diffeomorphic to T^3.

This example shows that we need to include not only abelian but also nilpotent Lie group to characterize almost flat manifold.

Theorem 10.2 (Gromov [69])**.** *There exists $\epsilon_n > 0$ such that if an n dimensional compact Riemannian manifold M satisfies $|K_M|Diam(M)^2 < \epsilon_n$ then M has a finite cover $\tilde M$ which is diffeomorphic to $\Gamma\backslash N$, where N is a nilpotent Lie group and Γ is its discrete subgroup.*

There is an improvement of Theorem 10.2 due to Ruh [133]. Let N be an nilpotent Lie group. There exists a connection ∇_{can} of TN which is invariant of both left and right actions of N. Let Γ be a discrete subgroup of N. ∇_{can} induces a connection on $\Gamma\backslash N$ which we denote by the same symbol. (We remark that ∇_{can} is not equal to the Levi-Civita connection.) Let Λ be a finite subgroup of $Aut(\Gamma\backslash N, \nabla_{can})$. We call $\Lambda\backslash(\Gamma\backslash N)$ an *infranilmanifold*.

Theorem 10.3 (Ruh)**.** *Under the assumption of Theorem 10.2, M is diffeomorphic to an infranilmanifold.*

Let us sketch some of the essential ideas behind the proof of Theorem 10.2. One important origin is a Margulis' lemma. Margulis' lemma first apeared in the study of discrete subgroup of Lie group.

Theorem 10.4 (Zassenhaus, see[70] 8.44)**.** *For each Lie group G there exists a neighborhood U of unit, such that if $\Gamma \subset G$ is a discrete subgroup then $U \cap \Gamma$ generates a nilpotent subgroup.*

The proof is based on the following fact. Let $g_1, g_2 \in G$ be in a neighborhood of unit 1, then

$$d(1, \{g_1, g_2\}) \le Cd(1, g_1)d(1, g_2) \tag{10.1}$$

Here $\{g_1, g_2\}$ is the commutator. This formula (10.1) is a consequence of the fact that the derivative of $(g_1, g_2) \mapsto \{g_1, g_2\}$ at 1 is zero. Once we have (10.1) we can prove Theorem 10.4 as follows. We choose U small enough such that if $g \in U$ then $d(1, g) < 1/(2C)$. Then (10.1) implies that if $g_i \in U$ then $d(1, \{g_1, g_2\})$ is strictly smaller than $d(1, g_i)/C$. We repeat this and find that N hold commutator between elements of U is in the $1/C^N$ neighborhood of 1. Since Γ is discrete, it implies the existence of N such that any N hold commutators between elements of $U \cap \Gamma$ are trivial. It follows that $U \cap \Gamma$ generates nilpotent group. □

There are various Riemannian geometry version of Theorem 10.4. The following, which is proved by Cheeger-Colding [31] (improving [59]) is one of the strongest version.

Theorem 10.5. *There exists ϵ_n with the following properties. Let M be an n dimensional complete Riemannian manifold with $Ricci_M \ge -(n-1)$ and $p \in M$. Then the image of $\pi_1(B_p(\epsilon_n, M)) \to \pi_1(B_p(1, M))$ has nilpotent subgroup of finite index.*

If we apply it to the situation of Theorem 10.2 we find that the fundamental group of M has nilpotent subgroup of finite index. (See §19 for more discussion on fundamental group.)

Another idea applied by Gromov to prove Theorem 10.2 is to use local fundamental pseudogroup, which we discuss briefly here. (See [57] §7 and [24] for its precise definition.) Let M be a complete Riemannian manifold. We assume $K_M \le 1$. Let $p \in M$. Then by Theorem 4.5 the exponential map $\mathrm{Exp}_p : T_pM \to M$ is an immersion on the ball $B_0(\pi; T_pM)$. Since $B_0(\pi; T_pM)$ has a boundary $\mathrm{Exp}_p : B_0(\pi; T_pM) \to M$ is not a covering map. So we can not consider its deck transformation group in the usual sense. But we can define "pseudogroup" in the following way. Let $\epsilon < \pi/10$. We consider the set of all loops $\ell : S^1 \to B_p(\epsilon, M)$ with $\ell(0) = p$ and $|\ell| < \epsilon$. We say $\ell \sim \ell'$ for such ℓ, ℓ' if there exists a based homotopy ℓ_t between them such that $|\ell_t| < \epsilon$ for each t. Let us denote the set of equivalence class by $\pi_1(M, p; \epsilon)$. The loop sum $*$ on $\pi_1(M, p; \epsilon)$ is not necessary defined. But when it is defined its $\sim$ equivalence class is well defined. (We need to use the fact that $\mathrm{Exp}_p : B_0(\pi; T_pM) \to M$ is an immersion to show this.) When loop sum is well defined it is associative. (Here the reader may find some fravor of Klingenberg's argument we mentioned at the end of §4.) Thus $(\pi_1(M, p; \epsilon), *)$ is something similar to group. We call it *fundamental pseudogroup*. The following pseudogroup version of Margulis' lemma is used in the proof of Theorem 10.2.

Lemma 10.6. *If $|K_M| \le 1$ and if $Diam(M) < \epsilon_n$, then there exists a subpseudogroup $(\pi_1^0(M, p; \epsilon), *)$ of $(\pi_1(M, p; \epsilon), *)$ such that $(\pi_1^0(M, p; \epsilon), *)$ is embeded (preserving $*$) into a nilpotent Lie group N, its image generates a discrete subgroup Γ and that the index $[\pi_1(M, p; \epsilon) : \pi_1^0(M, p; \epsilon)]$ is estimaged by a number depending only on n. Here $\epsilon_n \ll \epsilon \ll 1$.*

Lemma 10.6 consists main part of the proof of Theorem 10.2. (Actually we need a bit more. Namely we have to show that the action of $(\pi_1(M, p; \epsilon), *)$ to $B_0(\pi; T_p(M))$ is diffeomorphic to an action to $U \subset N$ of some subpseudogroup $\Gamma \cap U$, where N is a nilpotent Lie group and Γ is its discrete subgroup.) □

For the detail of the proof we refer [24, 57].

11. Collapsing Riemannian manifolds -I-

Using Theorems 3.3,3.4, we can descrive a sequence of n-dimensional Riemannian manifolds M_i with $|K_{M_i}| \le 1$ and $\mathrm{Vol}(M_i) \ge v > 0$ where v is independent of i. Namely the limit X (which exists after taking a subsequence) is a Riemannian manifold of $C^{1,\alpha}$ class and X is diffeomorphic to M_i for sufficiently large i.

In §10, we consider a sequence of Riemannian manifolds M_i with $|K_{M_i}| \le 1$ and $Diam(M_i) \to 0$. (The second condition is equivalent to say that M_i converges to a point.) Theorem 10.2 implies that M_i is an infranilmanifold for large i.

These are two extremal cases. We now discuss the intermediate case. Namely we consider the case when a sequence of Riemannian manifolds M_i converges to a metric space X (with respect to the Gromov-Hausdorff distance) such that $n > \dim X > 0$. We say that such sequence *collapses to* X. Here we discuss results under the assumption

$|K_{M_i}| \le 1$. (The study under weaker assumption is discussed in later sections.)

We first explain some examples of collapsing Riemannian manifolds. The first example is due to Berger and is called the Berger sphere.

Example 11.1. We consider Hofp fibration $\pi : S^3 \to S^2$. (Namely we regard $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$, and we associate to (z_1, z_2) the complex one dimensional space spaned by it, which is an element of $\mathbb{C}P^1 = S^2$.) We put the standard metri on S^3 and regard S^2 as a sphere of radius $1/2$. It is easy to see that π is a Riemannian submersion. (Namely if $V_h \in T_pS^3$ and V is perpendicular to the fiber of π containing p, then $g_{S^3}(V_h, V_h) = g_{S^2}(\pi_* V_h, \pi_* V_h)$.) We define a metric g_ϵ on S^3 as follows. Let $V, W \in T_pS^3$. We write

$$V = V_h + V_v, \qquad W = W_h + W_v,$$

where V_h, W_h are perpendicular to the fiber (with respect to g_{S^3}) and V_v, W_v are tangent to the fiber. We set

$$g_\epsilon(V, W) = g_{S^3}(V_h, W_h) + \epsilon^2 g_{S^3}(V_v, W_v).$$

It is easy to see that $\lim^{GH}_{\epsilon\to 0}(S^3, g_\epsilon) = (S^2, g_{S^2})$. We can check that the sectional curvature of (S^3, g_ϵ) is between 0 and 1 if $\epsilon \in (0, 1]$.

We can generalize this construction and prove the following :

Proposition 11.1. *Let M be a compact manifold on which a torus T^m acts. We assume that there is no point p on M such that p is fixed by all the elements of T^m. Then there exists a family of metrics g_ϵ on M such that K_{g_ϵ} is bounded from below and above and that $\lim^{GH}_{\epsilon\to 0}(M, g_\epsilon) = M/T^m$.*

To find such a sequence of metrics, we first take a T^m invariant Riemannian metric g_M on M. We next take X an element of Lie algebra of T^m such that the subgroup $\cong \mathbb{R}$ generated by X is dense in T^m. We regard X as a (Killing) vector field on M. We remark that X never vanishes on M. For $V, W \in T_pM$ we put

$$V = V_h + c(V)X(p), \qquad W = W_h + c(W)X(p),$$

where $g_M(V_h, X_p) = g_M(W_h, X_p) = 0$. We then define

$$g_\epsilon(V, W) = g_M(V_h, W_h) + \epsilon^2 c(V)c(W)g_M(X(p), X(p)).$$

We can prove that the limit of (M, g_ϵ) as $\epsilon \to 0$ is M/T^m with quotient metric and the sectional curvature of (M, g_ϵ) is bounded for $\epsilon \in (0, 1]$. $\square$

Let us take for example $M = S^3$. We can find an action of T^2 on S^3 satisfying the assumption of Proposition 11.1. Hence there exists a sequence of metrics on S^3 such that the limit is $S^3/T^2 = [0, 1]$ the interval. In particular the limit space is not a manifold.

This construction is further generalized in [38] (Theorem 12.1).

There are two approaches to study collapsing Riemannian manifolds under the assumption $|K_{M_i}| \le 1$. One is due to Cheeger-Gromov [39, 38] the other is due to the author [53, 55, 56]. These two approaches are unified in [34]. In this section, we discuss the second approach and in the next section we discuss the first (and the joined) approach.

Here we discuss the following two problems. For n,D, we denote by $\mathfrak{M}_n(D)$ the set of all isometry classes of n dimensional Riemannian manifolds M such that $|K_M| \le 1$, and $Diam(M) \le D$.

Problem 11.1. Let $M_i \in \mathfrak{M}_n(D)$ and $X = \lim^{GH}_{i\to\infty} M_i = X$.
(1) What kind of singularity can X has ?
(2) Describe the relation between X and M_i.

We remark that if we replace $\mathfrak{M}_n(D)$ by $\mathfrak{M}_n(D, v)$ the answers are Theorems 3.3, 3.4. Problem 11.1 will be studyed also under milder assumption on curvature later.

We first discuss Problem 3.4 (2) in the special case when X is a smooth manifold.

Theorem 11.2 (Fukaya [53, 56])**.** *Let $M_i \in \mathfrak{M}_n(D)$. Suppose $B = \lim^{GH}_{i\to\infty} M_i$ is a smooth Riemannian manifold. Then, for large i, there exists a fiber bundle $\pi_i : M_i \to B$ with the following properties.*
(1) *The fiber is diffeomorphic to an infranilmanifold F.*
(2) *The structure group is the group of affine transformations* $\mathrm{Aff}(F, \nabla_{can})$ *here we define the affine connection ∇_{can} on F as in the last section.*
(3) *π_i is an almost Riemannian submersion in the following sense. If $V \in T_p(M_i)$ is perpendicular to the fiber then*

$$1 - \epsilon_i < \frac{g_{M_i}(V, V)}{g_N(\pi_{i*}V, \pi_{i*}V)} < 1 + \epsilon_i$$

where $\epsilon_i \to 0$.

Yamaguchi [154] generalized Theorem 11.2 as follows.

Theorem 11.3 (Yamaguchi)**.** *If M_i is a sequence of n dimensional Riemannian manifold with $K_{M_i} \ge -1$. We assume $B = \lim^{GH}_{i\to\infty} M_i$ is a smooth Riemannian manifold. Then for large i there exists a fiber bundle $\pi_i : M_i \to B$. It satisfies* (3) *above.*

See §19 for more results on the fiber of $\pi_i : M_i \to B$ in Theorem 11.3.

The idea of the proof of Theorems 11.2, 11.3 is similar to the discussion in §9. Namely we embed the limit space B to $\mathbb{R}^X$ using distance function ($I_B : B \to \mathbb{R}^X$). (Here X is a net in B.) We then map M_i to the same space ($I_{M_i} : M_i \to \mathbb{R}^X$). We can not prove that I_{M_i} is an embedding since there is no bound of injectivity radius of M_i. However I_B is an embedding and $I_{M_i}(M_i)$ is contained in a tubular neighborhod $U(I_B(B))$ of $I_B(B)$ for large i. Hence we have a composition of three maps, I_{M_i}, projection of the normal bundle of $I_B(B)$, and I_B^{-1}. This map is our $\pi_i : M_i \to B$. To check that it safisfies (1),(2) we use parametrized version of the proof of Theorems 10.2, 10.3. □

In general, the limit space as in Question 11.1 has singularity. Hence Theorem 11.2 does not apply in the general case. However we can use its equivariant version and a trick (which we explain below) so that we can apply it to the general situation.

Let M be an n dimensional Riemannian manifold. We define its frame bundle by :

$$FM = \left\{ (o; e_1, \cdots, e_n) \,\middle|\, \begin{array}{l} p \in M, \\ (e_1, \cdots, e_n) \text{ is an orthonormal basis of } T_pM \end{array} \right\}.$$

There exists an $O(n)$ action on FM such that $FM/O(n) = M$. In other words $FM \to M$ is a principal $O(n)$ bundle. The Riemanian metric determines a connection of this principal bundle (that is the Levi-Civita connection.) Using it we can canonically define an $O(n)$ invariant Riemannian metric on FM such that $FM \to M$ is a Riemannian submersion and the fiber $\cong O(n)$ has given standard metric on $O(n)$. From now on we use this metric on FM.

Theorem 11.4 ([55])**.** *If $M_i \in \mathfrak{M}_n(D)$ and if $Y = \lim^{GH}_{i\to\infty} FM_i$. Then we have the following.*

(1) *Y is a smooth manifold.*
(2) *$O(n)$ acts by isometry on Y such that $\lim^{GH}_{i\to\infty} M_i = Y/O(n)$.*
(3) *There exists a sequece of $O(n)$ equivariant Riemannian metric on g_i on Y and $\epsilon_i \to 0$ such that*

$$1 - \epsilon_i < \frac{d_Y(x, y)}{d_{g_i}(x, y)} < 1 + \epsilon_i$$

for any $x, y \in Y$, where d_Y is the limit metric.
(4) *For each $p \in Y$ the connected component of the isotropy group $\{g \in O(n) | gp = p\}$ is abelian.*

To prove Theorem 11.4, we use the notion of fundamental pseudogroup we explained in the last section as follows. (The idea to use pseudofundamental group to study collapsing is initiated by Gromov in [70] Chapter 8.) Let $p \in X$. We take $p_i \in M_i$ which converges to p. We fix small ϵ and consider $\pi_1(M_i, p_i, \epsilon)$ which acts on $B_0(\epsilon, T_{p_i}M_i)$ such that the quotient space is isometric to an ϵ neighborhod of p_i in M_i. (We can define a notion of action of pseudogroup to a space and of the quotient space, in a reasonable way.) We can define a convergense of a pseudogroup action and can find a limit of $\pi_1(M_i, p_i, \epsilon)$, which we write N. The group N acts by isometry to the limit $\tilde{B}(p)$ of $B_0(\epsilon, T_{p_i}M_i)$. Here we put a Riemannian metric on $B_\epsilon(0, T_{p_i}M_i)$ which is induced by on M_i by the exponential map. Since the injectivity radius of $B_0(\epsilon, T_{p_i}M_i)$ is bound away from 0, it follows that we can apply

Theorem 3.4 to find that $\tilde{B}(p)$ is a Riemannian manifold of $C^{1,\alpha}$ class. The point here is that N is in general not discrete and collapsing occurs exactly when N has positive dimension. We can show that the group germ of the origin of N is a Lie group germ. Note that this is easy in case when the metric on Y is smooth. To avoid using metric which is not smooth, we approximate g_{M_i} by $g_{M_i,\epsilon}$ such that

$$|\nabla^k R_{g_{M_i,\epsilon}}| < C_k(\epsilon), \tag{11.1a}$$

$$e^{-\epsilon} g_{M_i} < g_{M_i,\epsilon} < e^{\epsilon} g_{M_i}. \tag{11.1b}$$

here the left hand side of (11.1b) is the norm of the k-th derivative of the curvature tensor and the right hand side is a constant depending on k and ϵ but is independent of i. The existence of such approximation is proved by [15], (and generalized by [1] to complete manifolds).

Then the limit of the ball $(\tilde{B}(p), \tilde{g}_{M_i,\epsilon})$ is smooth. Replacing G by its quotient we may assume that the action of G on $\tilde{B}(p)$ is effective.

We now consider the frame bundle $F\tilde{B}(p)$. Using the fact that G is effective and isometry on $\tilde{B}(p)$, it follows that the action of N on $F\tilde{B}(p)$ is *free*. Therefore $F\tilde{B}(p)/N$ is a manifold. We can easily see that $F\tilde{B}(p)/N$ is an open set of the limit Y of FM_i and by changing $p \in X$ it covers Y. Thus Y is a manifold as required. Using Margulis' lemma we find that the connected component of N is nilpotent. Since the isotropy group of $O(n)$ action on Y can be identified to the isotropy group of N action on $\tilde{B}(p)$, it follows that the connected component of the isotropy group is both companct and nilpotent. Hence it is abelian. □

Using Theorem 11.4, we can improve Theorem 11.2 as follows.

Theorem 11.5 ([55])**.** *Let M_i, Y be as in Theorem* 11.4*. Then there exists $\tilde{\pi}_i : FM_i \to Y$ for large i with the following properties.*

(1) *$\tilde{\pi}$ is a fiber bundle satisfying* (1),(2),(3) *of Theorem* 11.2*.*
(2) *$\tilde{\pi}$ is $O(n)$ equivariant and hence induces a map $\pi : M_i \to Y/O(n)$.*

The proof is an equivariant version of the proof of Theorem 11.4. (Compare Remark 9.1 (B).)

12. Collapsing Riemannian manifolds - II -

As we mentioned before there are two approaches to study collapsing Riemannian manifolds and we discuss another approach [39, 38] in this section. One advantage of this approach (compared with one we discussed in the last section) is that we do not need to assume diameter bound. Let us first give an example to illustrate a new phenomenon which occurs when we do not assume diameter bound.

Example 12.1 (See [135].)**.** Let Σ_k be a Riemann surface of genus $k > 0$. For each ϵ there exists a Riemannian metric $g_{k,\epsilon}$ on $\Sigma_{k,0} = \Sigma_k \setminus \operatorname{Int} D^2$ with the following properties.

(1) $0 \ge K_{g_{k,\epsilon}} \ge -1$.
(2) A neighborhood of the boundary of $\Sigma_{k,0}$ is isometric to $S^1(\epsilon) \times [0,1)$. (Here $S^1(\epsilon)$ is a circle with radius ϵ.)

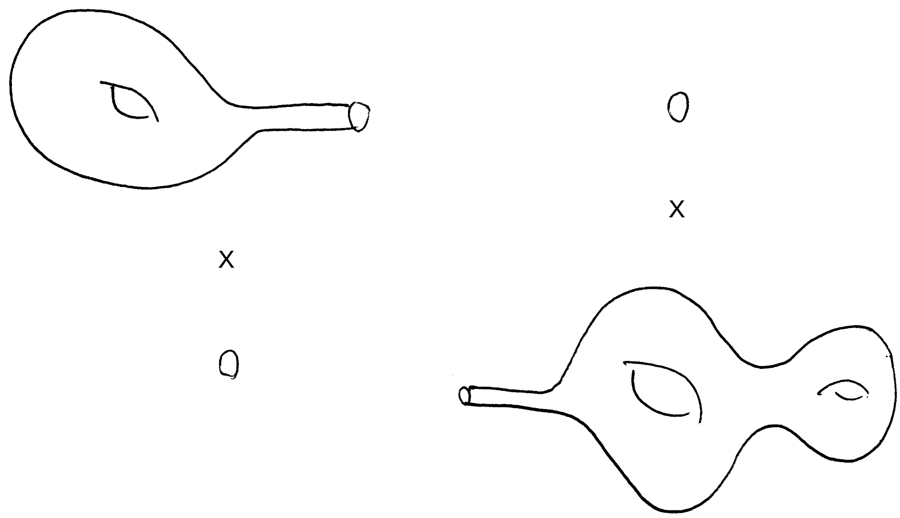

Figure 12.1

Now we consider $(\Sigma_{k_1,0}, g_{k_1,\epsilon}) \times S^1(\epsilon)$ and $(\Sigma_{k_2,0}, g_{k_2,\epsilon}) \times S^1(\epsilon)$ and glue them at their boundaries by the isometry $(s,t) \mapsto (t,s)$, $S^1(\epsilon) \times S^1(\epsilon) \to S^1(\epsilon) \times S^1(\epsilon)$. We thus obtain a family of 3 dimensional Riemannian manifolds $M_{k_1,k_2,\epsilon}$, which satisfies the curvature condition $0 \ge K \ge -1$. The injectivity radius of it goes to zero everywhere as $\epsilon \to 0$. It is however easy to see that $M_{k_1,k_2,\epsilon}$ is not a S^1 bundle over surface.

See [4] for a more sophisticated constructin.

We remark that the diameter of $\Sigma_{k,\epsilon}$ goes to infinity as ϵ goes to zero. The point of this example is that in each piece $\Sigma_{k_i} \times S^1(\epsilon)$ there is one direction (the direction of second factor) which collapses. But in the domain we glue metrics there are two factors which collapse. Theorem 11.2 implies that such a phenomenon does not occur. Namely the dimension of collapsing direction is constant in the case when limit space is compact. Thus to describe collapsing Riemannian manifolds without curvature bound, we need a language to describe the situation where the dimension of the collapsing direction changes. Cheeger-Gromov [38] used a notion of local action of group for this purpose. They call it F-structure.

Definition 12.1. *F-structure* on M is an open cover $M = \cup U_i$ together with an action of T^{n_i} on $\tilde U_i$, which is a finte cover of U_i, with the following properties.
(1) There exists no point on $x \in \tilde U_i$ which is fixed by all the elements of T^{n_i}.
(2) If $U_i \cap U_j \ne \emptyset$ then there exists a covering space $\pi_{ij} : \tilde U_{ij} \to U_i \cap U_j$, maps $\pi_{ij,i} : \tilde U_{ij} \to \tilde U_i$, $\pi_{ij,j} : \tilde U_{ij} \to \tilde U_j$ such that :
(3) $\pi_i \circ \pi_{ij,i} = \pi_j \circ \pi_{ij,j} = \pi_{i,j}$.
(4) There exists an action of $T^{n_{ij}}$ on $\tilde U_{ij}$ with property (1).
(5) There exists an n_i dimensional subtorus $T^{n_i}_{ij} \subset T^{n_{ij}}$ and a locally isomorphic group homomorphism $T^{n_i}_{ij} \to T^{n_i}$, such that $\pi_{ij,i}$ is equivariant. The same holds when we replace i by j.

Let us consider Example 12.1. We may split M into two pieces $U_i \cong \Sigma_{k_i} \times S^1$. On U_i we have an S^1 action. These two actions do not coincide to each other on the overlapped part $U_1 \cap U_2 \cong S^1 \times S^1 \times (-C, C)$. Namely the S^1 action on U_1 is an action to the first factor while the S^1 actoin on U_2 is an action to the second factor. However we have T^2 action which contains both actions. This is a typical situation of F-structure.

The main theorem in [38] is as follows.

Theorem 12.1 (Cheeger-Gromov)**.** *If M has an F-structure then there exists a sequence of metrics g_i on M such that $|K_{g_i}| \le 1$ and that the injectivity radius of (M, g_i) converges to zero everywhere as $i \to \infty$.*

The proof is a kind of generalization of the proof of Proposition 11.1. The new point which appears in the proof of Theorem 12.1 is that we need to controll the curvature at the points where the dimension of the torus acting there changes. Roughly speaking to keep the curvature bounded from above and below, we expand the direction normal to the action. □

The converse to Theorem 12.1 is the main theorem of [39]. Namely :

Theorem 12.2 (Cheeger-Gromov)**.** *There exists a positive constant ϵ_n such that if M is an n dimensional complete Riemannian manifold such that $|K_M| \le 1$ and the injectivity radius is everywhere smaller than ϵ_n then there exists an F-structure on M.*

Remark 12.1. We can modify Theorem 12.2 so that we do not need to assume that the injectivity radius is small everywhere. Namely we consider any M with $|K_M| \le 1$, and construct the F-structure on $\{p \in M | i_M(p) < \epsilon\}$.

Let us sketch the proof of Theorem 12.2 very biefly. We assume $|K_{M_i}| \le 1$ and $\sup i_{M_i} \to 0$ where i_{M_i} is an injectivity radius. We need to construct an F-structure on M_i for large i. There are two steps to do so. One is to construct a torus action on the finite cover locally. The other is to glue them. We explain the first step only. The following is the basis of this step.

Lemma 12.3. *If a Riemannian manifold X is complete and flat then there exists a compact flat submanifold S without boundary in X such that X is diffeomorphic to the normal bundle of S.*

This lemma is a special case of soul theorem 16.7 which we will discuss in §16. By using Theorem 10.1, we find that S has a finite cover which is a flat torus. So we can find a torus action on the finite cover of X in Lemma 12.3. To use Lemma 12.3 in our situation, we proceed as follows. Let $p_i \in M_i$ and $\epsilon_i = i_{M_i}(p_i)$. We consider the metric $g_i' = g_{M_i}/\epsilon_i$ and consider the limit (M_i, g_i'). (The limit is taken with respect to the pointed Hausdorff distance which we define in §16.) Since the curvature of (M_i, g_i') goes to zero and since the injectivity radius of (M_i, g_i') at p_i is 1 we have a flat manifold X as a limit. Also a neighborhood of p_i is diffeomorphic to a compact subset of X for large i.

This is very rough sketch. Actually gluing part (which we do not discuss here) is harder. □

In the case when we do not assume diameter bound, there are several possible ways to define collapsing. One definition is that injectivity radius becomes small everywhere. The other is that volume becomes small. (Note Theorem 5.6 implies that they are equivalent in the case when the diameter and the absolute value of the sectional curvature are bounded.) We say the first one (injectivity radius is small) the collapse and the second one (volume is small) the volume collapse. There is an example of manifold which admits an F-structure but does not admit volume collpased metric. Actually $\mathbb{C}P^2$ admits F-structure but we can use the fact that its Euler number is nonzero to prove the nonexistence of volume collpased metric. (This example is due to Januszkiewicz. See [57] p229 or [39].) Cheeger-Gromov defined a notion polarized F-structure which implies existence of volume collpased metric. However we do not know whether volume collapsed manifold has polarized F-structure. So the following problem is still open.

Definition 12.2 ([73])**.** A *minimal volume* $\mathrm{MinVol}(M)$ of a compact manifold M without boundary is the infinimum of the volume (M, g) where g is a Riemannian metric on M such that $|K_g| \le 1$.

Problem 12.1. Does there exists a positive number ϵ_n depending only on n with the following properties ? If n dimensional compact manifold M satisfies $\mathrm{MinVol}(M) < \epsilon_n$ then $\mathrm{MinVol}(M) = 0$.

There are several partial results toward Problem 12.1.

Theorem 12.4 (Rong [130, 131])**.** *In case dimension of M is 3 or 4 Problem 12.1 is affirmative.*

There is a very sharp result in the case when M admits a metric of constant negative curvature.

Theorem 12.5 (Besson-Courtrois-Gallot [20]). *Let an n dimensional manifold M admits a metric of constant curvature g_0. Then if g is any metric on M with Ricci $\geq -(n-1)$ we have* $\mathrm{Vol}(M,g) \geq \mathrm{Vol}(M,g_0)$.

Theorem 12.5 in particular implies $\mathrm{MinVol}(M) = \mathrm{Vol}(M,g_0)$.

The answer to Problem 12.1 is affirmative under an additional assumption on diameter.

Theorem 12.6 (Cheeger-Rong [40]). *There exists a positive number $\epsilon(n,D)$ depending only on n and D with the following properties. If n dimensional compact manifold M has a Riemannian metric g such that $|K_g| \leq 1$, Diam $\leq D$ and* $\mathrm{Vol}(M,g) < \epsilon_n$, *then for any ϵ there exists a Riemannian metric g_ϵ on M such that $|K_{g_\epsilon}| \leq 1$ and* $\mathrm{Vol}(M,g_\epsilon) < \epsilon$.

We next describe the result of [34]. We remark that the results in the last section do not give enough description in the case when diameter is not bounded. On the other hand, if we consider the case of almost flat manifold, for example, the F-structure corresponds to the action of the center of the nilpotent group, and hence only a part of the collpased direction is described by F-structure. So we need a local action of nilpotent group to describe collapsing Riemannian manifolds in the general case. Such a structure may be called N-structure. One trouble to define it rigorously is that the noncommutativity of group makes it harder to describe compatibility condition. To have a simplified description we remark the following fact. In the situation of Theorem 12.2, we can approximate the metric by one invariant of the F-structure. (Actually original metric is "almost invariant" by the action and we can take average so that it is strictly invariant.) So in place of writing compatibility of actions, we may state the actions are isometry with respect to the metric nearby (which is independent of the chart).

Note the fact that we can approximate the metric by invariant one, is also true in a modified sense for the almost flat manifold and in the situation of Theorem 11.2. Namely we can make the metric "invariant" of the action of nilpotent group. We need to remark however the following. In case of $\Gamma\backslash N$ (where N is a nilpotent group and Γ is its discrete subgroup) for example, the almost flat metric is *not* invariant of the *right* action of N. Since the induced metric on $\Gamma\backslash N$ is well deinfed only if we start with the *left* invariant metric on N. It means that the group acting on $\Gamma\backslash N$ (equipped with almost flat metric) by isometry is only the centor of N. In other words, we can find an isometric action of N only after taking infinite (universal) cover. This point is different from the case of abelian group (torus).

Now we are going to state the main result of [34]. Let M be a manifold and $p \in M$. Let $U_p \subset M$ be an open neighborhood of p. We denote by ∇^g the Levi-Civita connection of g.

Theorem 12.7 (Cheeger-Fukaya-Gromov [34]). *For each $\epsilon > 0$ and $n \in \mathbb{Z}_+$, there exists $\rho = \rho(\epsilon, n) > 0$ with the following properties. Let (M, g) be a complete n dimensional Riemannian manifold with $|K_g| \le 1$. Then there exists a metric g_ϵ and $U_p, \tilde{U}_p$, Γ_p, N_p for each $p \in M$ such that :*

(1) *N_p is nilpotent. $\Gamma_p \subset N_p$ is a discrete subgroup such that $\pi_0(N_p)$ is finite and that N_p is generated by its connected component $N_{p,0}$ and Γ_p.*
(2) *U_p is a neighborhood of p and $U_p \supseteq B_p(\rho, M)$.*
(3) *N_p acts on $(\tilde{U}_p, \tilde{g}_p)$ by isometry. Here $\tilde{U}_p$ is a covering space of U_p and $\tilde{g}_p$ is the metric induced by g_ϵ.*
(4) *If $\tilde{p} \in \tilde{U}_p$ and $[\tilde{p}] = p$ then $i_{\tilde{U}_p}(\tilde{p}) > \rho$.*
(5) *$[\Gamma_p : \Gamma_p \cap N_{p,0}] < k$.*
(6) *For any $x \in \tilde{U}_p$ $Diam(\Gamma_p \backslash N_p x) < \epsilon$. Here $N_p x$ is an N_p orbit.*

Moreover we have :

(7) *$e^{-\epsilon} g < g_\epsilon < e^\epsilon g$.*
(8) *$|\nabla^g - \nabla^{g_\epsilon}| < \epsilon$.*
(9) *$|\nabla^{g_\epsilon} R_{g_\epsilon}| < c(n, i, \epsilon)$, where R_{g_ϵ} is the curvature tensor of g_ϵ and $c(n, i, \epsilon)$ depends only on n, i, ϵ.*

Remark 12.2. The existence of g_ϵ satisfying (7)(8)(9) is proved by [15, 1]

Remark 12.3. The metric satisfing (1)(2)(3)(4)(5)(6) is called (ρ, k) round metric in [34].

We remark that at the point where $i_M(p) > \rho$ we may take $N_p = 1$ and $\tilde{U}_p = U_p$. Hence the statemet above is obvious.

On the other hand, Condition (4) implies that at the point p where injectivity radius is small, the group N_p is nontrivial. Hence, together with (1), we obtain an local action of torus by restricting the action of N_p to the center. Using (6) and the fact the the local action of torus is compatible with the metric g_ϵ, we can prove that these actions are compatible in the sense of Theorem 12.2.

Moreover, in the case when the diameter of M is smaller than a constant depending only on ϵ and n, we can prove that the group N_p is independent of the choice of p. Hence its orbits defines a foliation on the frame bundle of M. It implies Theorem 11.5. Thus Theorem 12.7 unifies two approaches to collapsing Riemannian manifolds.

The proof of Theorem 12.7 is a combination of the proofs of Theorem 12.2 and of Theorem 11.5. We use Theorem 12.7 and its proof (together with some improvement) to find N_p locally. We then glue them in a way similar to the proof of Theorem 12.7. Finally we take average and obtain the required metric g_ϵ. □

Example 12.2. Let Γ be a lattice of a semisimple Lie group G of noncomapct type and G/K be a symmetric space. We assume $\Gamma\backslash G/K$ is noncompact. Then for each $p \in G/K$ the group $\Gamma_p = \{g \in \Gamma | d(p, gp) < \epsilon\}$ has nilpotent subgroup $\Gamma_{p,0}$ of finite index $[\Gamma_p : \Gamma_{p,0}] < k$ by Theorem 10.4. (We remark that Γ_p may not be contained in a small neighborhood of the unit in G. But its subgroup of finite index is in a small neighborhood of the unit.) The Zariski closure $N_p \subset G$ of $\Gamma_{p,0}$ is a nilpotent group. This is our N_p. The original metric (the metric of symmetric space) is invariant of left N_p action.

Hattori [84] found the following. Let $M = \Gamma\backslash G/K$ be a locally symmetric space of noncompact type. We assume that it is noncompact and of finite volume. Then the limit $(M, g_M/R)$ as R goes to infinity is a cone of a simplicial complex T which is called the Tits building. (Here the limit is taken with respect to the pointed Gromov-Hausdorff distance (Definition 16.3).) Now if we take a simplex Δ of T then a "neighborhood" of it in M is diffeomorphic to $\Delta \times [0, \infty) \times \Gamma(\Delta)\backslash N(\Delta)$. The dimension of the nilmanifold $\Gamma(\Delta)\backslash N(\Delta)$ depends on Δ. They are glued appropriately, which gives a structure as in Theorem 12.7.

The following addendum to Theorem 12.7 is useful for various applications.

Proposition 12.8 ([132])**.** *If $a \ge K_M \ge b$ in Theorem 12.7 then we may choose g_ϵ so that $a + \epsilon \ge K_M \ge b - \epsilon$.*

13. Collapsing Riemannian manifolds - III -

In this section, we review some of the applications of the collapsing Riemannian manifolds. We recall that $\mathfrak{M}_n(D)$ is the set of isometry classes of n dimensional Riemannian manifold M with $Diam(M) \le D$, $|K_M| \le 1$.

Theorem 13.1 (Fang - Rong [51], Petrunin-Tuschmann [126])**.** *For each n, D the number of diffeomorphis classes of simply connected manifolds M in $\mathfrak{M}_n(D)$ with finite $\pi_2(M)$ is finite.*

Theorem 13.2 (Fang - Rong [51], Petrunin-Tuschmann [126])**.** *There exists $i(n, \delta) > 0$ such that if M is simply connected, $\pi_2(M)$ is finite and if $1 \ge K_M \ge \delta > 0$, then the injectivity radius of M is larger than $i(n, \delta)$.*

We remark that, in case dimension is even, Theorem 13.2 follows from Theorem 4.3 without assumption on π_2.

Example 13.1. We first consider the Lens space $S^3/\mathbb{Z}_p$ where $\mathbb{Z}_p \subset S^1$ is a cyclic group of order p. Its curvature is 1 and its limit is $S^2 =$

S^3/S^1. This example shows the assumption on $\pi_1(M)$ is necessary both in Theorems 13.1, 13.2.

The three examples below show that the assumption on $\pi_2(M)$ is also necessary in Theorems 13.1, 13.2.

We consider the Lie group $SU(3)$. It has a metric with positive sectional curvature. We consider its maximal torus $T^2 \subseteq SU(2)$. Let p_i, q_i be coprime integers such that $\lim p_i/q_i = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. We identify $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let x, y be coordinate of $\mathbb{R}^2$. We consider $S^1_i = \{[x,y] \in T^2 | y = p_i x/q_i\}$. We put $M_i = S^1_i \backslash SU(2)$ eqiupped with quotiemt Riemannian metric. M_i is a sequence of 7 dimensional manifolds of positive curvature. Using the fact $\lim p_i/q_i$ is irational, we can easily find that the limit of M_i with respect to the Gromov-Hausdorff distance is $T^2 \backslash SU(2)$. We can also prove that the sectional curvature of M_i is uniformly positive. Namely $C \ge K_{M_i} \ge \delta > 0$ for some δ, C independent of i. (This is a consequence of the fact that p_i/q_i converges. We remark that $\pi_2(M_i) \cong \pi_1(S^1) = \mathbb{Z}$.)

In a similar way we can use the T^2 action to $S^3 \times S^3$ to get a sequence of metrics g_i on $S^2 \times S^3$ with $C \ge K_{g_i} \ge \delta > 0$ such that $(S^2 \times S^3, g_i)$ converges to $S^2 \times S^2$.

We next consider an action of $T^2 \times T^2$ on $SU(3)$ where the first factor acts by left multiplication and the second factor acts by right multiplication. Using appropriate family of $S^1_i \cong S^1 \subseteq T^2 \times T^2$, Petrunin-Tuschmann [126] (using Eschenburg [50]) found an example of $M_i = S^1_i \backslash SU(3)$ with $C \ge K_{M_i} > \delta > 0$ such that M_i converges to $T^2 \backslash SU(3)/T^2$.

Remark 13.1. A similar π_2 assumption as in Theorem 13.2 was proposed by the author in [57] Remark 15.10. However [57] Conjecture 15.7 (by the author) turns out to be false. A counter example (due to Petrunin-Tuschmann) is the last example in Example 13.1.

We now sketch the proof of Theorem 13.1. We start with the following :

Lemma 13.3 (Rong [132])**.** *If we assume that $\pi_1(M)$ is finite in the situation of Theorem* 11.5 *in addition, then the fiber of $\pi : FM_i \to Y$ in Theorem* 11.5 *is diffeomorphic to a flat manifold.*

Using the fact that the fundamental group of M_i is finite (here we assume $\dim M_i > 2$), it follows easily that the fundamental group of the fiber has index finite abelian subgroup. Since the fiber is an infranilmanifold the lemma follows immediately. □

Lemma 13.3 implies that we have an F-strucuture whose orbits are fibers. (Here our F-structure is one called pure F-structure by Cheeger-Gromov [38]. Pure F structure is an F-structure such that all the orbits of the local action has the same dimension.) We next apply the averaging process in the proof of Theorem 12.7 to the situation of Theorem 11.5 and of Lemma 13.3. Then we have :

Lemma 13.4. *In the situation of Lemma 13.3, we can approximate the Riemannian metric on FM_i by g_ϵ in the same sense as Theorem 12.7* (7) (8) (9) *so that g_ϵ is invariant of local T^k action and of $O(n)$ action.*

Now we start the proof of Theorem 13.1. We assume that Theorem 13.1 is false. Then there exists a sequence $M_i \in \mathfrak{M}_n(D)$ such that M_i is simply connected, $\pi_2(M_i)$ is finite, and M_i is not diffeomorphic to M_j for $i \neq j$. We take FM_i and may assume that it converges to Y. Since we approximate the metric by one satisfying Theorem 12.7 (7) (8) (9), it follows that Y is a smooth Riemannian manifold. We may replace FM_i by its finite cover $\tilde FM_i$ so that it has global $T^k \times G$ action, where G is a compact group[20] and T^k orbits are the fibers of the fibration $\tilde FM_i \to Y$. We modify metric of $\tilde FM_i$ so that it is $T^k \times G$ equivariant. The next lemma is the place where we use the key assumption that $\pi_2(M_i)$ is finite.

Lemma 13.5. *If $\tilde FM_i/T^k$ is G diffeomorphic to $\tilde FM_j/T^k$ then $\tilde FM_i$ is $T^k \times G$ diffeomorphic to $\tilde FM_j$.*

In fact the torus bundle $T^k \to E \to B$ is determined by the (T^k analogue of) Euler class $\in Hom(H_2(B), \pi_1(T^k))$ (which is well defined up to $Aut(\pi_1(T^k))$. In our case where $\pi_2(\tilde FM_j)$ is finite and $\pi_1(\tilde FM_j)$ is trivial, Euler class is an isomorphism $H_2(B)/Tor \to \pi_1(T^k)$ hence is unique up to $Aut(\pi_1(T^k))$. To obtain the T^k equivariant diffeomorphism $\tilde FM_i \to \tilde FM_j$ which is G equivariant also, we use center of Mass technique (Proposition 8.3). □

We remark that $\tilde FM_j/T^k$ is the same dimension as Y and $\tilde FM_j/T^k$ converges to Y with respect to the G-Gromov-Hausdorff topology (which was introduced in [52]). Estimate (9) of Theorem 12.7 implies that Y is a smooth manifold. On the other hand, the sectional curvature of $\tilde FM_j/T^k$ is bounded from below. Hence Theorem 11.3 implies that $\tilde FM_j/T^k$ is diffeomorphic to Y for large i. We can use G equivariant version of Theorem 11.3 (which can be proved in the same way as Theorem 11.3 using an embedding to Hilbert space as in [55][21]), $\tilde FM_i/T^k$ is G diffeomorphic to Y for large i. Hence Lemma 13.5 implies that $\tilde FM_j$ is G diffeomorphic to $\tilde FM_j$ for i, j large. Namely M_i is diffeomorphic to M_j. This is a contradiction. □

To prove Theorem 13.2 we need another result by Petrunin-Rong-Tuschmann.

Theorem 13.6 ([117])**.** *Let M be a compact manifold. We assume that M admit a sequence of metrics g_i. We assume that $\Lambda \geq K_{g_i} \geq \lambda$ and that the metric space $X = \lim^{GH}_{i\to\infty}(M, g_i)$ is of dimension strictly*

[20] Actuall it is finite covering group of $O(n)$. (It may be disconnected.)

[21] See Remark 9.1.

smaller than M. We also assume that the distance function $d_i : M \times M \to \mathbb{R}$ induced by g_i converges to a function d which determines a pseudometric [22] *on M.*

Then $\lambda \leq 0$.

Remark 13.2. Klingenberg and Sakai conjectured a similar statement, but their conjecture does not assume additional assumption that d_i converges to a pseudometric.

To prove Theorem 13.2 using Theorem 13.6 we proceed as follows. We assume that there exists M_i with $1 \geq K_{K_i} \geq \delta > 0$ and injectivity radius goes to 0. We can discuss in the same way as the proof of Theorem 11.3 to show M_i is diffeomorhpic to M_j[23]. By looking the proof carefully we may assume that the diffeomorphism almost preserves distance function. Namely if we identify M_i with M_j then the sequence $M = M_i = M_j$ satisfies the assumption of Theorem 13.6. The conclusion of Theorem 13.6 contradicts to $K_{M_i} \geq \delta > 0$. □

One of the ideas of the proof of Theorem 13.6 is the following observation. If the collapsing occurs in the same way as the proof of Proposition 11.1 then the sectional curvature of the plain spanned by X and other vector is always converges to zero. To make this simple idea works we need a lot of delicate work which is not described here. □

We next discuss some other applications of collapsing theory.

Theorem 13.7 (Rong [132])**.** *There exists $w(n, \delta)$ such that if a compact n dimensional Riemannian manifold M satisfies $1 \geq K_M \geq \delta$ then there exists a cyclic subgroup C of the fundamental group $\pi_1(M)$ such that $[\pi_1(M) : C] < w(n, \delta)$.*

Remark 13.3. If we assume $1 \geq K_M \geq 0$, $Diam(M) < D$, then there exists an abelian subgroup C of $\pi_1(M)$ such that $[\pi_1(M) : C] < w(n, D)$ ([132]). There are results under milder assumption that is the case when M is of almost of nonnegative curvature. See §19.

The following is another application of collapsing theory. This time we apply to manifold of almost nonpositive curvature.

Theorem 13.8 (Fukaya-Yamaguchi [58])**.** *There exists $\epsilon(D, n)$ such that if a compact n dimensional Riemannian manifold M satisfies $Diam(M) \leq D$, $\epsilon(D, n) \geq K_M \geq -1$ then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*

This is a generalization of Hadamard-Cartan's theorem (Theorem 4.6) which is the case when $K_M \leq 0$.

[22] namely it satisfies axioms of metric except "$d(x, y)$ implies $= y$".

[23] We use Proposition 12.8 to show $\Lambda + \epsilon \geq K_{M_i} \geq \lambda - \epsilon$

14. Morse theory of distance function

So far we mainly discussed results assuming the curvature to be bounded from above and below. From this section, we consider the case when we assume curvature is bounded from below only.

The next theorem is a corollary of Theorem 4.1.

Theorem 14.1 (Rauch). *Let M be a compact manifold without boundary. If there exists a Morse function on M with two ciritical points, then M is homeomorphic to a sphere.*

In §4, we start with Theorem 4.1 and show the way to prove sphere theorems, finiteness theorems and compactness theorems by estimating the number of balls we need to cover a manifold. The number of contractible open subsets one needs to cover the space (plus one), is called the Lusternik-Shnirel'man category and is important in Morse theory. In this section we will try to apply Morse theory directly.

For a given Riemannian manifold M, a function which is determined automatically from the metric is a distance function $d_p(x) = d(p, x)$ from a point. (Note we can use the fact that $p = x$ is the unique critical point of $x \mapsto d_p(x)$ with $d(x, p) < i_M(p)$ to prove that $B_p(r, M)$ is diffeomorphic to a sphere if $r < i_M(p)$.)

The difficulty to apply Morse theory to distance funciton is that $x \mapsto d_p(x)$ is not differentiable for $d(x, p) > i_M(p)$. (d_p is not differentiable at p either. But this does not cause serious trouble. We may consider d_p^2 instead, for example.) During the proof of Theorem 2.5, Grove-Shiohama applied Morse theory away from the ball with radius $= i_M(p)$. After that, their method is used in many other places. The main idea of them is the following definition.

Definition 14.1. We say q is a *regular point* of d_p if there exists a nonzero vector $\vec{V} \in T_qM$ such that for $\underline{\text{any}}$ minimal geodesic $\ell : [0, d(p, q)] \to M$ joining p and q, the angle between $\frac{d\ell}{dt}(0)$ and $\vec{V}$ is not greater than $\pi/2$.

For example let p,q be as in the Figuer ? below. It is not clear how many minimal geodesics are there joining p with q. But it is easy to see that direction of any of them is downwards at q. Hence q is a regular point of d_p.

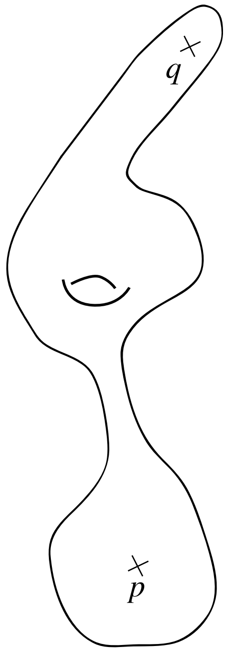

Figure 14.1

Remark 14.1. We may consider various situations similar to Definition 14.1. For example let us consider a continuous function f which is an infinimum of finitely many differentiable functions f_α locally, (namely $f = \inf f_\alpha)^{24}$. In this case we say q is a regular point of f if there exists a vector $\vec{V} \in T_qM$ such that, for each α with $f(q) = f_\alpha(q)$, we have $\vec{V}(f_\alpha) > 0$. We can apply a similar argument to a linear combination of finitely many d_p's or infinimum of them also. Proposition 14.2 holds for such cases.

Based on Definition 14.1, we can prove the following analogue of Morse lemma for d_p.

Proposition 14.2. *If arbitrary q with $a \le d_p(q) \le b$ is a regular point of d_p, and if $B_p(b, M)$ is compact, then $B_p(b, M) \setminus B_p(a, M) = \{q \in M | a \le d(p, q) \le b\}$ is homeomorphic to a direct product of $\partial B_p(b, M) = \{q \in M | d(p, q) = b\}$ and $[0, 1]$.*

The proof is similar to the proof of the following famous :

Theorem 14.3 (Morse lemma)**.** *We assume that $f : M \to \mathbb{R}$ is differentiable, and arbitrary q with $f(q) \in [a, b]$ is a regular point of f, and that $f^{-1}([a, b])$ is compact. Then, $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(\{a\}) \times [0, 1]$.*

The proof of Morse lemma uses an integral curve of grad f. (See [104].) Since d_p is not differentiable, the vector field grad d_p does not make sense. Instead, we will use the vector field V construted below.

[24] We remark that d_p may not satisfy this condition in general.

For $q \in B_p(b, M) \setminus B_p(a, M)$ let $V_q = V$ be the vector $\in T_q M$ as in Definition 14.1. If we can take V_q depending smoothly on q, then we can take the vector field $V(q) = V_q$ in place of $-\operatorname{grad} f$. (The condition in Definition 14.1 implies that d_p decreases along the integral curve of V.)

To find V_q depending smoothly on q, we proceed as follows. We first take $\tilde{V}_q$ which may not depend smoothly on q. We extend it to its neighborhood and denote it by the same symbol $\tilde{V}_q$. Then if q' is in a small neighborhod $U(q)$ of q, then the vector $\tilde{V}_q(q') \in T_{q'}M$ satisfies the condition of Definition 14.1. We cover $B_p(b, M) \setminus B_p(a, M)$ by finitely many of $U(q_i)$'s. We then take a partition of unity χ_i and put

$$V(q) = \sum \chi_i(q) \tilde{V}_{q_i}(q).$$

It is easy to see that this V has required properties.

Using this vector field V the proof of Propositioin 14.2 goes in the same way as the proof of Morse lemma. $\square$

To apply Morse theory of d_p to the proof of Theorem 2.5 we need the following lemma.

Lemma 14.4. *We assume that M satisfies the assumption of Theorem* 2.5*. Let $p, q \in M$ with $d(p, q) = Diam(M)$, and $x \in M$ be a point different from p, q. Let $\ell_p : [0, d(p, x)] \to M$, $\ell_q : [0, d(q, x)] \to M$ be minimal geodesics joining x to p, and x to q, respectively. (In case there are several of them, we assume any of them have the property below.)*

Then the angle between two tangent vectors $\frac{d\ell_p}{dt}(0)$ and $\frac{d\ell_q}{dt}(0) \in T_x M$ is greater than $\pi/2$.

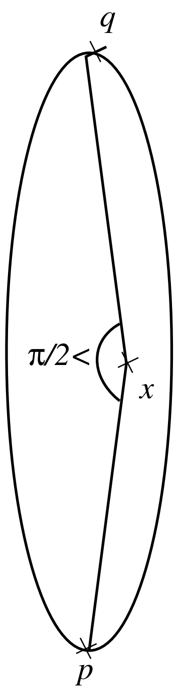

Figure 14.2

The proof of the Lemma 14.4 uses Toponogov's comparison theorem (Theorem 4.7). Under the assumption of Lemma 14.4 (that is $K_M \ge 1/4$) Theorem 4.7 implies the following Sublemma 14.5. Let $x, y, z \in M$. We conside the geodesic triangle whose vertices are those three points. We denote the lengh of its edges by $|xy|$ etc. and angles by $\angle xyz$ etc. We put $X = |yz|$, $Y = |zx|$, $Z = |xy|$.

Sublemma 14.5. *If $\angle zxy \le \pi/2$, then* $\cos\frac{X}{2} \ge \cos\frac{Y}{2}\cos\frac{Z}{2}$.

Note we have

$$s(Y/2, Z/2, \theta, 1) \le s(Y/2, Z/2, \pi/2, 1) = \cos^{-1}(\cos Y/2 \cos Z/2),$$

where $s(\cdot,\cdot,\cdot,\cdot)$ is as in Theorem 4.7.

We start the proof of Lemma 14.4. We put $|\ell_p| = t$, $|\ell_q| = s$, $d(p,q) = D$. Since d_p attains its maximum at q it follows that q is not a regular point of d_p. Hence there exists a geodesic ℓ joining p and q such that the angle between ℓ and ℓ_q is not greater than $\pi/2$. We applly Sublemma 14.5 to the geodesic triangle consisted of ℓ, ℓ_p, ℓ_q and obtain

$$\cos\frac{t}{2} \ge \cos\frac{s}{2}\cos\frac{D}{2}. \tag{14.1}$$

Since $D/2 > \pi/2$ we have $\cos\frac{D}{2} < 0$. Therefore one of $\cos\frac{s}{2}$, $\cos\frac{t}{2}$ is positive. We may assume $\cos\frac{s}{2} > 0$.

If the angle between ℓ_p and ℓ_q is not greater than $\pi/2$, Then we can again apply Sublemma 14.5 and obtain :

$$\cos\frac{D}{2} \geq \cos\frac{s}{2}\cos\frac{t}{2}. \tag{14.2}$$

Since $\cos\frac{s}{2} > 0$, (14.1), (14.2) implies

$$\cos\frac{D}{2} \geq \cos^2\frac{s}{2}\cos\frac{D}{2}.$$

We remark $0 < D/2, s < \pi$[25]. This is then a contradiction. □

Now Lemma 14.4 implies that if $x \neq p, q$ then x is a regular point of d_p, d_q. In fact, let V be the tangent vector of ℓ_q at x. It follows from Lemma 14.4 that the vector field V satisfies the condition in Definition 14.1. Namely x is a regular point of d_p. Now we can use Proposition 14.2 to prove that M is homeomorphic to sphere. Namely we proved Theorem 2.5. □

We remark that we proved Proposition 4.4 during the proof of Lemma 14.4. In fact, we proved $\cos t/2 > 0$ or $\cos s/2 > 0$ there. It implies $t < \pi$ or $s < \pi$. □

The method we explained above is very useful to study Riemannian manifolds under the bounds of sectional cuvature from below. It is also useful to study Alexandrov space (see §17, 18.)

Theorem 2.5 is a sphere theorem. There are several finiteness theorems corresponding to it. The first one is the following, which is called Gromov's Betti number estimate.

Theorem 14.6. (Gromov [71]) *There exists $C(n)$ such that if an n dimensional compact Riemanian manifold M satisfies $K_M \geq -\kappa(\kappa \geq 0)$ and if its diameter is D then*

$$\sum_k \operatorname{rank} H_k(M;F) \leq C(n)^{1+\kappa D}.$$

Here F is an arbitrary field.

Note in the case when $\kappa = 0$ the right hand side is independen of D.

It follows from Theorem 14.6 that connected sum of sufficiently many copies of $\mathbb{C}P^2$ does not carry a metric of nonnegative sectional curvature.

The proof of Theorem 14.6 is based on Morse theory of a kind of distance function. Namely we use an idea similar to Morse inequality to estimate the Betti number in terms of the number of critical points. However the proof is more involved since Morse theory of distance function itself does not work. The actual proof requires more complicated argument, which we omit here.

[25] This is a consequence of Myers' theorem (Theorem 5.4).

There are many other applications of Morse theory of distance function to metric Riemannian geometry. For example Gromov used it to show that complete manifold M such that $0 > -a^2 \ge K_M \ge -b^2$ and of finite volume is diffeomorphic to an interior of compact manifold with boundary ([68]).

Let us add a few more remarks to Theorem 2.5. If we assume $1 \ge K_M \ge 1-\epsilon$ in addition in Theorem 2.2 then we can show that M is not only diffeomorphic but is also close to a sphere as a Riemannian manifold. Namely if a sequence of n dimensional simply connected Riemannian manifolds M_i satisfies $1 \ge K_{M_i} \ge 1 - 1/i$ then M_i converges to S^n with standard metric with respect to the Gromov-Hausdorff distance.

On the contrary, corresponding statement in the stuation of Theorem 2.5 does not hold. Namely let us consider a sequence of Riemannian manifolds M_i such that $K_{M_i} \ge 1$ and that the diameter of M_i converges to π as i goes to infinity. Then Theorem 2.5 implies that M_i is homeomorphic to a sphere. However it is *not* true that the limit of M_i with respect to the Gromov-Hausdorff distance is isometric to the sphere with standard metric. We remark however a Riemannian *manifold* with diameter $= \pi$ and $K_M \ge 1$ (actually weaker assumption Ricci $\ge n-1$ is enough) is isometric to the sphere. (Theorem 21.11 [148, 41].)

In fact let us consider the quotient of S^2 by the action of $S^2/\mathbb{Z}_p$ generated by the rotation of angle $2\pi/p$ around the fixed axis. The quotient is a Riemannian manifold with constant curvature 1 except two points where the axis intersects with S^2. We approximate the quotient space by a Riemannian manifolds with curvarure ≥ 1 and obtain a sequence of Riemannian manifolds M_i whose diameter converges to π and $K_{M_i} \ge 1$. The limit is $S^2/\mathbb{Z}_p$ and is not isometric to the sphere with standard metric. The essential point here is that the Alexandrov space X with $= \pi$ and $K_M \ge 1$ is not necessary isometric to a sphere with standard metric. (Compare Theorem 23.11.)

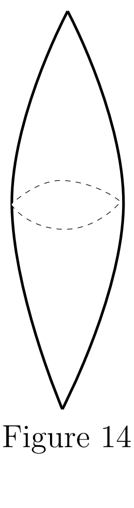

Figure 14.3

This is related to the fact that the limit of Riemannian manifolds M_i with $K_{M_i} \geq$ const is rather different from a Riemannian manifolds even in the case when the limit has the same dimension. For example, we consider a boundary S of a convex set in $\mathbb{R}^3$. There is a point of S where it has no tangent plane. In the situation when the absolute value of the sectional curvature is bounded, the Gromov-Hausdorff convergence is equivalent to the $C^{1,\alpha}$ convergence of the metric tensor (in the situation when the limit has the same dimension), by Theorem 3.4. Therefore the limit space has a tangent space everywhere.

By the reason we explained above the following question is yet open.

Problem 14.1. Are there any $\epsilon_n > 0$ such that if M is an n dimensional complete Riemannian manifold with $K_M \geq 1$ and $Diam(M) \geq \pi - \epsilon_n$, then M is *diffeomorphic* to a sphere ?

We remark that in the proof of Theorem 2.5 we consider distance function d_p, d_q simitaniously where p, q lies in the different side from x. This is similar to the notion strainer used in Alexandrov space. (See §17.)

15. Finiteness theorem by Morse theory

In this section, we explain idea of the proof of Theorem 3.5. The first half of it, which was proved in [78], asserts that the number of homogopy classes represented by an element of $\mathfrak{M}'_n(D,v)$ is finite. (We recall that $M \in \mathfrak{M}'_n(D,v)$ if $K_M \geq 1$, $Diam(M) \leq D$, and $\mathrm{Vol}(M) \geq v$, $\dim M = n$.) In this section we mainly explain this part. The key of the proof is the following proposition.

Proposition 15.1. *There exists $\epsilon = \epsilon(n,D,v) > 0$ such that the following holds for each $M \in \mathfrak{M}'_n(D,v)$. Let $p,q \in M$ with $d(p,q) < \epsilon$, $p \neq q$. Then q is a regular point of d_p.*

Moreover we have the following. We put $\Delta = \{(x,,x) \in M \times M | x \in M\}$, $\Delta(\epsilon) = \{(x,,y) | d(x,y) < \epsilon\}$. Then Δ is a deformation retract of $\Delta(\epsilon)$. The deformation retraction $H : \Delta(\epsilon) \times [0,1] \to \Delta(\epsilon)$ can be chosen so that the length of the curve $t \mapsto H(p,q,t)$ is not greater than $Cd(p,q)$. Here C depends only on n,D,v.

Using Proposition 15.1, the proof of Theorem 3.5 goes in a way similar to the proof of Proposition 5.5. Namely, from the first half of the Proposition 15.1, we find that the metric balls $B_p(\epsilon,M)$ of radius ϵ is contractible in M. On the other hand, the number of the metric balls $B_p(\epsilon,M)$ we need to cover M is estimated in the same way as §5 by using Proposition 5.2. However since it is not clear whether the intersection of finitely many metric balls $B_p(\epsilon,M)$ is contractible or not in our case, so we need to modify the proof of Proposition 5.5 a bit. The second half of Proposition 15.1 is used for this purpose. We omit this part of the proof. □

The proof of Proposition 15.1 is closely related to the proof of Proposition 5.6. So we first sketch the proof of Proposition 5.6. By Theorem 4.9 we only need to estimate the length ϵ of closed geodesic of minimal lengh from below for $M \in \mathfrak{M}_n(D, v)$. Let $\ell : S^1 \to M$ be the closed geodesic of length ϵ. We take an arbitrary point $x \in M$, and let $\ell(t) \in \ell(S^1)$ be the point of smallest distance from x. Then ℓ is orthogonal to $\overline{x\ell(t)}$ at $\ell(t)$[26]. (Here $\overline{x\ell(t)}$ is a minimal geodesic joining x and $\ell(t)$.) We put $\ell(0) = p$. Since $d(p, \ell(t)) \le \epsilon$, it follows that if $d(x, p)$ is sufficiently larger than ϵ, then the angle betweem ℓ and $\overline{xp}$ is close to $\pi/2$. We thus have proved the following Lemma 15.2.

Lemma 15.2. *Let $\delta, \rho > 0$. Then there exists ϵ depending only on n, D, v, δ, such that if ℓ is a closed geodesic with length $< \epsilon$ and if $\ell(0) = p$ then M is contained in the image of the exponential map of the domain $\subset T_pM$ in the figure below.*

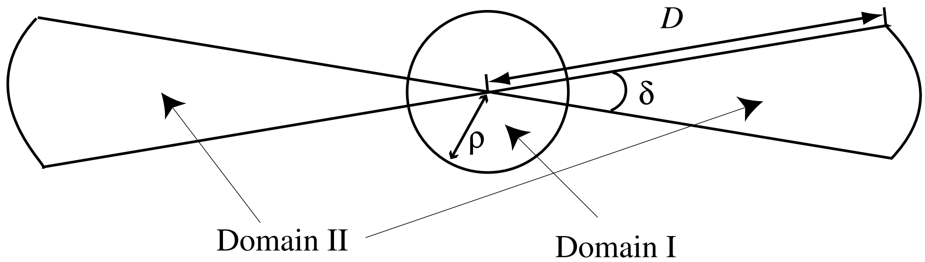

Figure 15.1

We can choose δ sufficiently small compared to D the diameter, so that the volume of the image of the domain II in the figure is smaller than $v/2$. By choosing ρ small we may assume the volume of the image of the domain I in the figure is smaller than $v/2$ also. Therefore if there exists a closed geodesic of length $< \epsilon$ then the volume of M is smaller than v. $\square$

We turn to the proof of Proposition 15.1. It suffices to show the following Lemma 15.3.

Lemma 15.3. *There exist $\theta = \theta(n, v, D) > 0$ and $\epsilon = \epsilon(n, v, D) > 0$ with the following properties. Let $M \in \mathfrak{M}'_n(D, v)$, $p, q \in M$, $d(p, q) < \epsilon$. Let ℓ_1 and ℓ_2 are minimal geodesics joining p and q. Then the angle between ℓ_1 and ℓ_2 at p or q is smaller than $\pi - \theta$.*

Lemma 15.3 implies that q in the lemma is a regular point of d_p. The first half of Proposition 15.1 follows from Proposition 14.2. The

[26]More precisely in case x is a cut point with respect to the geodesic ℓ (The notion of the cut point with respect to a submanifold is defined in a similar way to the notion of cut point from a point. See for example [33].) ℓ may not be orthogonal to $\overline{x\ell(t)}$. However this does not cause a trouble for the proof of Lemma 15.2 since the measure of the set of cut points is zero.

second half can also be proved in the same way by examining the proof of Proposition 14.2 carefully. □

The proof of Lemma 15.3 is similar to the proof of Proposition 5.6. Namely we replace Figure 15.1 by the following Figure 15.2.

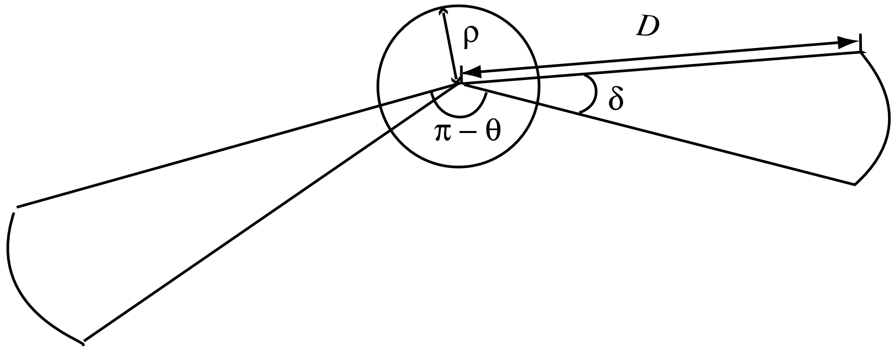

Figure 15.2

We thus explained an outline of the first half of the proof of Theorem 3.5. The other half that is the finiteness of the number of homeomorphism classes requires another deep argument. The main new technique required is the idea from controlled surgery. □

16. Soul theorem and splitting theorem

Typical results on nonocomact complete Riemannian manifolds of nonnegative curvature are soul theorem and splitting theorem. They also are very useful to study local structure of the Gromov-Hausdorff limit of Riemannian manifolds or its limit.

We first explain why the study of noncompact manifolds is useful to study local structure of the limit space. Let us begin with introducing some notations. Let X be a metric space and $\ell : [a, b] \to X$ be a continuous map (that is a curve). The *lenght* $|\ell|$ of ℓ is by definition a supremum of the sum

$$\sum d(\ell(t_i), \ell(t_{i+1}))$$

where $a = t_0 < t_1 < \cdots t_N = b$ runs over all partition, (N moves also).

Definition 16.1. We say that X is a *length space* if for each $p, q \in X$ there exists a curve joining p, q and of length $d(p, q)$.

Complete Riemannian manifold is a length space. Gromov-Hausdorff limit of length spaces is also a length space.

Definition 16.2. A complete metric space is said to be *compactly generated* if all of its metric balls are compact.

The set of all isometry classes of compact metric spaces is complete with respect to the Gromov-Hausdorff distance. A natural metric to put on the set of all isometry classes of complete compactly generated spaces is pointed Gromov-Hausdorff distance, which we define below.

Definition 16.3. Let X, Y be metric spaces and $x \in X$, $y \in Y$. We say that the *pointed Gromov-Haudsorff distance* $d_{pGH}((X,x),(Y,y))$ between (X,x) and (Y,y) is not greater than ϵ, if the Gromov-Hausdorff distance between the metric balls $B_{1/\epsilon}(x,X)$ and $B_{1/\epsilon}(y,Y)$ is not greater than ϵ. We write $\lim^{pGH}_{i\to\infty}(X_i,x_i) = (X,x)$ if $\lim_{i\to\infty} d_{pGH}((X_i,x_i),(X,x)) = 0$.

The following can be proved in the same way as Theorem 3.2 .

Theorem 16.1. *The set of all isometry classes of a pair (M,p) of an n dimensional Riemanniam manifold M with $Ricci_M \ge -(n-1)$ and a point p on it is relatively compact with respect to the pointed Gromov-Haudsorff distance.*

Now we can define the tangent cone. Let (X,d_X) be a length space and $x \in X$.

Definition 16.4. If the limit $\lim^{pGH}_{c\to\infty}((X,cd_X),x)$ exists, we call it the *tangent cone* (at $x \in X$) and write it as T_xX.

If X is an n dimensional Riemannian manifold then the tangent cone of X is isometric to $\mathbb{R}^n$ at each point.

Example 16.1. Let $\Omega \subset \mathbb{R}^n$ be a compact convex set. We put $X = \partial\Omega$ and define a length metric on it. (Namely the distance between $x, y \in X$ is the infinimum of the length of all curves joining x and y in X.)

Then tangent cone T_xX is described as follows. We consider all the ray (half of the strainght line) $\ell : [0,\infty) \to \mathbb{R}^n$ such that $\ell(t) \in \Omega$ for small $t > 0$. The set of such ℓ is an open subset of $\mathbb{R}^n$. Its boundary in $\mathbb{R}^n$ is the tangent cone T_xX.

If the space X is not so wild then we may expect the tangent cone T_xX exists and a neighborhood of x in X is homeomorphic to a neighborhood of the origin (base point) in T_xX. (This holds for Alexandrov space for example. See Theorem 18.1.) Namely we can study local structure of X by studying the tangent cone T_xX.

If X is a Gromov-Hausdorff limit of a sequence of Riemannian manifolds M_i and if the sectional curvature of M_i is bounded from below by a constant independent of i, then we may regard the limit X as the space with "curvature bounded from below". Then the infinimum of the "curvature" of family of length spaces (X, cd_X) as c goes to infinity will become nonnegative. Note if we multiply metric by c then the curvature is multiplied by c^{-2}.) It means that if tangent cone of X exists then it is of "nonnegative curvature". (The discussion here is informal and heuristic. So for a moment the curvature may either Ricci or sectional curvature.) This is one of the reasons why the study of the noncompact space with nonnegative curvature is important in the local theory of space which is a limit of Riemannian manifolds.

By using Gromov's precompactness theorem (Theorem 16.1) we have the following :

Proposition 16.2. *Let M_i be a sequence of Riemannian manifolds with $Ricci_{M_i} > -(n-1)$. Let $X = \lim^{GH}_{i\to\infty} M_i$. Let $x \in X$ and c_k be a sequence of positive numbers with $\lim c_k = +\infty$. Then there exists a subsequence of $((X, c_k d_X), x)$ which converges in pointed Gromov-Hausdorff distance.*

In general $((X, c_k d_X), x)$ itself may not converge. (Namely we need to take a subsequence.) Hence X may not have a tangent cone. This is one of the difficulties to study family of Riemannian manifolds with Ricci curvature bounded from below.

In case when X is a limit of Riemannian manifolds with *sectional* curvature bounded from below (or more generaly if X is an Alexandrov space), $\lim_{c\to\infty}((X, cd_X), x)$ converges without taking subsequence (Theorem 17.14).

Let us now state soul theorem and splitting theorem. We first define line and ray. Let X be a length space. A curve $\ell : (a, b) \to X$ is called a geodesic if is length minimizing locally. Namely ℓ is a geodesic if, for each $t \in (a, b)$, there exists ϵ such that $d(\ell(t-\epsilon), \ell(t+\epsilon))$ is equal to the length of the restriction of ℓ to $(t-\epsilon, t+\epsilon)$. We use arc length as a parameter in the next definition.

Definition 16.5. Let X be a length space. A geodesic $\ell : [0, \infty) \to X$ is called a *ray* if $d(\ell(t), \ell(s)) = |t - s|$ for any t, s. A geodesic $\ell : (-\infty, \infty) \to X$ is called a *line* if $d(\ell(t), \ell(s)) = |t - s|$ for any t, s.

(The difference between line and ray is the domain of its definition.)

If there exists a tangent cone $T_x X = \lim_{c\to\infty}((X, cd_X), x)$ then it is a union of its rays ℓ such that $\ell(0)$ is the base point. We also have the following :

Lemma 16.3. *Let X be a length space and $\ell : (-\epsilon, \epsilon) \to X$ be a minimal geodesic with $\ell(0) = x$. If the tangent cone $T_x X$ exists then it contains a line.*

In fact, since in (X, cd_X) there exists a minimal geodesic of length $c\epsilon$ containing the origin, its limit in $T_x X$ will be a line.

We assume that a complete metric space X is a length space and satisfies one of the following conditions.

Condition 16.1.
(a) X is a Riemannian manifold of nonnegative sectional curvature.
(b) $X = \lim^{pGH}_{i\to\infty} M_i$ such that $K_{M_i} \ge -\epsilon_i$, $\lim_{i\to\infty} \epsilon_i = 0$ and $\dim X = \dim M_i$.
(c) X is a Riemannian manifold with nonnegative Ricci curvature.
(d) $X = \lim^{pGH}_{i\to\infty} M_i$ such that $Ricci_{M_i} \ge -\epsilon_i$, $\lim_{i\to\infty} \epsilon_i = 0$ and $\mathrm{Vol}(M_i) \ge v > 0$.

The next theorem is called the splitting theorem.

Theorem 16.4. *If X satisfies one of the Conditions 16.1 and contains a ray then X is isometric to a direct product $\mathbb{R} \times X_0$.*

Theorem 16.4 is due to Toponogov [149] in case (a), to Cheeger-Gromoll [36] in case (c), Grove-Petersen [79] and Yamaguchi [154] in case (b) and Cheeger-Colding ([28]) in case (d).

We will explain an idea of the proof of the case (a),(c) later in this section. (Case (b) is similar to case (a). Case (d) is discussed in §23.)

We explain more how to apply it to study local structure of the limit space. Note that we can use Theorem 16.4 repeatedly. Namely if X_0 contains a line then we can again apply theorem and show that it is a direct product. Therefore if we can repeat it $\dim X$ times, then we can prove that $X = \mathbb{R}^n$. Lemma 16.3 implies that if x is an interior point of a minimal geodesic then T_xX contains a line. Therefore if we can find $n(= \dim X)$ "independent" geodesic for which x is an interior point, then the tangent cone T_xX is isometric to $\mathbb{R}^n$. This may imply that X is a manifold in a neighborhood of x. This argument appears in §17,18 and in §20,22,23.

We next explain an outline of the proof of splitting theorem. The main tool we use is covexity of Busemann function, (it is used also in the proof of soul theorem). Let X be a length space and $\ell : [0, \infty) \to X$ be its ray.

Definition 16.6. The *Busemann function* is the limit $b_\ell(x) = \lim_{t\to\infty}(t - d(x, \ell(t)))$.

Proposition 16.5. *If X satisfies either (a) or (b), then Busemann function of its ray ℓ is convex.*

If X satisfies (c), then Busemann function of its ray is subharmonic.

In the situation (d) we can not define subharmonicity in the usual way. So the argument is more involved. See §23 and [26, 31].

The proof of Proposition 16.5 is by comparison theorem. Namely it follows immediately from the Laplacian and Hessian comparison theorem (Theorem 16.6) for distance function. We remark that Hessian $\mathrm{Hess} f$ of a function f on a Riemannian manifold is defined by

$$(\mathrm{Hess}_x f)(V, W) = V(W(f)) - (\nabla_V W)(f) \tag{16.1}$$

and is a symmetric bilinear map $T_xM \otimes T_xM \to \mathbb{R}$. A function f is convex if its Hessian $\mathrm{Hess} f$ is nonnegative everywhere.

Laplacian Δf is its trace. Namely

$$\Delta f(x) = \sum_{i=1}^{n} (\mathrm{Hess} f)(e_i, e_i) \tag{16.2}$$

where e_i is an orthonormal basis of T_pM. (We remark that we are using nonpositive Laplacian. Namely $\Delta = -(d^*d + dd^*)$.) We say a smooth function is subharmonic if its Laplacian is nonnegative.

Theorem 16.6. *Let M be a Riemannian manifold and $p \in M$. We consider the function $d_p(x) = d(p,x)$.*
(1) *If $K_M \ge \kappa$ then*

$$\mathrm{Hess}_x d_p \le \frac{s'_\kappa(d(p,x))}{s_\kappa(d(p,x))}(g_x - dd_p \otimes dd_p). \tag{16.3}$$

Here $dd_p : T_xM \to \mathbb{R}$ is the exterior derivative of d_p.
(2) *If $Ricci_M \ge \kappa$ then*

$$\Delta f(x) \le (n-1)\frac{s'_\kappa(d(p,x))}{s_\kappa(d(p,x))}. \tag{16.4}$$

Here s_κ is as in (4.1).

Remark 16.1. We remark that d_p is not differentiable outside the ball $B_p(i_M(p), M)$. So we we need to be more careful to state Thereom 16.6. Precisely speaking (16.4), (16.6) holds in barrier sense. See for example [26].

We omit the proof of Theorem 16.6. We remark that (16.6) implies Corollary 21.5. In fact

$$\begin{aligned}
\frac{d}{dt}\mathrm{Vol}(B_p(t,M)) &= \int_{\partial B_p(t,M)} \langle \mathrm{grad}\, d_p, \mathrm{grad}\, d_p\rangle \Omega_{\partial B_p(t,M)} \\
&= \int_{B_p(t,M)} \mathrm{div}\,\mathrm{grad} d_p \Omega_M \\
&\le (n-1)\int_{B_0(t,T_pM)} \frac{s'_\kappa(d(p,x))}{s_\kappa(d(p,x))}\Omega_{\mathbb{R}^n} \\
&\le \frac{d}{dt}\int_{B_0(t,T_pM)} s_\kappa(d(p,x))^n \Omega_{\mathbb{R}^n} \\
&\le \frac{d}{dt}\mathrm{Vol}(B_{p_0}(t,\mathbb{S}^n_\kappa)).
\end{aligned}$$

Let us explain how we use Proposition 16.5 to prove Theorem 16.4, in case (a)(b). We assume that X contains a line $\ell : \mathbb{R} \to X$. We then have two rays $\ell_\pm : [0,\infty) \to X$ by :

$$\ell_+(t) = \ell(t), \qquad \ell_-(t) = \ell(-t)$$

We study their Buseman functions $b_{\ell_\pm}$. The triangle inequality implies

$$b_{\ell_+}(t) + b_{\ell_-}(t) \le 0. \tag{16.5}$$

By Proposition 16.5 the right hand side is convex. Since bounded convex function is constant it follows that $\ell_+(t) + \ell_-(t)$ is constant. (Actually it is 0). It follows that $\ell_+(t) = \mathrm{const} - \ell_-(t)$ is convex and is concave. Hence its level surface if totally geodesic. (Here we say $S \subset M$ is totally geodesic if any minimal geodesic of M joininig two points of S are contained in S.) This implies Theorem 16.4. In case (c) we use subharmonicity in place of convexity. $\square$

We next discuss soul theorem.

Theorem 16.7 (Cheeger-Gromoll [35], [128]). *If a complete Riemannian manifold M has nonnegative sectional curvature then there exists a compact submanifold $S \subseteq M$ without boundary, such that M is diffeomorphic to a normal bundle of S. Moreover S is totally geodesic.*

We call N the *soul* of M. The basis of the proof of Theorem 16.7 is Proposition 16.5. It asserts that, for each ray $\ell : [0, \infty) \to M$, the Buseman function b_ℓ is convex. In particular for any c the closed set

$$H(\ell, c) = \{x \in M | b_\ell(x) \le c\}$$

is convex. The next lemma is the key of the proof of Theorem 16.7. We fix $p \in M$ and let $Ray(p)$ be the set of all rays of M such that $\ell(0) = p$.

Lemma 16.8. *The set* $C_c(p) = \bigcap_{\ell \in Ray(p)} H(\ell, c)$ *is compact.*

The proof is by contradiction. Namely we assume that $C_c(p)$ is not compact and let $p_i \in C_c(p)$ be a divergent sequence. We put $d(p, p_i) = t_i$, and let $\ell_i : [0, t_i] \to M$ be a minimal geodesic such that $\ell_i(0) = p$, $\ell_i(t_i) = p_i$ and that it is parametrized by arc length. Since $\frac{d\ell_i}{dt}(0) \in T_pM$ is a unit vector, we may take subsequence so that it converges. Let $\ell : [0, \infty) \to M$ be a geodesic such that

$$\lim_{i\to\infty} \frac{d\ell_i}{dt}(0) = \frac{d\ell}{dt}(0).$$

Since $\lim_{i\to\infty} t_i = \infty$, it follows that ℓ is a ray. On the other hand, we have

$$\lim_{i\to\infty} b_\ell(p_i) = \infty.$$

This contradicts to $p_i \in C_c(p)$. $\square$

Thus, we obtained a compact convex sub<u>set</u> $C_c(p)$ of M. We can find a compact covex sub<u>manifold</u> S in it. The argument to do so is rather technical and is omitted. (See [33] Chapter 8.)

Perelman [122] proved that if, in the situation of Theorem 16.7, there exists a point where $K_M > 0$, then the soul S is a one point. We refer [63] for other topics related to soul theorem.

We remark that we already applied Theorem 16.7 in §12 to construct F-structure.

17. Alexandrov space - I -

In this and the next sections, we discuss recent development [22, 119, 120] of the theory of Alexandrov space. A good text book on the contents of this section is [138]. (See also [127].) In this and the next sections, we study compactly generated length space of finite Hausdorff dimension only. So we always assume that length space has this property.

The Alexandrov space is a length space with curvature bounded from below. To define notion of curvature for length space, we use Toponogov type comparison theorem in the opposite direction. Namely we *define* the condition $K_X \ge 1$ by using the conclusion of comparison theorem. However the conclusion of Theorem 4.7 does not (yet) make sense for length space, since it uses angle. So we consider the following slightly different version.

We use the notation of Theorem 4.7. Let M be a Riemannian manifold and $x, y, z, v, w \in M$. Let $x', y', z', v', w' \in \mathbb{S}^n(\kappa)$. We assume $v \in \overline{xy}$, $w \in \overline{xz}$, $v' \in \overline{x'y'}$, $w' \in \overline{x'z'}$.

Theorem 17.1 (Alexandrov-Toponogov)**.** *We assume $K_M \ge \kappa$ and $d(x,y) = d(x',y')$, $d(x,z) = d(x',z')$, $d(y,z) = d(y',z')$, $d(x,v) = d(x',v')$, $d(x,w) = d(x',w')$. $u \in \overline{xy}$, $u' \in \overline{x'y'}$, $v \in \overline{xz}$, $v' \in \overline{x'y'}$. Then we have $d(v,w) \ge d(v',w')$.*

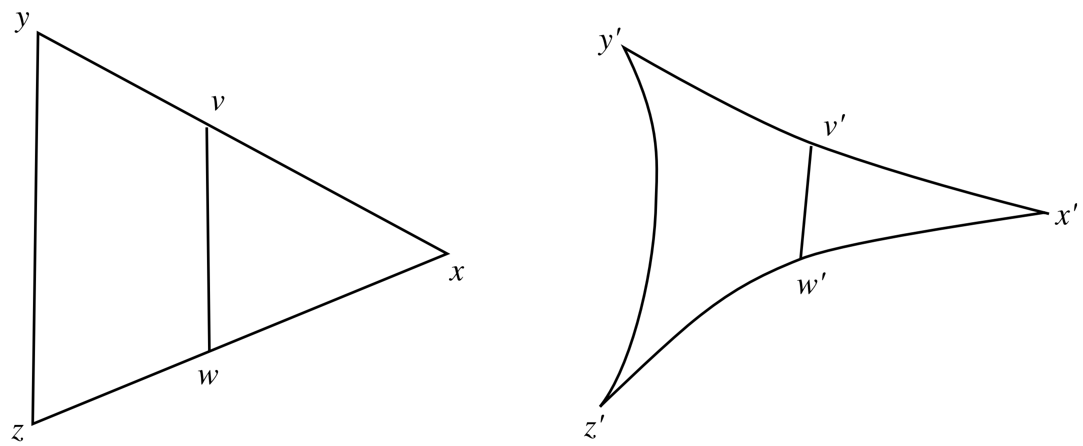

Figure 17.1

Definition 17.1 (Alexandrov)**.** A length space of finite dimension is said to be an *Alexandrov space* with $K \ge \kappa$ if the conclusion of Theorem 17.1 holds for X.

Remark 17.1. There are several other definitions equivalent to Definition 17.1. We will explain them later (Theorems 17.9 and 17.10).

Remark 17.2. There is a notion of Alexandrov space with curvature bounded from *above*. We do not discuss it in this article. It is proved by Beretovskij that if a length space is an Alexandrov space with curvature bounded from above and below then it is a C^0 Riemannian manifold. This result is related to Theorem 3.4 but was proved earier than that. See [16].

Hereafter we say Alexandrov space for Alexandrov space with $K \geq \kappa$ with some κ.

The notion of Alexandrov space was introduced by Alexandrov [6] more than 50 years ago. There are several related pioneering works around those old days, like Buseman [23]. In [22], Burago-Gromov-Perelman proved several fundamental theorems on Alexandrov spaces. After that the study of Alexandrov space becomes very active and important in metric Riemannian geometry. Their main results are :

Theorem 17.2 (Burago-Gromov-Perelman [22])**.** *Let X be an Alexandrov space. Then there exists a dense open subset X_0 such that, for each $p \in X_0$, there exists a neighborhood U_p and a Lipschitz homeomorphism $U_p \to V_p$ where $V_p \subset \mathbb{R}^n$ is an open set.*

Theorem 17.3 (Burago-Gromov-Perelman)**.** *Hausdorff dimension of Alexandrov space is an integer and is equal to its topological dimension.*

Remark 17.3. There are several ways to define topological dimension, that is covering dimension, (big and small) inductive dimension etc. Theorem 17.3 also implies that they coincide to each other for Alexandrov space.

We do not discuss the proof of Theorem 17.3. (It will follow from Corollay 18.3 in the next section.) Before explaining some of the ideas of the proof of Theorems 17.2, we give some examples of Alexandrov space.

Example 17.1. (0) Riemannian manifold (M, g) is an Alexandrov space with $K \geq \kappa$, if and only if the sectional curvature of (M, g) is greater than κ everywhere.

(1) Let $\Omega \subseteq \mathbb{R}^n$ be a compact and convex domain. Let $S = \partial\Omega$. We define the length metric d on S. Namely distance between $x, y \in S$ is the minimum of the length of the curves in S joining x with y. Then we can prove that (S, d) is an Alexandrov space of curvature ≥ 0.

(2) Let M be a Riemannian manifold with $K_M \geq \kappa$ and G be a compact group acting on M by isometry. Then the quotient space M/G equipped with quotient metric is Alexandrov space.

An important example of Alexandrov space is a Gromov-Hausdorff limit of Riemannian manifold. Actually we have

Proposition 17.4. *Let X_i be a sequence of compact length spaces and $X = \lim^{GH}_{i\to\infty} X_i$. If X_i are Alexandrov spaces with $K \geq \kappa$ then so is X. (Here κ is independent of i.)*

The proof is elementary.

Remark 17.4. Yamaguchi [154] proved that if M a C^∞ manifold and G is a compact Lie group acting smoothly on M then there exists a sequence of metrics g_i on M such that $K_{g_i} \ge \kappa$ for some κ independent and (M, g_i) converges to M/G.

Another source of examples is a cone, which we define below.

Definition 17.2. Let (Y, d) be a metric space. We consider the product $T \times [0, \infty)$ and identify $(x, 0)$ and $(y, 0)$. We thus obtain a space CY. We define a cone metric on it as follows :

$$d((x,t),(y,s)) = \sqrt{t^2 + s^2 - 2st\cos d(x,y)}.$$

We denote by $\mathbf{o} \in CY$ the equivalence class of $(x, 0)$.

Example 17.2. If $Y = S^n$ with $K_{S^n} \equiv 1$ then CS^n is isometric to $\mathbb{R}^{n+1}$.

Lemma 17.5. *If Y is a length space and $Diam(Y) \le \pi$ then CY is a length space.*

We can prove an analogue of Myers' Theorem (Theorem 5.4) for Alexandrov space. Namely :

Theorem 17.6 ([22])**.** *If M is an Alexandrov space with $K \ge 1$ then the $Diam(Y) \le \pi$.*

Theorem 17.7 ([22])**.** (1) *If CY is an Alexandrov space then Y is an Alexandrov space of $K \ge 1$.* (2) *If $\dim Y > 1$ and Y is an Alexandrov space of $K \ge 1$ then CY is an Alexandrov space of $K \ge 0$.* (3) *In case $\dim Y = 1$, the cone CY is an Alexandrov space of $K \ge 0$ if and only if $Diam(Y) \le \pi$.*

We do not discuss the proof.

We next discuss an example of length sapce which is not an Alexandrov space.

Example 17.3. Let us consider simplicial complex X consisting three arcs which are joined at one point o. We can define a metric on it such that the length of each arc is 1. Let x, y, z be interior points of each of the three simplexes, respectively. We can choose $v = w$ on $\overline{xy} \cap \overline{xz} = \overline{xo}$. Then $d(v, w) = 0$. But if we choose x', y', z', v', w' $d(v', w')$ as in Theorem 17.1 then $d(v', w') > 0$. (For any κ.) So the conclusion of Theorem 17.1 does not hold. Namely X is not an Alexandrov space.

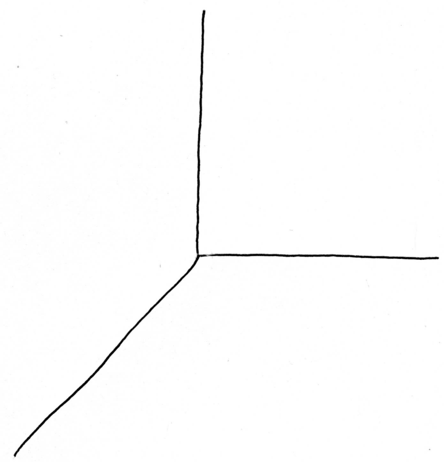

Figure 17.2

The arugument of Example 17.3 implies the following. We denote a map $\ell : (a,b) \to X$ a *geodesic* if for each $c \in (a,b)$ there exists ϵ such that the length of the restriction of ℓ to $(c-\epsilon, c+\epsilon)$ is $d(\ell(c-\epsilon), \ell(c+\epsilon))$.

Lemma 17.8. *If ℓ_1, ℓ_2 be a geodesic on an Alexandrov space X and if it coinsides on an open set, then there union is also an geodesic.*

In other words geodesic can never branch.

We next explain some other equivalent definitions of Alexandrov space.

Theorem 17.9. *Let X be a length space. We assume that for each $p \in X$ there exists a neighborhood U such that the conclusion of Theorem 17.1 holds for any $x, y, z, u, v \in U$. Then X is an Alexandrov space of $K \ge \kappa$. In other words, the same conclusion holds globally.*

In fact, usually the assumption of Theorem 17.9 is the definition of Alexandrov space.

We discuss another equivalent definition. Let X be a length space and $x, y, z \in X$. Let $\kappa \in \mathbb{R}$. In case $\kappa > 0$, we assume $d(x,y), d(x,z), d(y,z) < \pi/\sqrt{\kappa}$. We choose $x', y', z' \in \mathbb{S}^n(\kappa)$ such that $d(x,y) = d(x',y')$, $d(y,z) = d(y',z')$, $d(x,z) = d(x',z')$. We define

$$\angle_\kappa yxz = \angle y'x'z'$$

Theorem 17.10. *Let X be a length space.*

(1) *If, for each $p \in X$, there exists a neighborhood U of p such that*

$$\angle_\kappa bac + \angle_\kappa cac + \angle_\kappa cab \le 2\pi$$

for and $a, b, c, d \in U$ then X is an Alexandrov space with $K \ge \kappa$.

(2) *Let X is an Alexandrov space with $K \ge \kappa$ and $a, b, c, d \in X$. Then*

$$\angle_\kappa bac + \angle_\kappa cac + \angle_\kappa cab \le 2\pi.$$

Remark 17.5. By Theorem 17.6 $\angle_\kappa bac$ etc. in (2) is well defined.

The idea that if comparison theorem holds locally then it holds globally is due to Alexandrov and Toponogov. Theorem 17.10 is proved in [22].

We next discuss the angle between geodesics. Hereafter we assume that geodesic is parametrized by arc length. Let X be an Alexandrov space with $K \ge \kappa$ and $\ell_1, \ell_2 : [0, c) \to X$ be geodesics such that $p = \ell_1(0) = \ell_2(0)$.

Lemma 17.11. *If $s_1 \le t_1$, $s_2 \le t_2$ then*

$$\angle_\kappa \ell_1(s_1)\, p\, \ell_2(s_2) \ge \angle_\kappa \ell_1(t_1)\, p\, \ell_2(t_2).$$

This follows easily from definition. Therefore we can define :

Definition 17.3. $\angle \ell_1\ell_2 = \lim_{t_1,t_2\to 0} \angle_\kappa \ell_1(t_1)\, p\, \ell_2(t_2)$.

In case ℓ_1, ℓ_2 are minimal geodesics joining p to x, y respectively, we write $\angle xpy = \angle \ell_1\ell_2$.

Remark 17.6. (1) The angle $\angle xpy$ is independent of κ. (2) Two geodesics ℓ_1, ℓ_2 coicide to each other if $\angle \ell_1\ell_2 = 0$.

Theorem 17.12. *If X is an Alexandrov space of $K \ge \kappa$ and $x, y, z \in X$, then we have $d(y, z) \le s(d(x, y), d(x, z), \angle yxz, \kappa)$.*

Here s is defined in (4.4) In other words, Theorem 4.7 holds for Alexandrov space. The other version of triangle comparison theorem also holds.

Theorem 17.13. *If X is an Alexandrov space of $K \ge \kappa$ and $x, y, z \in X$, then we have $\angle yxz \ge \angle_\kappa yxz$.*

Remark 17.7. Toponogov type comparison theorem holds in Alexandrov space. Hence we can generalize the argument of the last section to prove splitting theorem (Theorem 16.4) for Alexandrov space with $K \ge 0$.

As we mentioned before an Alexandrov space has tangent cone.

Theorem 17.14 ([22])**.** *If (X, d) is an Alexandrov space with $K \ge \kappa$ and $x \in X$ then $\lim_{k\to\infty}((X, kd), x)$ converges with respect to the pointed Gromov-Hausdorff distance.*

The limit in Theorem 17.14 is the tangenet cone T_xX. Tangent cone is related to the angle between geodesics as follows.

Definition 17.4. Let $\tilde{\Sigma}^0_x$ be the set of all geodesics (parametrized by arc length) $\ell : [0, c) \to X$ for some c such that $\ell(0) = x$. We identify ℓ_1 and ℓ_2 if they coincides to each other at a neighborhood of 0. We denote by Σ^0_x the set of this equivalence relation. We can easily show that the angle $\angle$ defines a mertic on it. We define the *space of directions* $\Sigma_x(X)$ as the completion of Σ^0_x.

Lemma 17.15. *If X is an Alexandrov space, then $\Sigma_x(X)$ is an Alexandrov space with $K \geq 1$ and T_xX is an Alexandrov space with $K \geq 0$.*

Theorem 17.16 ([22])**.** *The tangent cone T_xX of an Alexandrov space X is isometric to the cone $C\Sigma_x(X)$. If $\dim X = n$ then $\dim \Sigma_x(X) = n - 1$ and $\dim T_xX = n$.*

We remark that the second half of Theorem 17.16 is a consequence of Proposition 17.7.

Now we start the discussion of the proof of Theorem 17.1. As we mentioned in the last section, if $x \in X$ is an interior point of $n = \dim X$ "independent" minimal geodesics then T_xX is isometric to $\mathbb{R}^n$, and this may imply x has neighborhood homeomorphic to $\mathbb{R}^n$. However the condition about the existance of geodesic is a bit too strict. So we relax it a bit. This seems to be an idea of strainer. Let X be an Alexandrov space with $K \geq \kappa$.

Definition 17.5. Let $x \in X$ and $(a_i, b_i) \in X^2$, $i = 1, \cdots, n$. We say that $\{(a_i, b_i)\}_{i=1,2,\cdots,n}$ is a (n, δ) *strainer* at x, if

$$\angle_\kappa a_i x b_i \geq \pi - \delta,$$

and

$$\angle_\kappa a_i x a_j, \angle_\kappa a_i x b_j, \angle_\kappa b_i x b_j \leq \delta, \qquad \text{for } i \neq j.$$

A point $x \in X$ is said to be (n, δ) *strained* if there exists an (n, δ) strainer at x.

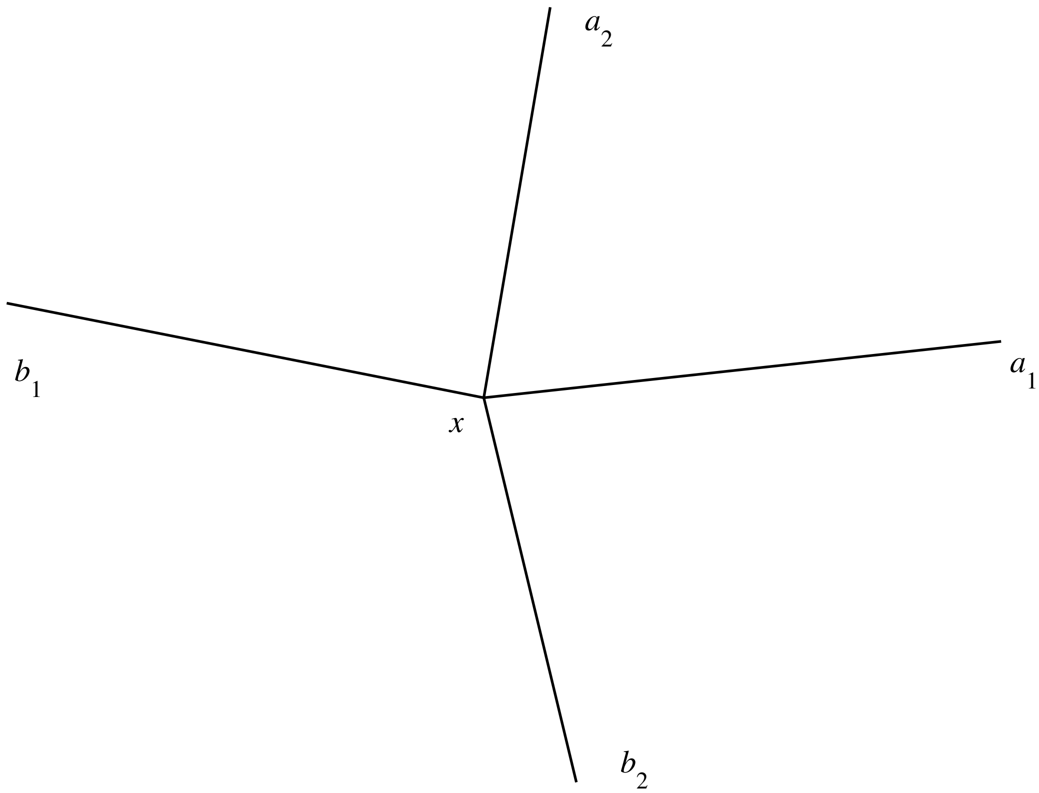

Figure 17.3

Remark 17.8. In [22] the strainer here is called "explosion" and strained point is called "brust point". The name strainer and strained point seems to be more popular now.

The main step of the proof of Theorem 17.2 is the following :

Proposition 17.17 ([22] Theorem 9.4 or [138] Theorem 7.4)**.** *If $p \in X$ is an (n, δ) strained point, then there exists $\rho > 0$, neighborhoods $U \subset V$ of p, and a map $\varphi : V \to \mathbb{R}^n$ with $\varphi(p) = 0$ with the following properties.*

(1) $d(\varphi(x), \varphi(y)) < 2d(x, y)$,
(2) *Let $x \in U$ and $X \in \mathbb{R}^n$, with $d(\varphi(x), X) < \rho$. Then there exists $y \in V$ such that $\varphi(y) = X$ and $d(x, y) \leq Cd(\varphi(x), X)$. Where C depends only on n and δ.*

We remark that (2) implies that φ is an open mapping in the neighborhood of x. Hence if φ is injective then φ gives a chart in a neighborhood of x. We can use the following to show φ is injective.

Lemma 17.18. *We may choose U small enough so that if φ is not injective then there exists an $(n + 1, 10\delta)$ strained point on a small neighborhood of U.*

We remark that the set of all the (n, δ) strained points is open. On the other hand Proposition 17.17 implies that if (n, δ) strained point exists then the Hausdorff dimension is not smaller than n.

Hence Proposition 17.17 and Lemma 17.18 imply the following. For each open set U, we can find n and a non empty open subset $U_0 \subset U$ consisting of (n,δ) strained points such that there are no $(n+1,10\delta)$ strained points on U. Then U_0 is an n dimensional manifold by Proposition 17.17 and Lemma 17.18. The proof of Theorem 17.2 then will be completed by using the next lemma.

Lemma 17.19 ([22] Corollary 6.5)**.** *We assume X is connected. If U, V are nonempty open subsets of X then Hausdorff dimension of U is equal to the Hausdorff dimension of V.*

We now sketch the proof of Proposition 17.17 and Lemmas 17.18, 17.19. We put

$$\mu = \inf\{d(p,a_1),\cdots,d(p,a_n),d(p,b_1),\cdots,d(p,b_n)\}.$$

We first explain the idea of the proof of Proposition 17.17. We put

$$\varphi(x) = -(d(x,a_1),\cdots,d(x,a_n)) + (d(p,a_1),\cdots,d(p,a_n)).$$

It is easy to see that (1) is satisfied. We show (2). For simplicity we consider the case $x = p$, $n = 2$. For each $X = (X_1,X_2) \in B_0(\rho,\mathbb{R}^2)$, we will find w with $\varphi(w) = (X_1,X_2)$, $d(p,w) \le Cd(0,X)$. We assume $X_1, X_2 > 0$. We first take the point $q_1 \in \overline{pa_1}$ such that $d(p,q_1) = X_1$. We first show :

$$\frac{|\varphi(q_1) - (X_1,0)|}{d(0,X)} \le \tau(\rho,\delta|n,\kappa,\mu) \tag{17.1}$$

In fact we can prove :

$$d(q_1,a_2) \ge d(p,a_2) - \tau(\rho,\delta|n,\kappa,\mu)X_1$$

by applying Theorem 17.1, where we put $x = a_1$ $y = p$, $z = v = a_2$, $u = q_1$.

To prove the oppsite inequality we take point $p' \in \overline{b_1q_1}$ such that $d(p',q_1) = X_1$. We have $d(p,p') \le X_1\tau(\rho,\delta|n,\kappa,\mu)$. In fact, since $\angle_\kappa b_1pa_1 > \pi-\delta$, it follows that $\angle b_1q_1a_1 \ge \angle_\kappa b_1q_1a_1 > \pi-\delta-\tau(\rho|n,\kappa)$. Hence $\angle pq_1b_1 < \delta + \tau(\rho|n,\kappa)$. Theorem 17.12 then implies $d(p,p') \le X_1\tau(\rho,\delta|n,\kappa,\mu)$.

We use $d(p,p') \le X_1\tau(\rho,\delta|n,\kappa,\mu)$ to show :

$$|\angle b_1q_1a_2 - \pi/2|, |\angle b_1p'a_2 - \pi/2| < \tau(\rho,\delta|n,\kappa,\mu). \tag{17.2}$$

We next apply Theorem 17.1 again by putting $x = q_1$ $y = b_1$, $z = v = a_2$, $u = p'$. Then using (17.2) have $d(p',a_2) \ge d(q_1,a_2) - \tau(\rho,\delta|n,\kappa,\mu)X_1$. Hence $d(p,a_2) \ge d(q_1,a_2) - \tau(\rho,\delta|n,\kappa,\mu)X_1$. We have proved (17.1).

We next take $w_1 \in \overline{a_2q_1}$ such that $d(w_1,q_1) = X_2$. Then we have

$$\frac{|\varphi(w_1) - (X_1,X_2)|}{d(0,X)} \le \tau(\rho,\delta|n,\kappa,\mu)$$

We repeat the process replacing p by w_1 and obtain w_2 such that $d(w_1,w_2) < C|\varphi(w_1) - (X_1,X_2)|$ and

$$\frac{|\varphi(w_2) - (X_1,X_2)|}{|\varphi(w_1) - (X_1,X_2)|} \le \tau(\rho,\delta|n,\kappa,\mu).$$

We can define $w_3,\cdots$ in a similar way. w_i is a Cauchy sequence whose limit w has the required property. $\square$

Let us prove Lemma 17.18. Let $\varphi(x) = \varphi(y)$. Let $z \in \overline{xy}$ with $d(x,z) = d(x,y)$. It is easy to see that (a_i,b_i) $i = 1,\cdots,n$ and (x,y) is an $(n+1,2\delta)$ strainer if $d(x,y)$ is small. $\square$

Finally we sketch the proof of Lemma 17.19. We may assume X is compact. Take $p \in V$ and put $D = \sup\{d(p,x)|x \in U\}$. We take R such that $B_p(D/R,X) \subset V$. We define $\Phi : U \to V$ as follows. For $x \in V$ we take a point $\Phi(x) \in \overline{px}$ such that $Rd(p,\Phi(x)) = d(p,x)$. (Note the minimal geodesic $\overline{px}$ may not be unique. So we need some technical argument to find Φ which is measurable.) Definition 17.1 implies that there exists $\rho > 0$ such that $d(\Phi(x),\Phi(y)) \ge \rho d(x,y)$. It follows that the Hausdorff dimension of $\Phi(U)$ is not smaller than the Hausdorff dimension of U. Therefore the Hausdorff dimension of V is not smaller than the Hausdorff dimension of U. We can prove the opposite inequality in the same way. $\square$

We thus finished a sketch of the proof of Theorem 17.2. $\square$

Definition 17.6. We define the *boundary* ∂X of an Alexandrov space X by induction of $\dim X$ as follows. If $\dim X = 1$ then X is either an arc or a circle. So we can define its boundary in an obvious way. Suppose ∂X is defined for X with $\dim X < k$. Let X be an Alexandrov space of $\dim X = k$. Then we say $x \in \partial X$ if $\partial\Sigma_x(X) \neq \emptyset$. (We remark that $\Sigma_x(X)$ is an Alexandrov space and $\dim \Sigma_x(X) = k-1$.)

Theorem 17.2 is improved by Otsu-Shioya [112]. To state their results we define notion of singular point set in Alaxandrov space more precisely.

Definition 17.7. Let X be an n dimensional Alexandrov space and $\delta > 0$. We put

$$\mathcal{S}_\delta(X) = \{x \in X|\mathrm{Vol}(\Sigma_x(X)) \le \mathrm{Vol}(S^{n-1}) - \delta\}.$$

$$\mathcal{S}(X) = \bigcup_{\delta>0} \mathcal{S}_\delta.$$

We remark that the Alexandrov space version of the following theorem is a motivation of Defintion 17.7.

Theorem 17.20 (Otsu-Shiohama-Yamaguchi [111])**.** *If an M dimensional Riemanian manifold M satisfies* $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \epsilon_n$*,* $K_M \ge 1$

then M is diffeomorpic to the sphere. Also M is close to S^n with respect to the Hausdorff distance[27]*.*

We discuss the idea of the proof of Theorem 17.20 in §21.

Theorem 17.21 (Burago-Gromov-Perelman, Otsu-Shioya)**.** *Let X be an Alexandrov space of dimension n. Then the Hausdorff dimension of $\mathcal{S}(X)$ is not greater than $n-1$. The Hausdorff dimension of $\mathcal{S}(X)\setminus\partial X$ is not greater than $n-2$.*

Theorem 17.22 (Otsu-Shioya[111])**.** *There exists a C^0 Riemannian metric on $X \setminus \mathcal{S}(X)$ which induces the metric on X. Moreover there exists $X_0 \subset X\setminus\mathcal{S}(X)$ such that the (n dimensional Hausdorff) measure of $X\setminus X_0$ is 0 and that there exists manifold structure of $C^{1.5}$ class and a Riemannain structure is of $C^{0.5}$ class on X_0.*

Remark 17.9*.* Actually we need to define $C^{1.5}$ structure etc. in the above theorem. This is because $X \setminus \mathcal{S}(X)$, X_0 are not open subset in general. Hence they are not manifold. See [112] for the precise statement.

Theorem 17.22 is used by Kuwae-Machigashira-Shioya [99] to develop analysis on Alexandrov space.
We also remark the following :

Theorem 17.23 (Fukaya-Yamaguchi [60])**.** *The isometry group of Alexandrov space is a Lie group.*

18. Alexandrov space - II -

In [119, 120] Perelman proved the following two fundamental results on Alexandrov space.

Theorem 18.1 (Perelman)**.** *Let X be an Alexandrov space with $K \ge \kappa$. Then, for any $x \in X$, there exists a neighborhood of if homeomorphic to T_xX the tangent cone.*

Theorem 18.2 (Perelman)**.** *Let X_i be a sequence of Alexandrov space with $K \ge \kappa$ where κ is independent of i. We assume $X = \lim^{GH}_{i\to\infty} X_i$ and $\dim X = \dim X_i$. Then X_i is homeomorphic to X for large i.*

Remark 18.1*.* Both of these theorems are proved in [119]. Later Perelman published another paper [120] where the proof of Theorem 18.1 is given in a simplified way. Perelman says in [120] that a similar method gives a slight simplification of the proof of Theorem 18.2, but the simpification is not so much big compared with one for Theorem 18.1. Unfortunately the paper [119] is not yet published.

[27] This theorem is improved later to Theorem 21.7 and to Corollary 22.4. Before [111], Shiohama [137] proved that M is homeomorphic to the sphere under similar but different assumption $K_M \ge -C$, $Ricci \ge (n-1)$, $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \epsilon(n, C)$
.

In fact Theorem 18.1 follows from Theorem 18.2 (and Theorems Theorems 17.14, 17.16). However the proof of Theorem 18.2 requires Theorem 18.1.

In this section we give a review of the proof of Theorem 18.1. Before that let us mention some of the corollaries of them.

We remark that T_xX is homeomorphic to $C\Sigma_x(X)$ by Theorems 17.16. Since $\Sigma_x(X)$ is again an Alexandrov space we can apply Theorem 18.1 again. We then find that the singularity of X is obtained locally by taking cones several times. Let us define it more precisely.

Definition 18.1. We define a connected metrizable space X to be *MCS space* of dimension n inductively on n as follows.
(1) An MCS space of dimension 2 is a 2 dimensional manifold with or without boundary.
(2) X is an MCS space of dimension n if, for each $x \in X$, there exists a neighborhood U of x and an MCS space Y_x of dimension $n-1$, such that there exists a homeomorphism F from the cone of to U such that F sends cone point to x.

The following is immediate from Theorems 18.1,

Corollary 18.3. *Every Alexandrov space is an MCS space.*

The following is also an immediate corollary.

Corollary 18.4. *For an Alexandrov space X, there exists X_k with $\cup X_k = X$ such that X_k is a k-dimensional topological manifold and that $\overline{X}_k = \cup_{i\le k}X_i$.*

Corollary 18.5. *Alexandrov space X is locally contractible. If it is compact then $\pi_1(X)$ and $H_k(X)$ are finitely generated.*

Hereafter we assume our Alexandrov space X has no boundary, for simplicity[28]. An idea used in [120] to prove these result is to generalize Morse theory of distance function to Alexandrov space. Let us start the following difinition. Hereafter X is an Alexandrov space with $K \ge -1$. Let $p \in X$. We put $d_p(x) = d(x,p)$.

Definition 18.2. x is said to be a *regular point* of d_p if there exists $\xi \in \Sigma_x(X)$ such that for each minimal geodesic ℓ joining x to p we have $\angle \xi\ell' > \pi/2$. Here $\ell' \in \Sigma_x(X)$ is the equivalence class of ℓ in $\Sigma_x(X)$.

Definition 18.2 is a generalization of Definition 14.1. We can generalize Proposition 14.2 also and we further generalize it to Theorem 18.7. For the proof of Theorem 18.1 we need to use a bit more general function than distance function and define the "regurality" of a map $X \to \mathbb{R}^k$ for $k > 1$ also. To state this generalization we need some notation.

[28]The general case can be handled by taking a doulbe $X \cup_{\partial X} X$ which is an Alexandrov space by [125].

Definition 18.3. Let U be an open subset of X.
(1) An *admissible function* $f : U \to \mathbb{R}$ is a function of the form

$$f(x) = \sum_{i=1}^{m} a_i \phi_i(d(A_i, x)) \tag{18.1}$$

where A_i is a compact subset of X, ϕ_i are smooth function with $0 \le \phi_i' \le 1$ and $a_i \ge 0$, $\sum a_i \le 1$.
(2) An *admissible map* $F : U \to \mathbb{R}^k$ is a composition $F = G \circ \vec{f}$, where G is a bi-Lipschitz homeomorphism and $\vec{f} = (f^1, \cdots, f^k)$ with admissible functions f_i.

Remark 18.2. In [120] more general function (map) is called admissible function (map). But only those in Definition 18.3 are used.

For $A \subset X$ and $x \in X$ we define $\Sigma^0_x(\overline{xA}) \subset \Sigma_x(X)$ by

$\Sigma^0_x(\overline{xA}) = (\{[\ell] | \ell$ is a minimal geodesic joining x to a point of $A.\})$

Let $\Sigma_x(\overline{xA}) \subset \Sigma_x(X)$ be the closure of $\Sigma^0_x(\overline{xA})$.
For $\Lambda_1, \Lambda_2 \subset \Sigma_x(X)$ we put

$$\angle \Lambda_1 \Lambda_2 = \inf\{\angle uv | u \in \Lambda_1, v \in \Lambda_2\}.$$

For an admissible function f as in (18.1) we can define its direction derivative $D_x f : \Sigma_x(X) \to \mathbb{R}$ by

$$(D_x f)(u) = \sum_i a_i \phi'(d(x, A_i)) \cos \angle(u, \Sigma_x(\overline{xA_i})).$$

In case X is a manifold $D_x f$ is the direction derivative in the usual sense.

If $f^{(1)}, f^{(2)}$ are admissible functions as in (18.1) we put

$$\langle D_x f^{(1)}, D_x f^{(2)} \rangle = \sum_{i,j} a_i^{(1)} a_j^{(2)} \phi^{(1)\prime}(d(x, A_i^{(1)})) \phi^{(2)\prime}(d(x, A_j^{(2)})) \cos \angle A_i^{(1)} A_j^{(2)}.$$

This again coincides with the usual inner product between derivatives in case when X is a manifold and $f^{(1)}, f^{(2)}$ are differentiable.

Definition 18.4. We say $F : U \to \mathbb{R}^k$ where $F = G \circ \vec{f}$ is ϵ-*regular* at $p \in U$ if the following conditions hold. Let us put $\vec{f} = (f^1, \cdots, f^k)$.

(1) For each $i \ne j$, we have $\langle D_p f^i, D_p f^j \rangle < -\epsilon$.
(2) There exists $\xi \in \Sigma_p(X)$ such that $(D_p f^i)(\xi) < -\epsilon$, for each i.

We say F is regular if it is ϵ-regular for some $\epsilon > 0$. We say F is ϵ-regular on U if is ϵ-regular at every point of U.

Example 18.1. (1) If $f : U \to \mathbb{R}$ is defined by $f(x) = d_p(x)$. It is an admissible function and hence is an admissible map. It is ϵ regular at x for some $\epsilon > 0$ if and only if x is a regular point of f in the sense of Definition 18.2.

(2) Let X be a two dimensional Alexandrov space. and $(a_1, b_1), (a_2, b_2)$ be a $(2, \delta)$ strainer at x. Let us define $\varphi : X \to \mathbb{R}^2$ by

$$\varphi(x) = -(d(x, a_1), d(x, a_2)) + (d(p, a_1), (p, a_2)).$$

as in the proof of Proposition 17.17. Then φ is a homeomorphism in a neighborhood of x. We put

$$p = \varphi^{-1}(r, 0), \quad q = \varphi^{-1}(-r/2, r\sqrt{2}/2), \quad r = \varphi^{-1}(-r/2, -r\sqrt{2}/2).$$

and define $F = \vec{f} = (d_p, d_q)$. We also set $\xi \in \Sigma_x(\overline{xr})$. We can prove (1)(2) for sufficiently small ϵ. We can generalize this construction to the case of higher dimension and prove that if x is an (k, δ) strained point, then there exists $F : U \to \mathbb{R}^k$ from a neighborhood of x which is ϵ regular at x.

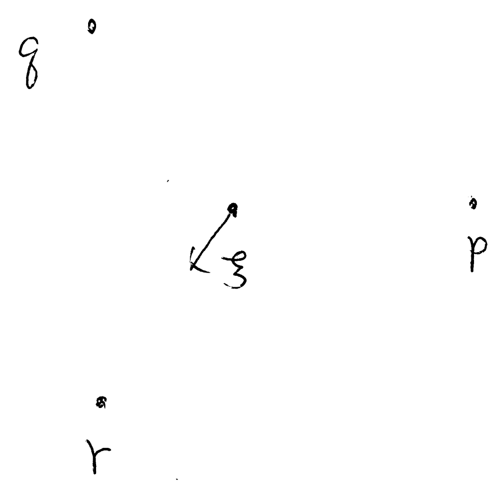

Figure 18.1

We can prove the following in a way similar to the proof of Proposition 17.17. (See [121] Lemma 2.3 and the argument just after that.)

Lemma 18.6. *Let $F : B_x(\rho, X) \to \mathbb{R}^n$ be an admissible map and is ϵ-regular at x. Then there exists a neighborhood $U \subseteq B_x(\rho, X)$ of x and $\delta > 0$, with the following property. If $y \in U, X \in \mathbb{R}^k$ with $d(F(v), X) \leq \delta$ then there exists $z \in B_x(\rho, X)$ such that $F(z) = X$ and $d(z, y) < Cd(F(v), X)$. Here C depends only on ρ, δ, ϵ.*

Lemma 18.6 implies that F is an open mapping. In case $\dim X = k$ and if there exists an ϵ-regular map at x, then Lemma 18.6 shows that a neighborhood of x is a manifold. In the general case, we have to study the situation where $k < \dim X$. The following Proposition 18.7 is the main result in such a case. We need a definition.

Definition 18.5. A map $F : X \to Y$ between topological space is called a *topological submersion* at $x \in X$ if there exists a neighborhood U of x, a neighborhood V of $F(x)$, and a topological space W such that there exists a homeomorphism $\Phi : U \cong V \times W$ satisfying $F = Pr_1 \circ \Phi$ on U.

In case X, Y are smooth manifolds and F is a smooth map, F is a topological submersion if its derivative is of maximal rank.

Theorem 18.7 ([120] Theorem 1.4)**.** *Admissible map $F : X \to \mathbb{R}^k$ is a topological submersion at regular point.*

The following result is also used in the proof of Theorem 18.1.

Theorem 18.8 (Siebennman [142] Corollary 6.14)**.** *Every proper topological submersion between MCS spaces is a locally trivial fiber bundle.*

Remark 18.3. (1) We remark that if M, N are smooth manifolds (without boundary) and $F : M \to N$ be a proper *smooth* submersion then F is a locally trivial fiber bundle. This fact can be proved much more easily than Theorem 18.8.

(2) The proof of Theorem 18.8 is based on isotopy extension theory. We remark that isotopy extension theory for manifolds (see [48]) was used by Cheeger for the proof of his finiteness theorem. (See §6.)

We next sketch the proof of Proposition 18.7. The difficult case is when X is of dimension greater k. We try to increase k as much as possible we then arrive the following situation.

Definition 18.6. Let $F : U \to \mathbb{R}^k$ be a regular admissible map from an open set U of an Alexandrov space X. We say $p \in X$ is *imcomplementable* if there exists no g such that $(f^1, \cdots, f^k, g)$ is regular at p.

The case $k = 0$ is included. Namely in that case $p \in X$ is imcomplementable if there exists no admissible function such that p is regular.

Example 18.2. (1) Let us consider the domain $\{(r\cos\theta, r\sin\theta) | \theta \in [-\alpha, +\alpha], r \geq 0\}$. We glue $(r\cos\alpha, r\sin\alpha)$ and $(r\cos -\alpha, r\sin -\alpha)$ to obtain a space X_α. We can show that $\mathbf{o} = [0,0]$ is imcomplementable if and only if $\alpha \leq \pi/2$. Actually we put $g = d_{[r,0]}$. Then g is regular if $\alpha > \pi/2$. On the other hand, if $\alpha \leq \pi/2$ then the diameter of $\Sigma_{\mathbf{o}} X_\alpha$ is not greater than $\pi/2$. Hence it is easy to see that (2) in Definition 18.4 can never be satisfied[29].

[29]It is ease to see from this argument that in case $k = 0$ the point $p \in X$ is imcomplementable if and only if $Diam\Sigma_p(X) \leq \pi/2$

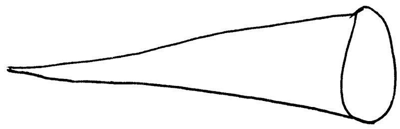

Figure 18.2

(2) Let us next take $X = X_\alpha \times \mathbb{R}$ where X_α is as above. We define $f : X \to \mathbb{R}$ by $f = d_{(\mathbf{o},-1)}$. It is easy to see that $(\mathbf{o}, 0)$ is a regular point. Actually we may take $\xi = D_{(\mathbf{o},0)} d_{(\mathbf{o},1)}$. We can show that f is imcomplementable $(\mathbf{o}, 0)$ if $\alpha \le \pi/2$.

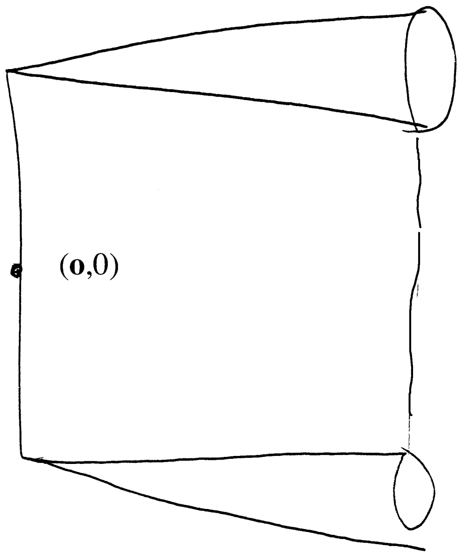

Figure 18.3

(2) Let

Now the main technical result in [120] is as follows.

Lemma 18.9 ([120] 1.3.). *If $F : U \to \mathbb{R}^k$ is admissible and regular at $p \in U$, and if p is imcomplementable, then there exists an admissible function $g : V \to \mathbb{R}$ defined on an open neighborhood V of p with the following properties. We write $F = G \circ \vec{f}$, $\vec{f} = (f^1, \cdots, f^k)$.*

(1) *$g \le 0$ on V and $g(p) = 0$.*
(2) *$F|_{g^{-1}(0)} : g^{-1}(0) \to \mathbb{R}^k$ defines a homeomorphism onto a neighborhood of $F(p)$.*
(3) *$(F, g) : V \to \mathbb{R}^{k+1}$ is regular on $V \setminus g^{-1}(0)$.*
(4) *There exists $\rho > 0$ such that $\{x \in V | d(F(x), F(p)) \le \rho, g(x) \ge -\rho\}$ is compact.*

Let us show how to choose such g in the case of Example 18.2 (2). Namely we have $U = X_\alpha \times \mathbb{R}$ and $F = f = d_{(\mathbf{o},-1)}$. We write a point of U as $([r\cos\theta, r\sin g\theta], t)$ and use r, θ, t as a coordinate. (We take $r \ge 0$, $\theta \in [-\alpha, \alpha]$.) Then $f(r, \theta, t) = \sqrt{(t+1)^2 + r^2}$. We take $q = (\delta, 0, \delta)$ and put $h = d_q$. Then $h(r, \theta, t) = \sqrt{(t-\delta)^2 + r^2 + \delta^2 - 2r\delta\cos|\theta|}$. It is easy to see that $(f, h) : U \to \mathbb{R}^2$ is regular outside on $(B_{(\mathbf{o},0)}(\rho, U) \setminus \{\mathbf{o}\}) \times \mathbb{R}$. (We remark $\{\mathbf{o}\} \times \mathbb{R}$ is the set of singular points.)

However if we put $g = h$ then (1)(2) are not satisfied. So we compose it with a homeomorphism of $\mathbb{R}^2$ so that (1)(2) will be satisfied.

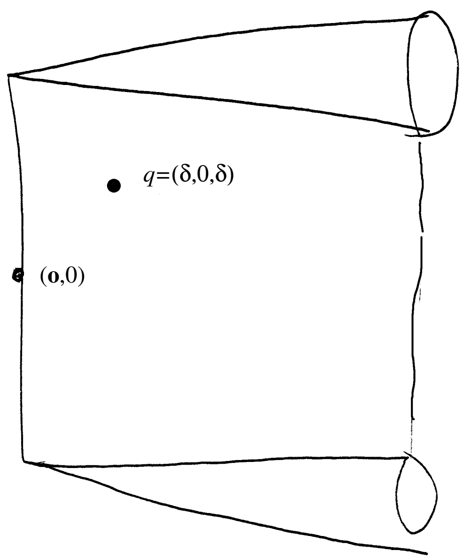

Figure 18.4

We consider the set $K(v) = \{x \in U | f(x) = v, r < \rho\}$ where $|\rho|$ and $|v|$ is small. We can easily check that

$$h(v, 0, 0) = \sup\{h(x) | x \in K(v)\}.$$

if $\alpha \le \pi/2$[30]. Namely

(*) Restriction of h to $K(v)$ takes its maximum at a unique point.

[30]This condition equivalent to the condition that $(\mathbf{o}, 0)$ is imcomplementable.

We remark that $f(v,0,0) = 1 + v$ and $h(v,0,0) = \sqrt{(v-\delta)^2+\delta^2}$. So if we put

$$g(r,v,\theta) = h(r,v,\theta) - \sqrt{(f(r,v,\theta)-1-\delta)^2+\delta^2}.$$

(1)(2) are satisfied. We define $G : B_{(1,\sqrt{2}\delta)}(\rho,\mathbb{R}^2) \to \mathbb{R}^2$ by

$$G(a,b) = (a, b - \sqrt{(a-1-\delta)^2+\delta^2})$$

where $\rho \ll \delta$. Since $(f,g) = G \circ (f,h)$, it follows that (f,g) is admissible. It also satisfies (3) since (f,h) satisfies (3). Thus we constructed g in the case of Example 18.2 (2).

In the general case, we need to choose h more carefully so that it is enough "concave". (Then (*) holds.) The proof of Lemma 18.9 is in [120] §3. □

We can use Lemma 18.9 to complete the proof of Proposition 18.7 as follows. We also proves the following at the same time.

Proposition 18.10. *If $F : X \to \mathbb{R}^k$ is an admissible map and if $p \in X$ is a regular point, then $F^{-1}(F(p))$ is a MCS space near p.*

The proof is by downward induction on k. If $k = \dim X$ then both Theorem 18.7 and Proposition 18.10 follows from Lemma 18.6. Let us assume that Theorem 18.7 and Proposition 18.10 are true for $k+1$ and prove the case k. We remark that both propositions are local statement on p[31]. In case p is not imcomplementable then we can increase k and use induction hypothesis. So it suffices to consider the case p is imcomplementable. We apply Lemma 18.9 to get g. Then $(F,g)|_{V\setminus g^{-1}(0)} : V \setminus g^{-1}(0) \to \mathbb{R}^{k+1}$ is regular. We can use the induction hypothesis to conclude that it is a topological submersion and the fibers are MSC spaces. Therefore $V \setminus g^{-1}(0)$ is a MSC space. Let $(F,g)^{-1}(B_0(\rho,\mathbb{R}^k) \times (-\rho,0)) = W$. Since

$$F : W \to B_0(\rho,\mathbb{R}^k) \times (-\rho,0) \tag{18.2}$$

is proper, Theorem 18.8 implies that (18.2) is a locally trivial fiber bundle. Since the base space is trivial it follows that (18.2) is a trivial bundle. Hence using Lemma 18.9 (2) we can prove Propositions 18.7 and 18.10 for $F : U \to \mathbb{R}^k$. Thus the induction works.

We remark that Proposition 18.10 implies Theroem 18.1 by putting $k = 0$.

We thus sketched the proof of Theroem 18.1. The proof of Theorem 18.2 uses a similar argument but more involved. See [119].

Let us compare the results we reviewed in the last and this section so far, to one in earier sections, where we consider the case $|K_M|$ is bounded. In §11, we asked two questions, Question 11.1 for a sequence M_i converging to X. (1) was on the singularity of X and (2) was on the relation between topologies of M_i and X.

[31]So we prove them by induction without assuming completeness of X.

In the case $|K_{M_i}| \le 1$, an answer to (1) was Theorem 11.4 and an answer to (2) was Theorems 11.5 and 12.7.

In our more general case where we assume $K_{M_i} \ge -1$ only, Theorem 18.1 and Corollary 18.3 give a satisfactory answer to (1).

However results on (2) is not satisfactory. In case $X = \lim_{i\to\infty}^{GH} M_i$ satisies $\dim X = \dim M_i$, Theorem 18.2 is a satisfactor answer. This is the non collpasing case. On the othe hand if X is a smooth Riemannian manifold, Theorem 11.3 by Yamaguchi, gives a nice answer. Namely there exists a fiber bundle $M_i \to X$ for large i [32]. However the trick (taking frame bundle) we explained in §11 does not work in our more general situation to reduce the problem to the case when X is a manifold. So the result is not yet satisfactory. There are however several interesting approach and partial results about the problem (2) in the case $M_i \ge -1$, which we review very briefly here.

First Theroem 11.3 is generalized to the case when the limit X has rather mild singularity. There are two papers about it. In [155], Yamaguchi assumed that for each $x \in X$ there exists a strainer (a_i, b_i) $i = 1, \cdots, n = \dim X$ with $d(x, a_i), d(x, b_i) > \mu > 0$ where μ is independent of x. Then he conclude that there exists a locally trivial Lipschtz fiber bundle structure $M_i \to X$ [33].

Perelman in [124] assumed that X has no proper extremal set. Here :

Definition 18.7. $F \subset X$ is extremal if for each $p \notin F$, $x \in X$, and $u \in \Sigma_x(X)$, we have $D_x d_p(u) \le 0$.

For example $F = \{x\}$ consisting of one point is not extremal if and only if there exists an admissible function f which is regular at x.

Perelman's theorem in [124] is that if there is no extremal set then there exists $f_i : M_i \to X$ such that $\pi_k(M_i, f_i^{-1}(p)) \cong \pi_k(X)$ for each $p \in X$. The plan proposed by Perelman [121] then is to stratify X using extremal set and construct fiber bundle structure stratawise. This plan is not yet completed.

Shioya-Yamaguchi [141] and Yamaguchi [156, 157] studied the case when $\dim M_i = 3, 4$ without extra assumption on X and gave satisfactory description in that case. In this article, we discuss 3 dimensional case only. Let M_i be 3 dimensional Riemannian maifold with $K_{M_i} \ge -1$, and $X = \lim_{i\to\infty}^{GH} M_i$. We assume $\dim X \le 2$. Then X is homeomorphic to a manifold with or without boundary. We assume that X is connected.

Theorem 18.11 (Shioya-Yamaguchi [141]). *We assume* $\dim X = 2$.

[32] Theorem 11.3 does not say much about the fibers. But there are various results which shows that the fibers are "of nonnegative curvature" in some sense.

[33] In the preprint version of [155] (which the author has), the locally triviality is not asserted. It is proved in [141].

(1) *If* $\partial X = \emptyset$*, then there exists a structure of Seifert fibered space* $M_i \to X$ *for large* i*.*
(2) *If* $\partial X \neq \emptyset$*, then* M_i *is homeomorphic to* $\mathrm{Sei}_i(X) \cup (\partial X \times D^2)$ *where* $\mathrm{Sei}_i(X)$ *is a Seibert fibered space over* $\mathrm{Int}X$*. We glue it with* $\partial X \times D^2$ *where the fibers of* $\mathrm{Sei}_i(X)$ *over the boundary point* x *is glued with* $\{x\} \times \partial D^2$*.*

In case $\dim X = 1$ there are two possibilities, $X \cong S^1$ or $[0,1]$. In the case $X = S^1$ there exists a fiber bundle $M_i \to S^1$ by Theroem 11.3.

Theorem 18.12 (Shioya-Yamaguchi [141])**.** *If* $X \cong [0,1]$ *then* M_i *is obtained by gluing* B_i *and* C_i *along their boundaries where each of* B_i, C_i *is homeomorphic to one of the following 4 manifolds.* (1) D^3*,* (2) *nontrivial* $[0,1]$ *bundle over* $\mathbb{R}P^2$*,* (3) $S^1 \times D^2$*,* (4) *nontrivial* $[0,1]$ *bundle over Klein bottle.*

The rough idea of the proofs of Theorems 18.11,18.12 are as follows. In either cases, we can apply generalization [155] of Theroem 11.3 except finitely many points (plus ∂X in case (2) of Theorems 18.11). In the neighborhood of those points we scale the metric to obtain noncompact nonpositively curved Alexandrov space. Then apply soul theorem (Alexandrov space analogue of Theorem 16.7). The soul S is an Alexandrov space of dimension ≤ 2 so is a manifold with or without boundary. Actually Shioya-Yamaguchi classified 3 dimensional noncompact complete Alexandrov space with $K \geq 0$. in this way, we can classify neighborhods $\subset M_i$ of a singular point of X locally. Then the last step is to glue those local neighborhoods. □

In the case when $\dim X = 0$ we can scale the metric of M_i and obtain limit of nonzero dimension. In this way [141] (improving [154, 59]) proved the following :

Theorem 18.13 (Shioya-Yamaguchi)**.** *There exists* ϵ *such that if* M *is a Riemannian 3 manifold with* $K_M Diam(M) \geq -\epsilon$ *then a finite cover of* M *is homeomorphic to* $S^1 \times S^2$*,* T^3*, nilmanifold or a simply connected Alexandrov space with* $K \geq 0$*.*

19. First Betti number and fundamental group

So far we discussed results about sectional curvarure. In this section we discuss also Ricci curvature. The recent progress mainly due to Cheeger-Colding will be discussed in later sections. In this section, we mainly concern with older results. To study Ricci curvature we need partial differential equation frequently. But we do not mention them so much.

We first review Theorem 2.3. The proof of Theorem 2.3 is based on Bochner trick. The most famous result in metric Riemannian geometry based on Bochner trick is the following :

Theorem 19.1 (Bochner [151]). *If an n dimensional compact Riemannian manifold M has nonnegative Ricci curvature then the first Betti number of M is not greater than n.*

The proof of Theorem 19.1 due to Bochner is as follows. Let u is a one form on M. Then we have the following equality of Weitzenböck type. (For proof see [151]. We remark that we use nonpositive Laplacian (16.2).)

$$\langle -\Delta u, u\rangle = -\frac{1}{2}\Delta\|u\|^2 + \langle \nabla u, \nabla u\rangle + \mathrm{Ricci}(u,u). \tag{19.1}$$

Let u be a harmonic one form. We integrate (19.1) over M. The left hand side is zero (since u is harmonic) and the integral of the first term in the right hand side vanish. Therefore we have :

$$\int_M \langle \nabla u, \nabla u\rangle \Omega_M + \int_M \mathrm{Ricci}(u,u)\Omega_M = 0. \tag{19.2}$$

(Here Ω_M is a volume element .) The first term of (19.2) is nonnegative. If we assume that the Ricci curvature is nonnegative then the second term also is nonnegative. Therefore the first and second term both are zero. Namely every harmonic one form is parallel. Since parallel one form is determined by its value at one point (here we are assuming that M is connected), it follows that the dimension of the space of harmonic one forms on M is at most n. Theorem 19.1 follows. □

When we try to apply a similar argument to the forms of higher degree and try to estimate higher Betti number by Ricci curvature, we will meet a trouble. In formula (19.1), the third term involves only Ricci curvature. This is true only for one form. A similar formula for forms of higher degree is much more comlicated. The assumption we need to apply a similar argument to forms of higher degree is exactly the assumption in Theorem 2.3, which is much stronger than one on Ricci curvature[34].

In §16, we discussed splitting theorem of Riemannian manifold of nonnegative Ricci curvature (Theorem 16.4). We can prove Theorem 19.1 by using it also. Actually we have the following :

Theorem 19.2 (Cheeger-Gromoll)**.** *If M is a compact manifold with nonnegative Ricci curvature then there exists a finite cover $\tilde M$ of M, such that $\tilde M$ is isometric to the direct product $X \times T^k$, where X is simply connected and T^k is a flat torus.*

To prove Theorem 19.2, we consider the universal covering $\hat M$. Since we may assume that the fundamental group of M is infinite (otherwise we may take $X = \hat M$), we can prove that $\hat M$ contains a line[35]. Now by applying Theorem 16.4, we find $\hat M = \mathbb{R}\times Y$. We may split $\hat M = \mathbb{R}^k\times Y'$ so that Y' has no $\mathbb{R}$ factor. If Y' is not compact, we can show Y'contains a line by the same argument. Then, by Theorem 16.4, Y' has an $\mathbb{R}$ factor, a contradiction. Namely Y' is compact. The $\pi_1 M$ action preserves the splitting $\hat M = \mathbb{R}^k \times Y'$. Theorem 19.2 follows easily. □

In case $k = n$ in Theorem 19.2, or in case when the first Betti number is equal to the dimension in Theorem 19.1, we can show that M is flat. (We can show this fact either by Bochner's proof using (19.1) or by Cheeger-Gromoll's proof based on splitting theorem.)

Theorem 19.2 is generalized by Gromov as follows.

Theorem 19.3 ([70] p73)**.** *There exists a continuous function $b(n, \rho)$ of $\rho \in \mathbb{R}$ with $b(n, 0) = n$, such that the following holds. If M is an n dimensional Riemannian manifold with diameter 1, Ricci cuvature $\geq \rho$, Then its first Betti number is not greater than $b(n, \rho)$.*

Corollary 19.4. *If M is an n dimensional Riemannian manifold with diameter 1 and* Ricci $> -\epsilon_n$*, then its first Betti number is not greater than n. Here ϵ_n is a positive number depending only on n.*

Gromov's proof is based on the estimate of growth function by using Bishop-Gromov inequality (Proposition 5.2) and is closer to the study of fundamental group we mention later in this section (Theorem 19.9, Theorem 19.10). The analytic proof using a similar idea to Bochner's is given by Gallot [61].

---

[34]On the other hand, if we write a formula similar to (19.1) for spinor and Dirac operator the second term involves only a Scaler curvature. (See text book of Atiyah-Singer index theorem.). A theorem by Lichnerowicz which asserts "The $\hat A$ genus of Riemannian manifold of positive scaler cuvature is zero" is obtained from this fact.

[35]Let $p_i, q_i \in M$ with $d(p_i, q_i) \to \infty$. Let x_i be the midpoint of a minimal geodesic joining p_i and q_i. Moving them by an action of $\pi_1(M)$, we may assume that there exists R independent of i such that $d(x, x_i) < R$. Then a subsequence of the sequence of geodesics joining p_i, x_i, q_i has a limit. This limit is a line.

As we mentioned before, the idea of the proof of Theorem 19.1 can not directly be applied to the study of second or higher Betti number. In fact a result similar to Theorem 19.3 does not hold for higher Betti number. Namely the statement such as :

" If M is an n dimensional compact Riemannian manifold with diameter 1, Ricci curvature $\ge \rho$, then its Betti number is smaller than a number depending only on ρ and n",

is false. See [134, 123] for counter examples. Note if we replace Ricci $\ge \rho$ by $K_M \ge \rho$ in the statement in the parenthesis then it is Theorem 14.6.

Let us consider the case when equality holds in Corollary 19.4, namely the case Ricci $> -\epsilon_n$ and first Betti number is n.

Theorem 19.5 (Yamaguchi [154])**.** *There exists a positive number ϵ_n such that if M is an n dimensional Riemannian manifold with diameter $Diam(M)K_M > -\epsilon_n$, and its first Betti number is b, then there exists a finite cover $\tilde M$ of M and a fiber bundle $\tilde M \to T^b$ over b dimensional torus.*

Moreover if $b = \dim M$ then M is diffeomorphic to the torus.

Remark 19.1. (1) Yamaguchi proved the same conclusion for the fiber of Theorem 11.3.

(2) Yamaguchi [152] proved the same conclusion under a different hypothesis $K_M \le 1$, $Diam \le D$, $Ricci_M \ge -\epsilon(D, n)$

To prove Theorem 19.5, Yamaguchi used Theorem 16.4 case (b). The second half of Theorem 19.5 is generalized by Colding [45] and Cheeger-Colding [29] as follows.

Theorem 19.6 ([45, 29])**.** *If M is an n dimensional Riemannian manifold with $Diam(M)Ricci_M > -\epsilon_n$, and its first Betti number is n, then M is diffeomorphic to a torus.*

Remark 19.2. The first half of the statement of Theorem 19.5 does not hold under the milder assumption $Diam(M)Ricci_M > -\epsilon_n$. Anderson [10] constructed an example of M with $Diam(M)Ricci_M > -\epsilon_n$ but has no fibration over $T^{b_1(M)}$.

We here explain some of the ideas used by Yamaguchi in [154] to show Theorem 19.5, which is also used in [45]. (The additonal ideas due to [45, 29] will be explained in later sections.)

For simplicity we consider the case $b = n = \dim M$ only. The proof is by contradiction. By scaling we may assume that there exists M_i with $Diam(M_i) = 1$, $K_{M_i} \ge -\epsilon_i$ but M_i are not diffeomorphic to T^n. We consider the coverline space $\hat M_i \to M_i$ whose covering transformation group is $\Gamma_i = \mathbb{Z}^b$. We study the limit of the pair $(\hat M_i, \Gamma_i)$. Here we define :

Definition 19.1 ([52])**.** A sequence of pairs $((X_i,p_i),G_i)$ of pointed metric spaces (X_i,p_i) and groups of isometries G_i is said to converge to $((X,p),G)$ with respect to the *equivariant pointed Hausdorff convergence* if there exists $\varphi_i : B_{p_i}(1/\epsilon_i,X_i) \to B_p(1/\epsilon_i,X)$, $\varphi'_i : B_p(1/\epsilon_i,X) \to B_{p_i}(1/\epsilon_i,X_i)$ $\psi_i : \Gamma_i \to G$, $\psi'_i : G \to \Gamma_i$ with $\epsilon_i \to 0$ such that :

(1) φ_i,φ'_i are ϵ_i Hausdorff approximations and

$$d(x,\varphi_i(\varphi'_i(x))) < \epsilon_i, \quad d(x,\varphi'_i(\varphi_i(x))) < \epsilon_i.$$

(2) If $x,\gamma(x) \in B_{p_i}(1/\epsilon_i,X_i)$, $\gamma \in \Gamma_i$ then

$$d(\varphi_i(\gamma(x)),\psi_i(\gamma)(\varphi_i(x))) < \epsilon_i.$$

(3) If $x,\gamma(x) \in B_{p_i}(1/\epsilon_i,X)$, $\gamma \in \Gamma$ then

$$d(\varphi'_i(\gamma(x)),\psi'_i(\gamma)(\varphi'_i(x))) < \epsilon_i.$$

We remark that we do not assume φ_i,φ'_i are homomorphism.

We can prove a similar compactness result as Theorem 16.1. Now let us go back to the proof of Theorem 19.5. Fix $p_i \in \hat M_i$. We may consider the limit $((\hat M_i,\Gamma_i),p_i)$ with respect to the equivariant pointed Hausdorff convergence. However then the limit may be a continuous group and is a bit hard to handle. So we use the following lemma.

Lemma 19.7 ([154])**.** *There exists subgroups $\Gamma'_i \subset \Gamma_i$ of finite index and η,η' (independent of i) such that*

(1) *For each $\gamma \in \Gamma'_i$ with $\gamma \ne 1$ we have $d(p_i,\gamma(p_i)) \ge \eta$.*
(2) *Γ'_i is generated by elements $\gamma_1,\cdots,\gamma_n$ such that $d(p_i,\gamma_k(p_i)) \le \eta'$. (Here $n = \dim M$.)*

Lemma 19.7 appeared in the proof by Gromov of Theorem 19.3. The fact that Γ_i is abelian plays an important role for its proof. We omit the proof of Lemma 19.7.

Now we can consider the limit of the sequence $((\hat M_i,\Gamma'_i),p_i)$. We denote it by $((X,G),p)$. Using Lemma 19.7 we can easily show that $G \cong \mathbb{Z}^n$ and its action is properly discontinuous. Now we apply splitting theorem to X and obtain $X = \mathbb{R}^k \times Y$ where Y is compact[36]. Since $\mathbb{Z}^n$ acts on it properly discontinuously, it follows that $k = n$. Since $\dim X \le \dim \hat M_i = n$ it follows that $X = \mathbb{R}^n$. We can also prove that $\hat M_i/\Gamma'_i$ converges to $X/G \cong T^n$. We put $\tilde M_i = \hat M_i/\Gamma'_i$. Since $\tilde M_i$ is n dimensional and converges to T^n it follows from Theorem 11.3 that $\tilde M_i$ is diffeomorphic to T^n. Using $H_n(M_i,\mathbb{Q}) = n$ again we can show that M_i is homeomorphic to T^n. Furthermore we can arrange covering index $\tilde M_i \to M_i$ so that "M_i is homeomorphic to T^N and M'_i is diffeomorphic to T^n" imply M_i is diffeomorphic to T^n, if ≥ 5. (This point is

[36] I think this was the first place where splitting theorem of the limit (singular) space was applied to study Riemannian manifold.

a standard application of nonsimplyconnected surjery.) The last step in the low dimension case is a bit complicated and is omitted. □

Remark 19.3. The above argument can be applied to the situation of Theorem 19.6. We only need to replace splitting theorem to one by Cheeger-Colding and Theorem 11.3 by Theorem 22.3. Colding's argument in [45] (though using Lemma 19.7) is slightly different. This is probably because splitting theorem we need for this purpose was not yet proved at that time.

We next remark the following corollaly of Theorem 19.2.

Corollary 19.8. *If compact Riemannian manifold M has nonnegative Ricci curvature, then its fundamental group $\pi_1(M)$ contains an abelian subgroup of finite index.*

It seems that series of results related to Corollaly 19.8 began with the following theorem.

Theorem 19.9 (Milnor [103])**.** *If a complete manifold M has a non-negative Ricci curvature and if G is a finitely generated subgroup of $\pi_1(M)$ then G has polynomial growth.*

The definition of group being polynomial growth is as follows. Let G be a finitely generated group and $g_1, \cdots, g_k$ generate G. Let $f_G(N)$ be the number of elements of G which can be written by a product of at most N of g_i or g_i^{-1}.

Definition 19.2. We say that G has *polynomial growth*, if there exists C, K such that $f_G(N) < C(N^K + 1)$.

It is easy to see that this definition is independent of the choice of generator of G.

The proof of Theorem 19.9 is based on Proposition 5.2 and proceed as follows. Let us assume M is compact for simplicity. Let $\tilde M$ be the covering space of M corresponding to G. Let $p \in \tilde M$. By Proposition 5.2 we have :

$$\mathrm{Vol}(B_p(R, \tilde M)) \le CR^n.$$

By an elementary argument using fundamental domain, we can show the existsnce of C with

$$C^{-1} < \frac{\mathrm{Vol}(B_p(R, \tilde M))}{f_{\pi_1(M)}(R)} < C.$$

Theorem 19.9 follows. □

Roughly speaking the growth function f_G evaluates how much G is far from being commutative. In fact, if G is free and nonabelian then there exists c, C such that :

$$f_G(R) > ce^{R/C}.$$

(We say that G has an exponential growth in this case). On the other hand, $G = \mathbb{Z}^k$ has polynomial growth.

Gromov[72] proved te following :

Theorem 19.10 (Gromov)**.** *A finitely generated group G has polynomial growth if and only if G has a nilpotent subgroup of finite index.*

Let us very briefly sketch its proof here. First we recall the following :

Theorem 19.11 (Tits [147])**.** *Let G be a finitely generated subgroup of $GL(n, \mathbb{R})$. Then either G contains a solvable subgroup of finite index or G contains a noncommutative free group.*

If G contains a noncommutative free group, we can show that G is not of polynomial growth. On the other hand, Milnor proved that solvable group is of polynomial growth if and only if it contains a nilpotent group of finite index. Hence to prove Theorem 19.10 it suffices to embed G to some Lie group. Gromov's idea is to do so by using Hilbert 5th problem. Let G a group of as in Theorem 19.10. We define a metric (the word metric) on G as follows. Let $\gamma_1, \cdots, \gamma_n$ be a generator. Let $\mu_1, \mu_2 \in G$. We define $d(\mu_1, \mu_2)$ to be the smallest number k such that $\mu_2 = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_k}^{\epsilon_k} \mu_1$. Here $i_j \in \{1, \cdots, n\}$, $\epsilon_j = \pm 1$.

Now we consider the limit $\lim^{GH}_{N\to\infty}(G, \frac{1}{N}d)$ as $N \to \infty$. The assumption that G is of polynomial growth is used to show that the limit exists. It is easy to see that the limit G' has a structure of group.

We then can use the fact that G' acts as isometry on itself preserving metric and a solution of Hilbert's 5th problem, to show that G' is a Lie group. So if we can embed G to G' we are done. But it is not so easy to embed G to G'. (Actually in case when G is discrete subgroup of a nilpotent Lie group N, then the limit is N but has a strange metric called Carnot-Carateodri metric (See [75]).) Therefore we need to discuss more carefully and some more technical argument is required. We omit it. □

Theorems 19.9 and 19.10 imply that the finitely generated subgroup of the fundamental group of complete manifold of nonnegative Ricci curvature has nilpotent subgroup of finite index. This fact is generalized by Fukaya-Yamaguchi [59][37] and further by Cheeger-Colding [45] as follows.

Theorem 19.12 (Cheeger-Colding)**.** *There exists a positive number ϵ_n such that if the an n dimensional Riemannian manifold satisfies $Diam(M)^2 Ricci_M \geq -\epsilon_n$, then $\pi_1(M)$ contains a nilpotent subgroup of finite index.*

[37] The result of [59] is the same conclusion as Theorem 19.12, but the assumption there is on sectional curvature instead of Ricci curvature.

We remark that Theorem 19.12 follows Theorem 10.5. We also remark that Theorem 10.5 implies Theorem 19.9. In Theorem 19.12 we can not replace the conclusion "nilpotent" by "abelian". Namely we can not replace the assumption Ricci ≥ 0 of Corollary 19.8 by $\ge -\epsilon_n$. The counter examle is an almost flat manifold (Example 10.1).

Some more results on fundamenta group is proved in [59] and [60] which we review here.

A group Γ is said to be polycyclic if there exists

$$1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma_k \tag{19.3}$$

such that Γ_i is a normal subgroup of Γ_{i+1} and Γ_{i+1}/Γ_i is cyclic. The smallest such number k is called *degree of polycyclicity* of Γ.

Theorem 19.13 ([59] Theorem 0.6, Corollary7.20 plus [45])**.** *There exists ϵ_n and w_n such that if an n dimenisional Riemannian manifold M satisfies $Ricci_M Diam(M) \ge -\epsilon_n$ then π_1 contains a normal subgroup Γ such that*

(1) $[\pi_1(M) : \Gamma] \le w_n$.
(2) Γ *is polycyclic and its degree of polycyclicity is not greater n.*

Theorem 19.14 (Fukaya-Yamaguchi [60])**.** *For each D, n there exists a finite set of groups $\mathfrak{G}$ with the following properties. Let M be a manifold with $K_M \ge -1$, $Diam(M) \le D$. Then there exists $G \in \mathfrak{G}$ and a surjective homomorphism $\pi_1 M \to G$ such that the kernel Γ satisfies* (1),(2) *of Theorem 19.13.*

Theorem 19.14 implies the following. For a group G let us put

$$D(G, n) = \inf\{Diam(M) | K_M \ge -1, \dim M = n, \pi_1 M \supseteq G\}.$$

Then, for any sequence of noncommutative simple groups G_i with $G_i \ne G_j$ for $i \ne j$, we have $\lim_{i\to\infty} D(G_i, n) = \infty$.

Remark 19.4. Theorem 17.23 plays a key role in the proof of Theorem 19.14. So far the author does not know the proof of the conclusion of Theorem 19.14 under milder assumption $Ricci_M \ge -(n-1)$. The trouble is a generalization of Theorem 17.23 to the limit X of manifolds M_i with $Ricci_{M_i} \ge -\delta_i$ where $\delta_i \to 0$. (Namely the problem whether the isometry group of such X is a Lie group or not.) Cheeger-Colding [30] proved that the group of isometries of X is a Lie group under additional assumption $\mathrm{Vol}(M_i) \ge v > 0$. Under this additional assumption there is a following result (Anderson [7]) : The number of isomorphism classes of $\pi_1 M$ where n dimensional Riemanniam manifold M with $Ricci_M \ge -(n-1)$, $\mathrm{Vol}(M_i) \ge v > 0$, $Diam(M) \le D$, is finite.

We here sketch a part of the proof of Theorem 19.12 given in [59]. Namely we assume splitting theorem and explain how to deduce Theorem 19.12 from it. Here we consider the case $Diam(M)K_M \ge -\epsilon_n$ [38] to simplify the argument.

We first need a lemma on the convergence of groups. If Γ acts on a metric space X by isometry and p is a base point of Xwe write

$$\Gamma(D) = \langle\{\gamma|d(\gamma(p),p) \le D)\}\rangle$$

Here $\langle A\rangle$ is a subgroup generated by A.

Lemma 19.15 ([59] Theorem 3.10)**.** *Let (X_i,Γ_i,p_i) converges to (X,G,q) in pointed equivariant Hausdorff distance. We assume that the connected component G_0 of G is a Lie group and G/G_0 is discrete and finitely presented. We also assume that X/G is compact. Moreover we assume that X_i is simply connected and Γ_i is properly discontinuous and free.*

Then there exists a sequence of normal subgroups $\Gamma_{i,0}$ converging to G_0 such that $\Gamma_i/\Gamma_{i,0} \cong G/G_0$ for large i[39].

We omit the proof. Now we prove the following :

Proposition 19.16. *Let (M_i,p_i) converges to $(\mathbb{R}^k,0)$ with respect to the pointed Hausdorff distance. Assume $Ricci_{M_i} \ge -(n-1)$. Then there exists $\epsilon > 0$ such that the image of $\pi_1(B_{p_i}(\epsilon,M_i))$ in $\pi_1(B_{p_i}(1,M_i))$ has solvble subgroup of finite index for large i.*

The solvability in Theorem 19.12 is the case $k = 0$ of Proposition 19.16. (The proof of more precise statement as in Theorem 19.13 and nilpotency (for which the argument of is omitted.)

The proof of Proposition 19.16 is by downward induction on k. The case $k = \dim M_i$ follows from Theorem 11.3. We assume Proposition 19.16 is correct for $k+m$ $(m>0)$ and show the case of k by contradiction.

Let (M_i,p_i) as in Proposition 19.16. We use Theorem 11.3 to find $V_i \subseteq M_i$ and a fiber bundle $f_i : V_i \to B_0(C_i,\mathbb{R}^n)$ with $C_i \to \infty$. Here $V_i \supseteq B_{p_i}(C_i/2,M_i)$. Let $\delta_i = Diam(f_i^{-1}(0))$. We take metric $g_{i,1} = g_i/\sqrt{\delta_i}$. The limit of $(V_i,g_{i,1})$ with respect to the pointed Hausdorff distance is $\mathbb{R}^k \times Z$ where Z is an Alexandrov space with $K \ge 0$. Let $\Gamma_i = \pi_1(F_i) = \pi_1(V_i)$. We take $((\tilde V_i,\tilde g_{i,1}),\Gamma_i,\tilde p_i)$ where $(\tilde V_i,\tilde g_{i,1})$ is the covering space of V_i equipped with metric induced from $g_{i,1}$. Let us

[38]If we use Cheeger-Colding's splitting theorem similar argument works. However we need several modification on the technical points to the arguments on [59] or one given below. Unfortunately the technical detail of such argument is not written in the literature. The author and T. Yamaguchi are planning to write it and make it public near future. But maybe it is too technical to be included in this article.

[39]This lemma is actually weaker than [59] Theorem 3.10. But it is enough for present purpose since we now have Theorem 17.23.

take a subsequence and let (V_∞, G, q) be the limit. We apply splitting theorem 16.4 to V_∞ and find $V_\infty = \mathbb{R}^\ell \times Y$ where Y is compact. Since $(\mathbb{R}^\ell \times Y)/G \cong \mathbb{R}^k \times Z$ we find that $V_\infty = \mathbb{R}^k \times \mathbb{R}^{\ell-k} \times Y$ such that G acts only on $\mathbb{R}^{\ell-k} \times Y$ and $(\mathbb{R}^{\ell-k} \times Y)/G = Z$.

Since G is a Lie group by Theorem 17.23 it follows that we can take its connected component G_0. Since G/G_0 is discrete and $(\mathbb{R}^{\ell-k} \times Y)/G$ is compact we can prove that G/G_0 has abelian subgroup of finite index. (This is easy to see if G acts effectively on $\mathbb{R}^{\ell-k}$. The compact factor Y only contributes a finite group.) To apply Lemma 19.15 we replace $V_\infty = \mathbb{R}^k \times \mathbb{R}^{\ell-k} \times Y$ by $X = B_0(D, \mathbb{R}^k) \times \mathbb{R}^{\ell-k} \times Y$ for large but fixed D and find a sequence $((X_i, d_{X_i}), \Gamma_i, p_i)$ converging to $((X, d_X), G, q)$. (We can find such $X_i \subset V_i$ easily by using fiber bundle f_i.)

We now apply Lemma 19.15 to obtain $\Gamma_{i,0}$.

Since (X, d_X) is an Alexandrov space, it follows from Theorem 17.2 that we can find q' near q and $r_i \to \infty$ such that $((X, r_i d_X), q')$ converges to $(\mathbb{R}^{k+m}, 0)$ with $m > 0$. (Note since $Diam(Z) = 1$ it follows that $\mathbb{R}^{\ell-k} \times Y$ is not a point.)

We may replace $((X_i, d_{X_i}), \Gamma_i, p_i)$ by a subsequence which converges to $((X_i, d_{X_i}), \Gamma_i, p_i)$ very quickley compared to $1/r_i$. Then we find q_i such that $((X_i, r_i d_{X_i}), \Gamma_i, q_i)$ converges to $(\mathbb{R}^{k+m}, G', 0)$ for some G'. Since we can use the fact that $\Gamma_{i,0}$ converges to G_0 the connected Lie group and the convergence is quick compared to r_i to show that $\Gamma_{i,0}$ is generated by $\Gamma_{i,0}(\delta_i) = \{\gamma \in \Gamma_{i,0} | d(\gamma(q_i), q_i) < \delta_i\}$ where $\delta_i \to 0$. Now we apply induction hypothesis. Then if $\epsilon > \delta_i$ (where ϵ is as in Proposition 19.16) we find that $\Gamma_{i,0}(\delta_i)$ has index finite solvable subgroup. This is a contradiction since $\Gamma_i/\Gamma_{i,0} \cong G/G_0$ has index finite abelian subgroup. $\square$

20. Hausdorff convergence of Einstein manifolds

In the last four sections, we discuss Gromov-Hausdorff convergence under the assumption $Ricci \geq -(n-1)$.

We first remark that, when we work under the assumption $Ricci \geq -(n-1)$, the topology can change when we go to the limit, even in the noncollapsing situation, namely in the situation where we assume $\mathrm{Vol} \geq v > 0$. Such a phenomenon was first observed in the study of 4 dimensional Einstein (or complex 2 dimensional Kähler-Einstein) manifold (at least around 20 years ago as far as I know).

Let us so start by a review of the case of Einstein manifolds. Let $\Gamma \subset SU(2)$ be a finite subgroup. We consider the quotient $\mathbb{C}^2/\Gamma$. It is a Kähler orbifold with isolated singularity at origin. (This singularity is called the Kleinian singularity.) There is a resolution called minimal resolution of Kleinian singularity which we denote by $\widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$. Eguchi-Hanson [49] and others constructed a Ricci flat Kähler metric

$g_{\widetilde{\mathbb{C}^2/\Gamma}}$ on $\widetilde{\mathbb{C}^2/\Gamma}$ which is asymptotically locally Euclidean (in the sense we define later in Definition 20.1). (Such a metric is called gravitational instanton.) Asymptotically locally Euclidean metrics on $\widetilde{\mathbb{C}^2/\Gamma}$ is classified by Kronheimer [98].

Suppose (X, g_X) is a 4 dimensional Riemannian orbifold locally of Kleinian type. (Namely X is locally a quotient of $\mathbb{C}^2$ by a finite group $\Gamma \subset SU(2)$. We assume also that the metric on X is a quotient metric with respect to certain Γ invariant metric locally.) We assume that X is Ricci flat Kähler. (Namely its Ricci curvature at regular points is 0 and the metric is Kähler at regular point.) We can locally glue metric g_X on X and the Ricci flat Kähler metric $\epsilon g_{\widetilde{\mathbb{C}^2/\Gamma}}$ on X to obtain a metric g'_ϵ on the resolution $\tilde X$ of X. g'_ϵ is almost Ricci flat. We can use the technique of Yau's proof of Calabi conjecture [158] to show that there exists a Ricci flat Kähler metric on $\tilde X$ near g_ϵ. (See [96, 13].) We remark that $(\tilde X, g'_\epsilon)$ and $(\tilde X, g_\epsilon)$ converges to (X, g) with respect to the Gromov-Hausdorff distance. Typical example is a Kummer surface where we take $X = T^4/\mathbb{Z}_2$ (and $\Gamma = \mathbb{Z}_2$).

Thus, we have

Observation 20.1. *There exists a famly of Riemannian manifolds (X, g_ϵ), such that $Ricci_{g_\epsilon} \equiv 0$, $\mathrm{Vol}(X, g_\epsilon) \ge v > 0$ and that the limit of (X, g_ϵ) as $\epsilon \to 0$ converges to a compact metric space X which is not a manifold.*

The construction here is an analogue of Taubes' constructon [145] of Anti-Self dual connection on 4 manifolds.

Later Joyce (see [89]) generalized this construction and used it to construct (higher dimensional) Riemannian manifolds with exceptional holonomy. (They are in particular Ricci flat.) Namely Joyce started, for example, with a 7 dimensional flat orbifold $X = T^7/\Gamma$, which is obtained by T^7 divided by a finite group of isometries Γ. In his example, the singular locus of X is codimension 4 totally geodesic smooth submanifold (actually it is a disjoint union of T^3). Then Joyce glued $T^3 \times \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ (equipped with direct product metric) along singularity to obtain a Riemannian manifold and use implicit function theorem to obtain a manifold with exceptional holonomy. In his construction, we also have a family of metrics g_ϵ which is of exceptional holonomy (and inparticular is Ricci flat) and which converges to X.

A converse to the Observation 20.1 is proved by Nakajima and others as follows.

Theorem 20.2 (Nakajima [105])**.** *Let g_i be a sequence of Einstein mertrics with $Ricci = \pm 1$ or 0, on a 4 manifolds M, such that $\mathrm{Vol}(M, g_i) \ge v > 0$. (Here v is independent of i.) Let $X = \lim^{GH}_{i\to\infty}(M, g_i)$. Then* *there exists a finite subset $S \subset X$ such that $X \setminus S$ is an Einstein 4 manifold.*

Moreover for ever $\delta > 0$ there exists a diffeomorphism $\Phi_i : X \backslash N_\delta S \to M$ such that the pullback Riemannian metric $\Phi_i^* g_i$ converges to the Riemannian metric on X in C^∞ topology.

Remark 20.1. In case of 4 dimensional Einstein manifold, the L^2 norm of the curvature

$$\int_{M_i} |R_{M_i}|^{n/2} \Omega_{M_i} = \int_M |R_M|^2 \Omega_M \tag{20.1}$$

is a topological invariant and is estimated by the Euler number. This fact is essential in the proof of Theorem 20.2[40].

In case $\dim M_i = 4$ the same conclusion as Theorem 20.2 holds under the additional hypothesis

$$\int_{M_i} |R_{M_i}|^{n/2} \Omega_{M_i} \le C. \tag{20.2}$$

(In case we assume (20.2) we do not need to fix a topological type of M.) Namely under Assumption (20.2) and $\mathrm{Vol}(M_i) \ge v > 0$, the limit space X of a sequence of Einstein manifolds M_i has only finitely many singular points.

We remark however the assumption (20.2) is too restrictive to handle limit of Einstein manifold. In the example of Joyce mentioned above the limit of a sequence of 7 dimensional Einstein manifolds is $T^7/\mathbb{Z}_2$ whose singularity is 3 dimensional. In this example, L^2 norm of the Ricci curavature is bounded but $L^{3.5}$ norm is not bounded.

To study the structure of M_i or X near a singular point $\in S$, we use the scaling argument as follows. For completeness we include the case when $\dim M$ is general. Namely we assume we have a sequence of Einstein manifolds M_i converging to X. We assume (20.2) and $\mathrm{Vol}(M_i) \ge v > 0$. (Then the singular point set S of X of finite order.) Let $p_i \in M_i$ which converges to $p_\infty \in S$. We scale the metric g_{M_i} to $R_i g_{M_i}$ so that $|K_{R_i g_{M_i}}|$ becomes 1 at p_i. We then consider the limit $((M_i, R_i g_{M_i}), p_i)$ with respect to the pointed Gromov-Hausdorff distance. Theorem 16.1 implies that it has a limit, which we denote by (X, g_X). Using injectivity radius bound we can show that (X, g_X) is a Ricci flat Riemannian manifold. (It is noncompact but complete.) It also satisfies the following condition :

[40](20.1) is scale invariant if and only if $\dim M = 4$. (We do not need Einstein condition for this.) In this sense also the situation is very much similar to the study of Yang-Mills equation in dimension 4. (Compare also the footnote at the beginig of §7.)

$$\left\{\begin{array}{l}\int_X |R_X|^{n/2}\Omega_X \le C_1\\ \mathrm{Vol}(B_p(R,X)) \ge C_2R^n.\end{array}\right. \tag{20.3}$$

(See [14].) We then can apply the following Theorem 20.3. We define :

Definition 20.1. A complete pointed Riemannian manifold $((X,g),p)$ is said to be *locally almost Euclidean* (abbreviated by ALE hereafter) of order $\tau > 0$, if there exists a finite group $\Gamma \subset O(n)$ and a diffeomorpism $\Phi : X \setminus B_p(R,X) \to (\mathbb{R}^n \backslash B_0(R,\mathbb{R}^n))/\Gamma$ such that

$$|(\Phi^{-1})^* g_X - g_{can})(x)| \le C|x|^{-\tau}, \tag{20.4a}$$

$$\frac{|(\nabla^k\Phi^{-1})^* g_X(x) - (\nabla^k\Phi^{-1})^* g_X(y)|}{|x-y|^\alpha} \le C\min(|x|,|y|)^{-1-\tau-\alpha}, \tag{20.4b}$$

holds for some α and R. Here g_{can} is the metric on $B_p(R,X) \ge C_2R^n$ induced by the Eulidean metric on $\mathbb{R}^n$.

Theorem 20.3 (Bando-Kasue-Nakajima [14])**.** *If (X,g_X) is an n dimensional Einstein manifold satisfying* (20.3) *then it is ALE of order $n-1$. If (X,g_X) is Einstein-Kähler and $n=4$ then it is ALE of order n.*

Combining them we have :

Theorem 20.4 ([14], Anderson [11])**.** *The limit space X in Theorem 20.2 is an Einstein orbifold*[41]*.*

In higher dimension Theorems 23.16, 23.17 give a natural generalization of the results we explained here. If we remove the assumption $\mathrm{Vol}(M_i) \ge v > 0$ (namely if we study collapsing situation), then even in the case of Einstein manifold, not so many thing is known. This problem is related to mirror symmetry in string theory and is calling attention of several differential geometers working on it. There is a result by Gross-Wilson [67] which discuss the case of K3 surface in the collapsing situation and obtain a singular torus fibration.

We now consider more general Riemannian manifolds under condition of Ricci curvature below. To obtain a result similar to Theorem 3.4 we need to avoid the phenomenon we described in Observation 20.1. There are several results assuming lower bound of injectivity radius, for this purpose. We denote by $\mathfrak{S}_n(D, i > \rho)$ the set of all isometry classes of n dimensional compact Riemannian manifold (without boundary) such that $Ricci_M \ge -(n-1)$, $Diam(M) \le D$ and $i_M \ge \rho$ everywhere. Let $\alpha \in (0,1)$.

[41] A similar results hold under an additional assumption (20.2)

Theorem 20.5 (Anderson-Cheeger [12]). *Let $M_i \in \mathfrak{S}_n(D, i > \rho)$ and $X = \lim^{GH}_{i\to\infty} M_i$. Then X is a Riemannian manifold of C^α class and there exists diffeomorphisms $\varphi_i : M_i \to X$ such that $(\varphi_i^{-1})^* g_{M_i}$ converges to g_X with respect to C^α norm.*

Remark 20.2. Under stronger assumption $|Ricci_M| \le (n-1)$, $Diam(M) \le D$ and $i_M \ge \rho$, Anderson [8] proved a stronger result. Namely the limit space X is a $C^{1,\alpha}$ Riemannian manifold and $(\varphi_i^{-1})^* g_{M_i}$ converges to g_X with respect to $C^{1,\alpha}$ norm. It was applied (in [8]) to prove a sphere theorems and pinching theorem for almost Einstein metric.

Corollary 20.6. *The number of diffeomorphism classes represeted by elements of $\mathfrak{S}_n(D, i > \rho)$ is finite.*

The proof of Theorem 20.5 is quite similar to the argument §6,7,8. Namely we construct harmonic coordinate and obtain an appropriate estimate then the proof is complete by using diffeotopy extension theorem (or center of mass technique which we can apply to a smooth metric near the limit C^α metric.) So the new result in [12] is the following :

Theorem 20.7 ([12] Theroem 0.1). *There exists $C(n,\rho), \epsilon(n,\rho) > 0$ with the following property. Let $M \in \mathfrak{S}_n(D, i > \rho)$. We can then cover M by harmonic corrdinate U_i such that the $C^{1,\alpha}$ norm of the coordinate transformation is smaller than $C(n,\rho)$ and the C^α norm of the metric tensor written in this coordinate is smaller than $C(n,\rho)$. Moreover for any $p \in M$, the metric ball $B_p(\epsilon(n,\rho), M)$ is contained in some U_i.*

21. Sphere theorem and L^2 comparison theorem

In the last three sections, we concern with the class of Riemannian manifolds with Ricci curvature bounded from below. Especially we discuss results obtained by Colding and Cheeger-Colding recently. The surveys [46, 47, 62] and a book [26] are recommended for their results. The basic tool to study such Riemannian manifolds is Theorem 5.2. So we first draw some of its consequences. We put

$$\begin{aligned} A_p(a,b;M) &= \{x \in M | a \le d(p,x) \le b\}, \\ S_p(a;M) &= \{x \in M | d(p,x) = a\}. \end{aligned} \tag{21.1}$$

Lemma 21.1. *If $Ricci_M \ge \kappa$, $a < b < c$, then*

$$\frac{\mathrm{Vol}(A_p(a,b;M))}{\mathrm{Vol}(A_{p_0}(a,b;\mathbb{S}^n(\kappa)))} \ge \frac{\mathrm{Vol}(A_p(b,c;M))}{\mathrm{Vol}(A_{p_0}(b,c;\mathbb{S}^n(\kappa)))} \tag{21.2}$$

and

$$\begin{aligned} &\frac{\mathrm{Vol}(S_p(a;M))}{\mathrm{Vol}(S_{p_0}(a;\mathbb{S}^n(\kappa)))} \\ &\ge \frac{\mathrm{Vol}(A_p(a,b;M))}{\mathrm{Vol}(A_{p_0}(a,b;\mathbb{S}^n(\kappa)))} \ge \frac{\mathrm{Vol}(S_p(b;M))}{\mathrm{Vol}(S_{p_0}(b;\mathbb{S}^n(\kappa)))}. \end{aligned} \tag{21.3}$$

(21.2) follows from

$$\frac{\mathrm{Vol}(A_p(0,a;M))+\mathrm{Vol}(A_p(a,b;M))}{\mathrm{Vol}(A_{p_0}(0,a;\mathbb{S}^n(\kappa)))+\mathrm{Vol}(A_{p_0}(a,b;\mathbb{S}^n(\kappa)))}$$
$$\geq \frac{\mathrm{Vol}(A_p(0,a;M))+\mathrm{Vol}(A_p(a,b;M))+\mathrm{Vol}(A_p(b,c;M))}{\mathrm{Vol}(A_{p_0}(0,a;\mathbb{S}^n(\kappa)))+\mathrm{Vol}(A_{p_0}(a,b;\mathbb{S}^n(\kappa)))+\mathrm{Vol}(A_{p_0}(b,c;\mathbb{S}^n(\kappa)))}.$$

By taking limit $b \to a$ and $b \to c$ in (21.2) we obtain (21.3). □

Lemma 21.2. *If $Ricci_M \geq (n-1) = \dim M - 1$ and if $p, q \in M$ with $d(p,q) > \pi - \epsilon$, then for each $x \in M$ we have*

$$d(p,x) + d(q,x) - d(p,q) \leq \tau(\epsilon|n).$$

To show Lemma Let $\delta = d(p,x) + d(q,x) - d(p,q)$, $r = d(p,x) - \delta/2$, $s = d(p,q) - r = d(q,x) - \delta/2$. Then $(B_p(r,M) \cup B_q(s,M)) \cap B_x(\delta/2,M) = \emptyset$. $B_p(r,M) \cap B_q(s,M) = \emptyset$. Therefore, by Theorem 5.2, we have

$$\frac{\mathrm{Vol}(B_p(r,M) \cup B_q(s,M))}{\mathrm{Vol}(M)} \geq 1 - \tau(\epsilon|n), \quad \frac{B_x(\delta/2,M)}{\mathrm{Vol}(M)} \geq C\delta^n.$$

Hence $\delta < \tau(\epsilon|n)$ as required. □

Corollary 21.3. *If $Ricci_M \geq (n-1) = \dim M - 1$ and if $p, q \in M$ with $d(p,q) > \pi - \epsilon$, then*

$$Diam(M \backslash B_p(\pi - \epsilon, M)) < \tau(\epsilon|n).$$

Corollay 21.3 is an immediate consequence of Lemma 21.2 and Myers' theorem 5.4. We remark that Corollay 21.3 is a version of Proposition 4.4. Namely the coclusion of Corollay 21.3 is weaker than that of Proposition 4.4 but it holds under milder assumption.

Lemma 21.4. *If $Ricci_M \geq (n-1) = \dim M - 1$ and if $p, q \in M$ with $d(p,q) > \pi - \epsilon$, then*

$$\frac{\mathrm{Vol}(S_p(\delta;M))}{\mathrm{Vol}(S_{p_0}(\delta;\mathbb{S}^n(1)))} \leq \frac{\mathrm{Vol}(S_p(\pi-\delta;M))}{\mathrm{Vol}(S_{p_0}(\pi-\delta;\mathbb{S}^n(1)))} + \tau(\epsilon|\delta,n). \tag{21.4}$$

We remark

$$\frac{\mathrm{Vol}(S_p(\delta;M))}{\mathrm{Vol}(S_{p_0}(\delta;\mathbb{S}^n(1)))} \geq \frac{\mathrm{Vol}(S_p(\pi-\delta;M))}{\mathrm{Vol}(S_{p_0}(\pi-\delta;\mathbb{S}^n(1)))}$$

is a consequence of (21.1). Hence (21.4) implies that the ratio or the volume, $\mathrm{Vol}(S_p(t;M))/\mathrm{Vol}(S_{p_0}(t;\mathbb{S}^n(1)))$ is almost constant for $t \in [\delta, \pi - \delta]$.

Let us prove Lemma 21.4. Let $\epsilon \ll \rho \ll \delta$. By Corollary 21.3, we have :

$$\begin{aligned} & A_q(\delta-2\rho,\delta-\rho;M) \\ & \subseteq A_p(\pi-\delta+\rho-\tau(\epsilon|n),\pi-\delta+2\rho+\tau(\epsilon|n);M). \end{aligned} \tag{21.5}$$

We may assume $\rho-\tau(\epsilon|n)\geq 0$. We remark

$$\mathrm{Vol}(A_{p_0}(\delta-2\rho,\delta-\rho;\mathbb{S}^n(1)))=\mathrm{Vol}(A_{p_0}(\pi-\delta+\rho,\pi-\delta+2\rho;\mathbb{S}^n(1))).$$

Therefore (21.5) (together with Lemma 21.1) implies the first inequality of :

$$\begin{aligned} & \frac{\mathrm{Vol}(A_q(\delta-2\rho,\delta-\rho;M))}{\mathrm{Vol}(A_{p_0}(\delta-2\rho,\delta-\rho;\mathbb{S}^n(1)))} \\ & \leq \frac{\mathrm{Vol}(A_p(\pi-\delta,\pi-\delta+\rho;M))}{\mathrm{Vol}(A_{p_0}(\pi-\delta,\pi-\delta+\rho;\mathbb{S}^n(1)))}+\tau(\epsilon|\delta,\rho,n) \\ & \leq \frac{\mathrm{Vol}(A_p(\delta-2\rho,\delta-\rho;M))}{\mathrm{Vol}(A_{p_0}(\delta-2\rho,\delta-\rho;\mathbb{S}^n(1)))}+\tau(\epsilon|\delta,\rho,n) \end{aligned} \tag{21.6}$$

Here the second inequlity is a consequence of Lemma 21.1. Changing the role of p and q we have

$$\begin{aligned} & \frac{\mathrm{Vol}(A_p(\delta-\rho,\delta;M))}{\mathrm{Vol}(A_{p_0}(\delta-\rho,\delta;\mathbb{S}^n(1)))} \\ & \leq \frac{\mathrm{Vol}(A_q(\delta-\rho,\delta;M))}{\mathrm{Vol}(A_{p_0}(\delta-\rho,\delta;\mathbb{S}^n(1)))}+\tau(\epsilon|\delta,\rho,n) \end{aligned} \tag{21.7}$$

Therefore by (21.6), (21.7) and Lemma 21.1 we have

$$\frac{\mathrm{Vol}(S_p(\pi-\delta;M))}{\mathrm{Vol}(S_{p_0}(\pi-\delta;\mathbb{S}^n(1)))}+\tau(\epsilon|\delta,\rho,n)\geq\frac{\mathrm{Vol}(S_p(\delta;M))}{\mathrm{Vol}(S_{p_0}(\delta;\mathbb{S}^n(1)))}$$

as required. □

Lemma 21.5. *If $Ricci_M\geq\kappa$ and $p\in M$ then*

$$\mathrm{Vol}(B_p(R,M))\leq\mathrm{Vol}(B_{p_0}(R,\mathbb{S}^n(\kappa))).$$

This is an immediate consequence of Theorem 5.2. □

We next discuss sphere theorems. The sphere theorem appearing here can be regarded as a generalizatin of Theorem 17.20. So we first sketch its proof. We first remark :

Lemma 21.6 ([80, 111])**.** *If $K_M\geq 1$, $\mathrm{Vol}(M)>\mathrm{Vol}(S^n)-\epsilon$, then the Gromov-Hausdorff distance between M and S^n is smaller than $\tau(\epsilon|n)$.*

Let $p\in M$ we identify T_pM with $T_{p_*}S^n$ for a point $p_*\in S^n$. We then define $\Phi:M\to S^n$ by $\Phi=\mathrm{Exp}_{p_*}\circ\mathrm{Exp}_p^{-1}$. Note Exp_p^{-1} is discontinuous. Using Corollary 21.3, Lemma 21.5, Toponogov comparison theorem 4.7, we can show that Φ is an $\tau(\epsilon|n)$-Hausdorff approximation. We omit the detail since we discuss sharper result (Theorem 21.8) later. □

Now under the assumption of Lemma 21.6 we can find points $p_0, \cdots, p_n$, $q_0, \cdots, q_n \in M$ such that

$$\begin{aligned} &|d(p_i,p_j) - \pi/2| < \tau(\epsilon|n), \qquad |d(p_i,q_j) - \pi/2| < \tau(\epsilon|n), \\ &|d(p_i,q_i) - \pi| < \tau(\epsilon|n). \end{aligned} \tag{21.8}$$

In fact, if $S^n \subseteq \mathbb{R}^{n+1}$, the points $p'_i = (0,\cdots,0,\overset{i}{1},0,\cdots,0)$, $q'_i = -p'_i$, satisfy (21.8). Hence we can choose $p_i = \Phi(p'_i)$ where $\Phi : S^n \to M$ is an ϵ-Hausdorff approximation.

Moreover, in case of S^n, the canonical embedding $I_{S^n} : S^n \to \mathbb{R}^{n+1}$ is obtained by

$$I_{S^n}(x) = (\cos d(p_0,x), \cdots, \cos d(p_n,x)). \tag{21.9}$$

Now the idea is to embed M in a neighborhood of S^n by using a formula similar to (21.9). Namely we first take a smooth function φ_i which is close to $d(x,p_i)$ up to first derivative, if $x \notin B_{p_i}(o(\epsilon),M) \cup B_{q_i}(o(\epsilon),M)$. We then define $I_M : M \to \mathbb{R}^{n+1}$ by

$$I_M(x) = (\varphi_0(x), \cdots, \varphi_n(x)). \tag{21.10}$$

We can then prove that $d(I_M\Phi(x), I(x)) < o(\epsilon)$ and

$$dist(T_{I(x)}S^n, T_{I_M\Phi(x)}(I_M(M)) < \tau(\epsilon|n).$$

Here *dist* in the above formula is a distance as a codimension one linear subspace in $\mathbb{R}^{n+1}$. We can use these two formulas to prove that M is diffeomorphic to S^n (in a similar way to §9.) □

Theorem 17.20 is generalized by Perelman as follows.

Theorem 21.7 ([118]). *There exists $\epsilon_n > 0$ such that if M satisfies $Ricci_M \geq (n-1)$, $\mathrm{Vol}(M) \geq \mathrm{Vol}(S^n) - \epsilon_n$, then M is homeomorphic to a sphere.*

Actually Perelman proved that $\pi_k(M) = 1$ for $k < n$ under the assumption of Theorem 21.7 and apply generalized Poincaré conjecture. The idea of the proof is hard to explain for the author in this kind of article. So we refer [118] or [159]. We will discuss a proof a sharper version Corollary 22.4 in §22.

Remark 21.1. We remark that a similar sphere theorem replacing volume by diameter does not holds. Actually Anderson [9] and Otsu [110] found examples of manifolds (M, g_i) such that $Ricci_{g_i} \geq (n-1)$, $\mathrm{Vol}(M, g_i) \geq v > 0$ and $Diam(M, g_i) \to \pi$ but $M \neq S^n$. (Otsu's example is $S^m \times S^{n-m}$ and Anderson's example is $\mathbb{C}P^n$ or $\mathbb{C}P^2 \sharp \mathbb{C}P^2$.

We remark that $\mathrm{Vol}(M_i) \to \mathrm{Vol}(S^n)$ implies $Diam M_i \to \pi$ (under the assumption $Ricci_{g_i} \geq (n-1)$) by Bishop-Gromov comparison theorem 5.2.

Now we start the review of the works of Colding, who began with the following theorem closely related to Theroems 17.20 and 21.7.

Theorem 21.8 (Colding [43, 44]). *Let M be an n dimensional Riemannian manifold with $Ricci_M \ge (n-1)$.*

(1) *If* $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \epsilon$ *then* $d_{GH}(M, S^n) < \tau(\epsilon|n)$.
(2) *If* $d_{GH}(M, S^n) < \epsilon$ *then* $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \tau(\epsilon|n)$.

The proof is somewhat similar to the proof of Theorem 17.20. However we need several new ideas. Especially we need to develop some method to compare I_M (21.10) with I_{S^n} (21.9). In the situation of the proof of Theorem 17.20, this was done by Toponogov's comparison theorem. In our situation, Toponogov's comparison theorem does not apply since there is no sectional curvature bound. Colding developped L^2 comparison theorem for this purpose. We describe it below.

We consider $p_* \in S^n$ and $\ell_* : [0, \alpha] \to S^n$ be a geodesic parametrized by arc length. We put $\ell_*(0) = q_*$, $(d\ell_*/dt)(0) = v_* \in T_{q_*}S^n$. We then put $h_{p_*,\alpha}(v_*, t) = \cos d(\ell_*(t), p_*)$. We can calculate it easily as

$$h_{p_*,\alpha}(v_*, t) = \frac{1}{\sin\alpha}\left(d(p_*, \ell(\alpha))\sin(\alpha - t) + d(p_*, \ell(0))\sin t\right). \tag{21.11}$$

Now we use (21.11) to define a function on M with which we compare the distance function. Let $p \in M$ and $\ell : [0, \alpha] \to M$ be a geodesic parametrized by arc length. We put $\ell(0) = q$, $(d\ell/dt)(0) = v \in T_qM$. ($\ell$ is determined by v so we write $\ell = \ell_v$.) Let $f : M \to \mathbb{R}$ be a function. We then define

$$h_{f,\alpha}(v, t) = \frac{1}{\sin\alpha}\left(f(\ell_v(\alpha))\sin(\alpha - t) + f(\ell_v(0))\sin t\right). \tag{21.12}$$

We remark that $h_{f,\alpha}$ may be regarded as a function of $(v, t) \in SM \times [0, \alpha]$, where SM is the unit tangent bundle $SM = \{v \in TM | |v| = 1\}$. In case $f(x) = d(p, x)$ we put $h_{f,\alpha} = h_{p,\alpha}$.

Now L^2 Toponogov theorem in [44] is as follows.

Theorem 21.9 ([44] Proposition 1.15). *Let $a_0 \in [\pi/2, \pi)$. We assume $Ricci_M \ge (n-1)$ and $p, q \in M$ with $d(p, q) \ge \pi - \epsilon$. Then, for $\alpha \le \alpha_0$, we have :*

(21.13)
$$\frac{1}{\alpha\mathrm{Vol}(SM)}\int_{v\in SM}\int_0^\alpha \left|\cos d(p, \ell_v(t)) - h_{p,\alpha}(v, t)\right|^2 \Omega_{SM}dt < \tau(\epsilon|n, \alpha_0).$$

(21.14)
$$\frac{1}{\alpha\mathrm{Vol}(SM)}\int_{v\in SM}\int_0^\alpha \left|\frac{d}{dt}\cos d(p, \ell_v(t)) - \frac{dh_{p,\alpha}}{dt}(v, t)\right|^2 \Omega_{SM}dt < \tau(\epsilon|n, \alpha_0).$$

Here Ω_{SM} is the Liouville measure. (Hereafter we omit the symbol of volume form in case it is clear which volume form we use.)

Remark 21.2. We remark that (21.15) means that the length $d(p, \ell_v(t))$ is close to the length of the corresponding triangle in S^n in L^2 sense.

(21.14) means that the angle $\angle p\ell_v(t)\ell_v(0)$ is close to the angle in corresponding triangle in S^n in L^2 sense.

Let us explain a part of the ideas of the proof of Theorem 21.9.

We first recall the following. Let $\lambda_1(M)$ denotes the first nonzero eigenvalue of Laplacian on (the functions of) M.

Theorem 21.10 (Lichnerowicz[100]-Obata[109]). *If an n dimensional Riemannian manifold satisfies $Ricci_M \ge (n-1)$ then $\lambda_1(M) \le -n$. The equality holds if and only if M is isometric to the sphere.*

The proof can be done by Bochner formula, in the same way the the argument of Step 1 below.

We also remark the following theorem by Cheng which is closely related to Theorem 21.10.

Theorem 21.11 ([41]). *Let M be a compact Riemannian manifolds with $Ricci_M \ge (n-1)$. If $Diam(M) \ge \pi$ then M is isometric to S^n.*

What is important for us is that the first eigenfunction of S^n is $\cos d(p, \cdot)$ and is exactly the function we want study in Theorem 21.10. So the idea of the proof of the Theorem 21.10 goes as follows.

Step 1: Let f satisfy $\|\Delta f + nf\| < \delta$, $\|f\| = 1$. (Here $\|\ \ \|$ is the L^2 norm.) We prove :

$$\frac{1}{\alpha \mathrm{Vol}(SM)} \int_{v \in SM} \int_0^\alpha \left| f(\ell_v(t)) - h_{f,\alpha}(v,t) \right|^2 < \tau(\delta|n), \tag{21.15}$$

and a similar estimate for the t derivative of $\cos f(\ell_v(t)) - h_{f,\alpha}(v,t)$ ([44] Lemma 1.4.)

This step uses Bochner-Weitzenbeck formula

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \langle \nabla \Delta f, \nabla f\rangle + \mathrm{Ricci}(\nabla f, \nabla f). \tag{21.16}$$

Here $\mathrm{Hess}(f)(X,Y) = X(Y(f)) - (\nabla_X Y)(f)$. (Note we are using positive Laplacian.) The proof is a kind of "almost version" of the proofs of Theorems 21.10 and 21.11. To clarify geometric ideas, avoiding analytic detail, we consider the case $\Delta f = \lambda f$, $n \ge -\lambda > 0$ and prove $f(\ell_v(t)) = h_{f,\alpha}(v,t)$. We integrate (23.13) and using $\int_M \langle \nabla f_1, \nabla f_2 \rangle = -\int_M \langle \Delta f_1, f_2\rangle$, we find

$$\int_M \left(|\mathrm{Hess}(f)|^2 - |\Delta f|^2 + (n-1)|\nabla f|^2 \right) \le 0$$

Since $\lambda \int_M |\nabla f|^2 = \int_M \langle \nabla \Delta f, \nabla f\rangle = -\int_M |\Delta f|^2$, it follows that

$$\int_M \left(|\mathrm{Hess}(f)|^2 - \frac{\lambda + n - 1}{\lambda} |\Delta f|^2 \right) \le 0.$$

By TraceHess$(f) = \Delta f$ and elementary linear algebra, we find $\lambda = -n$ and

$$\mathrm{Hess}(f) = -f g_M. \tag{21.17}$$

Using the fact $d^2 f(\ell_v(t))/dt^2 = \mathrm{Hess} f(\dot\ell_v(t), \dot\ell_v(t))$ we have

$$\frac{d^2}{dt^2} f(\ell_v(t)) = -f(\ell_v(t)). \tag{21.18}$$

$f(\ell_v(t)) = h_{f,\alpha}(v,t)$ follows. □

Step 2: Let $p, q \in M$ with $d(p,q) > \pi - \delta$. We consider $g(x) = \cos d(p,x)$. We then find f with $\|\Delta f + nf\| < \delta$ and $\|f - g\|_{L^2_1} < \delta$. ($\| \ \|_{L^2_1}$ is Sobolev norm, that is an L^2 norm up to first derivative.)([44] Lemma 1.10.)

The essential part of this step (which is explained below) is to show

$$\left| n\int_M g^2 - \int_M |\nabla g|^2 \right| \le \tau(\delta)\mathrm{Vol}(M), \tag{21.19}$$

$$|\int g| < \tau(\delta)\mathrm{Vol}(M), \tag{21.20}$$

In fact, (21.20) implies that g is almost perpendicular to the 0-th eigenfunction of Laplacian (the constant). Then we can use (21.19) and $\lambda_1 \ge n$ to get conclusion.

Let $a(v,t)$ be as in the proof of Theorem 5.2. We extend it as 0 outside V. (So precisely speaking $a(v,t)$ is the function which we wrote $a'(v,t)$ in the proof of Theoremm 5.2.) By Lemma 21.4 we have

$$\int_{v\in S^{n-1}} a(v,\delta) \le \int_{v\in S^{n-1}} a(v,\pi-\delta) + \tau(\epsilon|\delta,n) \tag{21.21}$$

On the other hand, the map $t \mapsto a(v,t)$ is nondecreasing by the proof of Theorem 5.2. It follows that

$$\left| \int_{v\in S^{n-1}} a(v,s) - \int_{v\in S^{n-1}} a(v,s') \right| \le \tau(\epsilon|\delta,n) \tag{21.22}$$

for $s, s' \in [\delta, \pi-\delta]$. Therefore

$$\begin{aligned} \left|\int_M g\right| &= \left| \int_{v\in S^{n-1}} \int_{t=0}^{\pi} a(v,t)\cos t \sin^{n-1} t \right| \\ &= \left| \int_{v\in S^{n-1}} \int_{t=0}^{\pi/2} (a(v,t) - a(v,\pi-t))\cos t \sin^{n-1} t \right| \\ &\le \tau(\epsilon,\delta|n)\mathrm{Vol}(M). \end{aligned} \tag{21.23}$$

Moreover using $|\nabla g|^2(x) = \sin^2 d(p,x)$ we have :

$$\int_M |\nabla g|^2 = \int_{v\in S^{n-1}} \int_{t=0}^{\pi} a(v,t)\sin^{n+1} t.$$

On the other hand

$$\int_M |g|^2 = \int_{v\in S^{n-1}} \int_{t=0}^{\pi} a(v,t)\cos^2 t \sin^{n-1} t.$$

We remark $\int_0^\pi \sin^{n+1} t dt = n\int_0^\pi \cos^2 t \sin^{n-1} t dt$. Hence using (21.22) we can easily show

$$\left| \int_M |\nabla g|^2 - n\int_M |g|^2 \right| < \tau(\epsilon,\delta|n)\mathrm{Vol}(M). \tag{21.24}$$

(21.23) and (21.24) complete this step as we mentioned before. □

These two steps and some more arguments imply Theorem 21.9. (The integral in Theorem 21.9 is taken with respect to the Liouville measure on the unit sphere bundle. In argument so far the measure is taken with respect to the measure on M itself (or its products). They are equivalent by Theorem 5.2.) □

We remark that in Theorem 21.9 we use only a weaker assumption $Diam(M) \sim \pi$ and not yet $\mathrm{Vol}(M) \sim \mathrm{Vol}(S^n)$. (Compare Remark 21.1 which shows that $Diam(M) \sim \pi$ does not imply $d_{GH}(M, S^n)$ is small.)

Now using Theorem 21.9, the proof of Theorem 21.8 goes roughly as follows.

We first explain (1). Let us assume $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \delta$. It then implies that for each $p \in M$ there exists $q \in M$ such that $d(p,q) > \pi - \tau(\delta)$[42]. (This follows from Bishop-Gromov Theorem 5.2.) Now we claim :

Lemma 21.12 ([44] Lemma 2.25)**.** *Under the assumption of Theorem 21.8 (1) there exists p_i, q_i ($i = 0, \cdots, n$) such that* (21.8) *holds.*

Once we have Lemma 21.12 we can construct a Hausdorff approximation $\Phi : M \to S^n$ by perturbing $x \to (\cos d(x,p_0), \cdots, \cos d(x,p_n))$. In fact, by Theorem 21.8, we can prove that the function $x \mapsto \cos d(x,p_0)$ behave in a similar way (modulo $\tau(\delta)$) outside the set of measure $\tau(\delta)$. This is enough to show that it is a Hausdorff approximation. □

Remark 21.3. As we mentioned before we can use L^2 comparison theorem directly to show that a map is a Hausdorff approximation. However we can not use it directly to find a homeomorphism. This is because L^2 comparison theorem does not tell what happens on a set of small measure. This point is very different from Toponogov comparison theorem, which however works only under the assumption of sectional curvarure. We can use several 'indirect' argument to obtain various topological information using L^2 comparison theorem. (See the next two sections.)

[42]Shiohama-Yamaguchi [140] introduced the notion of radius of M that is $\inf_p \sup_q d(p,q)$. This assertion means radius of M is close to π.

The proof of Lemma 21.12 uses Theorem 21.7 and goes as follows. We construct p_i, q_i $(i = 0, \cdots, k)$ satisfying (21.8) by induction on k. Suppose we have p_i, q_i $(i = 0, \cdots, k)$. We then construct a map $\Phi_k : M \to \mathbb{R}^k$ by $x \to (\cos d(x, p_0), \cdots, \cos d(x, p_k))$. We construct a set A_k from of $p_0, \cdots, p_k, q_0, \cdots, q_k$. In case $M = S^n$ and $p_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ $A_k = \mathbb{R}^{k+1} \cap S^n = S^k$ and obtained by joining p_i, q_i several times along minimal geodesics. We imitate the construction of A_k from p_i, q_j in M to obtain $A_k \subset M$. (Actually we need to join only by good geodesics ℓ_v that is a geodesic such that $\cos d(p, \ell_v(t)) - h_{p,a}(v, t)$ is small. Theorem 21.9 implies that there are enough such geodesics.)

Now the restriction of Φ_k to A_k is similar to one for S^n. Hence $\Phi_k(A_k)$ lies in a neighborhood of S^k and we may regard $A_k \cong S^k$. Since $k < n$, Theorem 21.7 implies that A_k is homotopic to zero in M. This implies that there exists $p_{k+1} \in M$ such that $\Phi_k(p_{k+1}) = 0$. We take q_{k+1} with $d(p_{k+1}, q_{k+1}) > \pi - \delta$. Thus induction works. $\square$

To prove (2) of Theorem 21.8 we proceed as follows. We take p_i, q_i $(i = 0, \cdots, n)$ such that (21.8) holds. (Since $d_H(M, S^n)$ is small we can take such p_i, q_i.) We use it to construct $\Phi : M \to S^n$ by $\tilde{\Phi}(x) = (\cos d(p_0, x), \cdots, \cos d(p_n, x))$, $\Phi(x) = \tilde{\Phi}(x)/|\tilde{\Phi}(x)|$. Using Theorem 21.9, we find that the determinant of the Jacobi matrix of Φ is almost everywhere close to 1. It follows that $|\mathrm{Vol}(M) - \mathrm{Vol}(\Phi(M))| < \tau(\delta|n)$. We need another idea to show that $\mathrm{Vol}(S^n \setminus \Phi(M)) < \tau(\delta|n)$. Actually for this purpose we need a "local version" of Theorem 21.15 ([43] Proposition 4.5). We omit it. $\square$

The argument of the proof of Theorem 21.8 is a prototype of the argument which are used by Coolding and Cheeger-Colding in several other places. We explain them more in the last two sections where the argument is combined with other arguments which are of more analytic nature.

22. Hausdorff convergence and Ricci curvature - I -

In §21, we compared the distance function of a manifold of positive Ricci curvature to one of round sphere, in the sense of L_1^2 norm. In this section, we compare the distance function of a manifold of almost nonnegative Ricci curvature to one of Eulidean space.

Theorem 22.1 (Colding [45] Theorem 0.1). *Let M_i be a sequence of n dimensional Riemannian manifolds with $\mathrm{Ricci}_{M_i} \geq -(n-1)$ and let M_∞ be another n dimensional Riemannian manifold. We assume $\lim^{GH}_{i\to\infty} M_i = M_\infty$. Then we have :*

$$\lim_{i\to\infty} \mathrm{Vol}(M_i) = \mathrm{Vol}(M_\infty).$$

Remark 22.1. Actually Colding proved the following stronger (local) result in [45]. Let M_i and M_∞ be complete Riemannian manifolds. We assume $Ricci_{M_i} \ge -(n-1)$. Let $p_i \in M_i$, $p_\infty \in M_\infty$, and $r > 0$. We assume that $\lim^{GH}_{i\to\infty} B_{p_i}(r, M_i) = B_{p_\infty}(r, M_\infty)$. Then $\lim_{i\to\infty} \mathrm{Vol}(B_{p_i}(r, M_i)) = \mathrm{Vol}(B_{p_\infty}(r, M_\infty))$.

Together with a result by Perelman and using results of controlled surgery, Theorem 22.1 implies the following.

Theorem 22.2 ([45])**.** *In the situation of Theorem 22.1, M_i is homotopy equivalent to M_∞ for large i. Moreover M_i is homeomorphic to M_∞ for large i if $n \neq 3$.*

Remark 22.2. In case the limit space is singular we can not prove a result similar to Theorem 22.2 because of Example 21.1 by Anderson and Otsu.

The Gromov-Hausdorff limit of the metrics Otsu constructed on $S^3 \times S^2$ is a suspension of $S^2 \times S^2$ and hence is not a topological manifold.

Theorem 22.1 follows from Theorem 22.2 roughly in the following way. Choose $p_j^\infty \in M_\infty$, $j = 1, \cdots, N$ and small $r > 0$ such that

$$\bigcup_{i=1}^{N} B_{p_j^\infty}(r, M_\infty) = M_\infty,$$

and

$$1 - \delta \le \frac{\mathrm{Vol}(B_{p_j^\infty}(r, M_\infty))}{\mathrm{Vol}_0(B_0(r, \mathbb{R}^n))} \le 1 + \delta \tag{22.1}$$

Let $\Phi_i : M_\infty \to M_i$ be ϵ_i Hausdorff approximation with $\epsilon_i \to 0$. We take $p_j^i = \Phi_i(p_j^\infty) \in M_i$. Since $d_H(B_{p_i^j}(r, M_i), B_{p_\infty^j}(r, M_\infty))$ is small it follows from Theorem 22.1 (more precisely its local version stated in Remark 22.1) together with (22.1) that

$$1 - 2\delta \le \frac{\mathrm{Vol}(B_{p_j^i}(r, M_\infty))}{\mathrm{Vol}_0(B_0(r, \mathbb{R}^n))} \le 1 + 2\delta. \tag{22.2}$$

We can then apply the method of Perelman appeared in the proof of Theorem 21.6. It may[43] imply that $B_{p_j^i}((1-\epsilon)r, M_\infty)$ is contractible in $B_{p_j^i}(r, M_\infty)$. This will imply that M_i is homotopy equivalent to M in a way similar to the proof of Theorem 3.5 in §15. Using controll surjery in a way similar to [113] we can prove that M_i is homeomorphic to M. □

[43] I wrote "may" here since Perelman did not state this result explicitely and only say that "The Main Lemma can obviously be modified … " at [118] p300. Indeed it is very likely so. But I did not check it in detail. By the way, Colding quote [119] in place of [118] at [45] p478 just before Theorem 0.4. I believe it is a misprint.

The proof of Theorem 22.2 is not worked out in so much detail in [45]. However we do not need to worry about it at all now, since Cheeger-Colding [29] improved Theorem 22.2 as follows.

Theorem 22.3 ([29] Theorem A.1.12)**.** *In the situation of Theorem 22.1, M_i is diffeomorphic to M_∞ for large i.*

We discuss its proof later in this section. Theorem 22.3 together with Theorem 21.8 immediately imply the following sharpning of Theorem 21.7. (We stated it as Theorem 2.6 in §2.)

Corollary 22.4 (Cheeger-Colding [29] Theorem A.1.10)**.** *There exists $\epsilon_n > 0$ such that if M satisfies $Ricci_M \ge (n-1)$, $\mathrm{Vol}(M) \ge \mathrm{Vol}(S^n) - \epsilon_n$ then M is diffeomorphic to a sphere.*

Remark 22.3. We remark Therem 21.8 is used in the proof of Corollary 22.4. The proof of Therem 21.8 we sketched in the last section uses Theorem 21.7. However we can avoid it as follows. Let $Ricci_{M_i} \ge (n-1)$, $\mathrm{Vol}(M_i) \ge \mathrm{Vol}(S^n) - \epsilon_i$, where $\epsilon_i \to 0$. We may assume that M_i converges to a metric space X. Then, by Theorem 23.11, X is isometric to a metric suspension SY, where SY is defined in Example 23.1 (3). Using the assumption on M_i and Theorem 22.5, we can show that the tangent cone T_xX of X at any point $x \in X$ is $\mathbb{R}^n$. Therefore, since $X = SY$, it follows that $Y = S^{n-1}$. Hence $X = S^n$ (isometric) as required.

Remark 22.4. The assumption of Theorem 22.2 plus an additional assumption $Ricci_{M_i} \le \lambda$ implies that the Riemannian metric of M_i converges to one of M in $C^{1,\alpha}$ topoology (after identifying manifolds by appropriate diffeomorphism). ([45] Theorem 0.6.)

We now explain some of the ideas of the proof of Theorem 22.1. The main part of the proof is the proof of (2) of the following theorem.

Theorem 22.5 ([45] Theorem 0.8 and Corollary 2.19)**.** *Let M be an n dimensional Riemannian manifold with $Ricci_M \ge -\lambda$ and $p \in M$.*

(1) *If $\mathrm{Vol}(B_p(1, M)) \ge \mathrm{Vol}(B_0(1, \mathbb{R}^n)) - \epsilon$ the we have*
$$d_{GH}(B_p(1, M), B_0(1, \mathbb{R}^n)) < \tau(\epsilon, \lambda|n).$$

(2) *If $d_{GH}(B_p(1, M), B_0(1, \mathbb{R}^n)) < \epsilon$ the we have*
$$\mathrm{Vol}(B_p(1, M)) \ge \mathrm{Vol}(B_0(1, \mathbb{R}^n)) - \tau(\epsilon, \lambda|n).$$

An argument to show Theorem 22.1 by using Theorem 22.5 (2) is omitted.

Let us sketch how to prove Theorem 22.5 (2). We will discuss the proof of Theorem 22.5 (1) in the next section. We only show the following version.

Lemma 22.6. *If M satisfies $d_{GH}(B_p(2R,B),B_0(2R,\mathbb{R}^n)) < \epsilon$ and $Ricci_M \ge -\lambda$ then we have*

$$\mathrm{Vol}(B_p(1,M)) \ge (1-\tau(\epsilon,\lambda,1/R|n))\mathrm{Vol}(B_1(1,\mathbb{R}^n)).$$

The argument to show Theorem 22.5 (2) using Lemma 22.6 is tricky but technical. (See [45] P 494.) (Note the inequality of oppsite direction

$$(1-\tau)\mathrm{Vol}(B_p(1,M)) \le \mathrm{Vol}(B_1(1,\mathbb{R}^n)).$$

is a consequence of Theorem 5.2.)

Theorem 22.5 looks similar to Theorem 21.8. The proof of Lemma 22.6 also is similar. We first need a result corresponding Theorem 21.15. In the proof of Theorem 21.15 we cosider the function $x \mapsto \cos d(p,x)$ in case when there exists q with $d(p,q) \ge \pi - \delta$. Here we consider the following function b^i_+ $i = 1,\cdots,n$ instead.

Let $\Phi : B_0(2R,\mathbb{R}^n) \to B_p(2R,M)$ be an ϵ-Hausdorff approximation. Let $q_i = \Phi(0,\cdots,0,\overset{i}{1},0,\cdots,0) \in M$. We put

$$b_i(x) = d(x,q_i) - d(p,q_i), \tag{22.3}$$

and study it in the ball $B_p(1,M)$. We remark that b_i may be regarded as an approximation of Busemann function (Definiton 16.6). In the proof of Cheeger-Gromoll splitting theorem 16.4, subharmonicity of Busemann function is the main point.

We choose ρ with $1 \ll \rho \ll R$. We consider $\mathbf{b}_i : B_p(\rho,M) \to \mathbb{R}$ such that

$$\Delta \mathbf{b}_i = 0, \tag{22.4a}$$

$$\mathbf{b}_i = b_i \qquad \text{on } \partial B_p(\rho,M). \tag{22.4b}$$

In the case of Euclidean space Busemann function is nothing but a linear function. So we compare b_i with a linear function. We put $g_i(v,t) = \mathbf{b}_i(\ell_v(t))$.

Proposition 22.7. *For $r \le \alpha < 1$, we have*

$$\|\mathbf{b}_i - b_i\|_{L^2_1(B_p(1,M))} \le \tau, \tag{22.5a}$$

$$\int_{v\in SB_p(1,M)} \left| \frac{dg_i(v,\cdot)}{dt}(r) - \frac{g_i(v,\alpha)-g_i(v,0)}{\alpha} \right| < \tau, \tag{22.5b}$$

$$\int_{B_p(1,M)} |\langle \nabla \mathbf{b}_i, \nabla \mathbf{b}_j\rangle - \delta_{ij}| < \tau, \tag{22.5c}$$

$$\int_{B_p(1,M)} |\mathrm{Hess}(\mathbf{b}_i)| < \tau. \tag{22.5d}$$

here $\tau = \tau(\lambda,\rho/R,1/\rho|n)$ and $\| \ \|_{L^2_1}$ is the L^2 norm up to first derivative.

The proof of (22.5a) is based on Li-Shoen's Poincaé inequality [101] (estimate of the first eigenvalue of $B_p(\rho, M)$), and the proof of (22.5d) is based on Bochner-Weitzenbeck formula (19.1) and Cheng-Yau's gradient estimate [42]. Then (22.5b) follows in a way similar to the proof of Theorem 21.9. We can use it to prove (22.5c). □

We put

$$\tilde{\Phi} = (\mathbf{b}_1, \cdots, \mathbf{b}_n) : B_p(1, M) \to \mathbb{R}^n.$$

(22.5a), (22.5b) imply that it induces an τ-Hausdorff approximation to $B_0(1, \mathbb{R}^n)$. (22.5c) implies that $\tilde{\Phi}$ almost preserces volume.

To complete the proof of Lemma 22.6 we need to show that $\mathrm{Vol}(B_0(1, \mathbb{R}^n) \setminus \tilde{\Phi}(B_p(1, M)))$ is small. We can prove it as follows[44]. Using (22.5) we can find a point $p_0 \in B_p(1/2, M)$ such that $\tilde{\Phi}^{-1}(\tilde{\Phi}(p_0)) = \{p_0\}$. (See [26]p53-54 for the proof of this fact.) On the other hand, since $\tilde{\Phi}$ is a τ-Hausdorff approximation, it follows that $\tilde{\Phi}(\partial B_p(1, M)) \subset B_{1+\tau}(0, \mathbb{R}^n) \setminus B_{1-\tau}(0, \mathbb{R}^n)$. Hence

$$\tilde{\Phi}_* : H_n(B_p(1, M), \partial B_p(1, M); \mathbb{Z}_2) \to H_n(B_{1+\tau}(0, \mathbb{R}^n), B_{1-\tau}(0, \mathbb{R}^n); \mathbb{Z}_2)$$

is well defined. Note

$$H_n(B_p(1, M), \partial B_p(1, M); \mathbb{Z}_2) \cong H_n(B_{1+\tau}(0, \mathbb{R}^n), B_{1-\tau}(0, \mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Using $\tilde{\Phi}^{-1}(\tilde{\Phi}(p_0)) = \{p_0\}$ we can show

$$\tilde{\Phi}_* : H_n(B_p(1, M), \partial B_p(1, M); \mathbb{Z}_2) \to H_n(B_{1+\tau}(0, \mathbb{R}^n), B_{1-\tau}(0, \mathbb{R}^n); \mathbb{Z}_2)$$

is non zero. This implies $\tilde{\Phi}(B_p(1, M))) \supset B_{1-\tau}(0, \mathbb{R}^n)$. This completes the proof of Lemma 22.6. □

We next sketch the proof of Theorem 22.3 given in [29] Appendix A. As is mentioned there this proof is similar to the proof by Cheeger [25] of his finiteness theorem using diffeotopy extension theorem (which we explained briefly in §6).

Let us begin with a definition. Let Z be a complete metric space and $\epsilon, r > 0$. (n is a positive integer.)

Definition 22.1. We say that Z satisfies $\mathcal{R}_{\epsilon,r,n}$ *condition* if for each $x \in Z$ there exists $s < r$ such that

$$d_{GH}(B_x(s, Z), B_0(s, \mathbb{R}^n)) < \epsilon s. \tag{22.6}$$

Theorem 22.8 (Cheeger-Colding [29] Theorems A.1.2, A.1.3)**.** *For each n there exists ϵ_n independent of r such that the following holds. If Z satisfies $\mathcal{R}_{\epsilon,r,n}$ condition with $\epsilon < \epsilon_n$, then, for each $s < r$, we can associate a smooth Riemannian manifold $Z(s)$ with the following properties.*

[44] Here we follows [26]p53-54. Colding's argument in [45] is a bit different.

(1) *There exists a homeomorphism* $\Phi_{Z,s} : Z \to Z(s)$ *which is of* $C^{1-\tau(\epsilon|n)}$*- Hölder continuous. Namely :*

$$C^{-1} \le \frac{(d(\Phi_{Z,s}(x),\Phi_{Z,s}(y)))^{1-\tau(\epsilon|n)}}{d(x,y)} \le C \tag{22.7}$$

for each $x, y \in Z$*. Moreover* $\Phi_{Z,s}$ *is an* $s\tau(\epsilon|n)$ *Hausdorff approximation.*

(2) $Z(s)$ *is 'well-defined' and 'independent' of* s *in the following sense. If* $u \le s$ *then there exists a diffeomorphism which is* $C^{1-\tau(\epsilon|n)}$*-Hölder continuous in a way independent of* t, u*. Namely we have*

$$C^{-1} \le \frac{(d(\Phi_{Z,u,s}(x),\Phi_{Z,u,s}(y)))^{1-\tau(\epsilon|n)}}{d(x,y)} \le C. \tag{22.8}$$

where C *is indepent of* u, s, x, y*. Moreover* $\Phi_{Z,u,s}$ *is an* $s\tau(\epsilon|n)$ *approximation and satisfies :*

$$d(\Phi_{Z,u,s}\circ\Phi_{Z,s}(x),\Phi_{Z,u}(x)) < s\tau(\epsilon|n). \tag{22.9}$$

(3) *If* Z *is a Riemannian manifold then we may choose* $\Phi_{Z,s}$ *to be a diffeomorphism for sufficiently small* s*.*

(4) *There exists* $\delta(n,r) > 0$ *depending* n *and* r *such that if* Z, Z' *both satisfy* $\mathcal{R}_{\epsilon,r,n}$ *condition with* $\epsilon < \epsilon_n$*, and if* $d_{GH}(Z,Z') < \delta(n,r)$ *then there exists a diffeomorphism* $\Psi : Z(r/2) \to Z'(r/2)$ *such that*

$$e^{-\tau(\epsilon,\delta|r,n)} \le \frac{d(\Psi(x),\Psi(y))}{d(x,y)} \le e^{\tau(\epsilon,\delta|r,n)} \tag{22.10}$$

$$d(\Psi\circ\Phi_{Z,r/2}(x),\Phi_{Z',r/2}(x)) < \tau(\epsilon,\delta|r,n). \tag{22.11}$$

To apply Theorem 22.8 for the proof of Theorem 22.3 we need the followig.

Proposition 22.9. *Let* M_i *be a sequence of* n *dimensional Riemannian manifolds and let* M_∞ *be another Riemannian manifold of the same dimension. We assume* $\lim^{GH}_{i\to\infty} M_i = M_\infty$*. Then for each* ϵ *there exists* r *such that* M_i *for large* i *and* M_∞ *satisfy* $\mathcal{R}_{\epsilon,r,n}$ *condition.*

Proposition 22.9 and Theorem 22.8 immediately imply Theorem 22.3.

Let us prove Proposition 22.9. Under the assumption we have $r = r(\mu)$ for each μ such that

$$1-\mu \le \frac{\mathrm{Vol}(B_p(r,M_i))}{\mathrm{Vol}(B_0(r,\mathbb{R}^n))} \le 1+\mu,$$

for large i and $i=\infty$ and any $p \in M_i$. (See (22.2).) Then we apply Theorem 5.2 to obtain

$$1-\mu \le \frac{\mathrm{Vol}(B_p(s,M_i))}{\mathrm{Vol}(B_0(s,\mathbb{R}^n))} \le 1+\mu,$$

for *any* $s \le t$. We now apply Theorem 22.5 (1) after scaled to obtain

$$d_{GH}(B_p(s, M_i), B_0(s, \mathbb{R}^n)) < s\tau(\mu|r, n).$$

as required. (Note since we scale the metric by factor $1/s > 1/r$ so the curvature will be $Ricci \ge -(1-n)r^2$. So the curvature assumption in Theorem 22.5 is satisfied if r is small enough.) □

We remark that independence of ϵ_n of r in Theorem 22.8 played a key role here.

We now prove of Theorem 22.8. Let $100s < r$. We will construct $Z(s)$ first. We remark that by using Assumption 22.6 we can find a subsets $\{x_i | i \in I\} \in Z$ such that

$$\bigcup_{i \in I} B_{x_i}(s, Z) = Z, \tag{22.12a}$$

$$\sharp\{i \in I | B_{x_i}(30s, Z) \cap B_{x_j}(30s, Z) \neq \emptyset\} \le N(n), \tag{22.12b}$$

for each $j \in I$. Here $N(n)$ is independent of i, s. Let $\varphi_i : B_{x_i}(100s, Z) \to B_0(100s, \mathbb{R}^n)$ be $\tau(\epsilon|n)s$ Hausdorff approximaion. We have $\tau(\epsilon|n)s$ Hausdorff approximation $\varphi'_i : B_0(100s, \mathbb{R}^n) \to B_{x_i}(100s, Z)$ such that $dist(\varphi'_i \circ \varphi_i, id) < \tau(\epsilon|n)s$ and $dist(\varphi_i \circ \varphi'_i, id) < \tau(\epsilon|n)s$. We consider

$$\varphi_{ji} = \varphi_j \circ \varphi'_i|_{B_0(10s, \mathbb{R}^n)} : B_0(10s, \mathbb{R}^n) \to B_0(35s, \mathbb{R}^n) \tag{22.13}$$

for $i \cap j$ with $B_{x_i}(30s, Z) \cap B_{x_j}(30s, Z) \neq \emptyset$. It satisfies

$$|d(\varphi_{ji}(x), \varphi_{ji}(x)) - d(x, y)| < \tau(\epsilon|n)s. \tag{22.14}$$

We here remark the following simple lemma.

Lemma 22.10. *If φ_{ji} satisfies* (22.14) *then there exists $\psi'_{ji} : B_0(10s, \mathbb{R}^n) \to B_0(35s, \mathbb{R}^n)$ satisfying* (22.14) *and*

$$e^{-\tau(\epsilon|n)} \le \frac{d(\psi'_{ji}(x), \psi'_{ji}(x))}{d(x, y)} \le e^{\tau(\epsilon|n)} \tag{22.15a}$$

$$d(\varphi_{ji}(x), \psi'_{ji}(x)) < s\tau(\epsilon|n) \tag{22.15b}$$

$$|\psi'_{ji}|_{C^k} < s^{-k} C_{k,n}. \tag{22.15c}$$

Here $C_{k,n}$ depends only on k and n.

The proof is an elementary smoothing argument.

We want to construct a smooth manifold by using ψ_{ji} as a coordinate transformation. It does *not* satisfy $\psi'_{kj} \circ \psi'_{ji} = \psi'_{ki}$ but the following holds if $B_{x_i}(20s, Z) \cap B_{x_j}(20s, Z) \cap B_{x_k}(20s, Z) \neq \emptyset$.

$$d(\psi'_{kj} \circ \psi'_{ji}(x), \psi'_{ki}(x)) \le s\tau(\epsilon|n), \tag{22.16}$$

for $x \in B_0(20s, \mathbb{R}^n)$. We can now use the argument of [25] to approximate ψ'_{ji} by ψ_{ji} which satisfies (22.15) and

$$\psi_{kj} \circ \psi_{ji} = \psi_{ki}. \tag{22.17}$$

(Note that the number of steps we need to take to achieve (22.17) is controlled by (22.12b).)

We thus constructed a manifold $Z(t)$ whose coordinate transformation is ψ_{ji}. We can use partition of unity to modify standard metric on $\mathbb{R}^n$ so that it is compatible with ψ_{ji}. Hence $Z(t)$ is a Riemannian manifold. We will construct $\Phi_{Z,s} : Z \to Z(s)$ later. At this stage we have $\Psi_{Z,s} : Z \to Z(s)$ which is an $s\tau(\epsilon|n)$ Hausdorff approximation.

We next show 'well-definedness' property (2). We first consider the case $u \in [s/2, s]$. Let us suppose we have $Z(u)$ for $u \ge s$. We use the symbol \tilde, for points, maps etc. used to construct $Z(u)$. (Namely we write $\tilde\varphi_{\tilde i}$, $\tilde x_{\tilde i}$, etc.)

Let $B_{\tilde x_{\tilde i}}(30u, Z) \cap B_{x_j}(30s, Z) \ne \emptyset$. We define $\Psi_{j\tilde i} : B_0(20u, \mathbb{R}^n) \to B_0(30s, \mathbb{R}^n)$ by $\Psi_{j\tilde i} = \varphi_j \circ \tilde\varphi'_{\tilde i}$. It satisfies

$$|d(\Psi_{j\tilde i}(x), \Psi_{j\tilde i}(y)) - d(x,y)| \le \tau(\epsilon)s \le 2\tau(\epsilon)u.$$

Hence we can approximate it by a smooth map $\Phi'_{j\tilde i}$ satisfying (22.15c). It is almost compatible with cooridinate transformations ψ_{ji},$\widetilde\psi_{\tilde j\tilde i}$. Hence again by an argument similar to [25] (or by using center of mass technique) we can approximate it by a diffeomorphism $\Phi_{j\tilde i}(x)$ which is exactly compatible with coordinate transformation. We thus obtain $\Phi_{Z,s,u}$, if $u \in [s/2, s]$. It is also an $s\tau(\epsilon|n)$ Hausdorff approximation. Let $\Phi_{Z,u,s}$ be the inverse of it. We remark that we have an inequality

$$e^{-\tau(\epsilon|n)} \le \frac{d(\Phi_{Z,u,s}(x), \Phi_{Z,u,s}(y))}{d(x,y)} \le e^{\tau(\epsilon|n)} \tag{22.18}$$

which is sharper than (22.10) in case $u \in [s/2, s]$.

We remark here that the proof of Theorem 22.8 (4) is almost the same as this argument. (So we do not discuss it.)

Now we continue the proof of (2) for the general u, s. We may assume $u = 2^{-k}s$. And put

$$\Phi_{Z,u,s} = \Phi_{Z,u,2u} \circ \cdots \circ \Phi_{Z,s/2,s}. \tag{22.19}$$

It is a diffeomorphism. We will check (22.10). Let $\rho > 0$. We first remark that $\Phi_{Z,a,b}$ is a $b\tau(\epsilon|n)$ Hausdorff approximation for $a \le b$. (This is because if $b = 2^k a$ then $\Phi_{Z,a,b}$ is $\sum_{j=0}^k \tau(\epsilon|n)2^{-j}b$ Hausdorff approximation.)

We first take ℓ such that

$$e^{-\rho} < \frac{1+2^{-\ell-1}}{1-2^{-\ell-1}} < e^{\rho}. \tag{22.20}$$

Now we take $x, y \in Z(s)$. We take k_1 such that $2^{-k_1-1}s < d(x,y) \le 2^{-k_1}s$. (In case $d(x,y) \ge s$ we put $k_1 = 0$.) Note, if $d(x,y) \le s$, we have

$$k_1 \le -C \log \frac{d(x,y)}{s} \tag{22.21}$$

We use (22.18) and (22.21) and obtain

$$\frac{d(\Phi_{Z,2^{-k_1}s,s}(x),\Phi_{Z,2^{-k_1}s,s}(y))}{d(x,y)} \le e^{k_1\tau(\epsilon|n)} \le Cd(x,y)^{-\tau(\epsilon|n)} \tag{22.22}$$

Combining inequality of the opposite direction which can be proved in a similar way, we have

$$C^{-1} \le \frac{(d(\Phi_{Z,2^{-k_1}s,s}(x),\Phi_{Z,2^{-k_1}s,s}(y)))^{1-\tau(\epsilon|n)}}{d(x,y)} \le C \tag{22.23}$$

We next put $k_2 = k_1 + \ell$. We may take ϵ so small that $\ell\tau(\epsilon|n) < \rho$. Then we have :

$$e^{-\rho} \le \frac{d(\Phi_{Z,2^{-k_2}s,s}(x),\Phi_{Z,2^{-k_2}s,s}(y))}{d(\Phi_{Z,2^{-k_1}s,s}(x),\Phi_{Z,2^{-k_1}s,s}(y))} \le e^{\rho} \tag{22.24}$$

Finally (22.20) inplies

$$e^{-\rho} \le \frac{d(\Phi_{Z,u,s}(x),\Phi_{Z,u,s})}{d(\Phi_{Z,2^{-k_2}s,s}(x),\Phi_{Z,2^{-k_2}s,s}(y))} \le e^{\rho} \tag{22.25}$$

since $\Phi_{Z,u,2^{-k_2}s}$ is an $2^{-\ell-k_1}s\tau(\epsilon|n)$ Hausdorff approximation and $d(x,y) \ge 2^{-k_1-1}s$. The proof of (2) is complete.

We remark here that once the well definedness property is established (3) is actually obvious. We only need to take s much smaller than injectivity radius of Z.

We finally construct $\Phi_{Z,s} : Z \to Z(s)$. We put $\Phi_{ji} = \Phi_{Z;2^{-j}s,2^{-i}s} : Z(2^{-is}) \to Z(2^{-js})$. By construction we have $\Phi_{kj} \circ \Phi_{ji} = \Phi_{ki}$. Thus we have an inductive system. Using (2), it is easy to see that the inductive limit $\lim Z(2^{-i}s)$ is isometric to Z and there exists a map $Z(s) \to \lim_{i\to\infty} Z(2^{-i}s)$ satisfying the condition of (1).

We thus finished the proof of Theorem 22.8. □

23. Hausdorff convergence and Ricci curvature -II -

In this section we continue the discussion about the Gromov-Hausdorff limit of a sequence of manifolds M_i with $Ricci_{M_i} \ge -(n-1)$. It is know that the limit space X can be very wild. For example it may not be locally contractible. ([102, 123].) Nevertheless various results are known for such limit space, some of which we describe in this section.

We first state Theroem 16.4 again.

Theorem 23.1 (Cheeger-Colding [28]). *Let M_i be a sequence of n dimensional Riemannian manifolds with $Ricci_{M_i} > -\lambda_i$ with $\lambda_i \to 0$ and let $(X,p) = \lim^{pGH}_{i\to\infty}(M_i,p_i)$. Suppose X contains a line. Then X is isometric to the direct product $\mathbb{R}\times X'$.*

Remark 23.1. The following a bit more general statement is proved.

Let $Ricci_{M_i} \ge -\lambda_i$ with $\lambda_i \to 0$, and $(X,p) = \lim^{pGH}_{i\to\infty}(M_i,p_i)$. We assume $X \cong \mathbb{R}^k \times Y$ and that Y contains a line. Then $Y \cong \mathbb{R}\times Y'$.

Before explaining the outline of the proof, we mention several of its applications. One important application is Theorems 19.12 and 10.5 which we explained already.

To state other applications, we need some definitions.

Definition 23.1. A *measured metric space* (X,μ) is a pair of metric space X and a Borel measure μ on it. In this article we always assume that $\mu(X)=1$. For pointed measured metric space (X,p,μ) we assume $\mu(B_p(1,X))=1$.

For a Riemannian manifold M we use renormalized volume form $\mu_M = \Omega_M/\mathrm{Vol}(M)$ and regard it as a measured metric space (unless other measure is specified explicitely). For a pointed Riemannian manifold (M,p), we use renormalized volume form $\mu_M = \Omega_M/\mathrm{Vol}(B_p(1,M))$.

Definition 23.2 ([54])**.** A sequence of measured metric spaces (X_i,μ_i) is said to converge to (X,μ) with respect to the *measured Gromov-Hausdorff topology* and write $\lim^{mGH}_{i\to\infty}(X_i,\mu_i) = (X,\mu)$, if there exists a sequence of ϵ_i-Hausdorff approximations $\varphi_i : X_i \to X$ with $\epsilon_i \to 0$, which are Borel measurable and such that, for any continuous function f on X, we have

$$\lim_{i\to\infty}\int_{X_i}(f\circ\varphi_i)d\mu_i = \int_X f d\mu.$$

The pointed measured Gromov-Hausdorff convergence is defined in the same way. (To be precise we need net in place of sequence to define a topology. It is obvious modification and is omitted.)

Remark 23.2. In [76] Chaper $3\frac{1}{2}$D Gromov defined a notion of $\square_\lambda$ convergence for measured metric space. It is similar to but is slightly different from measured Gromov-Hausdorff topology defined above. Namely there is a situation where the support $\mathrm{supp}\,\mu$ of limit measure is different from X. In that case $\square_\lambda$ convergence the limit of (X_i,μ_i) is $(\mathrm{supp}\,\mu,\mu)$, and is different from the limit (X,μ) of measured Gromov-Hausdorff topology. However if $(X_i,\mu_i) = (M_i,\mu_{M_i})$ is a Riemannian manifold and if $Ricci_{M_i} \ge -(n-1)$, then the support of the limit measure is always X itself. (We can prove it using Bishop-Gromov inequailty.) So the two definitions coincide to each other.

Measured Gromov-Hausdorff convergence was introduced to study spectra of Laplace operator. We mention it later.

Lemma 23.2 ([54])**.** *If $\lim^{GH}_{i\to\infty} X_i = X$, and if μ_i is a probability Borel measure on X_i, then there exists a subsequence k_i such that $\lim^{mGH}_{i\to\infty}(X_{k_i},\mu_{k_i}) = (X,\mu)$.*

The proof is elementary.

We remark that the limit measure μ depends on the choice of subsequence in general. In fact let us consider $T^2 = S^1\times S^1$ with metric

$g^f_\epsilon = dt^2 + \epsilon^2 f(t)^2 ds^2$, where $f : S^1 \to \mathbb{R}_+$ is a smooth function. Then (T^2, g_ϵ) converges to S^1 with standard metric and measure fdt with respect to the measured Gromov-Hausdorff topology. On the other hand, the limit in Gromov-Hausdorff distance is indepedent of f.

We denote by $\mathfrak{S}_n(D)$ the set of n dimensional Riemannian manifolds M with $Ricci_M \geq -(n-1)$, $Diam(M) \leq D$. We denote by $\mathfrak{S}_n(\infty)$ the set of n dimensional pointed Riemannian manifold (M, p) such that $Ricci_M \geq -(n-1)$. Let $\overline{\mathfrak{S}}_n(D)$, $\overline{\mathfrak{S}}_n(\infty)$ be the closure of $\mathfrak{S}_n(D)$, $\mathfrak{S}_n(\infty)$ with respect to the Gromov-Hausdorff distance, pointed Gromov-Hausdorff distance, respectively.

We next define the singularity set and the regular set of a length space $X \in \overline{\mathfrak{S}}_n(D)$. We recall that the sequence $(X, R_i d_X, x)$ with $R_i \to \infty$ always has a subsequence such that $(X, R_i d_X, x)$ converges with respcet to the pointed Gromov-Hausdorff distance (Proposition 16.2). However the limit is not unique. (Such an example is constructed in [29] §8.)

Definition 23.3. We say that T_xX is a *tangent cone* of X at x if there exists a sequence $R_i \to \infty$ such that $(X, R_i d_X, x)$ converges to $(T_xX, \mathbf{o})$ with respect to the pointed Gromov-Hausdorff distance[45].

Definition 23.4. Let $X \in \overline{\mathfrak{S}}_n(\infty)$. We say that a point $x \in X$ is in $\mathcal{R}_k$ if $\mathbb{R}^k$ is a tangent cone T_xX of x.

We say x is *regular* if it is in $\mathcal{R} = \cup_k \mathcal{R}_k$. Otherwise it is said to be *singular* and we denote by $\mathcal{S}$ the set of all singular points.

Remark 23.3. This definition coincides with $\mathcal{S}(X)$ in Definition 17.7 by Otsu-Shioya in case when X is an Alexandrov space, because of Theorem 22.5.

One of the main result by Cheeger-Colding on the limit space X (in the collapsin situation) is the following :

Theorem 23.3 (Cheeger-Colding [29])**.** *$\mu(\mathcal{S}) = 0$ for any limit measure μ.*

Remark 23.4. (1) We remark that Theorem 23.3 implies $\mu(X \setminus \cup_k \mathcal{R}_k) = 0$ but does not imply the existence of k such that $\mu(X \setminus \mathcal{R}_k) = 0$.

(2) In Theorem 23.3 the limit measure μ is used. We do not know how to uses Hausdorff measure since it is not known whether the Hausdorff dimension of $X \in \overline{\mathfrak{S}}_n(D)$ is integer or not.

Here is some very brief idea how a statement like Theorem 23.3 follows from Theorem 23.1. We want to find many points x on $X \in \overline{\mathfrak{S}}_n(D)$ such that T_xX is an Eulcidean space. A naive idea to find

[45] I am sorry that this terminology is inconsistent with one in Definition 16.4, where T_xX is called tangent cone when *any* such sequence $(X, R_i d_X, x)$ converges to $(T_x, 0)$. In this section we follow Cheeger-Colding and use this terminology. In Definition 16.4 we followed Burago-Gromov-Perelman.

such a point may be as follows. First we consider a minimal geodesic $\overline{xy_1}$ and take an interior point on it and put it x_1. Then any tangent cone $T_{x_1}X$ contains a line and hence split as $T_{x_1}X \cong \mathbb{R} \times X_1$. We may next take a point x_2 near $\mathbf{o} \in T_{x_1}X$ which is a midpoint of the minimal geodesic. Then $T_{x_2}X_1$ contans a line and hence $T_{x_2}X_1$ splits. This process stops after finitely many stages since we can estimate the dimensoin of tangent cone by Bishop-Gromov inequality. Thus we find near x some kind of 'point' for which a tangent cone is $\mathbb{R}^k$. This argument however is too much naive to prove Theorem 23.3. So we need to work more seriously. See [29] §2. □

We remark that Theorem 23.3 can be applied also to collapsing situation. Namely it can be applied to the limit X of M_i such that $\mathrm{Vol}(M_i) \to 0$. Several other results are proved by Cheeger-Colding in [29, 30]. Neverthless there are yet many things unclear in the collapsing situation. (In other words, the result in the collapsing situation does not seem to be in the final form.) So we do not discuss it here. (We will discuss one of the main results of [31] later.)

In the non-collapsing case, Cheeger-Colding obtained more precise results. We discuss some of them here. The following theorem is a generalization of Theorem 22.1. We denote by $\mathfrak{S}_n(D, v)$ the set of n dimensional Riemannian manifolds M with $Ricci_M \geq -(n-1)$, $Diam(M) \leq D$, $\mathrm{Vol}(M) \geq v$. We also denote by $\mathfrak{S}_n(\infty, v)$ the set of n dimensional pointed Riemannian manifold (M, p) such that $Ricci_M \geq -(n-1)$, $\mathrm{Vol}(B_p(1, M)) \geq v$. Let $\overline{\mathfrak{S}}_n(D, v)$, $\overline{\mathfrak{S}}_n(\infty, v)$ be the closure of $\mathfrak{S}_n(D, v)$, $\mathfrak{S}_n(\infty, v)$ with respect to the Gromov-Hausdorff distance, pointed Gromov-Hausdorff distance, respectively.

Theorem 23.4 ([29] Theorem 5.9)**.** *Let $M_i \in \mathfrak{S}_n(\infty, v)$ We assume that $\lim^{pGH}_{i\to\infty}(M_i, p_i) = (X, p)$. Then for any R we have*

$$\lim_{i\to\infty} \mathrm{Vol}(B_{p_i}(R, M_i)) = \mathcal{H}^n(B_p(R, X))$$

Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

Corollary 23.5 ([29])**.** *If $X \in \overline{\mathfrak{S}}_n(\infty, v)$ then the Hausdorff dimension of X is n. Moreover any limit measure μ is equal to a multiple of the n dimensional Hausdorff measure.*

Corollary 23.5 follows from Theorem 23.4 easily. We explain an idea of the proof of Theorem 23.4 later in this section.

Theorem 23.6 ([28] Theorem 5.2)**.** *Let $X \in \overline{\mathfrak{S}}_n(\infty, v)$ and $x \in X$. Then any tangent cone T_xX is isometric to a cone CY of some length space Y of diameter $\leq \pi$.*

Remark 23.5. This result is a kind of generalization of the corresponding result Theorem 17.16 on Alexandrov space. However it is not asserted that CY is unique. Actually there is a counter example [29]

8.41. The conclusion of Theorem 23.6 does not hold in the collapsing situation ([29] 8.95).

To prove Theorem 23.6 we need another kind of comparison theorem, which we will explain later.

To state the next result we need a definition.

Definition 23.5. Let $X \in \overline{\mathfrak{S}}_n(\infty, v)$. We say that $x \in \mathcal{R}_\epsilon$ if every tangent cone T_xX satisfies :

$$d_{GH}(B_0(1, T_xX), B_0(1, \mathbb{R}^n)) < \epsilon.$$

We put $\mathcal{S}_\epsilon = X \setminus \mathcal{R}_\epsilon$.

Remark 23.6. (1) Using Theorems 22.5,23.4 we can prove that there exists δ such that $\mathrm{Vol}(B_x(r, X)) \ge (1-\delta)\mathrm{Vol}(B_0(r, \mathbb{R}^n))$ implies $x \in \mathcal{R}_\epsilon$. Thus Definition 23.5 is equivalent to $\mathcal{S}_\delta(X)$ in Definition 17.7.

(2) We can easily see that if $\epsilon' < \epsilon$ then $\mathcal{R}_{\epsilon'}$ is contained in the interior $\mathrm{Int}\mathcal{R}_\epsilon$ of $\mathcal{R}_\epsilon$.

(3) In the case of $X \in \overline{\mathfrak{S}}_n(\infty, v)$, we can easily show $\mathcal{R} = \mathcal{R}_n$. Using it, we can easily prove $\mathcal{S} = \cup_{\epsilon>0}\mathcal{S}_\epsilon$, $\mathcal{R} = \cap_{\epsilon>0}\mathcal{R}_\epsilon$.

The following is an analogy of Thereoms 17.2 and 17.22.

Theorem 23.7 ([29] Theorem 5.14)**.** *There exists $\epsilon_0(n)$ such that if $X \in \overline{\mathfrak{S}}_n(\infty, v)$ and if $\epsilon < \epsilon_0(n)$ then there exists a smooth Riemannian manifold $Z(\epsilon)$ and a homeomorphism $\Phi_\epsilon : Z(\epsilon) \to \mathrm{Int}\mathcal{R}_\epsilon$ such that*

$$C^{-1} < \frac{d(\Phi_\epsilon(x), \Phi_\epsilon(y))^{1-\tau(\epsilon|n)}}{d(x, y)} < C.$$

Theorem 23.7 actually follows easily from Theorem 22.8. Namely we find, for each $\epsilon' > \epsilon$ and $x \in \mathcal{R}_\epsilon$, a positive number r such that $d_{GH}(B_x(r, X), B_0(r, \mathbb{R}^n))) < \epsilon'$. Therefore any compact subset of $\mathrm{Int}\mathcal{R}_\epsilon$ satisfies condition $\mathcal{R}_{\epsilon',r,n}$ for some r. Theorem 23.7 then follows from Theorem 22.8. □

We next prove of Theorem 23.4. For simplicity of notation we assume X is compact. We first prove $\mathcal{H}^n(\mathcal{S}) = 0$. Let μ be a limit measure. We remark that there exists C_1, C_2 such that for $0 < r \le 1$ we have :

$$C_1 r^n \le \mathrm{Vol}(B_p(r, M_i)) \le C_2 r^n, \tag{23.1a}$$

$$C_1 r^n \le \mu(B_p(r, X)) \le C_2 r^n \tag{23.1b}$$

In fact (23.1a) is a consequence of Bishop-Gromov inequality and $\mathrm{Vol}(M_i) \ge v > 0$. Then (23.1b) follows, since μ is a limit measure. By Theorem 23.3 we have $\mu(\mathcal{S}) = 0$. Therefore by (23.1b) and the definition of Hausdorff measure we have $\mathcal{H}^n(\mathcal{S}) = 0$.

It follows that :

$$\lim_{\epsilon \to 0} \mathcal{H}^n(\mathrm{Int}\mathcal{R}_\epsilon) = \mathcal{H}^n(X). \tag{23.2}$$

We can take disjoint union of finitely many balls $U_\epsilon = \cup_j B_{y_j}(r_j, X) \subset \mathcal{R}_\epsilon$ such that

$$\mathcal{H}^n(X \setminus U_\epsilon) < \tau(\epsilon), \tag{23.3a}$$

$$\left|\omega_n \sum_j r_j^n - \mathcal{H}^n(X)\right| < \tau(\epsilon) \tag{23.3b}$$

$$d_{GH}(B_{y_j}(r_j, X), B_0(r_j, \mathbb{R}^n)) < 2\epsilon r_j. \tag{23.3c}$$

where $\omega_n = \mathrm{Vol}(B_0(1, \mathbb{R}^n))$. Here (23.3c) is a consequence of $y_j \subset \mathcal{R}_\epsilon$. Then, for large i, we have disjoint union of balls $U_{\epsilon,i} = \cup_j B_{y_{j,i}}(r_j, M_i) \subset M_i$ with

$$d_{GH}(B_{y_{j,i}}(r_j, M_i), B_0(r_j, X)) < 3\epsilon r_j. \tag{23.4}$$

Thereofore, by (23.3b), (23.3c), (23.4), and Theorem 22.5 we have

$$|\mathcal{H}^n(U_\epsilon) - \mathrm{Vol}(U_{\epsilon,i})| < \tau(\epsilon, 1/i|n). \tag{23.5}$$

We thus proved :

$$\mathcal{H}^n(X) \le \mathrm{Vol}(M_i) + \tau(\epsilon, 1/i|n).$$

To prove the opposite inequality, we take finitely many balls $B_{z_a}(t_a, X)$ such that

$$X \subseteq \bigcup_j B_{y_j}(r_j, X) \cup \bigcup_j B_{z_a}(t_a, X) \tag{23.6a}$$

$$\sum_j t_a^n \le \tau(\epsilon). \tag{23.6b}$$

Then for large i we find $z_{a,i}$ such that

$$M_i = \bigcup_j B_{y_{j,i}}(r_j, M_i) \cup \bigcup_j B_{z_{a,i}}(t_{a,i}, M_i). \tag{23.7}$$

Since

$$\mathrm{Vol}\left(\bigcup_j B_{z_{a,i}}(t_{a,i}, M_i)\right) < C_n t_a^n$$

it follows that

$$\mathrm{Vol}(M_i \setminus U_{\epsilon,i}) \le \tau(\epsilon|n). \tag{23.8}$$

Therefore, $\mathcal{H}^n(X) \ge \mathrm{Vol}(M_i) - \tau(\epsilon, 1/i|n)$, as required. □

We now sketch the proof of Theorem 23.1. We start with the following situation.

(A) M is a Riemannian manifold with $Ricci_M \ge -\lambda$ with small λ.
(B) We assume $d_{GH}(B_L(z, M), B_L(z', X)) < \rho/10$ and there is a line containing z'. (Here L is large.)
(C) Let $p, q \in M$ with $d(p, q) = 2L$ with large L.
(D) $d(z, \overline{pq}) \le \rho/3$, $|d(z, p) - L| \le \rho/3$, $|d(z, q) - L| \le \rho/3$.

Here $M = M_i$, where M_i is as in Theorem 23.1 for large i. Such pair of points p, q exists because of (B).

We want to find a length space X' such that $B_z(R, M)$ is close to a D ball $B_{(x',0)}(R, \mathbb{R} \times X')$ in $\mathbb{R} \times X'$ with respect to the Hausdorff distance. (Here $1 \ll R \ll L$.)

We use the following function which is an approximation of the Busemann function.

$$b_+(x) = d(p, x) - d(p, z), \quad b_-(x) = d(q, x) - d(q, z). \tag{23.9}$$

The argument to controll them is similar to the proof of Proposition 22.7. However our problem is a bit different from the situation of Proposition 22.7 where Hausdorff approximation is given by assumption. Our situation is similar to Theorem 22.5 (1) where we use other assumption (which was the almost maximality of volume in case of Theorem 22.5 (1)) to find Hausdorff approximation. In our case, we use the following theorem by Abresch-Gromoll to obtain some information on $b_\pm$ and improve it by using a similar argument as the proof of Proposition 22.7. To state the result by Abresch-Gromoll we need a notation.

Definition 23.6. For $x, p, q \in M$, an *excess* $E(x; p, q)$ is by definition

$$E(x; p, q) = d(x, p) + d(x, q) - d(p, q).$$

Theorem 23.8 (Abresch-Gromoll[2])**.** *If* $Ricci_M \ge -(n-1)\lambda$, $d(z, p) \ge L$, $d(z, q) \ge L$ *and if* $E(z; p, q) < \rho$, *then*

$$E(x; p, q) < \tau(\rho, \lambda, 1/L|n, R)$$

for any $x \in B_z(R, M)$.

Remark 23.7. Abresch-Gromoll stated Theorem 23.8 in the case $E(z; p, q) = \rho$ namely the case $z \in \overline{pq}$. The above form is a modification by Cheeger-Colding [28] Proposition 6.2. ([26] Theorem 9.1.)

Remark 23.8. Abresch-Gromoll used Theorem 23.8 to show the following Theorem 23.9. It seems that Theorem 23.8 is the first comparison theorem established assuming condition on Ricci curvature only.

For a length space M and $B \subset A \subseteq M$ we write $Diam(B \subset A)$ the following number

$$\sup_{p,q \in B} \{\text{the length of the shortest curve joinng } p \text{ and } q \text{ in A}\}.$$

Theorem 23.9 ([Abresch-Gromoll[2])**.** *If* M *is a complete manifold with* $Ricci_M \ge 0$, $\inf K_M > -\infty$, *and*

$$Diam(S_p(3R, M) \subseteq A_p(2R, 4R; M)) \le C/R,$$

then M *is homotopy equivalent to an interior of a compact manifold with boundary.*

Theorem 23.9 is proved by Theorem 23.8 and Morse theory of distance function in a way similar to Theorem 14.6 .

Let us go back to the discussion of the proof of Theorem 23.1. Theorem 23.8 and $b_+ + b_- = E(x;p,q) - E(z;p,q)$ implies

$$-\rho \le b_+ + b_- \le \tau(L^{-1},\rho,\lambda|R,n) \tag{23.10}$$

Using the fact that $b_\pm$ is "almost subharmonic" we have the following formula (23.12). We define $\mathbf{b}_+ : B_z(R,M) \to \mathbb{R}$ by

$$\Delta \mathbf{b}_+ = 0, \tag{23.11a}$$

$$\mathbf{b}_+ = b_+ \qquad \text{on } \partial B_z(R,M). \tag{23.11b}$$

Then, we can prove

$$\|\mathbf{b}_+ - b_+\|_{L^2_1(B_z(R,M))} \le \tau(L^{-1},\rho,\lambda|R,n). \tag{23.12}$$

(Here the right hand side will become small by taking L large, λ,ρ small.)

We now consider the Bochner formula :

$$\frac{1}{2}\Delta(|\nabla \mathbf{b}_+|) = |\mathrm{Hess}\,\mathbf{b}_+|^2 + \mathrm{Ricci}(\nabla \mathbf{b}_+, \nabla \mathbf{b}_+). \tag{23.13}$$

We remark $|\nabla b_+| = 1$. Hence using (23.12) the integral of the left hand side of (23.13) is small. Since $Ricc_M \ge -(n-1)\lambda$ it follows from (23.13)

$$\int_{B_z(R,M)} |\mathrm{Hess}\,\mathbf{b}_+| \le \tau(L^{-1},\rho,\lambda|R,n) \tag{23.14}$$

$$\int_{B_z(R,M)} (|\nabla \mathbf{b}_+| - 1) \le \tau(L^{-1},\rho,\lambda|R,n). \tag{23.15}$$

We put $X' = \mathbf{b}_+^{-1}(0)$. Now we will use (23.14), (23.15) to show that $B_x(R,M)$ is close to a R ball in $X' \times \mathbb{R}$. with respect to the Gromov-Hausdorff distance as follows.

Let us take $y,z \in B_x(R,M)$. Let $y_0, z_0 \in X'$ such that

$$d(y,y_0) = d(y,X'), \qquad d(z,z_0) = d(z,X').$$

We will prove

$$\left|d(y,z)^2 - d(y_0,z_0)^2 - (\mathbf{b}_+(y) - \mathbf{b}_+(z))^2\right| \le \tau(L^{-1},\rho,\lambda|R,n). \tag{23.16}$$

(23.16) obviously implies that $y \mapsto (y_0, \mathbf{b}_+(y)) : B_x(R,M) \to B_{(x,0)}(R, X' \times \mathbb{R}))$ is a Hausdorff approximation and hence :

$$d_{GH}(B_x(R,M), B_{(x,0)}(R, X' \times \mathbb{R})) \le \tau(L^{-1},\rho,\lambda|R,n),$$

which is enough to comlete the proof of Theorem 23.1.

Let us sketch the proof of (23.16). For simplicity we take $\mathbf{b}_+(z) = 0$ and $z = z_0$.

Let $\ell : [0, l] \to M$ be a minimal geodesic joining y_0 to y. We put $Q(t) = d(\ell(t), z)$. Let $\gamma_t : [0, Q(t)] \to M$ be a minimal geodesic joining z to $\ell(t)$.

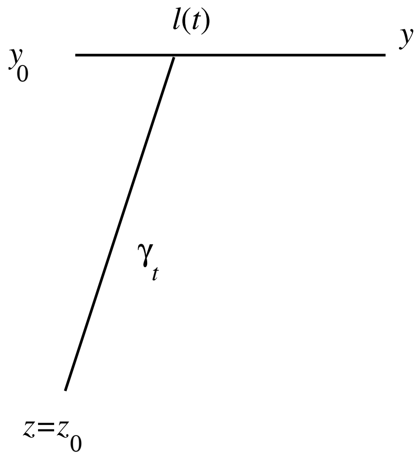

Figure 23.1

We put $h_t(s) = \mathbf{b}_+(\gamma_t(s))$. We remark that

$$\frac{d^2 h_t}{ds^2}(s) = (\mathrm{Hess}\,\mathbf{b}_+)(\dot\gamma_t(s), \dot\gamma_t(s)) \ll 1. \tag{23.17}$$

On the other hand $h_t(Q(t)) = \mathbf{b}_+(\gamma(t))$ is almost equal to t [46]. Hence

$$\left\| \frac{dh_t}{ds}(s) - \frac{t}{Q(t)} \right\| \ll 1 \tag{23.18}$$

(Here we remark that (23.17), (23.18) does not hold pointwise but only after integrating over some domain. We omit the technical difficulty which arises from this point.) By first variational formula, we have

$$\frac{dh_t}{ds}(s) = \langle \dot\gamma_t(s), \nabla \mathbf{b}_+ \rangle \doteq \cos \angle y_0 \gamma(t) z = \frac{dQ}{dt}(t). \tag{23.19}$$

(Here and hereafter $\doteq$ means "almost" equal.)

Hence $Q(t)$ "almost" satisfies the following differential equation.

$$\frac{dQ}{dt} \doteq \frac{t}{Q(t)}. \tag{23.20}$$

The solution of (23.20) with initial value $Q(0) = d(y_0, z)$ is $Q(t) = \sqrt{d(y_0, z)^2 + t^2}$. Hence at $t = b_+(y)$ we have (23.16) with $z = z_0$. (See [26] Chapter 9 or [28] §6 for the detail of the proof.) □

Here we say a few words about the proof of Remark 23.1. In this situation we can take not only p, q but also p_i, q_i $i = 1, \cdots, k$. Namely p, q is a point close to the line on Y and p_i, q_i are taken as a point close to the point on the coordinate axis of $\mathbb{R}^k$. Using them we obtain $\mathbf{b}_+$ together with $\mathbf{b}^i_+$, $i = 1, \cdots, k$. They all satisfy (23.14), (23.15).

[46] We can find $(d^2/dt^2)(\mathbf{b}_+ \circ \gamma)$ is small in the same way as (23.17). Moreover $(d^2/dt^2)(\mathbf{b}_+ \circ \gamma)(0)$ is close to 1 by definition and (23.15).

Moreover we have $\langle \nabla \mathbf{b}_+^i, \nabla \mathbf{b}_+^j \rangle \doteq \delta_{ij}$ where $\mathbf{b}_+^0 = \mathbf{b}_+$. We define $\Phi : X \to \mathbb{R}^{k+1}$ by $\Phi = (\mathbf{b}_+^0, \cdots, \mathbf{b}_+^k)$. Using it we can construct a pointed Hausdorff approximation $X \to \Phi^{-1}(0) \times \mathbb{R}^k$ in a way similar to the proof of Theorem 23.1. See [29] p 425 - 426 where similar argument appears. □

In §20 we reviewed several results obtained by L^2 comparison Theorem where we compared a manifold with round sphere. In §22 we used L^2 comparison theorem where the model space was flat Euclidean space. In the proof Theorem 23.1, we compared a manifold with direct product $\mathbb{R} \times X'$. In [28], Cheeger-Colding developped a comparison theorem where the model space is a warped product (hereafter we call it warped product comparison theorem) and gave various applications. We first review some of its applications.

One of its applications is Theorem 22.5 (1). The following is closely related to it. (Theorem 22.5 (1) corresponds to the case when $Y = S^{n-1}$.) Cheeger-Colding called this theorem 'volume cone implies metric cone theorem'.

Theorem 23.10 (Cheeger-Colding [28])**.** *For each ϵ there exists $\delta = \delta(\epsilon, n)$ with the following property. Let M be an n dimensional Riemannian manifold with $Ricci_M \ge -\delta(n-1)$. We assume*

$$\frac{\mathrm{Vol}(B_p(1,M))}{\mathrm{Vol}(S_p(1,M))} \le (1+\delta)\frac{\mathrm{Vol}(B_0(1,\mathbb{R}^n))}{\mathrm{Vol}(S_0(1,\mathbb{R}^n))}.$$

Then there exists a length space Y with $Diam(Y) \le \pi$ such that

$$d_{GH}(B_p(1,M), B_{\mathbf{o}}(1, CY)) \le \epsilon.$$

We remark that $CY = ([0,\infty) \times Y)/\sim$ where $(0,x) \sim (0,y)$ with metric defined in Definition 17.2.

Another application of warped product comparison theorem is Theorem 23.11. To state it we define warped product.

Definition 23.7. Let (X, g_X) be a Riemannia manifold and $f : (a,b) \to \mathbb{R}_+$ be a smooth function. Then the *warped product* $(a,b) \times_f X$ is by definition a product $(a,b) \times X$ equipped with the metric $dr^2 \oplus f(t)^2 g_X$, where r is the coordinate of the interval (a,b).

We need to define warped product for general length space also. Let X be a length space and $f : (a,b) \to \mathbb{R}_+$ be a smooth function. Let $\ell : [\alpha,\beta] \to (a,b) \times X$ be a path which is, say, Lipschitz continuous. We put $\ell(t) = (r(t), \ell_X(t))$. We may change parameter so that $\ell_X(t) : [\alpha,\beta] \to X$ is parametrized by arc length. We then define length $L(\ell)$ of $\ell : [\alpha,\beta] \times_f X$ by

$$L(\ell) = \int_\alpha^\beta \sqrt{((dr/dt)(t))^2 + f(r(t))^2} dt.$$

We thus defined the length space $(a,b) \times_f X$.

Example 23.1. (1) The simplest case is $f \equiv 1$. Then warped product is the direct product.

(2) If $(a,b) = (0,\infty)$ and $f(r) = r$, then the warped product $(0,\infty) \times_r X$ is the cone CX minus $\mathbf{o}$. If moreover $X = S^{n-1}$ then it is $\mathbb{R}^n \setminus \{0\}$.

(3) We take $(a,b) = (0,\pi)$ and $f(r) = \sin r$. In this case the warped product $(0,\pi) \times_{\sin r} X$ is called the *metric suspension* SX. In particular the metric suspension SS^{n-1} of round sphere S^{n-1} is the round sphere S^n.

Theorem 23.11 ([28] Theorem 5.14)**.** *If* $Ricci_M \geq (n-1)$, $\dim M = n$ *and if* $Diam(M) \geq \pi - \epsilon$ *then there exists a length space* X *such that* $d_{GH}(M, SX) < \tau(\epsilon|n)$.

Remark 23.9. It is not true in general that M is homemomorphic (or homotopy equivalent to SX). The counter examples are ones by Anderson and Otsu we mentioned already.

There are several other applications, for example to the study of cone at infinity. We omit it.

We now explain the idea of the proofs of these theorems. The main idea is to use warped product comparison theorem. To state it we need some preliminary discussion. We begin with a characterization of warped product. Let $f : (a,b) \to \mathbb{R}_+$ be a smooth function we put :

$$\mathcal{F}(r) = \int_a^r f(t)dt, \qquad k(r) = \frac{df}{dr}(r). \tag{23.21}$$

Lemma 23.12. *Let* X *be a Riemannian manifold and* $M = (a,b) \times_f X$*. Then we have :*

$$\mathrm{Hess}(\mathcal{F}) = k(r) g_M. \tag{23.22}$$

Example 23.2. (1) In case $M = \mathbb{R} \times_1 X$ the direct product. $\mathcal{F}$ is linear and $k = 0$.

(2) In case $M = \mathbb{R}^n = CX \setminus 0 = (0,\infty) \times_r X$, we have $\mathcal{F} = r^2/2$ and $k(r) = 1$. If $X = S^{n-1}, M = \mathbb{R}^n$ then $\mathcal{F}(x_1, \cdots, x_n) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$ and (23.22) is obvious.

(3) In case $M = (0,\pi) \times_{\sin r} S^{n-1}$ we have $\mathcal{F}(r) = -k(r) = \cos r$. Formula (23.22) is (21.17).

Let us prove Lemma 23.12. We put $\partial_r = \partial/\partial r$. $\mathrm{Hess}(\mathcal{F})(\partial_r, \partial_r) = k$ is obvious since $t \mapsto (t,p)$ is a geodesic. Let V be a vector filed of X, which we we regard a vector field on M. We have $[V, \partial_r] = 0$. Since $g_M(V,V) = f^2 g_X(V,V)$ it follows that

$$-g_M(\partial_r, \nabla_V V) = g_M(\nabla_V \partial_r, V) = g_M(\nabla_{\partial_r} V, V) = fk g_X(V,V).$$

On the other hand, $V(\mathcal{F}) = 0$, $\partial_r(\mathcal{F}) = f$. Therefore

$$\mathrm{Hess}(\mathcal{F})(V,V) = -(\nabla_V V)(\mathcal{F}) = f^2 k g_X(V,V) = k g_M(V,V),$$

as required. □

The warped product comparison theorem is an 'almost version' of the following converse to Lemma 23.12.

Proposition 23.13. *If M is a Riemannian manifold $\mathcal{F} : M \to (\alpha, \beta)$ with is a fiber bundle. Suppose We assume that there exists a function $k : M \to \mathbb{R}$ such that*

$$\mathrm{Hess}_x(\mathcal{F}) = k(x) g_M. \tag{23.23}$$

We put $X = \{x \in M | \mathcal{F}(p) = \mathcal{F}(x)\}$. Then there exists a function $f : (a,b) \to \mathbb{R}_+$ such that

$$M \cong (a,b) \times_f X \quad \textit{(isometry)}, \tag{23.24a}$$

$$\mathcal{F}(x) = \int_{r(p)}^{r(x)} f(t) dt \tag{23.24b}$$

$$k(x) = \frac{df}{dr}(r(x)). \tag{23.24c}$$

Here $r : M \cong (a,b) \times_f X \to (a,b)$ is the projection to the first factor.

We now state warped product comparison theorem. Let M be a complete Riemannian manifold and K be a compact subset. We put

$$r(x) = d(x,K) = \inf\{y \in K | d(x,y)\}, \tag{23.25a}$$

$$A_K(a,b,M) = \{x \in M | a < r(x) < b\}. \tag{23.25b}$$

Let $f : (a,b) \to \mathbb{R}_+$ be a smooth function and we define $\mathcal{F}(r)$ and $k(r)$ as in (23.21). We regard r as a function on $A_K(a,b,M)$ then $\mathcal{F}$ and k are functions on $A_K(a,b,M)$ as well. The following assumption are generalization of similar formulae we met several times already. For example (23.14),(23.14) where $k(r) = 0$, and (21.17) where $k(r) = \cos r$.

Assumption 23.1. There exists $\tilde{\mathcal{F}} : A_K(a,b,M) \to (a,b)$ such that

$$\sup |\tilde{\mathcal{F}} - \mathcal{F}| \le \epsilon, \tag{23.26a}$$

$$\frac{1}{\mathrm{Vol}(A_K(a,b,M))} \int_{A_K(a,b,M)} |\nabla \tilde{\mathcal{F}} - \nabla \mathcal{F}| \le \epsilon, \tag{23.26b}$$

$$\frac{1}{\mathrm{Vol}(A_K(a,b,M))} \int_{A_K(a,b,M)} |\mathrm{Hess}\tilde{\mathcal{F}} - k g_M| \le \epsilon. \tag{23.26c}$$

Theorem 23.14 asserts under Assumption 23.1 plus some more (which will follow), $A_K(a,b,M)$ is Gromov-Hausdorff close to some warped product $(a,b) \times_f X$.

Assumption 23.2. M is an n-dimensional complete Riemannian manifold with $K_M \geq -\Lambda$. $Diam(A_K(a,b,M)) \leq D$. $0 < \alpha' < \alpha$, $0 < \xi < \alpha - \alpha'$.

For each $x \in r^{-1}(a+\alpha')$ there exsits $y \in r^{-1}(b-\alpha')$ such that

$$d'(x,y) \leq b - a - 2\alpha' + \epsilon. \tag{23.27}$$

Theorem 23.14 (Cheeger-Colding[28] Theorem 3.6)**.** *Under the Assumptions 23.1 and 23.2, there exists a length space X such that*

$$d_{GH}((A_K(a+\alpha, b-\alpha, M), d'), (a+\alpha, b-\alpha) \times_f X) \leq \tau(\epsilon|\alpha', \xi, n, f, D).$$

Remark 23.10. In Assumption 23.2 and Theorem 23.14 we use the symbol d' for the metric of subsets of M. Note the space $A_K(a,b,M)$ is not complete. So when we define the metric function $d : A_K(a,b,M) \times A_K(a,b,M) \to \mathbb{R}$ using Riemannian metric, we need to be a bit careful. Namely for $p,q \in A_K(a,b,M)$ we need to take the infinimum of the length of the curves joining them in a slightly larger domain. The metric d' stands for such a metric. We do not define it since it is too much technical. See [28] p 205-206.

Let us explain the idea of the proof of Theorem 23.14. Actually the idea is quite similar to one of the proof of (23.16) we discussed already.

We take $X = r^{-1}(a+\alpha)$. To define a metric on it we consider broken geodesic on its small neighborhood and take the infinimum of the length of them. Now we construct Hausdorff approximation $\Phi : A_K(a+\alpha, b-\alpha, M) \to (a+\alpha, b-\alpha) \times_f X$. Let $y, z \in A_K(a+\alpha, b-\alpha, M)$. We take $y_0, z_0 \in X$ so that $d(y, y_0) = d(y, X)$, $d(z, z_0) = d(z, X)$. We remark $r(x) = d(x, X) - a - \alpha'$. We put

$$\Phi(y) = (r(y), y_0),$$

and will prove that Φ is an Hausdorff approximation.

We assume $z_0 = z$ for simplicity. Let $\ell : [0, l] \to M$ be a minimal geodesic joining y_0 to y. We put $Q(t) = d(\ell(t), z)$. Let $\gamma_t : [0, Q(t)] \to M$ be a minimal geodesic joining z to $\ell(t)$.

Actually there is a technical trouble here. Namely since $A_K(a+\alpha, b-\alpha, M)$ is not complete, we may not be able to take γ_t. By this reason, we need to take broken geodesic. (See the figure below.) However since this is a technical point, we forgot it and presume that we can take γ_t.

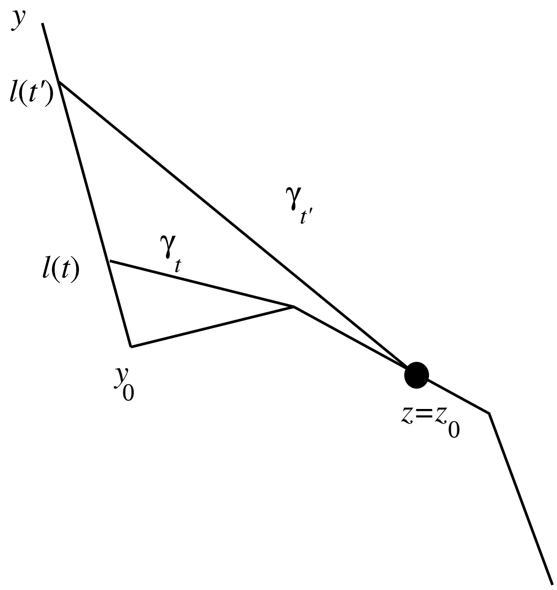

Figure 23.2

We put $h_t(s) = \tilde{\mathcal{F}}(\gamma_t(s))$. By (23.26), h_t 'almost' satisfies the differential equation

$$\frac{d^2 h_t}{ds^2}(s) \doteq H(h_t(s)). \tag{23.28}$$

where $H(c) = k(\tilde{F}^{-1}(c))$. We remark that (23.28) is an ordinary differential equation of second order and hence has unique solution under appropriate boundary condition. Note $h_t(0) \doteq \mathcal{F}(a+\alpha)$, $h_t(Q(t)) = \mathcal{F}(t+a+\alpha)$. Thus h_t is determined by $Q(t)$. (Precisely we have to say that h_t is 'almost' determined by $Q(t)$ since (23.28) is only 'almost' satisfied.) Moreover we have

$$\frac{dh_t}{ds}(Q(t)) \doteq \frac{dQ}{dt}(t) \tag{23.29}$$

by the same reason as (23.19). Thus, (23.29) becomes a differential equation of first order on Q and is determined by f. We remark that Q satisfies an initial value condition $Q(0) = d(z, y_0)$. Therefore, the value of Q at $t = r(y) - a - \alpha$ is determined by this equation and initial value $d(z, y_0)$. (Precisely speaking, we can only say the value of Q is almost determined.) By definition $Q(r(y) - a - \alpha) = d(y, z)$. Since it is almost determined by $d(z, y_0)$ and $r(y)$ and $r(z)$ (which we assumed to be zero for simplicity), it follows that Φ 'almost' preserves the length.

The fact that a small neighborhood of the image of Φ contains $(a + \alpha, b - \alpha) \times_f X$ follows from (23.27). This is a sketch of the proof of Theorem 23.14. □

We now discuss applications of Theorem 23.14.

We first show how we can use Theorem 23.14 to prove Theorem 22.5 (1). Let us assume $\mathrm{Vol}(B_p(1, M)) \geq \mathrm{Vol}(B_0(1, \mathbb{R}^n)) - \epsilon$ and $Ricci_M \geq$

$-\lambda$. We put $f(t) = t$. Then $k(t) \equiv 1$, $\mathcal{F}(t) = t^2/2$. We need to check Assumptions 23.1, 23.2. Put $r(x) = d(x,p)$. We calculate

(23.30)
$$(2-n)\int_{S_p(R,M)} r^{1-n} = \int_{S_p(R,M)} \mathrm{grad} r^{2-n}\cdot dn = \int_{B_p(R,M)} \Delta r^{2-n}.$$

Since $\mathrm{Vol}(B_p(1,M)) \ge \mathrm{Vol}(B_0(1,\mathbb{R}^n)) - \epsilon$ it follows from Lemma 21.1 that $\int_{S_p(R,M)} r^{1-n} = c_n \mathrm{Vol}(S_p(R,M))/\mathrm{Vol}(S_0(R,\mathbb{R}^n))$ is almost independent of R. Hence (23.30) implies $\int_{B_p(R,M)\setminus B_p(\delta,M)} \Delta r^{2-n}$ is small. Namely r^{2-n} is almost a harmonic function on $B_p(R,M)\setminus B_p(\delta,M)$. (We remark that r^{2-n} is harmonic on $\mathbb{R}^n$.) Then we have

$$\Delta r^{2-n} = (2-n)\mathrm{div}(r^{1-n}\mathrm{grad} r) = (2-n)r^{1-n}\Delta r + (2-n)(1-n)r^{-n}.$$

Hence

$$\Delta r \doteq (n-1)r^{-1}. \tag{23.31}$$

And hence

$$\Delta r^2 = 2\mathrm{div}(r\mathrm{grad} r) = 2 + 2r\Delta r \doteq 2n. \tag{23.32}$$

We now apply (23.13) to $r^2/2 = \mathcal{F}$ and obtain

$$n \doteq \frac{1}{2}\Delta r^2 \doteq |\mathrm{Hess}\,\mathcal{F}|^2 + r^2 Ricci(\nabla r, \nabla r). \tag{23.33}$$

Since $Ricci_M \ge -\lambda$ it follows from (23.33) and (23.32) that

$$\mathrm{Hess}\ \mathcal{F} \doteq g_M.$$

Hence if $\tilde{\mathcal{F}}$ is a harmonic function which approximate $\mathcal{F}$ we can check Assumptions 23.1, 23.2. Therefore Theorem 23.14 implies that there exists X such that

$$d_{GH}((A_p(2\delta, 1-2\delta, M), d'), (2\delta, 1-2\delta)\times_r X) \le \tau(\epsilon|\delta,\lambda,n).$$

To complete the proof it suffices to show that X is close to S^{n-1} with respect to the Gromov-Hausdorff distance. We can do it by looking the proof of Theorem 23.14 in our case a bit more carefully. Alternatively we can proceed as follows. Take $\rho \ll 1$, with $\delta \ll \rho^n$. By assumption and Bishop-Gromov inequality we can find p_1, q_1 such that $2d(p,p_1) = 2d(p,q_1) = d(p_1,q_1)$. We use it in the same way as the proof of Theorem 23.1 to find $V_1 \supset B_p(\rho,M)$ such that $d_{GH}(V_1, [-\rho,\rho]\times X_1) \le \rho\tau(\epsilon,\lambda|n)$. We then take points p_2, q_2 in a neighborhood of X_1 such that $2d(p_2,p) \doteq 2d(q_2,p) \doteq d(p_2,q_2)$. Then we use it in the same way as the proof of Remark 23.1 to find $V_2 \supset B_p(\rho^2,M)$ such that $d_{GH}(V_2, [-\rho,\rho]^2\times X_2) \le \rho^2\tau(\epsilon,\lambda|n)$. Repeating this n times, we obtain $V_n \supset B_p(\rho^n,M)$ such that $d_{GH}(V_n, [-\rho,\rho]^n) \le \rho^n\tau(\epsilon,\lambda|n)$. Since $\delta \ll \rho^n$ it then follows that $d_{GH}(X, S^{n-1}) \le \tau(\epsilon,\lambda|n)$. It implies Theorem 22.5 (1). $\square$

The proof of Theorem 23.10 is similar to the first half of the proof of Theorem 22.5 (1) and is omitted. $\square$

We next explain the proof of Theorem 23.11. Let $Ricci_M \geq (n-1)$ and $p, q \in M$ with $d(p,q) \geq \pi - \epsilon$. Put $f(r) = \sin r$, $r(x) = d(x,p)$, $\mathcal{F}(r) = -k(r) = -\cos r$. By (the proof of) Theorem 21.9 (See (21.17). We remark that f there is our $\mathcal{F}$.)

$$\mathrm{Hess}(\mathcal{F}) \doteq k(r) g_M.$$

In this way we can check Assumptions 23.1. Assumption 23.2 follows from Bishop-Gromov inequality in this case. We thus can apply Theorem 23.14 and prove Theorem 23.11. □

We next explain a idea of proof of Theorem 23.6. Let $((X, d_X), x) = \lim^{pGH}_{i\to\infty}(M_i, x_i)$ with $(M_i, x_i) \in \mathfrak{S}_n(\infty, v)$. We suppose that a tangent cone $T_x X = \lim^{pGH}_{i\to\infty}((X, r_i d_X), x)$ is not a cone.

Then there exists δ, R, ρ (and a subsequence of r_i which we denote by the same symbold) such that

$$d_{GH}(A_x(\delta, R; (X, r_i d_X)), A_{\mathbf{o}}(\delta, R; CY)) > \rho$$

for and cone CY. We can take $j_i \to \infty$ such that

$$d_{GH}(A_{x_{j_i}}(\delta, R; (M_{j_i}, r_i g_{M_{j_i}})), A_{\mathbf{o}}(\delta, R; CY)) > \rho/2. \tag{23.34}$$

for and cone CY. We now claim that

$$\begin{aligned}&\frac{\mathrm{Vol}(S_{x_{j_i}}(\delta/r_i, (M_{j_i}, g_{M_{j_i}})))}{\mathrm{Vol}(S_0(\delta/r_i, \mathbb{R}^n))}\\ &> (1+\epsilon)\frac{\mathrm{Vol}(S_{x_{j_i}}(R/r_i, (M_{j_i}, g_{M_{j_i}})))}{\mathrm{Vol}(S_0(R/r_i, \mathbb{R}^n))},\end{aligned} \tag{23.35}$$

for ϵ independent of i. In fact, if (23.35) does not hold, then we can apply the argument of the first half of the proof of Theorem 22.5 (1) to $(A_{x_{j_i}}(\delta, R; (M_{j_i}, r_i g_{M_{j_i}})$ and using Theorem 23.14 we can show that (23.34) does not holds.

Now it is easy to deduce a contradiction from (23.35). By taking a subsequence we may assume that $\delta/r_i > R/r_{i+1}$. Then (23.35) and Bishop-Gromov inequality implies

$$\begin{aligned}&\frac{\mathrm{Vol}(S_{x_{j_i}}(R/r_i, (M_{j_i}, g_{M_{j_i}})))}{\mathrm{Vol}(S_0(R/r_i, \mathbb{R}^n))}\\ &> (1+\epsilon)^{i-1}\frac{\mathrm{Vol}(S_{x_{j_1}}(R/r_1, (M_{j_1}, g_{M_{j_1}})))}{\mathrm{Vol}(S_0(R/r_1, \mathbb{R}^n))}.\end{aligned} \tag{23.36}$$

This is a contradiction since the left hand side is bounded as $i \to \infty$. □

In [31], Cheeger-Colding studied a convergence of eigenvalue of Laplace operator using the result eplained so far. We state their result (without outline of the proof) here.

We start with a simple example to illustrate that measured Hausdorff convergence is related to the eigen value of Laplace operator. Let us consider $T^2 = S^1 \times S^2$ with Riemannian metric $g^f_\epsilon = dt^2 + \epsilon^2 f(t)^2 ds^2$.

Here $f : S^1 \to \mathbb{R}_+$. We assume $\int f dt = 1$. As we mentioned before the limit of (T^2, g_ϵ^f) with respect to the measured Hausdorff topology is S^1 with standard metric and measure $f dt$. The Dirichlet integral on (T^2, g_ϵ^f) is

$$D(h,h) = \epsilon \int f(t) \left(\left(\frac{dh}{dt} \right)^2 + \frac{1}{\epsilon f(t)} \left(\frac{dh}{ds} \right)^2 \right) dtds.$$

In case we consider eigenvalue of Laplacian which stay bounded as $\epsilon \to 0$, it suffices to consider h which is constant along s direction. Hence we are to consider the bilinear form on $L^2(S^1)$ defined by

$$D(h,h) = \int f(t) \left(\frac{dh}{dt} \right)^2 dt.$$

In [54] the author proved that a similar phenomenon occures in the situation we discussed in §11. Cheeger-Colding generalized it much and proved the following Theorem 23.15.

Theorem 23.15 ([31] Theorem 7.9)**.** *Let $M_i \in \mathfrak{S}_n(D)$. We assume that it converges to (X, μ) with respect to the measured Hausdorff topology. Then there exists a (unbounded) symmetric bilinear form D on $L^2(X, \mu)$ with discrete spectrum $\lambda_0(D) = 0 < \lambda_1(D) \le \lambda_2(D) \le \cdots$ such that k-th eigenvalue $\lambda_k(-\Delta_{M_i})$ of Laplace operator (on functions) on M_i converges to $\lambda_k(D)$.*

Remark 23.11. (1) In case the multiplicity of eigenvalue $\lambda_k(D)$ is m then we put $\lambda_k(D) = \cdots = \lambda_{k+m-1}(D)$.

(2) The eigenfunction of $-\Delta_{M_i}$ converges to the eigenfunctions of D in an appropriate sense.

We finally remark the study of limits of Einstein manifolds (or manifolds with integral bounds of curvature tensor) we discussed in §20 is improved by [32, 27] etc. Here we restrict ourselves to quote the following Theorems 23.16, 23.17. Let M_i be a sequence of n dimensional Riemannian manifolds. We consider the following integral bounds of the curvature for $p_i \in M_i$.

$$\int_{B_{p_i}(1,M_i)} |R_{M_i}|^p \Omega_{M_i} < C \tag{23.37}$$

where C is independent of i. Let $\mathcal{S}$, $\mathcal{S}_k$ be as in Definitions 23.4. $\mathcal{H}^m$ is the m dimensional Hausdorff measure.

We say $x \in \mathcal{S}$ is $(n-4k)$-*nonexceptional* if there exists a tangent cone T_xX which is *not* isometric to $\mathbb{R}^{n-4k} \times C(S^{4k-1}/\Gamma)$ where $\Gamma \subset O(4k)$ is a finite group acting freely on S^{4k-1}. Otherwise x is said to be $n-4k$-exceptional. Let $\mathcal{N}_{n-4k} \subset \mathcal{S}_{n-4k}$ be the set of all $(n-4k)$-nonexceptional points.

Theorem 23.16 ([32] Theorems 1.15,1.20, [27] Theorem 6.10)**.** *Let $M_i \in \mathfrak{S}_n(\infty, v)$ and $\lim^{pGH}_{i\to\infty}(M_i, p_i) = (X, p)$. We assume* (23.37) *for $1 \geq p \geq n/2 = \dim M_i/2$.*

(1) *If p is not an integer then $\mathcal{H}^{n-2p}(\mathcal{S}) = 0$.*
(2) *The Hausdorff dimension of $\mathcal{S}$ is not greater than $n - 2p$.*
(3) *If $p = 2$, then $\mathcal{H}^{n-4}(\mathcal{N}_{n-4}) = 0$.*
(4) *If M_i are Kähler and p is an integer then $\mathcal{H}^{n-2p}(\mathcal{S} \cap B_p(R, X)) < \infty$ for any R.*

We remark that in case $n = 2$ and M_i are Einstein, Theorem 23.16 (3) is Theorem 20.4.

Theorem 23.17 ([27] Theorem 11.1)**.** *In the situation of Theorem 23.16 we have :*

(1) *If $p = 1$ then compact subsets of $\mathcal{S}$ are $n - 2$ rectifiable.*
(2) *If either $p = 2k$ is even an integer then $\mathcal{N}_{n-4k} \cap B_p(R, X)$ are $N - 4k$ rectifiable.*
(2) *M_i are Kähler and p is integer, then $\mathcal{S} \cap B_p(R, X)$ are $n - 2p$ rectifiable.*

We remark that as in 4 dimensional case, if $p = 2$ and M_i are Einstein, the condition (23.37) can be written in terms of Characteristic classes and hence is a topological one.

These results is parallel to the corresponding results in (higher dimensional) gauge theory [106, 146].

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