Interacting maps, symbolic dynamics and automorphisms in microscopic scale

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Introduction

Biological systems are consisted by hierarchy of various kinds of systems. Sometimes, or very often time scalings are completely different between reaction velocity of ensymes and oscillation of the molecules. Thus a particular feature is that global structure appears from many micro systems which have mutually completely different scalings. In the quantum level, the smallest scaling of time, molecules are interacting by sharing their electrons. Electron orbitals represent thier quantum states. On the molecular level, the micro level, systems are consisted by interactions between molecules. In general these will be quite complicated, however these huge systems show several patterns, for example cirrgadian rythms and feed-back structure, which are both particularly important to keep biological systems. Study of these phenomena will be the first step to try to understand a basic principle in molecular biology, *from random micro to macro patterns*.

Among such various time scalings, molecular biology tells us so rich structure of these systems. A basic direction for us is to formulate such biological aspects in mathematics, and study their mathematical structures in light of the corresponding biological phenomena.

In this paper, we will try to express some of these biological structures using maps between intervals. We will interpret orbit spaces of maps as oscillations of molecules. Molecular interactions are formulated by infinitely many times compositions of two maps. These correspond to the smallest time scalings. We will study analytic aspects of composition of maps from dynamical systems point of view, on the other hand, use of family of maps enables to have relations with other fields, automaton, geometric group theory, integrable systems and random graph theory.

Solutions of Lotka Vorterra equation describes some patterns of some growth populations of competiting animals (section 2). By making discretizations of the equation twice, one obtains the difference-difference equation, a cell automaton whose solutions are essentially described by some automaton. This corresponds to making time scalings more and more micro, and we show that the solutions of the cell automaton can be obtained from our formulation of comosition of maps. In particular we will realize flow of the solitions as the transition process in chemical reaction. Another important integrable system is the Toda equation, and one will see also the solutions of the cell automaton of Toda type are obtained as interaction of maps in our sense. One point here is that one can see relation of continuous maps as a purely flexible object with integrable systems which are very rigid ones, after their variables have chenged into very micro scale as cell automaton.

We also study the opposite direction, which makes time scalings more and more macro. Differential equations have been used to induce several biological phenomena, and they can produce some patterns appearing in nature. Thus it would be natural to seek for connections with micro interactions. One of the important directions will be to study dynamical properties for the latter, which are characteristic to the former. For example one may try to induce solutions of reaction diffusion equations from composition of maps as interaction of molecules. The point is that the latter has more micro time scaling than the former.

In section 5, we will connect these by time and distance scaling changes which makes micro interactions more macro. We will regard orbit spaces as functions on $\mathbf{N} = \{0, 1, 2, \dots\}$. Coming from probability ideas, we study a family of ODEs with the parametrization

$\{0, 1, \dots, N\}$. This is the feed back equations on molecules. Taking an orbit space as the initial condition, we study how it reflects to the solution when one makes $N \to \infty$ and at the same time changes the distance scaling. One will see that under such scaling, if the initial condition of the orbit space converges, then the solutions of the ODE also converge. In fact we will obtain the Turing's reaction diffusion equations from these micro motions by scaling limit. It is known that pattern formation appears from solutions of these non linear equations. Such macro patterns can be seen in many situations, like stripes of animal or fish skins.

0.A Molucular oscillations: In quantum level, molecular states are not deterministic. Thus when one tries to consider both aspects, the micro oscillation and more macro interacting one, one is inebitably forced to take into account on all possible states.

In this paper, we study dynamics of orbits $\{f^n(x)\}_{n \in \mathbf{N}}$, and regard them as though they are oscillations of molecules. If a molecule takes a state x, then one expresses all possible states of the molecule by *iteration* of f as:

$$f^k(x), \quad k \in \{0, 1, 2, \dots\}$$

and we call the orbit as the *oscillation*.

We will try to express interactions between molecules by some operations between $\{f^k\}_{k=0,1,\dots}$ and $\{g^k\}_{k=0,1,\dots}$. For families of maps, we will make them interact by themselves. In particular we will introduce several notions to express molecular interactions.

In general structure of the range of a map will be very complicated. Dynamics of a molecule is a long-time behaviour of the oscillation. Behaviours of states around period orbits are deeply studied:

$$P(n) = P(n)(f) = \{x \in [0, 1] : f^n(x) = x\}.$$

When a molecule is interacting with other molecules or is of phase transition, then its oscillation, or hybridization show chaostic behaviours in quantum chemistry. An element $x \in [0, 1]$ near the set $P(n)$ will behave either stable or unstable manners. Unstable points will cause chaostic phenomena.

0.B Micro dynamics of a single molecule: Before defining such operations, we will illustrate which types of molecular dynamics we will meet, if we use iteration of maps. We will consider such situations by using simple differential equations, where dynamical properties can be pictorically expressible in a simple way ([K2]).

Let $h : [0,1] \mapsto \mathbf{R}$ be a function. In order to express a large scale dynamics, one will consider the following family of the equations:

$$\dot{x}^k(t) \equiv -h(x^k), \quad x^k(0) = f^k(x)$$

where x^k represents a variable. We will call the above equation as the *feedback equation* for the oscillation. Iteration of the families of solutions:

$$\{f^l(x^k)(t)\}_{l,k}$$

will represent large scale dynamics of the molecule. Let us put $y_k(t) = \{f^l(x^k(t)\}_l$. We will call $y_k(t)$ as the *solution states*.

Let us consider dynamics of the above equations. Suppose x is a fixed point by f, and take a small $\delta > 0$ and put $x_\pm = x \pm \delta$.

Let us consider two cases that its derivative satisfies (a) $f'(x) > 1$ or (b) < 1. For (a), both $x_\pm$ are unstable points, and for (b), $x_\pm$ are stable points. Namely as $k \to \infty$, $f^k(x_\pm)$ will be away from x for (a), and approach to x for (b).

There are two cases;

$$(1) \quad h(x) \geq 0 \text{ or } \quad (2) \quad h(x) \leq 0.$$

Let us consider the case (a). After a small time, x will move to x_- for (1) and x_+ for (2). In both cases (1) and (2), their iterations $f^k(x_\pm)$ will move away from x. Since the effect of $-h(x)$ also helps to move on the same direction, These cases of dynamics will be *strongly unstable.* In these cases, the molecule will move to another states.

Suppose (b) holds. Then x will move to $x_{\pm}$ as above. Notice that the iterations $f^k(x_{\pm})$ will both converge to x as $k \to \infty$. If (1) holds, then the effect of the dynamics will try to make x_- to move away from x, so the dynamics will be in an *equilibrium state* for some appropreate h. If (2) holds, the dynamics will help to converge to x, and so it will be *strongly stable.* When two molecules are interacting in strongly stable states, then they will consist of a *polymer.* Self-feedback is working for the case (b) and (2). For more detail, see [K2].

We have shortly described very simple patterns, but in general the orbits of states will be much more complicated, and may show choastic behaviour.

0.B.2 Simple interactions: As in $0.B$, we will here illustrate which types of molecular interactions we will meet, if we use iteration of a map ([K2]).

In our framework, a state of an ensyme A is expressed by an element $x \in [0,1]$ near $P(n)$ for some n. Now let us take two proteins A_1 and A_2, and suppose A_2 acts on A_1 as a *repressor* (*activator* resp.). Suppose that at a time t these are in states x_1 and x_2 respectively. Repressing (resp. activating) dynamics are expressed by differential equations as:

$$\dot{x}_1^k(t) = -h(x_2^k) \quad \text{(repressor) or,} \quad \dot{x}_1^k(t) = h(x_2^k) \quad \text{(activator).}$$

Combinig with self-beedback, one can totally write the equations for the activator case as:

$$\begin{cases} \dot{x}_1^k(t) = h(x_2^k) - h(x_1^k), & x_1^k(0) = f^k(x_1), \\ \dot{x}_2^k(t) = -h(x_2^k), & x_2^k(0) = f^k(x_2). \end{cases} \tag{0.1}$$

Let us consider the patterns of this dynamics as in the previous section. In this simple case, there are no interaction from A_1 on A_2, and for the case of x_2, one can trace its behaviour as in the section 2. Let us take a fixed point x_1. The point is the sign of $h(x_2^k) - h(x_1^k)$. There are two cases:

$$(1) \quad h(x_2^k) - h(x_1^k) \geq 0, \quad (2) \quad h(x_2^k) - h(x_1^k) \leq 0.$$

If the molecule starts from x_1, then one may follow the dynamics of A_1 by the same way as section 2. Let us put $x_\pm = x_1 \pm \delta$, and take another $x_1 < x_1' < x_+$. suppose at the beggining, A_1 has the state x_1'. Then we have the following diagram; x_1' moves to:

$$\begin{cases} x_+ \text{ (strongly unstable)}, & f'(x_1) > 1, \quad h(x_2^k) - h(x_1^k) \geq 0, \\ x_1' \text{ (equilibrium state)}, & f'(x_1) > 1, \quad h(x_2^k) - h(x_1^k) \leq 0, \\ x_1' \text{ (equilibrium state)}, & f'(x_1) < 1, \quad h(x_2^k) - h(x_1^k) \geq 0, \\ x_1 \text{ (strongly stable)}, & f'(x_1) < 1, \quad h(x_2^k) - h(x_1^k) \leq 0. \end{cases} \quad (0.2)$$

For example suppose $f'(x_1) > 1$ and both $h(x_2^k)$ and $h(x_2^k) - h(x_1^k)$ are negative. Then as the quantity of A_2 is gradually increasing, the state of A_1 cannot stay near x_1. Namely A_1 will change from an equilibrium state into a strongly unstable one.

One may consider the case when A_1 starts at the state $x_- < x_1'' < x_1$ by a parallel way. For more detail, see [K2].

0.C Micro interaction: Lete us take two oscillations $\{f^n(x)\}$ and $\{g^n(x)\}$. For each half infinite sequence $\bar{k} = (k_0, k_1, \dots)$, $k_i = 1, 2, \dots$, we will associate with another oscillation $\{h^m\}_{m=0}^\infty$, where for $\Sigma_{i=0}^l N_i \leq m < \Sigma_{i=0}^{l+1} N_i$,

$$h^m(x) = \begin{cases} f^{m - \Sigma_{i=0}^l N_i} \circ g^{N_l} \circ f^{N_{l-1}} \circ \cdots \circ g^{N_1} \circ f^{N_0}(x) & l \text{ is odd}, \\ g^{m - \Sigma_{i=0}^l N_i} \circ f^{N_l} \circ g^{N_{l-1}} \circ \cdots \circ g^{N_1} \circ f^{N_0}(x) & l \text{ is even}. \end{cases}$$

We call it as the *interaction* between these two oscillations.

The first part of section 1 is devoted to study analytic properties of interactions from one dimensional dynamics point of view. For example we study the set of periodic sequences.

For a single map, there is an ordering of the natural numbers called Sarkovskii order. It is known that if $n \succ m$ holds for the ordering and if there exists a periodic point of period n of a map, then it also has period m. Thus the set of periodic points are closed under its ordering.

Each half infintie sequence as above corresponds to an element in the full shift X_2 with the alpabet $\{0,1\}$. Thus an interaction is determined by each $\bar{k} \in X_2$. Let $X^s(f,g) \subset X_2$ be a subset such that for any $\bar{k} \in X(f,g)$, the corresopnding periods of the interaction are closed under the ordering. The Sarkovskii's theorem is formulated in our setting as:

Proposition 0.1

$$X^s(f,f) = X_2.$$

In general it is not known even whether $X^s(f,g)$ might be closed under shift or be a closed subset. Here we will study a weaker version. One has a subset $X_0^{ly}(f,g)$ which requires some condition on cycles which we call the *Li-York property* and a common local maximum property on finite range data. These are automatically satisfied for a single map (f,f). Then one has the following:

Theorem 0.1 $X_0^{ly}(f,g) = X_2$.

By use of natural projections $\pi : [0,1] \backslash \frac{1}{2} \to \{0,1\}$, this has naturally led to various relations with symbolic dynamics. In fact one obtains continuous maps between rooted trees.

Let us take any subset $X \subset X_2$. Then combining with $\{h^n\}$, one obtains a map:

$$\Phi(x) : X \to Y \subset X_2, \quad \bar{k} \to \{\pi(h^k)\}$$

which we call the *interaction map*. We study several properties of Φ.

Firstly let us state an analytic result. A circle homeomorphism $F : S^1 \cong S^1$ corresponds to a monotone interval map $f : [0,1] \to [0,1]$ with a unique discontinuous point c. Thus there is a natural decomposition of $[0,1]$ into three intervals $[0,c) \cup \{c\} \cup (c,1]$. When one considers two circle homeomorphisms F and G, then one also has a natural decomposition into five intervals. By use of the corresponding projections $\pi : [0,1] \to \{0,\dots,5\}$, one can assign elements

$\Phi(f,g)(x) \equiv \{\pi \circ h^n(x)\}_n \in X_5$ from the interactions $\{h^n\}$. In particular we put:

$$K_i^+(f) = \lim_{x \downarrow c} \Psi(f,g)(c_i), \quad i = 1,2.$$

Let us denote:

$$O = O(f,g) : X_2 \times [0,1] \to [0,1]^{\mathbf{N}}$$

as $O(\bar{k}, x) = (h^0, h^1, \ldots)$. We say that (f,g) and $(\tilde{f}, \tilde{g})$ are *semi conjugate* on $X \subset X_2$, if there is a surjective and continuous map $H : [0,1] \times X \to [0,1]$ so that:

$$H \circ (h^m(f,g) \times \sigma^m) = h^m(\tilde{f}, \tilde{g}) \circ H : [0,1] \times X \to [0,1]$$

hold for all $m = 0, 1, \ldots$.

Let us take two pairs (f,g) and $(\tilde{f}, \tilde{g})$ which correspond to circle homeomorphisms, and satisfy $c_1 < c_2$ and $\tilde{c}_1 < \tilde{c}_2$. Then we have the following result which reduces the semi-conjugacy to symbolic dymanics $(1.A.3.1.2)$:

Theorem 0.2 *Let us take a surjectively invariant subshift X. Suppose (1) $K_i^+(f,g) = K_i^+(\tilde{f}, \tilde{g}) \in X_5$ holds on X and $i = 1,2$, and (2) $O(\tilde{f}, \tilde{g})(c_i, \bar{k})$ has X-dense orbits for one of $i = 1,2$.*

Then there is a semi conjugacy between (f,g) and $(\tilde{f}, \tilde{g})$ on X.

Recall that an automaton is consisted by two functions, transition and exist ones. One can use four interval maps to obtain similar structure as automata. In fact it generalizes and includes automata, which would be a 'perturbation' of automata structure.

One can follow the same construction as above when one uses four interval maps. In particular, one obtains continuous maps $\Phi(x) : X_2 \to X_2$ from a family of maps (f, g, α, β). Let $R = R(f, g, \alpha, \beta) \subset [0,1]$ be the set where $\Phi(x)$ gives a homeomorphism:

$$\Phi(x) : X_2 \cong X_2.$$

We denote by $G \subset \text{Aut } X_2$ the group generated by R. One can construct automata groups from four interval maps. For example, one has the following:

Proposition 0.2 *There is a family of smooth interval maps (f, g, α, β) such that the corresponding G is isomorphic to the lamplighter group. In particular G acts on ∂X_2 ergodically.*

0.D Systems: Oriented graphs are obtained from famillies of oscillations, called *interaction graphs.* Let f and g be two interval maps, and denote by $\Phi(x)$ as the corresponding interaction map. Now another interval map d. Then we say that the oscillation $O_3 = \{d^m(z)\}_m$ is the *interaction* in X_2 with $O_1 = \{f^k(x)\}_k$ and $O_2 = \{g^k(x)\}_k$ with respect to $\bar{k} \in X_2$, if the equality holds:

$$\{\pi \circ d^m(z)\}_m = \Phi(x)(\bar{k})$$

for some $z \in [0, 1]$.

Suppose the oscillation $\{h^l(z)\}_l$ is the interaction with $\{f^l(x)\}_l$ and $\{g^l(x)\}_l$ wrt $\bar{k}$. Then among three maps $\{f, g, h\}$, one can obtain a *marked oriented edge*:

$$(f, x) \xrightarrow{g} (h, z).$$

Let us choose families of maps $\{f_0, \dots, f_k\}$, points $\{x_0, \dots, x_l\}_l$ and an element $\bar{k} \in X_2$. By restricting $f, g, d \in \{f_i\}$ and $x, z \in \{x_j\}$, one can construct an oriented, marked and finite graph:

$$G(\{f_i\}_i; \{x_j\}_j, \bar{k})$$

which we call an *interaction graph.*

Cycles are particularly important in relation with cirggadiam rythms. By varying points $x \in [0, 1]$, one obtains families of interaction graphs. From random graph theory, it is expected that random interaction graphs would consiste of at most one cycles when their critical exponents are small in some sense, where the critical exponent $\lambda(G)$ is the rates of the numbers of vertices and edges.

Let us choose an infintie family of points $\{x_0, x_1, \dots\}$, and consider $\{G_N \equiv G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^N, \bar{k})\}_N$, the corresponding family of interaction graphs. In general it is not easy to estimate how the critical exponents $\lambda(G_N)$ will behave. We have the following:

Proposition 0.3 *Suppose a family $\{f_i\}$ consistes of l-modal maps which have no wandering intervals and no periodic attractors. Then for each $\bar{k} \in X_2$, there is a family of points $\{x_0, x_1, \dots\}$ so that the corresponding interaction graph*

$$\lim_{N\to\infty} \lambda(G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^N, \bar{k}) = 0$$

holds.

0.E Soliton cell automaton: Lotka Vorterra equation is known as describing the growth rate of competiting animals. One of the important point is existence of *soliton.* There is a corresponding difference-difference equation, *Lotka Vorterra cellular automaton*:

$$v_n^{t+1} - v_n^t = \max(0, v_{n+1}^t - L) - \max(0, v_{n-1}^{t+1} - L).$$

A solution consistes of a family of functions $v_n^t \in \{0, 1, \dots, L\}$. There is an exmple of a soliton for the discrete L-V equation that can be continuously deformed into another soliton for the L-V cellular automaton.

Using a family of interval maps, one can obtain an interaction map $\Phi(x) : X_L \to X_L$ by the same way as before, where one divides $[0, 1]$ into L intervals. Let us choose $x \in [0, 1]$, and put $(k_0^t, k_1^t, \dots) = \Phi(x)^t(k_0, k_1, \dots)$. We call them as the *flow* of Φ.

Proposition 0.4 *There is a family of smooth maps $\{f_{i,j}\}_{i,j=0,\dots,L}$ so that the corresponding two step interaction gives a flow of the e Lotka Vortarra cellular automaton.*

In particular one can realize soliton of the L-V cellular automaton as the flow of the interaction maps. In our formulation, the soliton flow can be interpreted as a transition process in chemical reaction.

We have also obtained another family of maps which gives flow of the Toda cell automaton. Structure of the flow contains some different points from the L-V case.

0.F Feedback system: When proteins play roles in bio-mechanisms, their dynamics seem to hold both aspects, deterministic and non-deterministic behaviours. The former is because proteins are polymers and whose dynamics apply classical mechanics. Quantum effect by electron enters into the latter. In particular in chemical reaction, molecular oscillation palys very important roles. One particular point is that these have very different time scalings mutually. Here one will try to represent *bio-interaction* dynamics by taking into account of these features.

Differential equations can be used to describe dynamics of molecules, which correspond to the former. Let us take simple differential equations including time parameter. Take a smooth function $h : [0,1] \to \mathbf{R}$ which might represent gradient of energy functional at a state $x \in [0,1]$. With respect to the time parameter, *feedback equation* for a single state is written as:

$$\dot{x} = -h(x).$$

This represents a large scale dynamics, and we need to take into account of small scale oscillations.

Electron orbital (EO) is not completely random. There is a hierarchy of orbits of spins, $1s, 2s, 2p$, hybridization, etc, from low energy states. So the strength of feed-back stresses will be various with respect to states of molecules. At the same time, the time paramter in the differntial equation corresponds to larger scaling compared with the oscillations of molecules. Thus taking into account of these, one will consider a family of feed-back equations for an oscillating molecule as:

$$\dot{x}^k = -h(k, x^k), \quad x^k(0) = f^k(x).$$

EO has interaction which is called *interorbital interaction* in quantum chemistry. When x^k has interaction with both x^{k-1} and x^{k+1}, the feed back equations will become: $\dot{x}^k = V(x^{k-1}, x^k, x^{k+1}) + h(k, x_k)$ where V is the diffusion term.

0.G Probabilistic limit and pattern formation: Inside cell, molecules are receiving various kinds of forces from others, e.g., H^2O, etc, which are in fact non deterministic. Thus in order to take into account with more microscopic motion of molecules, one will consider *Brownian motion*, and one can consider the following family of *stochastic differential equations:*

$$dx^k = [V(\{x^i\}_i) + h(k, x_k)]dt + dB_t$$

with the initial condition $x^k(0) = f^k(x)$. Here V is the interaction term, $h(k, \quad)$ is the feed-back term and dB_t is the Brownian motion.

The generator L of the above equation is a PDE. In this paper we study its finite state approximation, a finite state *feed-back processes.* Namely the distributions μ_t conrresponding to a finite range stochastic processes satisfiy a family of differential equations:

$$\frac{d}{dt}\mu_t = \mu_t L + (h(\mu_t^0), \ldots, h(\mu_t^{N-1}))$$

where L is a fintie matrix and $h : \mathbf{R} \to \mathbf{R}$ is a function with $h(0) = 0$.

As we have said, global structure appears from micro systems. In order to bridge two different time scalings, one takes a scaling limit, where one regards that an oscillation $x, f(x), \ldots, f^N(x), \ldots$ lives in a small time scale N, on the other hand t is in much larger scale.

Using the corresponding distribution, we will construct a family of measures $\mu_t^N(d\theta)$ on S^1, where N is the number of states. By letting $N \to \infty$, one obtains reaction diffusion equations:

Proposition 0.5 *Let η^N be a feed-back process with the feed back term h. Suppose $\mu_0^N(d\theta)$ converges to $\rho d\theta$. Then $\mu_t^N(d\theta)$ converges to $\rho(t, \theta)d\theta$ where ρ satisfies the reaction diffusion equation:*

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\frac{\partial^2}{\partial \theta^2}\rho + h(\rho).$$

When one considers interaction between molecules, one takes two stochastic processes which we call *interacting processes.* In particular taking scaling limit of two interacting processes, one obtains the Turing's reaction diffusion equations:

Proposition 0.6 *Suppose both $(\mu_0^N)_1(d\theta)$ and $(\mu_0^N)_2(d\theta)$ converge to $\rho_1 d\theta$ and $\rho_2 d\theta$ respectively. Then $(\mu_t^N)_1(d\theta)$ and $(\mu_t^N)_2(d\theta)$ also converge to $\rho_1(t,\theta)d\theta$ and $\rho_2(t,\theta)d\theta$ respectively, where ρ_1 and ρ_2 satisfies the Turing's reaction diffusion equation:*

$$\frac{\partial}{\partial t}\rho_1 = \frac{d_1}{2}\frac{\partial^2}{\partial \theta^2}\rho_1 + h_1(\rho_1, \rho_2),$$
$$\frac{\partial}{\partial t}\rho_2 = \frac{d_2}{2}\frac{\partial^2}{\partial \theta^2}\rho_2 + h_2(\rho_1, \rho_2).$$

The contents of the paper is as follows.

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2 From micro interaction to macro: 2.B Microscopic system, 2.C Cyclic system, 2.C.2 Cirrgadian rythms, 2.D Switching of genes, 2.D.2 Systems by binary enhansers, 2.E Soliton cell automaton, 2.E.1 Lotka Volterra cellular automaton, 2.E.1.2 Induction of L-V cellular automaton from interactions, 2.E.1.2.2 Solitons for L-V cellular automaton, 2.E.2 Two dimensional interactions and Toda cellular automaton, 2.E.2.2 Solitons for Toda cellular automaton,

3 Entropy: 3.A Interaction entropy, 3.B Topological entropy, 3.C Shift entropy, 3.C Distributed entropy,

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5 Time hierarchy, scaling limit and probability: 5.A Time hierarchies, 5.B Scaling change, 5.C Interaction differential equations, 5.D Finite state approximations of generators, 5.E Approximation of maps, 5.F Iterating maps and feed-back processes, 5.G Interacting proceeses and pattern formation.

1 Interaction

1.A Microscopic interaction: In quantum level, molecules interact each other by chemical bond, where electrons move on these orbitals.

Let us take two maps $f, g : [0,1] \to [0,1]$, and consider two oscillations:

$$O_1 = \{f^k(x)\}_{k=0,1,\dots}, \quad O_2 = \{g^k(x)\}_{k=1,\dots}, \quad x \in [0,1].$$

We will express interaction between these as follows. Let us take a sequence:

$$\bar{N} = (N_0, N_1, \dots), \quad N_i \in \{1, 2, \dots\}.$$

Definition 1.1 *$O = O(\bar{N})$ is an interaction between O_1 and O_2 with the interacting states $\bar{N}$, if for $x \in [0,1]$ and $\Sigma_{i=0}^{l} N_i \le m < \Sigma_{i=0}^{l+1} N_i$,*

$$O = O(x) = \{h^k(x)\}_{k=0,1,\dots},$$
$$h^m(x) = \begin{cases} f^{m-\Sigma_{i=0}^{l} N_i} \circ g^{N_l} \circ f^{N_{l-1}} \circ \dots \circ g^{N_1} \circ f^{N_0}(x) & l \text{ is odd}, \\ g^{m-\Sigma_{i=0}^{l} N_i} \circ f^{N_l} \circ g^{N_{l-1}} \circ \dots \circ g^{N_1} \circ f^{N_0}(x) & l \text{ is even}. \end{cases}$$

Let:

$$X_2 = \{(k_0, k_1, \dots) : k_i \in \{0,1\}\}$$

be the one sided full shift with the alphabet $\{0,1\}$. Then one may also assign $O(\bar{k}) = \{h^k\}_{k=0,1,\dots}$ for each $\bar{k} \in X_2$ as follows. We will associate 0 for f, and 1 for g below.

Let us divide the sequence $\bar{k}$ into succesive regions:

$$(k_0, k_1, \dots) = (\alpha_0, \dots, \alpha_1, \beta_0, \dots \beta_1, \alpha_2, \dots, \alpha_3, \beta_2, \dots, \beta_{i-1}, \alpha_i, \dots)$$

where $\alpha_{2i+1}+1=\beta_{2i}$ and $\beta_{2i+1}=\alpha_{2i+2}-1$. Moreover for all $i = 0,1,\dots$

$$k_l = \begin{cases} 1 & \alpha_{2i} \leq l \leq \alpha_{2i+1}, \\ 0 & \beta_{2i} \leq l \leq \beta_{2i+1}. \end{cases}$$

Then for all $m = 0,1,\dots,$ we put:

$$N_i = \begin{cases} \alpha_{i+1}-\alpha_i+1 & i = 2m, \\ \beta_i - \beta_{i-1}+1 & i = 2m+1. \end{cases}$$

Suppose $\alpha_0 \neq \phi$. Then one obtains another sequence $\bar{N} = (N_0, N_1, \dots)$ by positive numbers $N_i \in \{1,2,\dots\}$. Then by this assignment, we put:

$$O(\bar{k}) \equiv O(\bar{N}).$$

So in this case $O(\bar{k})$ begins from f.

When $\alpha_0 = \phi$ and $\bar{k}$ is of the form:

$$\bar{k} = (\beta_0, \dots \beta_1, \alpha_2, \dots, \alpha_3, \beta_2, \dots, \beta_{i-1}, \alpha_i, \dots)$$

then one obtains similarly $O(\bar{k}) = \{h^m\}_m$ where:

$$h^m \equiv h^m(x) = \dots \circ g^{N_a} \circ f^{N_{a-1}} \circ \dots \circ g^{N_1}(x).$$

We say that $\bar{N}$ is the *partition* of $\bar{k}$.

If we use $\bar{N}$, then the interaction always begins from f. On the other hand when one uses X_2, it contains the case when the interasction starts from g also. So the latter case will be more general and convenient. Later on we will always use X_2.

1.A.1 Cyclic sequences: Given a sequence $\bar{k} = (k_0, k_1, \dots)$ and choose $x \in [0,1]$. We say that $O = O(x) = \{h^k(x)\}$ has cycle l, if there is some m with $h^m(x) = h^{m+l}(x)$. We will say that $\{h^m(x)\}$ is a *cyclic sequence*, if there exists an increasing sequence $p_0 < p_1 < \dots$ so that each $\{h^{p_i}(x), h^{p_i+1}(x), \dots, h^{p_{i+1}}(x)\}$ consistes of a periodic orbit of length $l_i = p_{i+1} - p_i$.

Let $\{I_i\}_i$ be a family of intervals. We say that it is cyclic, if there exists an increasing sequence $p_0 < p_1 < \dots$ so that $I_{p_i} = I_{p_0}$ hold for all i. We say that $O(x) = \{h^m(x)\}$ is *almost cyclic* with respect to a periodic family of intervals $\{I_i\}_i$, if $h^m(x) \in I_{m+1}$ hold for all m.

Let us denote $d_i : [0,1] \to [0,1]$ as $d_{2m} = f$ and $d_{2m-1} = g$ for all m. Let us take a sequence $\bar{k} = (k_0, k_1, \dots) \in X_2$.

Lemma 1.1 *(1) Let $I \subset [0,1]$ be an interval. If $I \subset d_i(I)$, then there is a fixed point $x \in I$ for d_i in the closure.*

(2) Let $\{I_0, I_1, \dots\}$ be a family of closed intervals in $[0,1]$. Suppose

$$I_{i+1} \subset d_{k_i}(I_i).$$

Then there exists a decreasing family $I_0 \supset J_0 \supset J_1 \supset \dots$ and a point $x_0 \in I_0$ so that:

$$h^{m-1}(J_{m-1}) = I_m, \quad h^{m-1}(x_0) \in I_m$$

hold for all m.

Proof : (1) is standard, and follows from the mean value theorem.

We verify (2). Since $d_{k_0}(I_0) \supset I_1$ holds, there is an interval $J_0 \subset I_0$ with $d_{k_0}(J_0) = I_1$. We proceed by induction. Suppose we have $J_0 \supset \dots \supset J_{n-1}$ so that $h^{i-1}(J_{i-1}) = I_i$ hold for $1 \le i \le n$. By the hypothesis, $d_{k_n}(I_n) \supset I_{n+1}$ holds, and so there exists an interval $\tilde{I}_n \subset I_n$ with $d_{k_n}(\tilde{I}_n) = I_{n+1}$. On the other hand since $h^{n-1}(J_{n-1}) = I_n \supset \tilde{I}_n$ holds, ther exists $J_n \subset J_{n-1}$ with $h^{n-1}(J_n) = \tilde{I}_n$. Thus we have obtained the equalities:

$$h^n(J_n) = d_{k_n} h^{n-1}(J_n) = d_{k_n}(\tilde{I}_n) = I_{n+1}.$$

This completes the induction hypothesis.

Finally we choose $x_0 \in \cap_{i=1}^{\infty} J_i$. Then it satisfies the desired property. This completes the proof of the lemma.

Corollary 1.1 *If a cyclic family of intervals $\{I_i\}_i$ satisfies $I_{i+1} \subset d_{k_i}(I_i)$ for all i, then there is $x \in [0,1]$ so that $O(x) = \{h^m(x)\}_m$ is an almost cyclic sequence.*

1.A.2 Finite range sequences: Let us take a family of points $\{z_0, z_1, \dots, z_m\}$. We say that with respect to $\{h^k\}$, $\{z_0, z_1, \dots, z_m\}$

has *finite range* and the *essential range* is $\{x_0, \dots, x_l\}$, if there is some L_0 so that for all $L \geq L_0$ and $i = 0, 1, \dots, m$, the following holds:

$$\cup_{k=L}^{\infty} h^k(z_i) = \{x_0, \dots, x_l\}.$$

We say that $(f, g, \bar{k}, \{x_i\})$ gives a finite range data.

Let $\{z_0, \dots, z_m\}$ be a finite range sequence with the essential range $\{x_0, \dots, x_l\}$. We say that it is a *cyclic sequence*, if there is N_0 so that for all $0 \leq i \leq m$, the family $\{h^k(x_i)\}_{k \geq N_0}$ does not contain any cycles of length less than $l+1$.

We say that $l+1$ is the period of the cyclic sequence.

Lemma 1.2 *Let $\{z_0, \dots, z_m\}$ be a finite range sequence. Then for each i, there is some N_i so that $\{h^k(x_i)\}_{k \geq N_i}$ consistes of a sequence by cycles of length $l+1$.*

Proof: By finite range property, each family $\{h^k(x_i)\}_{k=0,1,\dots}$ contains cycles. By the definition, their lengths are longer than $l+1$. Suppose there could exist a cycle of length $l'+1 \geq l+2$, and write it as $(x_{i(0)}, \dots, x_{i(l')}) \subset \{x_0, \dots, x_l\}$. Then at least some x_j appears more than twice, and it implies that the cycle contains another whose length is less than $l+1$. This is a contradiction.

Example 1.1: Let $(f, g, \bar{k}, x)$ give a finite range data. It will happen that for some different $\bar{k}'$, the corresponding $\{(h')^k\}$ may not be of finite range.

Let us take a periodic point x of f of period 5, but it is not a periodic point of g. Suppose $f^l(x) = g^l(x)$ hold for $l \leq 3$, but for $k \geq 4$, $f^k(x) \neq g^k(x)$. Then for example

$$\bar{k} = (2, 1, 3, 1, 3, 1, 3, 1, \dots)$$

gives a finite range data with respect to $\{x\}$ with the essential range $\{x, f(x), f^2(x)\}$. It is a cyclic sequence with the period 3.

On the other hand if $k_{2m} \geq 4$ satisfies for some m, then they will not give finite range data in general.

Definition 1.2 *Let* $\{z_0, \dots, z_m\}$ *be a cyclic sequence with respect to* $(f, g, \bar{k})$ *such that* $h^k(\{z_0, \dots, z_m\}) = \{x_0, \dots, x_l\}$ *hold for all large* k.

We say that it is a periodic sequence, *if* $\{x_0, \dots, x_l\}$ *is a common periodic orbits for both* f *and* g.

Let us put the set of periods as:

$P(\bar{k}, f, g) = \{l \in \mathbf{N};$ there is some periodic sequence of period $l\}$.

Recall the Sarkovskii's order:

$$3 \succ 5 \succ 7 \succ \cdots \succ 6 \succ 10 \succ 14 \succ \dots$$

We say that a subset $A \subset \mathbf{N}$ is *closed* under Sarkovskii's order, if $m \in A$ holds when $n \succ m$ and $n \in A$.

Recall that X_2 is the one sided full shift with the alphabet $\{0, 1\}$. Let us put a subset $X^s(f, g) \subset X_2$ by:

$X^s(f, g) \equiv \{\bar{k} \in X_2 : P(\bar{k}, f, g)$ is closed under the Sarkovskii's order$\}$.

Proposition 1.1 (Sarkovskii)

$$X^s(f, f) = X_2.$$

Corollary 1.2 $X^2(f, g) \subset X_2$ *is dense.*

Proof: By Sarkovskii's result, $(0, 0, \dots)$ and $(1, 1, \dots) \in X^2(f, g)$. Since periodic sequences are characterized by properties at infinity, all sequences of the form:

$$(k_0, k_1, \dots, k_s, i, i, i, \dots), \quad i = 0, 1$$

are in $X^2(f, g)$. Such sequences consist of dense subset in X_2. This completes the proof.

In particular one can see $X^2(f, g) = X_2$, when $X^2(f, g)$ is closed in X_2.

Here we study a weaker version. Let $\{x_0, \dots, x_l\}$ be a periodic sequence with respect to $\bar{k} \in X_2$. We say that it has a *common local maxima*, if there is some a so that

$$x_a = \max\{x_i; f(x_i) > x_i\} = \max\{x_i; g(x_i) > x_i\}$$

hold. We call that it is an *admissible* periodic sequence.

We put a smaller subset

$$P_0(\bar{k}, f, g) \subset P(\bar{k}, f, g)$$

which consistes of periods of all admissible periodic sequences.

Let us put the odd minimun:

$$l_0 \equiv \inf\{l : l \in P_0(\bar{k}, f, g) \text{ is odd } \geq 3\}.$$

Let us say that $P_0(\bar{k}, f, g)$ satisfies *Li-York property*, if for any $M >> 0$, $\{h^m\}_{m \geq M}$ contains cycles of all length $l \geq l_0$.

Let us put:

$$X_0^{ly}(f, g) \equiv \{\bar{k} \in X_2 : P_0(\bar{k}, f, g) \text{ satisfy Li-York property}\}.$$

Theorem 1.1 $P_0(\bar{k}, f, g)$ *satisfy the Li-York property.*

Corollary 1.3 $X_0^{ly}(f, g) = X_2$.

Question: $P(\bar{k}, f, g)$ might satisfy Li-York property.

Let $\{I_1, I_2, \dots\}$ be a set of intervals in $[0, 1]$ and recall the corresponding *Markov graph* with respect to f. For each I_i, one associates a vertex, and two vertices i and j are connected by an oriented edge, if and only if $f(I_i) \supset I_j$ holds. Thus a Markov graph is an oriented finite graph.

Proof of the theorem: The proof goes in a parallel way, using lemma 1.1 to the one of the Sarkovskii's theorem. We outline the proof.

Let $d : [0, 1] \to [0, 1]$ be f or g. We will denote each interval $[x_i, x_{i+1}]$ by I_j for some j.

Step 1: Let us put $I_1 = [x_a, x_{a+1}]$. By the assumption (2), $d(x_a) \geq x_{a+1}$ and $d(x_{a+1}) \leq x_a$ hold. Thus we have $I_1 \subset d(I_1)$. In particular by lemma 1.1(1), there is a cycle of length 1.

Step 2: In the Markov graph consisted by $\{I_j\}_j$, for any I_j, there is a path which connects I_1 to I_j. This follows from the definition of finite range sequences.

Step 3: Let us put $J_l = \cup_{j<a}[x_j, x_{j+1}]$ and $J_r = \cup_{j>a}[x_j, x_{j+1}]$. Then when there are no I_j except I_1 which can be connected to I_1 by a path, then $d(J_l) \subset J_r$ and $d(J_r) \subset J_l$ hold respectively. Thus the periods outside I_1 are all even. Moreover in this case one can see that 2 is a period using lemma 1.1(1).

Step 4: Suppose $P_0(\bar{k}, f, g)$ contains an odd ≥ 3 period, and let $n \geq 3$ be the smallest such period. Then by step 2 and 3, there is a path of the form $I_1 \to I_2 \to \dots I_{m-1} \to I_1$, $m \leq n-1$. In particular by adding I_1 if necessarily, one can see that all $l \geq n$ are the periods of cycles for $\{h^m\}_{m \geq M}$ by lemma 1.1(1). This completes the proof.

1.A.3 Symbolic dynamics and iterations: Let $\pi : (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \to \{0, 1\}$ be the natural map. Then for any two maps f and g, one can obtain another maps defined for a.e. $x \in [0, 1]$:

$$\pi \circ f, \pi \circ g : [0, 1] \to \{0, 1\}.$$

Let $X \subset X_2$ be a subset. Then for each $\bar{k} \in X$ and $x \in [0, 1]$, one has the interaction $O(\bar{k}) = \{h^m\}_{m=0,1,\dots}$. Combining with $O(\bar{k})$ and π, one obtains a continuous map as follows.

Definition 1.3 *An interaction map $\Phi(x) : X \to X_2$ is given by:*

$$\Phi = \Phi(x) : X \mapsto Y(x) \subset X_2,$$
$$\Phi(\bar{k}) = (\pi \circ h^0(x), \pi \circ h^1(x), \dots)$$

where $Y(x)$ is the image of Φ.

We denote the shift by $\sigma : X_2 \to X_2$. Suppose $X \subset X_2$ is an invariant subshift. In general the image $Y(x)$ of $\Phi(x)$ is not invariant, since Φ is not always equivariant. But the total image:

$$Y = \cup_{x \in [0,1]} Y(x) \subset X_2$$

is an invariant subshift.

Question: When is Y of finite type ?

One may generalize this construction as follows. Let $\pi_l : [0, 1] \to \{0, 1, \dots, l\}$ be a map. For example one may devide $[0, 1]$ into $l + 1$

intervals as $I_0 = [0, c_1), I_1 = (c_2, c_3), \ldots, I_{l-1} = (c_{l-1}, c_l), I_l = (c_l, 1]$. Then one puts $\pi_l(I_i) = i$.

Let us take m maps $f_1, \ldots, f_m$, and let $X_m = \{(k_0, k_1, \ldots) : 0 \leq k_i \leq m-1\}$ be the full shift with the alphebet $\{0, 1, \ldots, m-1\}$.

Now using π_l and $(f_1, \ldots, f_m)$, one can generalize the construction above directly, and obtain a map:

$$\Phi(x) : X_m \to Y(x) \subset X_l.$$

1.A.3.0 Signed ordering: Any shift admits a natural ordering. Recall an ordering on X as $\bar{k} \succ \bar{k}'$, if and only if $k_i = k_i'$ for $0 \leq i \leq n \leq \infty$ and for all $i \geq n+1$, $k_i < k_i'$ hold.

Here we introduce ordering on the orbit spaces of interval maps. We will restrict on more treatable class of maps. Let $I = [0, 1]$ and $f : I \to I$ be a piecewise monotone maps with $f(\partial I) \subset \partial I$. We say that it is *$l$-modal*, if it has precisely l turning points, $0 < c_1 < \cdots < c_l < 1$.

Let us have a decomposition of I into $2l+1$ intervals:

$$\{I_1 = [0, c_1), I_2 = (c_1, c_2), \ldots, I_l = (c_{l-2}, c_{l-1}), \\ I_{l+1} = (c_l, 1], I_{l+2} = \{c_1\}, \ldots, I_{2l+1} = \{c_l\}\}.$$

We say that this is a canonical decomposition of the l-modal map f. Let us equip *sign* $\epsilon(I_i) = \epsilon_f(I_i) \in \{0, \pm 1\}$ for each I_i so that $\epsilon(I_i) = 1(-1)$ if $f|I_i$ is increasing (decreasing) respectively, and $\epsilon(c_i) = 0$.

Let $\bar{I} = \{I_1, \ldots, I_l\}$ and $\bar{J} = \{J_1, \ldots, J_m\}$ be two partitions of I. Then we put another refined partition by:

$$\bar{I} \vee \bar{J} \equiv \{I_i \cap J_j\}_{i,j}.$$

Let us take two modal maps f and g, and consider their canonical decompositions $\bar{I}$ and $\bar{J}$ respectively. We put $m = \sharp \bar{I} \vee \bar{J}$, and make reordering of the intervals as:

$$I_1 < I_2 < \cdots < I_m$$

where $I_i < I_j$ means any $x \in I_i$ is smaller than any $y \in I_j$. Also we put $-I_j < -I_i$ if $I_i < I_j$ formally. We equip sign for this case as

$\epsilon_0(I_i) = 1(-1)$, if $f|I_i$ is increasing (decreasing) respectively, and it is zero at the turning point of f. ϵ_1 is defined by the same way using g instead of f.

We put the corresponding projection by:

$$\pi_m : [0,1] \to \{1, 2, \ldots, m\}.$$

Now let us identify X_m with the full shift with the alphabet $\{I_1, \ldots, I_m\}$. With the above decomposition of the interval for f and g, let us consider:

$$\Phi(x) : X_2 \to Y(x) \subset X_m.$$

Then we introduce a signed ordering on $[0,1] \times X_2$ as follows. Let us put $\epsilon_0 = \epsilon_f$ and $\epsilon_1 = \epsilon_g$, and $\bar{k} = \Phi(x)(\bar{m})$ and $\bar{k}' = \Phi(y)(\bar{m}')$. Then $(x, \bar{m}) \prec (y, \bar{m}')$, if for some $n \geq 0$, $k_i = k_i'$ for $i = 0, \ldots, n-1$, and

$$(\Pi_{0 \leq j \leq n-1} \epsilon_{m_i}(k_i))k_n < (\Pi_{0 \leq j \leq n-1} \epsilon_{m_i'}(k_i'))k_n'.$$

We put $(x, \bar{m}) \preceq (y, \bar{m}')$, if $(x, \bar{m}) \preceq (y, \bar{m}')$ or $\bar{k} = \bar{k}'$.

Notice that when $f = g$, $\bar{k} = \Phi(x)(\bar{m})$ is independent of choice of $\bar{m}$.

Lemma 1.3 (MS) *For $f = g$, the following hold:*
(1) for any $x < y$, $(x, \bar{m}) \preceq (y, \bar{m}')$ holds.
(2) If $(x, \bar{m}) \prec (y, \bar{m}')$ holds, then $x < y$ also holds.

We say that $(x, \bar{m}) \in [0,1] \times X_2$ is *minimal*, if it is minimal with respect to the above ordering. By the above lemma, when $f = g$, $(0, \bar{m})$ are all minimal element for any $\bar{m} \in X_2$.

Perhaps there will exist an algorithm to find out a minimal element for f and g.

1.A.3.1 Shift equivariant maps: Let $\pi : [0,1] \to \{1, \ldots, N\}$ be a map and take an invariant subshift $X \subset X_2$ and two maps f and g. As we have seen above, $\Phi : X \to Y(x) \subset X_N$ is not equivariant. In order to obtain the shift map, we will consider a family of maps. Let us define a shift map:

$$\sigma : X \times [0,1] \to X \times [0,1],$$
$$\sigma((\bar{k}, x)) = (\sigma(\bar{k}), d_{k_0}(x)))$$

where $\sigma(k_0, k_1, \dots) = (k_1, k_2, \dots)$ and

$$d_i(x) = \begin{cases} f(x) & i = 0, \\ g(x) & i = 1. \end{cases}$$

Now combinig with π, the family of the interaction maps:

$$\Psi = \Psi(f,g) : X \times [0,1] \to X_N, \quad \Psi(\bar{k}, x) \equiv \Phi(x)(\bar{k})$$

becomes continuous and equivariant with respect to σ.

Let us define *conjugacy*. Let us choose two pairs of maps (f, g) and (f', g'). Then correspondingly one has two families of the interaction maps $\Psi(f, g)$ and $\Psi(f', g')$.

We say that $\Psi(f, g)$ and $\Psi(f', g')$ are *weakly semi-conjugate*, if there is a shift equivariant continuous map $h : X \times [0,1] \to X \times [0,1]$ so that

$$h \circ \Psi(f,g) = \Psi(f',g') \circ h$$

holds.

A weak semi-conjugacy is a *weak conjugacy*, if h is a homeomorphism.

A weak semi-conjugacy is *semi-conjugacy*, if there is a continuous surjection $H : X \times [0,1] \to [0,1]$ so that h is of the form:

$$h(\bar{k}, x) = (\bar{k}, H(\bar{k}, x)).$$

vspace3mm

1.A.3.1.2 Circle homeomorhisms: Let us consider two circle homeomorphisms F and $\tilde{F}$. They can be identified with increasing interval maps $f, \tilde{f} : [0,1] \to [0,1]$ with unique discontinuous points $c, \tilde{c} \in (0,1)$ respectively.

We take a projection $\pi : [0,1] \to \{L, c, R\}$

$$\pi(x) = \begin{cases} L & x \in [0,c) \\ c & x = c \\ R & x \in (c,1]. \end{cases}$$

$\tilde{\pi} : [0,1] \to \{\tilde{L}, \tilde{c}, \tilde{R}\}$ is similar. Thus one obtains the corresponding maps:

$$\Psi(f,f), \tilde{\Psi}(\tilde{f}, \tilde{f}) : [0,1] \to X_3$$

Let us put $K^+(f) = \lim_{x \downarrow c} \Psi(f,f)(c)$, and $K^+(\tilde{f})$ is similar.

Poincaré verified that any orientation preserving circle homeomorphism with no periodic point is semi conjugate to a unique irrational rotation, where the corresponding irrational number is equal to the rotation number. A part of its proof shows the following:

Lemma 1.4 *Suppose (1) $K^+(f) = K^+(\tilde{f}) \in X_3$, and (2) $\tilde{c}$ has a dense orbit on $[0,1]$ for $\tilde{f}$.*

Then there exists a semi conjugacy between f and $\tilde{f}$. Namely there is a surjecive monotone map $h : [0,1] \to [0,1]$ with

$$h \circ f = \tilde{f} \circ h.$$

Now we study this lemma in our formulation. Recall that one has associated with the interaction $O(\bar{k}) = \{h^m\}_m$ for $\bar{k} \in X_2$ and f, g. We denote this assignment as:

$$O = O(f,g) : [0,1] \times X_2 \to [0,1]^{\mathbf{N}}$$

as $O(\bar{k}, x) = (h^0, h^1, \dots)$.

Let us take an invariant subshift $X \subset X_2$, and denote the shift by $\sigma : X \to X$. We say that X is *surjectively invariant*, if $\sigma(X) = X$ holds.

In our case, the orbit is given after determining an element $\bar{k} \in X$. Let us have a condition corresponding to (3) above.

Definition 1.4 *$x \in [0,1]$ has X-*dense orbit*, if the following set:*

$$O^{-1}(f,g)(x, \bar{k}) \equiv \cup_{m=0}^{\infty} \cup_{\bar{k}' \in \sigma^{-m}(\bar{k})} \{h^m(f,g)(x, \bar{k}')\}$$

is dense in $[0,1]$.

The next follows from an easy calculation:

Lemma 1.5 *(f,g) and $(\tilde{f},\tilde{g})$ are semi conjugate on $X \subset X_2$, if and only if there is a surjective and continuous map $H : [0,1] \times X \to [0,1]$ so that:*

$$H \circ (h^m(f,g) \times \sigma^m) = h^m(\tilde{f},\tilde{g}) \circ H : [0,1] \times X \to [0,1]$$

holds for all $m = 0,1,\dots$.

. Namely for $O(f,g) = \{h^m\}_m$ and $O(\tilde{f},\tilde{g}) = \{\tilde{h}^m\}_m$,

$$H(h^m(x,\bar{k}), \sigma^m(\bar{k})) = \tilde{h}^m(H(x,\bar{k}), \bar{k})$$

hold for all $m = 0,1,\dots$

Let us take two circle homeomorphisms F and G, and the corresponding intertval maps as $f,g : [0,1] \to [0,1]$ with the unique discontinuous points c_1 and c_2 respectively. Suppose $c_1 < c_2$.

We divide the interval $[0,1]$ as:

$$\{I_1 = [0,c_1), I_2 = \{c_1\}, I_3 = (c_1,c_2), I_4 = \{c_2\}, I_5 = (c_2,1]\}.$$

Then we have the corresponding projection as $\pi : [0,1] \to \{1,2,3,4,5\}$, and the corresponding interaction map:

$$\Psi(f,g) : X \times [0,1] \mapsto X_5.$$

Let us put:

$$K_i^+(f,g) = \lim_{x \downarrow c_i} \Psi(f,g)(c_i) : X \mapsto X_5$$

for $i = 1,2$. $K_i^+(\tilde{f},\tilde{g})$ is similar.

Let us take two pairs (f,g) and $(\tilde{f},\tilde{g})$ which correspond to circle homeomorphisms, and suppose $c_1 < c_2$ and $\tilde{c}_1 < \tilde{c}_2$.

Theorem 1.2 *Let us take a surjective invariant subshift X. Suppose (1) $K_i^+(f,g) = K_i^+(\tilde{f},\tilde{g}) \in X_5$ holds on X and $i = 1,2$, and (2) $O(\tilde{f},\tilde{g})(\tilde{c}_i,\bar{k})$ has X-dense orbits for one of $i = 1,2$.*

Then there is a semi conjugacy between (f,g) and $(\tilde{f},\tilde{g})$ on X.

Proof: Let us put:

$$H(x,\bar{k}) = \sup\{y : \Phi(\tilde{f},\tilde{g})(y,\bar{k}) \preceq \Phi(f,g)(x,\bar{k})\}$$

where $\preceq$ is the usual lexicographic ordering on X_5.

Clearly $H(c_i,\bar{k}) = \tilde{c}_i$ hold for $i = 1, 2$. By the construction,

$$H(h^m(x,\bar{k}),\sigma^m(\bar{k})) \geq \tilde{h}^m(H(x,\bar{k}),\bar{k})$$

hold for all $m = 0, 1, \ldots$ We show that the commutativity:

$$H(h^m(x,\bar{k}),\sigma^m(\bar{k})) = \tilde{h}^m(H(x,\bar{k}),\bar{k}), \quad m = 0, 1, \ldots$$

hold by induction.

Firstly suppose no discontinuous points $\tilde{c}_1$ and $\tilde{c}_2$ are contained in $\{\tilde{h}^m(H(x,\bar{k}),\bar{k})\}_{m=0}^{n-1}$.

Suppose the equalities hold for $m = 0, \ldots, n-1$ and $H(h^n(x,\bar{k}),\sigma^n(\bar{k})) > \tilde{h}^n(H(x,\bar{k}),\bar{k})$ could hold. Then by the above inequality, $y = H(x,\bar{k}) + \epsilon$ also preserves the ordering

$$\Phi(\tilde{f},\tilde{g})(y,\bar{k}) \preceq \Phi(f,g)(x,\bar{k})$$

for a small $\epsilon > 0$. Thus contradicts to the definition of H.

Thus the above commutativity holds under the above assumption. Moreover since $H(c_i,\bar{k}) = \tilde{c}_i$ hold, it follows totally that the commutativity $H(h^m(x,\sigma^m(\bar{k})),\bar{k}) = \tilde{h}^m(H(x,\bar{k}),\bar{k})$ hold for all x and m.

Let (x,y) be a connected component of $[0,1]\setminus$ closure $O(f,g)(c_i)$. Since H is order preserving, $H(x,\bar{k}) \leq H(y,\bar{k})$ holds. Suppose $H(x,\bar{k}) < H(y,\bar{k})$ could hold. Then by the assumption (3), $H(x,\bar{k}) < \tilde{h}^n(\tilde{c}_i,\bar{k}') < H(y,\bar{k})$ holds for some $n \geq 1$ and $\bar{k}'$ with $\sigma^n(\bar{k}') = \bar{k}$. Thus $H(x,\sigma^n(\bar{k})) < \tilde{h}^n(\tilde{c}_i,\bar{k}) < H(y,\sigma^n(\bar{k}))$ holds. Since the following:

$$H(x,\sigma^n(\bar{k})) < H(h^n(c_i),\sigma^n(\bar{k})) = \tilde{h}^n(\tilde{c}_i,\bar{k}) < H(y,\sigma^n(\bar{k}))$$

hold, and $h^n(c_i) \leq x$ or $\geq y$ should hold, this is a contradiction, since H is order preserving.

Thus $H(x,\bar{k}) = H(y,\bar{k})$ hold for any $\bar{k}$, and one may extend H on all the interval. This completes the proof.

1.A.3.1.3 Rotation numbers: Let us take f and g corresponding to circle homeomorphisms F and G. Then we define the rotation family by:

$$\rho = \rho(f,g) : X_2 \to \mathbf{R},$$
$$\rho(\bar{k}) = \lim_{n\to\infty} \frac{1}{n}(\hat{h}^n(x) - x)$$

where $\hat{h}^n : \mathbf{R} \cong \mathbf{R}$ is the lift of the homeomorphism $h^n = \dots F^{N_i} \circ G^{N_{i-1}} \circ \dots : S^1 \cong S^1$.

1.A.3.2 Regularity and automorphisms: Let X_2 be the full shift and denote $\Phi = \Phi(x) : X_2 \to X_2$ be as before. We say that $x \in [0,1]$ is a *regular* point, if $\Phi(x)$ is a homeomorphism. We denote the set of regular points by

$$R = R(f,g) = \{x \in [0,1] : x \text{ is regular }\}.$$

Example 1.1: Let us choose:

$$f(x) = \begin{cases} \frac{1}{2} + x & 0 \le x < \frac{1}{2}, \\ x - \frac{1}{2} & \frac{1}{2} < x \le 1. \end{cases}$$

and $g(x) = x$. We take $I_1 = [0, \frac{1}{2})$ and $I_2 = (\frac{1}{2}, 1]$. Thus for any $x \in I_i$, $i = 1,2$, $\pi \circ f(x) \in I_{i+1}$ and $\pi \circ g(x) \in I_i \mod 2$. Thus the set of regular points is $R(f,g) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$.

Each element $x \in R$ gives an automorphism $\Phi(x)$ of X_2, and we put a subgroup of the automorphisms of X_2 generated by $\Phi(x)$:

$$G = G(f,g) = \text{ gen } \{\Phi(x) : x \in R(f,g)\} \subset \text{ Aut } X_2.$$

Let T_2 be the infinite binary tree with a base vertex x_0. Then each element in X_2 corresponds to a path starting at x_0.

Lemma 1.6 *Any $x \in R(f,g)$ gives an automorphism of the tree T_2:*

$$\Psi(x) = \Psi(x,f,g) : T_2 \cong T_2.$$

This follows from a property that any geodesic in T_2 is unique. Thus one can regard G as a subgroup of the automorphisms of T_2:

$$G = G(f,g) = \text{ gen } \{\Psi(x) : x \in R(f,g)\} \subset \text{ Aut } T_2.$$

Lemma 1.7 (GZ) *The action of G on T_2 is spherically transitive, if and only if the induced action on the boundary of T_2 is ergodic.*

When the orbits of f and g were very random, then most of maps $\Phi(x)$ will degenerate and will be far from homeomorphisms. On the other hand if these were quite simple, then the action of the tree will not be spherically transitive. Thus ergodicity measures suitable complexity of f and g. Later we will see more examples of G which arise from automata structure, and we will have an example with ergodic G $(1.D)$.

1.A.3.3 Convergence of shifts: Let us equip with a metric on X_2, and denote the shift by $\sigma : X_2 \to X_2$.

Let $X_i \subset X_2$ be a family of subsets $i = 0, 1, \dots$. We say that $\{X_i\}_i$ *converges* to $X' \subset X_2$, if

$$\lim_i d(X_i, X') = 0$$

holds.

Let $X' \subset X_2$ be a subset. We say that X' converges to $Y \subset X_2$ *at infinity*, if $X_i \equiv \sigma^i(X')$ converges to Y.

1.A.3.4 Renormalization: Let us choose (f,g,x). We say that a subshift $Y \subset X_2$ is *renormalizable* with respect to (f,g,x), if there exists a finite family $(Y_0, \dots, Y_l)$ of subshifts, $Y_0 = Y_l = Y$, so that one obtains the corresponding interaction maps:

$$Y \xrightarrow{\Phi_0} Y_1 \xrightarrow{\Phi_1} Y_2 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_{l-2}} Y_{l-1} \xrightarrow{\Phi_{l-1}} Y. \qquad (1.1)$$

1.B Interval dynamics: Let us take two maps $f, g : [0,1] \to [0,1]$, and a subshift $X \subset X_2$. Choose $\bar{k} = (k_0, k_1, \dots) \in X$. Correspondingly one has a sequence by positive numbers $(N_0, N_1, \dots)$ $(1.A)$. Then one obtains an interaction $O = O(\bar{k}) = \{h^k\}_{k=0,1,\dots}$ with the interacting states $\bar{k}$.

Let us define a map:

$$\tau(m) = \tau(f, g; \bar{k})(m) : [0,1] \to [0,1], \quad \tau(x) = h^m(x).$$

Let $T = (a, b) \subset [0, 1]$ be a proper and open interval. Namely $\bar{T}$ is a proper open subset in $[0, 1]$.

We say that T is an *invariant interval* for τ, if $\tau(m)(T) = T$ holds for all $m = 0, 1, \dots$. It is *locally maximal* invariant interval, if for any small ϵ_0, there is some $\epsilon_0 \geq\geq \epsilon > 0$ so that $(a-\epsilon, b+\epsilon)$ is not invariant.

Proposition 1.2 *Let us take $\bar{k} \in X_2$. Suppose both f and g have negative Schwarzian derivatives:*

$$Sf(x) \equiv \frac{d^3 f(x)}{df(x)} - \frac{3}{2}\left(\frac{d^2 f(x)}{df(x)}\right)^2, Sg(x) < 0$$

and h^m are orientation preserving for all m. If T is a locally maximal invariant interval, then it contains critical points of f or g.

We need the Minimum principle (see [MS] p154):

Lemma 1.8 *Let $T = (a, b) \subset [0, 1]$, and suppose f has negative Schwarzian derivatives. Then for any $x \in (a, b)$,*

$$|df(x)| > \min\{|df(a)|, |df(b)|\}$$

holds.

Proof of proposition: Suppose T does not contain critical points for both f and g. Thus boundaries of T are both fixed points for h^m. From maximality, both of $df(b)$ or $dg(b)$ must be more than 1. Then by the minimum principle, $df(x) > 1$ and $dg(x) > 1$ for any $x \in (a, b)$. Then T cannot be invariant. This completes the proof.

One may consider the above situation when T is maximal with respect to a property $\tau(m)(T) \subset T$.

1.B.1 Uniformly bounded intervals: Let $T \subset [0,1]$ be a proper and open interval. We say that T is an ϵ *invariant interval*, if

$$d(\tau(\bar{k})(m)(T), T) < \epsilon$$

holds for all $\bar{k} \in X$ and $m = 0, 1, \dots$.

Let us have an infinitesimal version. Let us take a point $x \in [0,1]$ which are fixed by both f and g. Take $\bar{k} \in X_2$. We say that x is an *uniformly invariant* fixed point with respect to $\bar{k}$, if the following holds; for any decreasing sequence $\epsilon_0 > \epsilon_1 > \dots > \epsilon_i > \dots > 0$ with $\epsilon_i \to 0$, there is another family of intervals $T_0 \supset T_1 \supset \dots \supset T_i \supset \dots \ni x$ so that T_i are all ϵ_i uniformly invariant intervals.

Now we define *invariant radius* $r(x, \bar{k})$ by:

$$r(x, \bar{k}) = \sup_{\{(T_i, \epsilon_i)\}} \liminf_i \{ \frac{\epsilon_i}{|T_i|} : \\ \{(T_i, \epsilon_i)\}_i \text{ give uniformly bounded data}\}.$$

Let $X \subset X_2$ be a subshift. Then we say that X is a *bounded subshift* at x, if $r(x, \bar{k}) < \infty$ for all $\bar{k} \in X$. Let us put:

$$r(x, X) = \sup_{\bar{k} \in X} r(x, \bar{k}).$$

We say that X is a *uniformy bounded* subshift at x, if $r(x) < \infty$. Let us put:

$$X(x) \equiv \{\bar{k} \in X_2 : r(x, \bar{k}) < \infty\}.$$

$X(x)$ is a maximal uniformly bounded subshift. $X(x)$ is an invariant subshift.

Question: When $X(x)$ is a closed subshift in X_2 ?

For integers M, N, we define an invariant subshift $\Sigma(R) \subset X_2$ as follows; for any sequence $\bar{k} = (k_0, k_1, \dots)$, we have the partition as a sequence of integers by $1 \leq n_0, n_1, \dots,$ where $k_0, \dots, k_{n_0 - 1}$ are all 0, $k_{n(0)}, \dots, k_{n(1)-1}$ are all 1, $k_{n(1)}, \dots, k_{n(2)-1}$ are all 0, and so on.

Then $\bar{k} \in \Sigma(R)$ if and only if for the corresponding partition data

$(n_0, n_1, \dots)$ satisfies

$$\lim_{m\to\infty} \frac{\Sigma_{i=0}^m n_{2i}}{\Sigma_{i=0}^m n_{2i-1}} = R$$

hold for all i. Clearly $\Sigma(R)$ are all invariant subshift of X_2.

Example 1.2: Suppose x is a fixed point for both f and g with $0 < df(x) < 1 < dg(x)$. We define a *rate* $R = R(x)$ at x by:

$$dg(x) = \frac{1}{df(x)^R}.$$

Then $\Sigma(R)$ is a bounded subshift at x and $r(x, \Sigma(R)) = \infty$. In order to obtain finite radius $r(x)$, one needs to restrict growth rates of asymptotics of partition data $(n_0, n_1, \dots)$.

1.B.2 Koebe principle: Koebe principle saids that for C^3 interval map f with negative Schwarzian derivetives $Sf < 0$, there is a positive function $F : (0, \infty) \to (0, \infty)$ so that for any pair of intervals $M \subset T \subset [0, 1]$, if for some n, $f^n(T)$ contains ϵ neighbourhood of $f^n(M)$, then T contains $F(\epsilon)$ neighbourhood of M.

Let us take two maps $f, g : [0, 1] \to [0, 1]$ and choose $\bar{k} = (k_0, k_1, \dots) \in X_2$ with the corresponding partition $(N_0, N_1, \dots)$. We denote the interaction $O = O(\bar{k}) = \{h^k\}_{k=0,1,\dots}$, and $\tau(m) = \tau(f, g; \bar{k})(m) : [0, 1] \to [0, 1]$ by $\tau(x) = h^m(x)$.

We say that the triple $(f, g, \bar{k})$ satisfies *Koebe property*, if there is a positive function F which depends on the triple, so that for any pair of intervals $M \subset T \subset [0, 1]$, if for some n, $h^n(T)$ contains ϵ neighbourhood of $h^n(M)$, then T contains $F(\epsilon)$ neighbourhood of M. Let us define a subshift:

$$\begin{aligned} X_2(K) &= X_2(K)(f, g) \\ &= \{\bar{k} \in X_2 : (f, g, \bar{k}) \text{ satisfies the Koebe property.}\}. \end{aligned}$$

$X_2(K)$ is an invariant subshift.

Example 1.3: For f, g with $Sf, Sg < 0$, $(0, 0, \dots), (1, 1, \dots) \in X_2(K)$. More generally, any periodic sequence $(\bar{k}_m, \bar{k}_m, \bar{k}_m, \dots) \in X_2(K)$, where

$\bar{k}_m = (k_0, k_1, \dots, k_m)$ is any sequence. This follows since compositions of maps of negative Schwarzian derivatives have also the same property. Thus $X_2(K) \subset X_2$ is dense.

1.C Molecular interactions:

1.C.1 Selection maps: Let $\mu : [0,1] \to \{0,1\}$ be a measurable map. We will say that μ is a *selection map* in the following sense.

Let us choose two maps $f, g : [0,1] \to [0,1]$ as above and take $x \in [0,1]$. We will say that a sequence $(k_0, k_1, \dots)$, $k_i = 0, 1$, is a *configuration data* of interaction with $\{g^k(x)\}$ and $\{f^l(x)\}$, with respect to μ, if k_i is determined inductively by the following rule. Let us put $d_0(x) = f(x)$ and $d_1(x) = g(x)$. Then we denote $z^i = z^i(x)$ by:

$$z^i(x) = d_{k_i} \circ d_{k_{i-1}} \circ \cdots \circ d_{k_0}(x)$$

where we determine k_i inductively as:

$$k_{i+1} = \mu(z^i(x)).$$

We will say that $\{z^i\}_{i=1,2,\dots}$ is an *interaction* with respect to μ. We will say that an interaction is *active*, if $(z^i, z^{i+1}, \dots)$ is not a constant sequence for any i.

Example 1.4: Let us consider the previous example where f is the tent map and g is its reverse. Let $x = \frac{1}{4}$ or $\frac{3}{4}$. Then if an interaction, $\{z^i\}$ is active, then for any choice of μ, it is eventually periodic with period 2.

1.C.1.2 Wandering intervals: Let f and g be interval maps, and $\bar{k} \in X \subset X_2$ be a subshift. We write the corresponding interaction $\{h^m\}_m$ as:

$$h^m(x) = d_{k_m} \circ d_{k_{n-1}} \circ \cdots \circ d_{k_0}(x)$$

where $d_0 = f$ and $d_1 = g$.

We say that an interval $J \subset [0,1]$ is *separating* with respect to $\bar{k}$, if the corresponding interaction $\{h^m(J)\}_m$ are all disjoint.

An separating interval is *wandering*, if the ω-limit set of the family $\{h^m(J)\}_m$ is not a fintie set of points.

A separating interval J is *maximal*, if for any m and T which properly contains $h^m(J)$, $\{d_{k_n} \circ \cdots \circ d_{k_{m+1}}(T)\}_{n \geq m+1}$ is not separating.

A wandering interval J is *maximal*, if it is maximal in the above sense, and moreover for any T which properly contains J, there is some m so that $h^m : T \to h^m(T)$ is not homeomorphism.

A separating interval is *essentially contracting*, if there is a positive $\delta > 0$ so that for all large M:

$$\lim_{n \to \infty} |d_{k_n} \circ \cdots \circ d_{k_{M+1}}((1+\delta)h^M(J))| = 0$$

holds.

Let us define the *distorsion* of f over an interval T:

$$\text{Dist } (f, T) = \sup_{x,y \in T} \log \frac{|df(x)|}{|df(y)|}.$$

We need some elementary facts. Let $\{h^m\}$ be an interaction with respect to $(f, g, \bar{k})$.

Lemma 1.9 *Let f, g be C^1. Then the inequalities hold for all n:*

$$\textit{Dist } (h^n, T) \leq \Sigma_{i=0}^n \textit{ Dist } (d_{k_i}, h^i(T)).$$

If both $x \to \log|df(x)|$ and $\to \log|dg(x)|$ extend to Lipschitz functions, then

$$\textit{Dist } (h^n, T) \leq C\Sigma_{i=0}^{n-1}|h^i(T)|$$

hold for all n, where C is the Lipschitz constant. Moreover when all $h^i(T)$ are pairwise disjoint, then the inequalities hold:

$$\textit{Dist } (h^n, T) \leq C.$$

Proposition 1.3 *Let $f, g : [0,1] \to [0,1]$ be two modal maps such that $x \to \log|df(x)|, \log|dg(x)|$ extend to Lipschitz maps. Then for any $\bar{k} \in X_2$ and the corresponding interaction $\{h^m\}_m$, any separating interval is essentially contracting.*

The proof is parallel to $f = g$ case, where the theorem holds for wandering intervals (Schwartz). There a *contraction principle* plays an important role. Namely for an interval T with $\lim_n |f^n(T)| = 0$, it is a wandering interval, or $\omega(T)$ is a periodic orbit. One may try to generalize this to multi-function case.

Corollary 1.4 *Let f, g be two modal maps, and take $\bar{k} \in X_2$. If a separating interval J is not essentially contracting, then $\omega(J)$ contains a critical point for f or g.*

1.C.1.2.1 Duplication in shift: Let us take $\bar{k} = (k_0, k_1, \dots) \in X_2$, and put $\bar{k}_m^n \equiv (k_{m+1}, k_{m+2}, \dots, k_n)$. We say that $\bar{k}$ has a *duplication* at (m, n), if:

$$\bar{k}_n^{2n-m} = \bar{k}_m^n$$

holds.

Let J be a separating interval for $(f, g, \bar{k})$. We put inductively that J_m is a maximally separating interval containing $d_{k_m}(J_{m-1})$, where $h^m = d_{k_m} \circ d_{k_{m-1}} \circ \dots \circ d_{k_0}$.

Lemma 1.10 *If $\bar{k}$ has a duplication at (m, n), then $J_m \cap J_n = \phi$ holds.*

Proof: Let us put $h_m^n \equiv d_{k_n} \circ \dots \circ d_{k_{m+1}}$. Suppose $J_m \cap J_n \neq \phi$ might hold. Then since J_n contains $h_m^n(J_m)$,

$$d_{k_{2n-m}} \circ \dots d_{k_{n+1}}(J_n) \cap J_n = h_m^n(J_n) \cap J_n \supset h_m^n(J_n \cap J_m) \neq \phi$$

holds. But this is a contradiction, since J_n is a separating interval.

Corollary 1.5 *Suppose $f = g$ and choose a separating interval J. (1) If any T strictly containing J satisfies that $\{f^n(T)\}_n$ is not separating, then J is maximal in our sense.*

(2) Suppose J is a maximal separating interval. If it is wandering, then it is also a maximal wandering interval in our sense.

Proof: (1) follows from the above lemma. For (2), see [MS] $p94$.

1.C.1.2.2 Koebe principle and wandering intervals: Let J be an interval and suppose for a subindices $\{n(k)\}_k$, $h^{n(k)}(J)$ converge to a point c. Here we study a converging rate:

$$\lim_k \frac{\text{dist } (c, h^{n(k)}(J))}{|h^{n(k)}(J)|}.$$

We say that $\{h^n(J)\}_n$ *converges rapidly* to c, if for a subsequence $\{n(k)\}_k$, it converges to 0.

Let J be an interval. We say that it is *proper* with respect to $(f, g, \bar{k})$, if for any interval T containing J, there are no pairs (m, n), $m < n$, so that:

$$d_{k_n} \circ \ldots d_{k_{m+1}} : h^m(T) \mapsto h^n(T)$$

satisfy both (1) it is homeomorphism and (2) $h^n(T) \subset h^m(T)$ holds.

Lemma 1.11 *Let J be a wandering interval with respect to (f, f). Then it is proper.*

Proof: Otherwise J would be attracted by a periodic attractor, a contradiction to wandering property.

When both f and g have negative Schwarzian derivatives, then so are their compositions $f \circ g$ or $g \circ f$.

Let us say that f and g are *unimodal pair*, if they are both unimodal maps with the same unique critical point c, and there is a map $\tau : [0,1] \cong [0,1]$ so that

$$f(\tau(x)) = f(x), \quad g(\tau(x)) = g(x),$$
$$\tau(x) \neq x \text{ for } x \neq c$$

hold.

we say that a unimodal pair (f, g) is *non degenerate*, if $\frac{d^2 f}{dx^2}(c), \frac{d^2 g}{dx^2}(c) \neq 0$ at the unique critical point. If (f, g) is non degenerate, then the corresponding τ is Lipschitz.

For example $Q_\mu = \mu x(1 - x)$ and $Q_{\mu'}$ are all unimodal pair, $\mu, \mu' \in [0, 4]$.

We say that a separated interval J is *symmetric-proper*, if for any interval T containing J, there are no pairs (m, n), $m < n$, so that:

$$d_{k_n} \circ \ldots d_{k_{m+1}} : h^m(T) \mapsto h^n(T)$$

satisfy both (1) it is homeomorphism, and (2) $h^n(T) \subset h^m(T) \cup \tau(h^m(T))$ hold. Any wandering interval is also symmetric-proper for $f = g$.

Recall the invariant subshift $X_2(K) \subset X_2$ in the previous section.

Theorem 1.3 *Let (f, g) be a non degenerate unimodal pair and choose an element $\bar{k} \in X_2(K)$. If a maximally symmetric-proper wandering interval J contains the critical point c in $\omega(J)$, then $\{h^n(J)\}_n$ converges rapidly to c.*

Proof: Let us choose subindices $\{n(k)\}_k$ inductively so that:

$$n(k+1) = \min\{j > n(k); h^j(J) \subset (h^{n(k)}(J), \tau(h^{n(k)}(J))\}$$

where $(K, \tau(K))$ is the smallest interval which connects K and $\tau(K)$ with empty intersection with K and $\tau(K)$. Also we put $[K, \tau(K)] = K \cup (K, \tau(K)) \cup \tau(K)$.

A basic direction of the proof is followed by the $f = g$ case by Guckenheimer (see [MS]). We sketch it.

Step 1: Let $T_{n(k)} \supset J$ be the largest interval on which $h^{n(k)}$ is a homeomorphism. Then for all large $k \geq k_0$, either:

$$h^{n(k)}(T_{n(k)}) \supset [c, h^{n(k-1)}(J)] \text{ or } h^{n(k)}(T_{n(k)}) \supset [c, \tau(h^{n(k-1)}(J))]$$

holds. The proof uses symmetric-properness.

Step 2: We check

$$\lim_{n \to \infty} \frac{|(h^{n(k)}(J), c)|}{|h^{n(k)}(J)|} \to 0.$$

By contradiction argument, if otherwise one might find some interval T which strictly contains J so that $h^m|T$ are homeomorphisms for all m, which contradicts to maximality of J. To find T, one will use Koebe Principle for maps of negative Schwarzian derivatives. Elements in $X_2(K)$ guarantee uniformity of T.

1.C.2 Chemical reaction rate:

1.C.2.1 Chemical reaction rate: Let us take $f, g : [0, 1] \to [0, 1]$ and $\bar{k} \in X_2$. For $x \in [0, 1]$, one obtains the corresponding interaction map:

$$\Phi(x) : X_2 \to X_2.$$

One will interpret iteration of the map:

$$\Phi(x)^t \equiv \Phi(x) \circ \dots \Phi(x) \ (t \text{ times}) \ : X_2 \to X_2$$

as process of *chemical reaction* in time t. If:

$$\lim_{t \to \infty} \phi(x)^t(\bar{k}) \in X_2$$

exists, then we say that it is a *product of chemical reaction.*

One may seek for relation with theory of chemical reaction rate, where several diffusion equations play important roles.

1.C.2.2 Transition state: When molecules react and produce another molecules:

$$A + B \to C + D$$

the process will follow along the *potential energy surface.* During the chemical process, it takes an intermediate state. This is unstable, which immediately changes into another state, and produces chemical products C and D. This intermediate state is called an *transition state.*

Let us consider a chemical reaction in our sense:

$$\Phi(x)^t : X_2 \to X_2.$$

We formulate the transition process as follows. Let $\sigma : X_2 \to X_2$ be the shift operator. We say that it passes through a transition state during the chemical reaction, if there is a sequence $n_0 < n_1 < \cdots < n_t < \to \infty$ satisfying the followings:

(1) A product of chemical reaction $C \equiv \lim_t \Phi(x)^t(\bar{k}) \in X_2$ exists, and

(2) $M \equiv \lim_{t \to \infty} \sigma^{n_t}(\Phi(x)^t(\bar{k})) \in X_2$ exists.

We say that M above is a transition state.

Above we have formulated the transition state as something like *soliton.* In fact in $2.E.1.2$, we will see that the flow $\Phi(x)^t$ with respect to some family of maps produces a transition state as a soliton of the

Lotka Vorterra cell automaton.

1.D Automata as interactions: Let us generalize the interactions between orbits of maps. Let us choose two maps:

$$\phi : [0,1] \times \{0,1\} \to [0,1],$$
$$\psi : [0,1] \times \{0,1\} \to \{0,1\}.$$

We say that ϕ is a *transition function*, and ψ is an *exit function*.

For each $x \in [0,1]$, we construct a continuous map:

$$\Phi(x) : X_2 \to X_2$$

as follows; choose $\bar{k} = (k_0, k_1, \dots) \in X_2$. Then we have $\Phi(\bar{k}) = (k'_0, k'_1, \dots)$, where we inductively put:

$$k'_0 = \psi(x, k_0), \quad k'_1 = \psi(\phi(x, k_0), k_1), \dots, k'_i = \psi(s_i, k_i),$$
$$s_0 = x, \quad s_{i+1} = \phi(s_i, k_i).$$

We say that the pair (ϕ, ψ) is an *automaton*.

Let us have four maps α, β and f, g. Let $\pi : [0,1] \to \{0,1\}$ be as before. Then we put:

$$\phi(x,i) = \begin{cases} \alpha(x), & i = 0, \\ \beta(x), & i = 1, \end{cases}$$
$$\psi(x,i) = \begin{cases} \pi \circ f(x), & i = 0, \\ \pi \circ g(x), & i = 1. \end{cases}$$

By this way one obtains an automaton. We say that the corresponding maps $\Phi(x)$ on X_2 are *automata interactions*. We say that two automata are *isomorphic*, if the corresponding $\Phi(x)$ are isomorphic. Let us put the group generated by $\Phi(x)$ for all regular points x $(1.A.3.2)$:

$$G = G(\phi, \psi) = \text{ gen } \{\Phi(x) \in \text{ Aut } X_2\}.$$

Suppose these four maps satisfy the following properties:

$$
\begin{aligned}
&0 < \alpha(x) < \frac{1}{2}, \quad \frac{1}{2} < \beta(x) < 1, \\
&\frac{1}{2} < f(x) \le 1, \quad 0 \le x < \frac{1}{2}, \\
&0 \le f(x) < \frac{1}{2}, \quad \frac{1}{2} < x \le 1, \\
&0 \le g(x) < \frac{1}{2}, \quad 0 \le x < \frac{1}{2}, \\
&\frac{1}{2} < g(x) \le 1, \quad \frac{1}{2} < x \le 1.
\end{aligned}
$$

Lemma 1.12 *For any $x \in [0,1]\backslash\frac{1}{2}$, the corresponding automaton is isomorphic to a two state automaton A described below:*

where $\epsilon \in S_2$ is the non trivial element in the symmetric group.

Proposition 1.4 (GZ) *For A above, the corresponding $G \subset Aut\ X_2$ is isomorphic to the lamplighter group $\oplus_{\mathbf{Z}}\mathbf{Z}/\mathbf{Z}_2 \quad \mathbf{Z}$.*

Corollary 1.6 *For f, g, f', g' above, the group G acts on ∂X_2 ergodically.*

So far we have seen the following diagram passing through two steps:

$$\text{family of maps} \to \text{finite state automata} \to \{\Phi(x) : X_2 \to X_2\}.$$

Some cases of $\{\Phi(x) : X_2 \to X_2\}$ are obtained from two maps using the interaction of two maps we have defined in $1.A.3$. It is not known how to obtain an algorithm to construct two maps for a given family $\{\Phi(x) : X_2 \to X_2\}$. One may find out some obtructions for such reduction of the numbers of maps.

1.D.2 Depth of automorphisms: Let $G \subset \text{Aut } X_2$ be an automorphism group. We say that G has *finite depth*, if for each $g \in G$, there is $m = m(g) \geq 0$ so that for all $\bar{k} = (k_0, k_1, \dots)$, $g(\bar{k})$ is of the form:

$$(k'_0, \dots, k'_{m-1}, k_m, k_{m+1}, \dots).$$

Lemma 1.13 (GZ) *The lamplighter group contains an infinitely generated free subgroup which has finite depth.*

In fact [GZ] found some canonical generators and verified their depths more explicitly.

Let us have four partition of the interval:

$$I_1 = (0, \frac{1}{4}), \quad I_2 = (\frac{1}{4}, \frac{1}{2}), \quad I_3 = (\frac{1}{2}, \frac{3}{4}), \quad I_4 = (\frac{3}{4}, 1).$$

We choose the tent map as f and its inverse, $g(x) = 1 - f(x)$. For the transition function, we choose:

$$\alpha(x) = \begin{cases} x & 0 < x < \frac{1}{2}, \\ 1 - x & \frac{1}{2} < x < 1, \end{cases}$$
$$\beta(x) = \begin{cases} 1 - x & 0 < x < \frac{1}{2}, \\ x & \frac{1}{2} < x < 1. \end{cases}$$

Then both α, β can be regarded as maps:

$$\alpha, \beta : \bar{I} \equiv \{I_1, I_2, I_3, I_4\} \to \{I_1, I_2, I_3, I_4\}.$$

Thus one has two maps from f, g and α, β:

$$\phi : \bar{I} \times \{0, 1\} \to \bar{I},$$
$$\psi : \bar{I} \times \{0, 1\} \to \{0, 1\}.$$

Let A be the automaton before whose group is isomorphic to the lamplighter group. In general for any two automata A and B with the same alphabets, there is a composition $A \circ B$ (see [GZ]).

Lemma 1.14 *The above automaton is isomorphic to $A \circ A^{-1}$.*

Next we take the reverse of the tent map as f, and the tent map as g. For the transition function, we choose:

$$\begin{aligned}
\frac{1}{2} < \alpha(x) < \frac{3}{4}, &\quad 0 < x < \frac{1}{4}, \\
\frac{1}{4} < \alpha(x) < \frac{1}{2}, &\quad \frac{1}{4} < x < \frac{1}{2}, \\
\frac{1}{2} < \alpha(x) < \frac{3}{4}, &\quad \frac{1}{2} < x < \frac{3}{4}, \\
\frac{1}{4} < \alpha(x) < \frac{1}{2}, &\quad \frac{3}{4} < x < 1,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4} < \beta(x) < \frac{1}{2}, &\quad 0 < x < \frac{1}{4}, \\
\frac{1}{2} < \beta(x) < \frac{3}{4}, &\quad \frac{1}{4} < x < \frac{1}{2}, \\
\frac{1}{4} < \beta(x) < \frac{1}{2}, &\quad \frac{1}{2} < x < \frac{3}{4}, \\
\frac{1}{2} < \beta(x) < \frac{3}{4}, &\quad \frac{3}{4} < x < 1.
\end{aligned}$$

Then as above, one obtains the corresponding automaton.

Lemma 1.15 *The corresponding automaton is isomorphic to $A^{-1} \circ A$.*

$A^{-1} \circ A$ contains automorphisms of *depth* 1.

1.D.3 Automata sequences: In the above examples, both α and β have some nice property so that it enebles one to reduce transition functions as: $\phi : \bar{I} \times \{0,1\} \to \bar{I}$.

In order to treat more general class of maps, one proceeds as follows. Let $\bar{I} = \{I_1, \dots, I_i\}$ and $\bar{J} = \{J_1, \dots, J_j\}$ be two partitions of $[0,1]$. We say that $\bar{I}$ *dominates* $\bar{J}$, if each $J_l \subset [0,1]$ is contained in some I_k, and we denote $\bar{I} \geq \bar{J}$.

An *automata sequences* consiste of families $A = \{\bar{I}^k, \phi_k, \psi_k\}_k$ where:
(1) dominated sequences:

$$\bar{I}^0 \geq \bar{I}^1 \geq \cdots \geq \bar{I}^i \geq \dots$$

(2) families of transition and exit functions:

$$\phi_i : \bar{I}^i \times \{0,1\} \to \bar{I}^{i-1},$$
$$\psi_i : \bar{I}^i \times \{0,1\} \to \{0,1\}$$

for all $i = 1, 2, \dots$.

Let us take $\bar{k} = (k_0, k_1, \dots)$. We say that a family $\{I^i_{m_i} \in \bar{I}^i\}_i$ is *acceptable* with respect to $\bar{k}$, if

$$\phi_i(I^i_{m_i}, k_i) = I^{i-1}_{m_{i-1}}$$

holds for all $i = 0, 1, 2, \dots$. We say that an admissible sequences is acceptable, if there is an acceptable family for all $\bar{k} \in X_2$.

Let us take an acceptable family $\{I^i_{m_i}\}_i$. Now one constructs a map:

$$\{I^i_{m_i}\}_* : X_2 \to X_2$$

by $\{I^i_{m_i}\}_*(k_0, k_1, \dots) = (k'_0, k'_1, \dots)$, where:

$$\phi_i(I^i_{m_i}, k_i) = I^{i-1}_{m_{i-1}},$$
$$\psi_i(I^i_{m_i}, k_i) = k'_i$$

hold for all $i = 0, 1, 2, \dots$.

Let us have four maps α, β and f, g and $\pi : [0,1] \to \{0,1\}$ be the projection. Then as before we put:

$$\phi(x,i) = \begin{cases} \alpha(x), & i = 0, \\ \beta(x), & i = 1, \end{cases}$$

$$\psi(x,i) = \begin{cases} \pi \circ f(x), & i = 0, \\ \pi \circ g(x), & i = 1. \end{cases}$$

Let us take dominated sequences $\{\bar{I}^i\}_i = \bar{I}^0 \geq \bar{I}^1 \geq \cdots \geq \bar{I}^i \geq \dots$. we say that $\{\bar{I}^i\}_i$ is acceptable with respect to the pair (α, β), if both α, β can be regarded as maps:

$$\alpha, \beta : \bar{I}^i = \{I_0^i, \dots, , I_{n(i)}^i\} \to \bar{I}^{i-1} = \{I_0^{i-1}, \dots, , I_{n(i-1)}^{i-1}\}$$

for all $i = 1, 2, \dots$ and a.e. $x \in [0,1]$. Then for a.e. x, one obtains naturally automata sequences $\{\bar{I}^k, \phi_k, \psi_k\}_k$ with respect to these four maps.

Lemma 1.16 *Suppose the above conditions. Then for a.e. $x \in [0,1]$ and any $\bar{k} \in X_2$, there is a unique acceptable family $\{I_{m_i}^i\}_i$.*

One may try to construct actions on $\mathbf{R}$ trees passing through the above procedure.

Eample 1.5: Let α be the tent map, and β be its reverse. Then one can choose families $\{\bar{I}^i\}_i$ as:

$$\bar{I}^i = \{I_0^i, \dots, I_{2^i-1}^i\}, \quad I_l^i = (\frac{l}{2^i}, \frac{l+1}{2^i}).$$

One can take any irrational x.

1.D.3.2 Distribution of automata sequences: Let $A = \{\bar{I}^k, \phi_k, \psi_k\}_k$ be an automata sequences with $\bar{I}^0 = \{I_0 = (0, \frac{1}{2}), I_1 = (\frac{1}{2}, 1)\}$. We denote the probability by:

$$P_0 = \lim_k \frac{\sharp\{I_l \in \bar{I}^k \cap (0, \frac{1}{2})\}}{\sharp \bar{I}^k}.$$

We denote $P_1 = 1 - P_0$ similarly.

We define the *distributed entropy* by:

$$h_0(A) = \log \lim_k \frac{1}{k} [\sharp\{I_l \in \bar{I}^k \cap (0, \frac{1}{2})\}]^{\frac{1}{k}}.$$

We have $h_1(A)$ similarly.

When one of P_i are large (and so the other is small), then the automorphism group G will be likely small.

In section 3, we will study more on entropy of interaction.

1.E Fractal groups and spectral functions: In [BG], spectra of fractal groups are computed. Let T_3 be the 3-regular rooted tree. Let e_0, e_1, e_2 be the three edges connected to the root. Then we put the automorphism $a \in \text{Aut } T_3$ by $a(e_i) = e_{i+1} \bmod 3$. We put another automorphism s by:

$$s(0x\sigma) = 0a(x)\sigma, \quad s(1x\sigma) = 1x\sigma, \quad s(2\sigma) = 2s(\sigma).$$

Let $\Gamma \subset \text{Aut } T_3$ be the group generated by a and s.

Similarly, let t be another automorphism by:

$$t(0x\sigma) = 0a(x)\sigma, \quad t(1x\sigma) = 1a(x)\sigma, \quad t(2\sigma) = 2t(\sigma).$$

Then we put by $\bar{\Gamma} \subset \text{Aut } T_3$ as a group generated by a and t.

Let $\pi : \Gamma \to U(L^2(\partial T_3))$ be the regular representation. Let us put Stab $(n) = \{g \in \Gamma : g(i_1 i_2 \dots i_n i_{n+1} \dots) = (i_1 \dots i_n i'_{n+1} i'_{n+2} \dots)\}$. Then $\Gamma_n = \Gamma /$ Stab (n) is the parmutation of $\{0,1,2\}^n$, and we put by π_n as the induced representations. It is known:

$$\text{Spec } \pi = \text{ closure } \cup_{n \geq 0} \text{ Spec } \pi_n.$$

The combinatorial Laplacian $\Delta_n = a_n + a_n^{-1} + s_n + s_n^{-1}$. Then we put the *spectral function* by:

$$Q_n(\lambda, \mu) = \lambda(a_n + a_n^{-1}) + s_n + s_n^{-1} - \mu.$$

Let:

$$\begin{aligned} H_\theta &= \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda, \\ F(x) &= 4 - 2x - x^2. \end{aligned}$$

Lemma 1.17 (BG)

$$|Q_n(\lambda, \mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^{3^{n-1}+1} \Pi_{2 \leq m \leq n, \theta \in X_M} H_\theta^{3^{n-m}+1}$$

where $X_n = F^{-1}(X_{n-1})$ *and* $X_2 = -1$.

Moreover spectrum of π *is the set closure* $\{\mu : |Q_n(1, \mu)| = 0\}$.

There is a parallel expression for $\bar{\Gamma}$.

Let us put:

$$|Q_n(\lambda,\mu,\epsilon)| = (2+2\lambda-\mu)(2-\lambda-\mu)^{3^{n-1}+1}\Pi_{2\le m\le n,\theta\in X_M(\epsilon)} H_\theta^{3^{n-m}+1}$$

where $X_n = F^{-1}(X_{n-1})$ and $X_2 = -1+\epsilon$. Then we obtain an ϵ perturbed spectrum:

$$\text{Spec } \pi_n(\epsilon) \equiv \{\mu : |Q_n(1,\mu,\epsilon)| = 0\},$$
$$\text{Spec } \pi(\epsilon) \equiv \text{ closure } \cup_{n=0}^{\infty} \text{ Spec } \pi_n(\epsilon).$$

In general let $F : [0,1] \to [0,1]$ be a map. Then similarly one can obtain $|Q(F : \lambda,\mu,\epsilon,x_0)|$ using $X_n = F^{-1}(X_{n-1})$ and $X_2 = x_0+\epsilon$. Let us put Spec $\pi_n(F:\epsilon,x_0) = \{\mu : Q(F:1,\mu,\epsilon,x_0) = 0\}$ and Spec $\pi(F,\epsilon,x_0) =$ closure $\cup_n \pi_n(F:\epsilon,x_0)$.

We say that Spec π is *continuous* at x_0, if $\lim_{\epsilon\to 0}$ Spec $\pi(F:\epsilon,x_0) =$ Spec $\pi(F:0,x_0)$ might hold.

2 From micro interaction to macro

Let us consider a system where several ensymes are mutually interacting:

$$\{A_1,\dots,A_N\}.$$

Here one will express *routes* of interaction by *oriented signed graphs* with self-loops, using families of maps.

2.B Microscopic system: Let us take three maps $f,g,d : [0,1] \to [0,1]$ and choose two points $x,z \in [0,1]$. For any $\bar{k} \in X_2$, one has associated with the interaction $O = \{h^m(x)\}_m$:

$$h^m(x) = \cdots \circ g^{N_m} \circ f^{N_{m-1}} \circ \cdots \circ g^{N_1} \circ f^{N_0}(x)$$

where $(N_0, N_1, \dots)$ is the partition corresponding to $\bar{k} \in X_2$. By composition with the projection $\pi : [0,1] \to \{0,1\}$, one obtains $\Phi(x) : X_2 \to X_2$ as before.

Now choose another oscillation $O_3 = \{d^m(z)\}_m$. Let us say that O_3 is the *interaction* in X_2 with $O_1 = \{f^k(x)\}_k$ and $O_2 = \{g^k(x)\}_k$ with the state $\bar{k} \in X_2$, if the equality holds:

$$\{\pi \circ d^m(z)\}_m = \Phi(x)(\bar{k}) \in X_2$$

for some $z \in [0,1]$.

Suppose the oscillation $\{h^l(z)\}_l$ is the interaction with $\{f^l(x)\}_l$ and $\{g^l(x)\}_l$. Then among three maps $\{f, g, h\}$, one can obtain a *marked oriented edge*:

$$(f, x) \xrightarrow{g} (h, z).$$

Let us choose families of maps $\{f_0, \dots, f_k\}$, points $\{x_0, \dots, x_l\}$ and $\bar{k} \in X_2$.

Definition 2.1 *An interaction graph is a marked oriented graph:*

$$G(\{f_i\}_i; \{x_j\}_j, \bar{k})$$

where:

$$V = \{(f_i, x_j) : 0 \leq i \leq k, 0 \leq j \leq l\} \text{ (the set of vertices),}$$
$$E = \{e_{i,j,k} : (f_i, x_h) \xrightarrow{f_j} (f_k, x_v)\} \text{ (the set of edges).}$$

Let us choose an infinite family of points $\{x_0, x_1, \dots, x_j, \dots\}$, and put:

$$G_j \equiv G(\{f_i\}_{i=0}^k; \{x_0, \dots, x_j\}, \bar{k}).$$

We are interested in graphical properties for:

$$G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^\infty, \bar{k}) \equiv \lim_{j \to \infty} G_j.$$

Let G be a graph and denote by n and T as the numbers of vertices and edges respectively. We put:

$$\lambda = \lambda(G) = \lambda(n, T) \equiv \frac{2T}{n}$$

and call it the *critical exponent* of G. Let us put

$$\lambda(G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^\infty, \bar{k})) \equiv \liminf_j \lambda(G_j).$$

One can generalize this as follows. Let f be an l-modal map. Then there are l critical points $c_1 < c_2 < \cdots < c_l$. Let us divide $[0,1]$ into $2l+1$ intervals as:

$$I_1 = [0, c_1), I_2 = \{c_1\}, I_3 = (c_1, c_2), \dots, I_{2l} = (c_{l-1}, c_l), I_{2l+1} = \{c_l\}.$$

Then there is the associated projection:

$$\pi : [0,1] \to \{0, \dots, 2l+1\}.$$

For any pair of l-modal maps f and g (namely they have the same critical set), one obtains the corresponding interaction map:

$$\Phi(x) : X_2 \to X_{2l+1}.$$

Let us have another l-modal map d with the same critical set as f, g. Then one saids also that $\{d^m(z)\}_m$ is the interaction in X_{2l+1} with $O_1 = \{f^k(x)\}_k$ and $O_2 = \{g^k(x)\}_k$ with the state $\bar{k} \in X_2$, if the equality holds:

$$\{\pi \circ d^m(z)\}_m = \Phi(x)(\bar{k}) \in X_{2l+1}$$

for some $z \in [0,1]$.

Proposition 2.1 *Suppose a family $\{f_i\}$ consistes of l-modal maps which have no wandering intervals and no periodic attractors. Then for each $\bar{k} \in X_2$, there is a family of points $\{x_0, x_1, \dots\}$ so that the corresponding interaction graph satisfies:*

$$\lambda(G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^\infty, \bar{k}) = 0.$$

Later we will see that small critical exponents will reflect to numbers of small cycles in graphs.

Sublemma 2.1 (MS) *(p105) If f satisfies the assumption, then there is a subshift $X' \subset X_{2l+1}$ independent of f so that $\Phi(f,f)$gives a canonical bijection $\Phi(f,f) : [0,1] \cong X'$.*

Proof of proposition: We proceed by induction. First we take any $x_0^i \in [0,1]$ and put $T(1) = T$. Suppose we have chosen $\{x_0, \dots, x_s\}$ so that $G_s \equiv G(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^s, \bar{k})$ satisfies $\lambda \leq \frac{2T}{(k+1)s}$.

Let us put by $T(s)$ the number of the edges for G_s. By the above sulemma, there is some $x_{s+1} \in [0,1]$ so that for each $i = 0, \dots, k$, $\{f_i^m(x_{s+1})\}_m$ is not interaction in X_{2l+1} with any of pairs $\{f_{i'}^m(x_j)\}_m$ and $\{f_{i''}^m(x_{j'})\}_m$ for $j, j' \leq s$, since the number of edges in $G(s)$ is finite. Thus $\lambda(G_{s+1}) = \frac{2T(s)}{(k+1)(s+1)} \leq \frac{2T}{(k+1)(s+1)}$ holds. This completes the proof.

2.C Cyclic system: Let us consider a system consisted by N ensymes $(A_1, \dots, A_N)$. Suppose each A_i is being activated by A_{i-1}, and catalyzes on A_{i+1}, where we consider mod N. We call this as a *cyclic system.* Each A_i takes a state $x_+(i)$ when being silence, and $z_-(i)$ when activating.

We have a picture for this situation ([K2]). Suppose there exists a pair $x(i) < z(i)$ for each i such that both $x(i)$ and $z(i)$ are fixed points, and there are no other periodic points between them and $f(x) < x$ hold for $x \in (x(i), z(i))$.

We take $x(i) < x_+(i) << z_-(i) < z(i)$. The dynamics can be written as:

$$\dot{x}_i^k = -x_i^k + x_{i-1}^k.$$

Thus when A_{i-1} is in a low state, then A_i will also be in a low state and near $x_+(i)$. If the state of A_{i-1} is high and near $z_-(i-1)$, then the state of A_i will be also high and near $z_-(i)$.

2.C.2 Cirrgadian rythms: Molecular systems have various kinds of rythms. These are called *cirrgadian rythms.* One of the most important discovery in molecular biology is that some particular genes are govering such cycles. Namely once they stop working in the system, then the system will cause serious influence on cycles, sometimes they will disappear. They are called *clock genes.*

One cycle in a graph will show a structure of cirrgadian rythm. More than two cycles may induce many routes of interactions, which will prevent from occuring rythm.

Let us see that under some situation, controlled by some critical exponent, that for each one system represented by a graph, there could exit at most one cycle from probability point of view.

Let G be an interaction graph. This is an oriented marked graph. Let G' be the corresponding graph which forgets orientation and marking. G' has a property that:

(1) it may have self-loop, and no more than one loop at each vertex, and

(2) each pair of vertices may be connected by no more than two edges.

For a graph, we will denote the number of vertices by n and of edges by T. Let

$$\mathfrak{G}_{n,T}$$

be the set of graphs with n vertices and T edges such that they satisfy the above two properties. Let

$$\mathfrak{A}_{n,T} \subset \mathfrak{G}_{n,T}$$

be the subset consisted by graphs with at most one cycles.

Let us take a family of interval maps $\{f_0, \dots, f_k\}$ and an infinite set of points $\{x_0, x_1, \dots\}$. Consider the set of the corresponding interaction graphs and its subset:

$$\begin{aligned}
&\bar{G} = \{G_j = \{G(\{f_i\}_i; \{x_0, \dots, x_j\}, \bar{k})\}_{j=0,1,\dots}, \\
&\mathfrak{G}(\{f_i\}; \{x_j\}_j, \bar{k})_{n,T} \equiv \bar{G} \cap \mathfrak{G}_{n,T}.
\end{aligned}$$

Question: Let us make $n, T \to \infty$ so that the critical exponent λ satisfies:

$$(1-\lambda)^3 n \to \infty.$$

Then can one choose an infinite set of points $\{x_0, x_1, \dots\}$ so that the probabilities of random one cycles:

$$P(\{f_i\}_i; \{x_j\}_j, \bar{k})_{n,T} \equiv \frac{\mathfrak{G}(\{f_i\}_i; \{x_j\}, \bar{k})_{n,T} \cap \mathfrak{A}_{n,T}}{\mathfrak{G}(\{f_i\}_i; \{x_j\}, \bar{k})_{n,T}} \to 1$$

converges to one ?

This comes from the following result in random graph theory:

Proposition 2.2 (Ko) *Under the above assumption on the critical exponent, the probabilities:*

$$P_{n,T} \equiv \frac{\mathfrak{A}_{n,T}}{\mathfrak{G}_{n,T}} \to 1$$

converges to one.

Such less than one cycle phenomena heavily depends on the behaviour of the critical exponent $\lambda(n, T) = \lambda(G)$.

2.D Switching of genes: Ensymes play roles of *switching* in a biological systems. Let us choose an ensyme in a system. The system contains two states, namely when the ensyme is activating and catalyzing the other molecules, and another is when being silence. Let us have pictures of this situation.

Let $(A_1, \dots, A_N)$ be the set of molecules which consists of an interaction system. When A_N is being silence, and in a state z, its interaction system may be expressed by:

$$(\{f_i\}; \{x_1, \dots, x_N, z\}).$$

Suppose A_{N-1} works as a co-ensyme on A_N and A_N begins activating. Then the system will change as:

$$(\{f_i\}; \{x'_1, \dots, x'_N, z'\}).$$

The corresponding interaction graphs may be mutually very different.

One may construct both micro and macro bifurcation phenomana.

2.D.2 Systems of binary enhansers: Let $(A_1, \dots, A_N)$ be a system consisted by ensymes. Each A_i is catalyzed by another ensyme $A_i' \in \{A_1, \dots, A_N\}$, and it takes two states x_i or z_i according to whether it is catalyzed by A_i' or not. Suppose each A_i catalyzes another two ensymes B_i or $C_i \in \{A_1, \dots, A_N\}$ when A_i is in states x_i or z_i respectively. Namely A_i changes the partoners to calalyze whether it is activating or not. Then we will say that A_i is a *binary enhanser* and works as a *switching.*

Example 2.1: We have a picture from [K2]. Let us consider a system by (A_1, A_2, A_3), and suppose A_2 takes two states between the interval $(x(2), z(2))$ where $x(2)$ and $z(2)$ are both fixed points, there are no other periodic points and $f(x) < x$ hold in the interval. A_3 takes two states between $(x(3), z(3))$ which has the same properties except that $f(x) > x$ hold in the interval. A_1 takes two states, x_1 and z_1 satisfying $h(x_1) << 0$ and $h(z_1) >> 0$.

Let us consider a pre-interaction system $(f; x, x_+(2), z_-(3))$ where x may be arbitrary. A_2 and A_3 admit the dynamics:

$$\dot{x}^k(i) = -x^k(i) + h(x_1), \quad i = 2, 3.$$

When $x = x_1$, then A_1 catalyzes only on A_2, and when $x = z_1$, it does only on A_3.

2.E Soliton cell automaton: So far we have studied micro scopic interaction by use of compositions of maps. Later in section 5, one will obtain reaction diffusion equations from ordinary differential equations with the initial conditions $\{f^n(x)\}_n$. They will appear by making the micro interaction scaling change so that such interaction can sit in a larger scaling time. In particular in a sense it saids that macro feature will be deterministic by the initial condition.

In this section, we study a converse direction. Namely we will consider *difference-difference* equations which appear by making differential equations difference which corresponds to changing time scale into more micro.

One of the important direction will be to unify these two approaches to mutually differnt time scalings.

Here we will induce some *soliton equations* from automata interaction we have introduced in $1.D$. In particular we will treat Lotka Vorterra equation and Toda equation which are both very important in relation with integrable systems.

2.E.1 Lotka Vorterra cellular automaton: Let us recall the *Lotka Vorterra* equation:

$$\frac{d}{dt}u_n = u_n(u_{n+1} - u_{n-1}) \quad (1)$$

where $\{u_n\}_n$ is a family of variables. This equation is famuous one which describes population growth of fishes.

The Lotka Vorterra *cellular automaton*, or the *ultra-discrete* Lotka Vorterra equation is given by:

$$v_n^{t+1} - v_n^t = \max(0, v_{n+1}^t - L) - \max(0, v_{n-1}^{t+1} - L). \quad (2)$$

These two equations are related with each other passing through the *discrete* Lotka Vorterra equation:

$$\frac{V_n^{t+1}}{V_n^t} = \frac{1 + \delta V_{n+1}^t}{1 + \delta V_{n-1}^{t+1}}. \quad (3)$$

These are known to have *solitary* solutions. We will shortly describe relations between discrete and the L-V cell automaton ([HT]). Let us put:

$$V_n^t = \exp(\frac{v_n^t}{\epsilon}), \quad \delta = \exp(-\frac{L}{\epsilon}).$$

Then one obtains the equation:

$$v_n^{t+1} - v_n^t = \epsilon \log(1 + \exp(\frac{v_{n+1}^t - L}{\epsilon})) - \epsilon \log(1 + \exp(\frac{v_{n-1}^{t+1} - L}{\epsilon})).$$

Notice that $\lim_{\epsilon \to +0} \epsilon \log(\exp(\frac{a}{\epsilon}) + \exp(\frac{b}{\epsilon}) = \max(a, b)$. Thus letting $\epsilon \to 0$, one obtains the equation (2).

2.E.1.2 Induction of L-V celluar automaton from interactions: Let us introduce another generalization of micro interactions as follows. Let us fix a set of alphabets $\{0, 1, \dots, L\}$, and choose a family of maps $\{f_{i,j}\}_{i,j=0,1,\dots,L}$ and $\bar{k} \in X_{L+1}$. Then we define inductively:

$$h^m(x) = f_{k_m, k_{m+1}} \circ h^{m-1}(x), \quad h^{-1}(x) \equiv x.$$

We say that $\{h^m(x)\}_m$ is a *two step* interaction.

Let $\pi : [0, 1] \to \{0, 1, \dots, L\}$ be the projection. Then as before one obtains a map:

$$\Phi(x) : X_{L+1} \to X_{L+1}, \quad \Phi(x)(\bar{k}) = \bar{k}',$$
$$\bar{k}'_m = \pi \circ h^m(x).$$

Let $\Phi : X_{L+1} \to X_{L+1}$ be a map, and take an initial condition:

$$(k_0^0, k_1^0, k_2^0, \dots)$$

and a boundary condition:

$$(k_0^1, k_0^2, k_0^3, \dots).$$

Then we inductively put:

$$(k_0^t, k_1^t, \dots) = \Phi(k_0^{t-1}, k_1^{t-1}, \dots)) = \Phi^t((k_0^0, k_1^0, k_2^0, \dots)) \in X_{L+1}.$$

We call $(k_0^t, k_1^t, \dots)$ as the *flow* of Φ.

Proposition 2.3 *There is a family of smooth maps $\{f_{i,j}\}_{i,j=0,\dots,L}$ so that the corresponding two step interaction gives a flow of the Lotka Vortarra cell automaton.*

Proof: Recall that the n-step value at t, $v_n^t \in \{0, \dots, L\}$ is determined by the triple $(v_{n-1}^t, v_n^{t-1}, v_{n+1}^{t-1})$. Thus there is a map:

$$\tilde{\Phi} : \{0, \dots, L\} \times \{0, \dots, L\}^2 \to \{0, \dots, L\}$$

so that:

$$\tilde{\Phi}(v_{n-1}^t, v_n^{t-1}, v_{n+1}^{t-1}) = v_n^t \in \{0, , \dots, L\}.$$

Now let us divide the interval $[0,1]$ into $L+1$ intervals, and denote them as $I_0 < I_1 < \cdots < I_L$. Then we choose $f_{i,j}$ so that

$$f_{i,j}(I_l) \subset I_{\tilde{\Phi}(l,i,j)}$$

are satisfied for all $i,j,l \in \{0,1,\dots,L\}$. Then the induced map $\Phi(x) : X_{L+1} \to X_{L+1}$ coincide with $\tilde{\Phi}$ above, where one takes the initial point $x \in I_{k_0}$.

Question: (1) As the case of automatic interaction, one may try to obtain an algorithm to construct $L+1$ maps for a given family $\{\Phi(x) : X_{L+1} \to X_{L+1}\}$, or try to find out some obtructions for such reduction.

(2) By letting each length of the divided interval smaller and smaller, and increasing the number of intervals $L \to \infty$, one may try to recover the original Lotka Vorterra equation. At the same time one may pursue how solitons behave under such deformation.

2.E.1.2.2 Solitons for L-V cellular automaton: One can write down *solitons* explicitly as the solutions of the L-V cell automaton ([TTMS]). Here we write down one soliton only.

Let us put:

$$\begin{aligned} V_n^t &= \frac{f_{n+2}^t f_{n-1}^{t+1}}{f_{n+1}^t f_n^{t+1}}, \\ f_n &= 1 + \exp(kn - \omega t), \\ \omega &= \log\left(\frac{1+\delta(1+\exp(-k))}{1+\delta(1+\exp(k))}\right). \end{aligned}$$

V_n^t is a one soliton for the discrete Lotka Vorterra equation.

Let us put $V_n^t = \exp(\frac{v_n^t}{\epsilon})$, $\delta = \exp(-\frac{L}{\epsilon})$, $k = \frac{K}{\epsilon}$ and $\omega = \frac{\Omega}{\epsilon}$. Then letting $\epsilon \to 0$, one obtains the following soliton for the Lotka Vorterra cell automaton:

$$\begin{aligned} v_n^t &= \rho_{n-1}^{t+1} + \rho_{n+2}^t - \rho_n^{t+1} - \rho_{n+1}^t, \\ \rho_n^t &= \max(0, Kn - \Omega t + \phi), \\ \Omega &= \max(0, -K-L) - \max(0, K-L). \end{aligned}$$

2.E.2 Two dimensional interactions and Toda cell automaton: Let us consider a differntial equation:

$$\frac{d^2 r_n}{dt^2} = \exp(r_{n+1}) - 2\exp(r_n) + \exp(r_{n-1}). \quad (1)$$

This is called the *Toda equation*. This describes motion of one dimensional lattice, and the motion at n position is governed by the one of the both sides.

The *Toda cell automaton* is given by:

$$\begin{aligned} &u_n^{t+1} - 2u_n^t + u_n^{t-1} \\ &= \max(0, u_{n+1}^t - L) - 2\max(0, u_n^t - L) + \max(0, u_{n-1}^t - L). \quad (2) \end{aligned}$$

As before these are connected by another difference equation:

$$\begin{aligned} U_n^{t+1} &- 2U_n^t + U_n^{t-1} \\ &= \log(1 + \delta^2(\exp(U_{n+1}^t - 1)) - 2\log(1 + \delta^2(\exp(U_n^t) - 1)) \\ &\quad + \log(1 + \delta^2(\exp(U_{n-1}^t) - 1)). \quad (3) \end{aligned}$$

When one puts $U_n^t = r_n(\delta t)$ and make $\delta \to 0$, then one obtains (1). On the other hand when one puts $\delta = \exp(-L/2\epsilon)$ and $u_n^t = U_n^t/\epsilon$, and make $\epsilon \to 0$, then one obtains (2).

Let us introduce another generalization of micro interactions as follows. Let us fix a set of alphabets $\{0, 1, \ldots, L\}$, and choose a family of maps from $[0,1]^2$ to $[0,1]$:

$$\{f_{i,j} : [0,1]^2 \to [0,1]\}_{i,j=0,1,\ldots,L}.$$

As before one considers X_{L+1}, however in this case t of k_t implies time, rathar than the position n for $\bar{k} = \{k_t\}_{t=0,1,\ldots} \in X_{L+1}$.

We will consider three positions $(0,1,2)$. Toda equation tells us that if one knows the motion at both sides $(0,2)$, then one can determine the one at the middle position 1. We will find a map:

$$\Phi : X_{L+1} \times \{0,2\} \to X_{L+1}$$

where unlike to the previous case, $\{0,2\}$ appears, which corresponds to the boundary conditions. Namely if one gives motion of boundaries $(\bar{k}^0,\bar{k}^2)$, then $\bar{k}^1 \equiv \Phi(\bar{k}^0,\bar{k}^2) \in X_{L+1}$ gives the motion of the middle position.

Let us take $(\bar{k}^0,\bar{k}^2) \in X_{L+1} \times \{0,2\}$. Now we define inductively:

$$h^{t+1}(x) = f_{k_t^0,k_t^2}(h^{t-1}(x), h^t(x)), \quad h^0(x) = h^1(x) = x.$$

Let $\pi : [0,1] \to \{0,1,\dots,L\}$ be the projection. Thus we obtain a map:

$$\Phi(x) : X_{L+1} \times \{0,2\} \to X_{L+1}, \quad \Phi(x)(\bar{k}^0,\bar{k}^2) = \bar{k}^1$$

by $\bar{k}_t^1 = \pi \circ h^t(x)$. We call this as the *two dimensional* interaction.

Proposition 2.4 *There exists a family of smooth maps $f_{i,j} : [0,1]^2 \to [0,1]$, $i,j = 0,1,\dots,L$, so that the corresponging two dimensional interaction:*

$$\Phi(x) : X_{L+1} \times \{0,2\} \to X_{L+1}, \quad \Phi(x)(\bar{k}^0,\bar{k}^2) = (k_0^1,k_1^1,\dots,k_t^1,\dots),$$
$$h^{t+1}(x) = f_{k_t^0,k_t^2}(h^{t-1}(x), h^t(x)), \quad k_t^1 = \pi \circ h^t(x)$$

give the solutions of the Toda cell automata equation with the boundary conditions $(\bar{k}^0,\bar{k}^2)$.

Proof: Recall that the n-step value at t, $u_n^t \in \{0,\dots,L\}$ is determined by $(u_{n-1}^{t-1}, u_n^{t-1}, u_{n+1}^{t-1}, u_n^{t-2})$. Thus there is a map:

$$\tilde{\Phi} : \{0,\dots,L\}^4 \to \{0,\dots,L\}$$

so that:

$$\tilde{\Phi}(u_{n-1}^{t-1}, u_n^{t-2}, u_{n+1}^{t-1}, u_n^{t-1}) = u_n^t \in \{0,,\dots,L\}.$$

Now as above let us divide the interval $[0,1]$ into $L+1$ intervals, and denote them as $I_0 < I_1 < \dots < I_L$. Then we choose $f_{i,j} : [0,1]^2 \to [0,1]$ so that

$$f_{i,j}(I_l, I_k) \subset I_{\tilde{\Phi}(i,l,j,k)}$$

are satisfied for all $i,j,k,l \in \{0,1,\dots,L\}$. Then the induced map $\Phi(x) : X_{L+1} \times \{0,1\} \to X_{L+1}$ coincide with $\tilde{\Phi}$ above, where one chooses the initial point $x \in I_{k_0^1}$.

Let us regard:

$$F_{i,j}^{i',j'} \equiv (f_{i,j}, f_{i',j'}) : [0,1]^2 \to [0,1]^2, \quad 0 \le i,j \le \frac{L}{2} - 1, \frac{L}{2} \le i', j' \le L.$$

Identifying $[0,1]^2 \subset \mathbf{C}$, one may consider the above interaction by *holomorphic maps*. Sometimes molecules can change the states into very different quantum states. This is the *tunnel effect*. One way to express this phenomena will be to use the energy potential surfaces by introducing extra parameter, for example complex coordinate.

2.E.2.2 Solitons for Toda cell automaton: One can write down *solitons* explicitly as the solutions of the Toda cell automaton ([MSTTT]). Here we write down one soliton only.

Suppose Ω and P satisfy $\sinh\Omega = \pm\delta \sinh P$. Then the following:

$$\begin{aligned} U_n^t &= \Delta_n^2 \log(1 + \exp(2(Pn - \Omega t + \xi^0)) \\ &= \log(1 + \sinh^2 P sech^2(Pn - \Omega t + \xi^0) \end{aligned}$$

gives one soliton for the discrete Toda equation.

Let us put $P = p/2\epsilon$, $\Omega = \omega/2\epsilon$ and $\xi^0 = \eta^0/2\epsilon$. Then $u_n^t \equiv U_n^t/\epsilon$ satisfies:

$$u_n^t = \epsilon\Delta_n^2 \log(1 + \exp(\frac{pn - \omega t + \eta^0)}{\epsilon}).$$

Now letting $\epsilon \to 0$, one obtains one soliton of the Toda cell automaton:

$$\begin{aligned} u_n^t &= \Delta_n^2 \max(0, pn - \omega t + \eta^0) \\ &= \max(0, |p|(1-n) - \omega t + \eta^0), \\ \omega &= \pm\{\max(0, p - L) - \max(0, -p - L)\}. \end{aligned}$$

3 Entropy

We will define two different entropies for interaction. One is to use $\Phi(x) : X_2 \to X_2$ we have studied so far. The other is to use the shift operator. The former is new which can appear only if one uses pair of maps.

3.A Interaction entropy: We recall the *measure entropy* as follows. Let (X, μ) be a measure space and $\sigma : X \to X$ be a measurable map. Let $\mathfrak{C} = \{C_0, C_1, \dots\}$ be a measurable partition of X. Then we have the entropy of the partition $\mathfrak{C}$ as:

$$H_\mu(\mathfrak{C}) = \Sigma_i \mu(C_i) \log \mu(C_i).$$

The time n-wedge of $\mathfrak{C}$ is another partition defined as:

$$\wedge_{t=0}^n \sigma^{-t}(\mathfrak{C}) = \{C_{i_0} \cap \sigma^{-1}(C_{i_1}) \cap \cdots \cap \sigma^{-n}(C_{i_n})\}.$$

Then the entropy of σ with respect to $\mathfrak{C}$ is given by:

$$h_\mu(X, \mathfrak{C}) = \lim_{n\to\infty} \frac{1}{n+1} H_\mu(\wedge_{t=0}^n \sigma^{-t}(\mathfrak{C}))$$

and finally the entropy of σ is:

$$h_\mu(X, \sigma) = \sup_{\mathfrak{C}} h_\mu(X, \mathfrak{C}).$$

Let X_2 be the full shift with the alphabet $\{0, 1\}$. Take two maps $f, g : [0, 1] \to [0, 1]$ as before. Then one obtains $\Phi(x) : X_2 \to X_2$. Let us equip the standard metric and measure (X_2, d, μ). We define a pointwise *interaction entropy* by:

$$h(x, f, g) \equiv h_\mu(X_2, \Phi(x)).$$

By considering family, one obtains a map:

$$\Phi : X_2 \times [0, 1] \to X_2 \times [0, 1]$$

by $\Phi(\bar{k}, x) = (\Phi(x)(\bar{k}), x)$. Then we define the interaction entropy by:

$$h(f, g) \equiv h_\mu(X_2 \times [0, 1], \Phi)$$

where μ is the product measure on $X_2 \times [0, 1]$ and we equip the standard measure on $[0, 1]$.

For any invariant subshift $X \subset X_2$, $\Phi(x)(X) \subset X$, one obtains the pointwise interaction entropy $h(X; x, f, g)$. For $\Phi(X \times [0, 1]) \subset X \times [0, 1]$, one also obtain the interaction entropy $h(X; f, g)$.

3.B Topological entropy: For an irreducible subshift of finite type Σ_A in symbolic dynamics, where A is a finite matrix, the measure entropy is equal to the topological entropy where the latter is defined by a combinatoric way from the matrix data.

Let us consider the interaction map $\Phi(x) : X_2 \to X_2$. We put $X_2(l) = \{\bar{k}_l = (k_0, \dots, k_l) : k_i \in \{0,1\}\}$ and:

$$W(l,x) = \{\Phi(x)(\bar{k}_l) : \bar{k}_l \in X_2(l)\} \subset X_2(l).$$

Now we define the pointwise topological entropy of interaction by:

$$h^t(x,f,g) \equiv \lim_{l\to\infty} \frac{1}{l} \log \sharp W(l,x).$$

For a subset $X' \subset X_2$, let us put $n(l) \equiv \inf_{x \in X'} \sharp W(l,x)$. Then we define the *minimum entropy* for (X', f, g) by:

$$h^m(X', f, g) \equiv \lim_{l\to\infty} \frac{1}{l} \log n(l).$$

Conjecture: There is a combinatoric way of definition of topological entropy of interaction using combinatorics of maps.

3.C Shift entropy: Let us take two maps f and g. For $\bar{k} \in X_2$ and its l-prefix $\bar{k}_l \in X_2(l)$, we denote the the interaction by $O(\bar{k}_l) = \{h^0(x), \dots, h^l(x)\}$.

Let us take $X' \subset X_2$ and put:

$$\begin{aligned} n_l(\epsilon) = \{(x_1, \dots, x_m) : \quad & \text{for } 0 \le i \le l, \\ & (h^i(x_1), \dots, h^i(x_m)) \text{ consistes of } \epsilon \text{ net in } [0,1] \text{ for all } \bar{k}_l \in X'(l)\}. \end{aligned}$$

Then we put:

$$N_l(\epsilon) = \min_{(x_1,\dots,x_l) \in n_l(\epsilon)} \Sigma_{i=1}^l \sharp\{\Phi(\bar{k}_l, x_i) : \bar{k}_l \in X'(l)\},$$

$$h(X', f, g, \epsilon) = \limsup_{l\to\infty} \frac{1}{l} \log N_l(\epsilon).$$

We now define the *entropy* for (X', f, g) by:

$$h(X', f, g) = \lim_{\epsilon\to 0} h(X', f, g, \epsilon).$$

When $f = g$, then $h(X', f, f) = h(X') + h(f)$ holds, where $h(X')$ is the topological entropy for subshift X'.

Lemma 3.1

$$h(X', f, g) \geq h^m(X', f, g) + \inf\{h(f), h(g)\}.$$

Now we define the *interaction entropy* $H(X', f, g)$, which will measure how the oscillations $\{f^n(x)\}$ and $\{g^n(x)\}$ are well interacting:

$$H(X', f, g) = h(X', f, g) - h^m(X', f, g),$$
$$H(f, g) = \inf_{h(X', f, g) > 0} H(X', f, g).$$

On the other hand, one obtains a *universal entropy*:

$$H(X') = \inf_{f,g} (h(X', f, g) - \inf\{h(f), h(g)\}).$$

Question: When $H(X') \equiv 0$ might hold ?

Let f and g be modal maps and choose any $\bar{k} \in X_2$. Then the corresponding interaction $O(\bar{k}) = \{h^m(x)\}_m$ can be regarded as:

$$h^i : [0, 1] \to [0, 1]$$

which is also a modal map. Let $l_i(\bar{k})$ be the *lap number* of h^i. Namely the number of maximal intervals on which h^i is monotone.

Let $X' \subset X_2$ be a subset. Then we put:

$$l_i(X') = \inf_{\bar{k} \in X'} l_i(\bar{k}).$$

Now we define the *lap number of the interaction* as:

$$s(X', f, g) \equiv \lim_{i \to \infty} l_i(X')^{\frac{1}{i}}.$$

Lemma 3.2 *The equality* $h(X', f, f) = \log s(X', f, f)$ *holds.*

Question: When $h(X', f, g) = s(X', f, g)$ might hold ?

Recall that codon correspondence assignes one amino-acid for each triple of the alphabets $\{A, T, G, C\}$. In order to compare systems represented by DNA or proteins, it would be important to study *communicational entropy.*

3.D Distributed entropy: Let f and g be modal maps and choose any $\bar{k} \in X_2$. For the corresponding interaction $O(\bar{k}) = \{h^m(x)\}_m$, let us divide $[0,1]$ into smaller disjoint intervals $\{I_1, \dots, I_{i(k)}\}$ such that each I_j is a maximal interval on which h^i is monotone.

We define the *distributed lap number* as:

$$l_i^0(\bar{k}) = \sharp\{j : I_j \subset [0, \frac{1}{2})\}.$$

One defines $l_i^1(\bar{k})$ similarly.

Similarly as before, one defines $l_i^a(X') = \inf_{\bar{k} \in X'} l_i^a(\bar{k})$. Now the em distributed lap number of the interaction is given by:

$$s_0(X', f, g) \equiv \lim_{i \to \infty} l_i(X')^{\frac{1}{i}}.$$

$s_1(X', f, g)$ is similarly defined.

4 Mutation

Let us take two families $\{f_0, \dots, f_k\}$ and $\{x_0, \dots, x_l\}$. Then one ontains the corresponding family of the oriented marked graphs, the interaction graphs parametrized by $\bar{k} \in X_2$:

$$G(\{f_i\}; \{x_j\}) = \{G(\{f_i\}; \{x_j\}, \bar{k})\}_{\bar{k} \in X_2}.$$

We interpret this as a biological system. When mutation occurs, then the system will also change, and it will correspond to replacement of the graphs.

We define that a *mutation* in an interaction system $(\{f_i\}; \{x_j\})$ is a change of both maps and points as

$$(\{f_0, \dots, f_k\}; \{x_0, \dots, x_l\}) \to (\{f'_0, \dots, f'_{k'}\}; \{x'_0, \dots, x'_{l'}\}).$$

The corresponding interaction graphs also change as:

$$G(\{f_i\}; \{x_j\}) \to G(\{f'_i\}; \{x'_j\}).$$

4.B Random graphs: Suppose a mutation occur:

$$(\{f_0, \dots, f_k\}; \{x_0, \dots\}) \to (\{f'_0, \dots, f'_{k'}\}; \{x'_0, \dots\})$$

where the numbers of points are infinite. Then we put the corresponding interaction graphs as:

$$G_j \equiv G(\{f_i\}_{i=0}^k; \{x_0, \dots, x_j\}),$$
$$G'_j \equiv G(\{f'_i\}_{i=0}^{k'}; \{x'_0, \dots, x'_j\}).$$

We put $\bar{G} = \{G_j\}_{j=0}^\infty$, and $\bar{G}'$ is similar.

Let R be a property on a graph. For example conditions on the number of cycles, etc. Then we denote the set of oriented graphs with a property R by $\mathfrak{G}(R)$. Let us put:

$$\bar{G}(R, j) \equiv \{G_m; G_m \in \mathfrak{G}(R), 0 \le m \le j\} \subset \bar{G}.$$

$\bar{G}'(R, j)$ is similar.

Now we define a *rate of a property* R under mutation by:

$$R((\{f_i\}; \{x_j\}) : (\{f'_j\}; \{x'_j\})) = \lim_j \frac{\sharp \bar{G}(R, j)}{\sharp \bar{G}'(R, j)}.$$

4.C Duplication of genes: Let $f_i^s : [0, 1] \to [0, 1]$ be a family of parametrized maps, $i = 0, \dots, k$. Then one obtains a family of interaction graphs:

$$G_j^s \equiv G(\{f_i^s\}_i; \{x_0, \dots, x_j\}).$$

This is one representation of an *evolution* of interaction systems.

In *neutral theory* in evolution, bifurcation of spicies is explained by mutation by *duplication* of genes. When duplication occurs, then in order to keep the system, it will be enough if one of them plays the role of the original function. So the other may be free to mutate if the result would not affect the system itself.

In our setting we interpret the doubling genes as *doubling of numbers of points* as $s \to s' = s + \epsilon$:

$$(x_0^s, \dots, x_l^s) \to (x_0^{s'}, y_0^{s'}, x_1^{s'}, y_0^{s'}, \dots, x_l^{s'}, y_l^{s'})$$

where s plays a role of the *evolutional time parameter.*

Let $f_s : [0,1] \to [0,1]$ be a parametrized map, and suppose stable periodic orbits have *pitchfork bifurcation*, e.g., the logistic map $f_s(x) = sx(1-x)$. Then there is a family of bifurcation points $s_0 < s_1 < \dots$, $\lim_i s_i = 3.5699456...$, where the numbers of stable periodic points at $s_i + \epsilon$ become twice of that of $s_i - \epsilon$ for a small $\epsilon = \epsilon(i) > 0$.

The limit:

$$\delta = \lim_i \frac{s_i - s_{i-1}}{s_{i+1} - s_i} = 4.669201...$$

is known as the *Faigenbaum constant.*

Let $(x_0^s, \dots, x_{l(s)}^s)$ be the set of stable periodic points for f_s. Then the interaction graph:

$$G(f_s; \{x_j^s\})$$

is the union of the vertices $\{x_i^s\}_{i=0}^{l(s)}$ with a self-loop at each vertex. The image of the interaction map $\Phi(x_i^s)$ consistes of a periodic sequence with the period $l(s)$.

5 Time hierarchy, scaling limit and probability

5.A Time hierarchies: Biological systems have some hierarchy of time scalings from micro to macro. One knows at least four steps of time hierarchies; (1) Quantum stetes of electrons, (2) Interaction between molecules, (3) Life time (day by day) and (4) Evolution.

Let $f : [0,1] \to [0,1]$ be a map. An oscillation $\bar{x} = \{x, f(x), \dots\}$, as we have defined, lives in (1).

The time scales between the n step oscillation $f^n(x)$ and the time t in molecular level are mutually in completely different hierarchies. In (2), we want to express molecular activity in time t, and so we will consider at the same time, a family of movement in t:

$$x^k : [0,\infty) \to [0,1], \quad x^k(0) = f^k(x), \quad k = 0, 1, \ldots.$$

Thus in a sense the family $\{x^k(t)\}_{k,t}$ will contain informations on the molecular activity in both time scales (1) and (2). In micro scale, in the presence of fluctuation, one cannot trace its rigorous t-time development. So we will consider t-time activity by probabilistic method. In order to make compatible such different scalings, we will take some scaling limits later.

5.B Scaling change: In this section we will illustrate a basic idea of scaling limit.

Let $f : [0,1] \to [0,1]$ be a map, and choose $x \in [0,1]$. We consider the orbit space, the oscillation, $\{f^n(x)\}_{n\geq 0}$. Then we rescale the behaviour of the orbit as $\alpha_N : [0,1] \to \mathbf{R}$ by the following way:

$$\alpha_N(\frac{k}{N}, x) = \frac{f^k(x) + f^{k+1}(x) + \cdots + f^{k+N}(x)}{N}$$

where $k = 0, 1, \ldots, N-1$ and we extend it piecewise-linearly. We put $\alpha_N(0) = \alpha_N(1)$, and regard $\alpha_N : S^1 \to \mathbf{R}$.

Remark 5.1: When f acts ergodically on the interval, and if we fix k, then $\lim_{N\to\infty} \alpha_N(\frac{k}{N}, x)$ is a constant a.e. x. It will correspond to '$(\lim_N \alpha_N)(0, x)$'. So in this case just taking average will be less interesting, and so here we will consider the function α_N over S^1.

$\lim_{N\to\infty} \alpha_N$ is in time level (2). When one considers time development $\{x^k(t)\}_{k,t}$, one can rescale each oscillation $\{x^k(t)\}_k$ for each t by the same way:

$$\alpha_N^t(\frac{k}{N}, x) = \frac{x^k + x^{k+1} + \cdots + x^{k+N}}{N}.$$

$$\alpha^t = \lim_{N\to\infty} \alpha_N^t$$

will stand for molecular activity with the initial condition $\alpha = \lim_N \alpha_N$ in time t.

Later we will approximate α_N^t so that the domains are finite sets of points. The method is to use the projections $\pi : [0,1] \to \{0, 1, \ldots, L\}$ we have used so far. Then we will see that they are governed in time t by the reaction diffusion equations.

Examples 5.1: Suppose x_0 is a fixed point for f and $|f'(x_0)| > 0$. So x_0 is unstable, and for simplicity suppose $x_0 - \epsilon$ will approach to another fixed point $x_1 < x_0$ and $x_0 + \epsilon$ goes to $x_2 > x_0$ for all $0 < \epsilon < \epsilon_0$. Then we obtain three constant functions:

$$\lim_{N \to \infty} \alpha_N([x_0 - \epsilon_0, x_0)) \equiv x_1,$$
$$\lim_{N \to \infty} \alpha_N(x_0) = x_0,$$
$$\lim_{N \to \infty} \alpha_N((x_0, x_0 + \epsilon_0]) \equiv x_2.$$

5.C Interaction differential equations: Let $h : \mathbf{N} \times [0,1] \to [0,1]$ be a map, and consider a family of differential equations:

$$\dot{x}^k(t) = \Sigma_{j \neq k}^{N-1}(x^k - x^j) + h(k, x^k), \quad x^k(0) = f^k(x).$$

This is a model equation for motion of molecules, where the former in the r.h.s. is the interaction term, and the latter is the feed-back one. Molecules are fluctuating among mutual interaction. For this we will replace the above equation by *stochastic differential equations.*

Let $t \in [0, \infty)$. A family of random variables $\{X_t\}_{t \geq 0}$ parametrized by t is called a stochastic process. A *Brownian motion* is a stochastic process $B = (B_t)_{t \geq 0}$ over a probability space $(\Omega, \mathfrak{F}, P)$ which satisfies some properties that will represent random motion of interaction among molecules.

For a variable X_t, the following is called a *stochastic differential equation*:

$$dX_t = \alpha(t, X_t) dB_t + b(t, X_t) dt$$

where α and b are mutually called a diffusion and a drift term respectively. The solution X_t with the initial value $x \in \mathbf{R}$ is characterized by the stochastic integral equation:

$$X_t = x + \int_0^t \alpha(t, X_t) dB_t + \int_0^t b(t, X_t) dt.$$

Under some mild assumptions, the existence and the uniqueness are known, where uniqueness means $P(X_t = X'_t, \forall t \geq 0) = 1$.

Now let us introduce the *feed-back equation* between molecules as a family of stochastic differential equations:

$$dx^k = [\Sigma_{j \neq k}^{N-1}(x^k - x^j) + h_k(x^k)]dt + dB_t,$$
$$x^k(0) = f^k(x).$$

Using the solution set $x^k(t)$, we have a family of functions $\alpha_N = \alpha_N(\quad,\quad,x) : \mathbf{R}_+ \times [0,1] \to \mathbf{R}$ by:

$$\alpha_N(t, \frac{k}{N}, x) = \frac{x^k(t) + x^{k+1}(t) + \cdots + x^{k+N}(t)}{N}$$

where $k = 0, 1, \ldots, N$ and we extend it linearly. At limits $t, N \to \infty$;

$$\{\alpha_\infty(\infty, \quad)\}$$

will show some macro feature.

5.D Finite state approximations of generators: Here we will treat *generator* of the diffusion process of the solutions.

Let us consider the stochastic differential equations:

$$dx^k = [\Sigma_{j \neq k}^{N-1}(x^k - x^j)]dt + dB_t,$$
$$x^k(0) = f^k(x), \quad k = 0, \ldots, N-1.$$

The solutions $(x^0(t), \ldots, x^{N-1}(t))$ gives a diffusion process on $\mathbf{R}^N$.

Let F be a function over $\mathbf{R}^N$, and define a family of operators T_t given by the expectation of the values of F after t-step:

$$T_t F(x^0, \ldots, x^{N-1})$$
$$\equiv E[F(x^0(t), \ldots, x^{N-1}(t)) | (x^0(0), \ldots, x^{N-1}(0)) = (x^0, \ldots, x^{N-1})].$$

It is a semi-group on functions over $\mathbf{R}^N$.

Let us put a differential operator:

$$L_N \equiv \frac{1}{2}\Sigma_{k=0}^{N-1} \exp[-\frac{1}{2}\Sigma_{j\neq k}^{N-1}(x^k - x^j)^2]\frac{\partial}{\partial x^k} \exp(\frac{1}{2}\Sigma_{j\neq k}^{N-1}(x^k - x^j)^2])\frac{\partial}{\partial x^k}.$$

Then L_N is the generating operator for T_t, where

$$\exp(tL_N) = T_t$$

holds for all $t \geq 0$.

Let $S = \{0, 1, \ldots, N-1\}$ be a finite set and (Ω, J) be a probability space. Let $X_t : \Omega \to S$ be a stochastic process, and we denote by

$$\mu_t = (\mu_t^0, \ldots, \mu_t^{N-1})$$

the distribution of X_t over S.

Let $P(t)$ be a N by N matrix positive valued continuous family such that

$$\Sigma_{j=0}^{N-1} P(t)_{i,j} = 1$$

for all i. A martix satisfying this property is called a *probability matrix.*

We denote $P(t) = (P(t)_{i,j}(t))_{i,j=0,\ldots N-1}$. Recall that $X_t : \Omega \to S$ is a *Markov process* if:

$$\begin{aligned} P(X_0 = a_0, &X_{t_1} = a_1, \ldots, X_{t_n} = a_n) \\ &= P(X_0 = a_0)P_{a_0,a_1}(t_1 - t_0)\ldots P_{a_{n-1},a_n}(t_n - t_{n-1}) \end{aligned}$$

holds for any $t_1 < t_2 < \ldots t_n$ and any $a_0, \ldots, a_n$.

Let us take a matrix $L = (L_{i,j})_{i,j=0,1,\ldots,N}$ of the following form:

$$L_{i,j} \begin{cases} \geq 0, & i \neq j, \\ = -\Sigma_{k\neq i} L_{i,k}, & i = j. \end{cases}$$

We will call L as a *Markov generating operator*. Then

$$P(t) \equiv P(t)_L = \exp(tL) = \Sigma_{k=0}^{\infty}\frac{1}{k!}(tL)^k$$

gives a probability matrix for each t.

If X_t is a Markov process with respect to $P(t)_L$, then the distribution μ_t of X_t satisfies the equation:

$$\frac{d}{dt}\mu_t = \mu_t L.$$

5.E Approximation of maps: Let us choose $N > 0$. Then we consider the configuration space:

$$\{0,1\}^N.$$

Let $f : [0,1] \to [0,1]$ be an interval map, and $\pi : (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \to \{0,1\}$ be $\pi((0,\frac{1}{2})) \equiv 0$ and $\pi((\frac{1}{2},1)) \equiv 1$. By composition, one obtains a measurable map:

$$\pi \circ f : [0,1] \to \{0,1\}.$$

Let us explain a basic idea of approximation of stochastic process by finite dimensional states. Let $\{x^k(t)\}_{k,t}$ be as in 5.C where $x^k(0) = f^k(x)$, $x \in [0,1]$. By changing the notation, one will denote $\pi \circ x^k(t) = X_t^k(x)$ and $X_t(x) = \{X_t^k(x)\}_k$. By restriction on $k = 0, 1, \dots, N-1$, one will obtain a stochastic process:

$$X_t : [0,1] \to \{0,1\}^N, \quad X_t(x) = (X_t^0(x), \dots, X_t^{N-1}(x)),$$
$$X_0(x) = (\pi(x), \pi \circ f(x), \dots, \pi \circ f^{N-1}(x)).$$

One may generalize this to obtain another stochastic processes:

$$X_t^m : [0,1] \to \{0, \dots, m-1\}^N$$

as follows. For example, let us consider $m = 2$ case and put:

$$\pi^2 : (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1) \to \{0,1,2,3\}$$

by $\pi((\frac{l}{4}, \frac{l+1}{4})) \equiv l$, $l = 0,1,2,3$. Then one obtains naturally a partition:

$$\pi^2 \circ f : [0,1] \to \{0,1,2,3\}.$$

One can do a parallel construction, and obtain X_t^2.

5.F Iterating maps and feed-back processes: Let $H : [0,1] \to \{0, 1, \dots, N-1\}$ be a measurable map.

Let us denote discrete tori $T_N = \mathbf{Z}/N\mathbf{Z} = \{0,1,\dots,N-1\}$, and put the *configuration spaces* by:

$$\chi_N \equiv \{0,1\}^{T_N}.$$

On χ_N, we have operators $\pi_{x,y}$ for all $x, y \in T_N$:

$$\pi_{x,y} : \chi_N \to \chi_N$$

by

$$\pi_{x,y}(\eta)(m) = \begin{cases} \eta(y) & m = x, \\ \eta(x) & m = y, \\ \eta(m) & m \neq x, y. \end{cases}$$

One obtains also an induced map $\pi^*_{x,y} : C(\chi_N) \to C(\chi_N)$ by:

$$\pi^*_{x,y}(F)(\eta) = F(\pi_{x,y}(\eta)) - F(\eta).$$

Let us take $x \in [0, 1]$ and put $\eta_0 \in \chi_N$ by:

$$\eta_0(m) = X_0(x)$$

where $X_0(x)$ is in $5.E$. We want to consider a stochastic process with the initial condition $\eta_0(m)$. Let $J \subset I = [0, 1]$ be a measurable set and choose a probability measure (J, P). We denote by $E[\quad]$ as the expectation by P over J.

We put $L_N : C(\chi_N) \to C(\chi_N)$ by:

$$L_N F(\eta) = \Sigma_{x=0}^{N-1} \frac{1}{2} \pi_{x,x+1} F(\eta).$$

Let us take a stochastic process $\eta^N(t) : J \to \chi_N$:

$$\eta^N(t) = (\eta^N_m(t))_{m \in T_N}$$

and denote the distribution by $p^N(t, x) = (p^N_0(t), \ldots, p^N_{N-1}(t))$. Let us take a function $h : \mathbf{R} \to \mathbf{R}$ with $h(0) = 0$.

Definition 5.1 *$\eta^N(t)$ is called a feed-back process, if the distribution $p^N(t, \bar{x})$ satisfies the equation:*

$$\frac{d}{dt} p^N(t) = p^N N^2 L_N + (h(p^N_0(t)), \ldots, h(p^N_{N-1}(t))).$$

We will say that the feed-back process is generated by $(N^2 L_N, h)$.

Now we have a sample distribution of the correlation space over S^1:

$$\alpha_t^N(d\theta) \equiv \frac{1}{N}\Sigma_{m=0}^{N-1}\eta_m^N(t)\delta_{m/N}(d\theta)$$

where δ is the delta function.

For $F \in C(S^1)$, we define a measure:

$$\mu_t^N(F) \equiv E[\alpha_t^N(F)] = \frac{1}{N}\Sigma_{m\in T_N}p_m^N(t)F(\frac{m}{N})$$

where $\alpha_t^N(F) = \int_{S^1} F d\alpha_t^N$.

Then the *feed back measure* β_t^N is given by:

$$\beta_t^N(F) = \frac{1}{N}\Sigma_{m\in T_N}h(p_m^N(t))F(\frac{m}{N}).$$

Thus combining with the above, one obtains the following:

Lemma 5.1

$$\frac{d}{dt}\mu_t^N(F) = \frac{1}{2}\mu_t^N(\Delta_N F) + \beta_t^N(F).$$

Proposition 5.1 *Let η^N be a feed-back process with the feed back term h. Suppose $\mu_0^N(d\theta)$ converges to $\rho d\theta$ as $N \to \infty$. Then $\mu_t^N(d\theta)$ converges to $\rho(t,\theta)d\theta$ where ρ satisfies the reaction diffusion equation:*

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\rho + h(\rho).$$

Proof : Let us have the Taylor expansion $\rho(t) = \Sigma_{n\in\mathbf{Z}}a_n(t)e_n$. For $e_k(\theta) = \exp(i2\pi k\theta)$, we have the equalities:

$$\Delta_N(e_k) = -2\lambda_k^N e_k, \quad \lambda_k^N = N^2[1-\cos(2\pi k/N)] \to 2\pi^2 k^2.$$

Thus letting $N \to \infty$, one obtains the desired equality:

$$\frac{d}{dt}\rho = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\rho + h(\rho).$$

This completes the proof.

5.G Interacting processes and pattern formation: Patterns appear both in macro and micro scopic levels, e.g., stripes, or functions of molecular interactions, etc. Such patterns can be found among solutions of the so called *Turing's reaction diffusion equations.*

Let us consider two stochastic processes:

$$\eta_i^N(t) : J \to \chi_N, \quad i = 1, 2$$

and the corresponding distributions $p_i^N(t) = (p_0^{N,i}(t), \ldots, p_{N-1}^{N,i}(t))$ respectively. Suppose these satisfy the following equations:

$$\frac{d}{dt}p_1^N(t) = d_1 p_1^N N^2 L_N + (h_1(p_0^{N,1}(t), p_0^{N,2}(t)), \ldots, h_1(p_{N-1}^{N,1}(t), p_{N-1}^{N,2}(t))),$$
$$\frac{d}{dt}p_2^N(t) = d_2 p_2^N N^2 L_N + (h_2(p_0^{N,1}(t), p_0^{N,2}(t)), \ldots, h_2(p_{N-1}^{N,1}(t), p_{N-1}^{N,2}(t)))$$

where $d_1, d_2 > 0$ are positive constants, and $h_1, h_2 : \mathbf{R}^2 \to \mathbf{R}$ are functions with $h_i(0) = 0$. We say that they are *interacting processes*, and it has the *interacting generating set*

$$(d_1, d_2, N^2 L_N, h^1, h^2).$$

One obtains two measures on S^1 by the same way as before $(\mu_t^N)_1$ and $(\mu_t^N)_2$.

We have the following:

Proposition 5.2 *Suppose both $(\mu_0^N)_1(d\theta)$ and $(\mu_0^N)_2(d\theta)$ converge to $\rho_1 d\theta$ and $\rho_2 d\theta$ respectively. Then $(\mu_t^N)_1(d\theta)$ and $(\mu_t^N)_2(d\theta)$ also converge to $\rho_1(t,\theta)d\theta$ and $\rho_2(t,\theta)d\theta$ respectively, where ρ_1 and ρ_2 satisfies the Turing's reaction diffusion equation:*

$$\frac{\partial}{\partial t}\rho_1 = \frac{d_1}{2}\frac{\partial^2}{\partial \theta^2}\rho_1 + h_1(\rho_1, \rho_2),$$
$$\frac{\partial}{\partial t}\rho_2 = \frac{d_2}{2}\frac{\partial^2}{\partial \theta^2}\rho_2 + h_2(\rho_1, \rho_2).$$

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