Homotopy type of gauge groups
of
SU(3)-bundles over $S^6$

by

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1 Introduction

Let $G$ be a compact Lie group, $\pi : P \to B$ a principal $G$-bundle over a finite complex $B$. We denote by $\mathcal{G}(P)$, the group of $G$-equivariant self-maps covering the identity map of $B$. $\mathcal{G}(P)$ is called the gauge group of $P$.

Denote by $P_{n,k}$, the principal SU($n$)-bundle over $S^4$ with $c_2(P_{n,k}) = k$. In [?], the second author shows $\mathcal{G}(P_{2,k})$ is homotopy equivalent to $\mathcal{G}(P_{2,k'})$ if and only if $(12, k) = (12, k')$, where $(12, k)$ is the G.C.D. of 12 and $k$. Recently in [?], we show $\mathcal{G}(P_{3,k}) \simeq \mathcal{G}(P_{3,k'})$ if and only if $(24, k) = (24, k')$. On the other hand in [?] M.Crabb and W.Sutherland prove as $P$ ranges over all principal $G$-bundles over $B$, the number of homotopy types of $\mathcal{G}(P)$ is finite if $B$ is connected and $G$ is a compact connected Lie group. If $B$ is $S^4$ and $G = SU(2)$, then there are precisely six homotopy types of $\mathcal{G}$ ([?]).

The purpose of this paper is to show the following:

**Theorem 1.1.** Denote by $e'$ a generator of $\pi_6(\text{BSU}(3)) \cong \mathbb{Z}$ and by $\mathcal{G}_k$, the
gauge group of the principal SU(3) bundle over $S^6$ classified by $ke$. Then $G_k \simeq G_{k'}$ if and only if $(120, k) = (120, k')$.

By Atiyah-Bott [?], the classifying space $BG(P)$ of $G(P)$ is homotopy equivalent to $\text{Map}_P(B, BG)$, the connected component of maps from $B$ to $BG$ containing the classifying map of $P$. Consider the fibre sequence

\begin{equation}
(1.1) \quad G_k \rightarrow SU(3) \xrightarrow{\alpha_k} \text{Map}^{\infty}_{k, e}(S^6, BSU(3)) \rightarrow \text{Map}^{\infty}_{k', e}(S^6, BSU(3)) \xrightarrow{\epsilon_k} BSU(3).
\end{equation}

By Lang [?], $\text{Map}^{\infty}_{k, e}(S^6, BSU(3))$ is homotopy equivalent to $\text{Map}^{\infty}_{e}(S^6, BSU(3))$ and $\alpha_k$ can be identified with $\langle 1_{SU(3)}, k \epsilon \rangle = k \langle 1_{SU(3)}, e \rangle$ in $\text{Map}^{\infty}_{e}(S^6, BSU(3)) \cong [\Sigma^6 SU(3), BSU(3)] \cong [\Sigma^6 SU(3), SU(3)]$, where $\epsilon$ is the adjoint of $e'$ and $\langle \cdot , \cdot \rangle$ denotes the Samelson product.

In §3 we show $\Sigma^6 SU(3) \simeq \Sigma^7 CP^2 \vee S^{14}$, and therefore

$$[\Sigma^6 SU(3), BSU(3)] \cong [\Sigma^6 CP^2, SU(3)] \oplus \pi_{13}(SU(3)).$$

In §2 we prove the unstable $K^1$-group $[\Sigma^6 CP^2, SU(3)]$ is isomorphic to $Z/120 \oplus Z/3$ and $[\Sigma^6 CP^2, SU(3)]/G_k = \{ (120, k) \langle 3, k \rangle, k \in \{ \alpha \in [\Sigma^6 CP^2, SU(3)] \mid \langle \alpha, k \epsilon \rangle = 0 \} \}$. Put $Y = \text{Map}^{\infty}_{0}(S^6, BSU(3))$. $Y$ is a loop space and $\pi_j(Y)$ is finite for all $j$. Since $\pi_{13}(SU(3)) = Z/6, 120 \alpha_1 = 0$. By [?], if $(120, k) = (120, k')$ then there exists a self homotopy equivalence $h$ of $Y$ satisfying $h \circ (k \alpha_1) \simeq k' \alpha_1$. Therefore if $(120, k) = (120, k')$ then $G_k \simeq G_{k'}$. On the other hand applying the functor $[\Sigma^6 CP^2, \cdot]$ to (1.1), we get if the order of $[\Sigma^6 CP^2, BG_k]$ is equal to $[\Sigma^6 CP^2, BG_{k'}]$ then $(120, k) = (120, k')$ and prove Theorem ??.

2 $[\Sigma^6 CP^2, SU(3)]$

First we determine $[\Sigma^6 CP^2, U(4)]$. Put $X = \Sigma^6 CP^2 = S^8 \cup_\eta e^{10}$ where $\eta$ is the generator of $\pi_6(S^8) \cong Z/2$ and $W_4 = U(\infty)/U(4)$. Recall that as an algebra

$$H^*(BU(\infty)) = Z[c_1, c_2, \ldots]$$

where $c_j$ is the $j$-th universal Chern class and

$$H^*(U(\infty)) = \bigwedge (x_1, x_3, \ldots)$$

where $x_{2j-1} = \sigma(c_j)$. Consider the projection $\pi : U(\infty) \rightarrow W_4$. As an algebra

$$H^*(W_4) = \bigwedge (\bar{x}_9, \bar{x}_{11}, \ldots)$$

and $\pi^*(\bar{x}_{2j+1}) = x_{2j+1}$. Put $a_{2j} = \sigma(\bar{x}_{2j+1})$. $a_8$ and $a_{10}$ are generators of $H^8(\Omega W_4) \cong H^{10}(\Omega W_4) \cong Z$. Note that $Sq^2 \rho x_9 = 0$ where $\rho$ is the mod 2 reduction and therefore

$$W_4 \simeq (S^8 \vee S^{11}) \cup e^{13} \cup \cdots,$$

$$\Omega W_4 \simeq (S^8 \vee S^{10}) \cup e^{12} \cup \cdots.$$
Since \( \dim X = 10 \), \([X, \Omega W_4] = [X, S^8] \oplus [X, S^{10}] \). Using the fact that \( \eta^2 \) generates \( \pi_{10}(S^8) \cong \mathbb{Z}/2 \) we get

\[
i^* : [X, S^8] \to [S^8, S^8] \cong \mathbb{Z}
\]
is monic and \( \text{Im} i^* = 2\mathbb{Z} \), where \( i : S^8 \subset X \) is the inclusion. Define a homomorphism \( \lambda : [X, \Omega W_4] \to H^8(X) \oplus H^{10}(X) \) by \( \lambda(\alpha) = (\alpha^* a_8, \alpha^* a_{10}) \) for \( \alpha \in [X, \Omega W_4] \). Then we have

**Lemma 2.1.** \( \lambda \) is monic and \( \text{Im} \lambda = \{(n, m) | n \equiv 0 \pmod{2}\} \).

Consider the fibre sequence

\[
\Omega U(\infty) \xrightarrow{\Omega \pi} \Omega W_4 \xrightarrow{j} U(4) \xrightarrow{i} U(\infty).
\]

Put \( u = (2, 5) \) and \( v = (0, 1) \). Then \( u, v \in \text{Im} \lambda \) and \( u \) and \( v \) generate \( \text{Im} \lambda \).

**Lemma 2.2.** \( \text{Im} \lambda \circ (\Omega \pi)_* \) is generated by \( 12u \) and \( 120v \).

Note that as an algebra \( H^*(\mathbb{C}P^2) = \mathbb{Z}[t]/(t^3) \) for \( |t| = 2 \) and \( K^*(\mathbb{C}P^2) = \mathbb{Z}[x]/(x^3) \) where \( ch x = t + \frac{t^2}{2} \). Therefore \( ch x^2 = t^2 \). Denote by \( \zeta_3 \) a generator of \( K(S^8) \). \( \tilde{K}(X) \) is a free abelian group generated by \( \zeta_3 \hat{\otimes} x \) and \( \zeta_3 \hat{\otimes} x^2 \). Since

\[
(\Omega \pi)_* (\sigma(x_{2j+1})) = j! ch_j
\]

(see [?]) we have

\[
(\lambda \circ (\Omega \pi)_*)(\zeta_3 \hat{\otimes} x) = (24, 60),
\]

\[
(\lambda \circ (\Omega \pi)_*)(\zeta_3 \hat{\otimes} x^2) = (0, 120)
\]

and Lemma ?? is obtained.

Since \( \tilde{K}^1(X) = 0 \), we have the following:

**Theorem 2.3.** \([X, U(4)] \cong \mathbb{Z}/12 \oplus \mathbb{Z}/120 \).

Denote the commutator of \( U(n) \) by \( \gamma \) and the lift of \( \gamma \) constructed in [?] by \( \tilde{\gamma} : U(n) \wedge U(n) \to \Omega W_n \). In [?] using \( \tau(x_{2n+1}) = c_{n+1} \) we get

\[
\tilde{\gamma}^*(a_{2n}) = \sum_{j+k=n-1} x_{2j+1} \oplus x_{2k+1}
\]

where \( \tau \) is the transgression with respect to the fibering

\[
W_n \to BU(n) \to BU(\infty).
\]

Using \( \tau(x_{2n+3}) \equiv c_{n+2} \mod (c_{n+1}) \) we can prove

\[
\tilde{\gamma}^*(a_{2n+2}) = \sum_{j+k=n} x_{2j+1} \oplus x_{2k+1}
\]
quite similarly.

Denote the inclusion $\Sigma \mathbb{C}P^2 \subset SU(3)$ by $\kappa$, a generator of $\pi_5(SU(3))$ by $\epsilon$, the projection $\Sigma \mathbb{C}P^2 \to S^5$ by $q$ and a generator of $H^3(S^5)$ by $s$. Put $\kappa' = \epsilon \circ q$.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
SU(3) \wedge SU(3) & \xrightarrow{i \wedge i} & U(4) \wedge U(4) \\
\gamma \downarrow & & \gamma \downarrow \\
SU(3) & \xrightarrow{i} & U(4)
\end{array}
\]

where $i$ is the inclusion. Note that $\kappa^* (x_3) = \sigma(t)$, $\kappa^* (x_5) = \sigma(t^2)$, $\kappa'^* (x_3) = 0$, $\kappa'^* (x_5) = 2\sigma(t^2)$, $\epsilon^* (x_3) = 0$ and $\epsilon^* (x_5) = 2s$. Therefore we have

\[
\lambda(\tilde{\gamma} \circ (i \circ \kappa \wedge i \circ \epsilon)) = (2, 2) = \alpha,
\]

\[
\lambda(\tilde{\gamma} \circ (i \circ \kappa' \wedge i \circ \epsilon)) = (0, 4) = \beta.
\]

Since $\alpha + \beta = u + v$ and $4\alpha + 3\beta = 4u$, we have the following:

**Lemma 2.4.** The subgroup of $[X, U(4)]$ generated by $i \circ \langle \kappa, \epsilon \rangle$ and $i \circ \langle \kappa', \epsilon \rangle$ is isomorphic to $\mathbb{Z}/120 \oplus \mathbb{Z}/3$.

On the other hand consider the exact sequence

\[
(\ast) \quad \pi_{10}(SU(3)) \to [X, SU(3)] \to \pi_8(SU(3)).
\]

Since by [$?\], $\pi_{10}(SU(3)) \cong \mathbb{Z}/30$ and $\pi_8(SU(3)) \cong \mathbb{Z}/12$, the order of $[X, SU(3)]$ is a divisor of 360. By Lemma $??$, $(\ast)$ is a short exact sequence, $i_* : [X, SU(3)] \to [X, U(4)]$ is monic and $\text{Im} i_*$ is the subgroup generated by $i \circ \langle \kappa, \epsilon \rangle$ and $i \circ \langle \kappa', \epsilon \rangle$.

Therefore we have the following:

**Theorem 2.5.** As a group $[X, SU(3)] \cong \mathbb{Z}/120 \oplus \mathbb{Z}/3$. $\mathbb{Z}/120$ is generated by $\langle \kappa + \kappa', \epsilon \rangle$ and $\mathbb{Z}/3$ is generated by $\langle 4\kappa + 3\kappa', \epsilon \rangle$.

For an integer $k$ define

\[
G_k = \{ a \in [\Sigma \mathbb{C}P^2, SU(3)] | \langle a, k \kappa \rangle = 0 \}.
\]

Since $[\Sigma \mathbb{C}P^2, SU(3)] \cong \tilde{K}(\Sigma \mathbb{C}P^2)$, $[\Sigma \mathbb{C}P^2, SU(3)]$ is generated by $\kappa$ and $\kappa'$. Since

\[
\begin{vmatrix}
1 & 4 \\
1 & 3
\end{vmatrix}
= -1, \quad \kappa + \kappa'
\]

and $4\kappa + 3\kappa'$ are also generators of $[\Sigma \mathbb{C}P^2, SU(3)]$. Therefore we have the following:

**Lemma 2.6.** $|[\Sigma \mathbb{C}P^2, SU(3)]/G_k| = (120, k)(3, k)$.

3 Proof of Theorem 1.1

First we show $\Sigma^6 SU(3) \simeq \Sigma^7 \mathbb{C}P^2 \vee S^{14}$. Consider the cofibering

\[
S^{13} \xrightarrow{\theta} \Sigma^7 \mathbb{C}P^2 \to \Sigma^6 SU(3).
\]
Since $\Sigma^7\mathbb{C}P^2$ is 8-connected,

$$\Sigma^\infty : [S^{13}, \Sigma^7\mathbb{C}P^2] \to \{S^{13}, \Sigma^7\mathbb{C}P^2\}$$

is isomorphic (See [?]). Note that $\Sigma^\infty(\theta) = 0$, we get $\theta = 0$. Therefore $\Sigma^6\text{SU}(3) \simeq \Sigma^7\mathbb{C}P^2 \vee S^{14}$ and

$$[\Sigma^6\text{SU}(3), BSU(3)] \cong [\Sigma^7\mathbb{C}P^2 \vee S^{14}, BSU(3)]$$

$$\cong [\Sigma^6\mathbb{C}P^2, SU(3)] \oplus \pi_{14}(SU(3))$$

$$\cong \mathbb{Z}/120 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/6$$

(See [?] and [?]). Put $Y = \Omega^6\mathbb{C}P^2$. $Y$ is a loop space. Since $\pi_j(Y)$ is finite for any $j$, $Y = \Pi Y(p)$. For a non zero integer $n$, the exponent of $n$ at a prime $p$ is denoted by $\nu_p(n)$.

Let $k$ and $k'$ be non zero integers satisfying $(120, k) = (120, k')$. Define $h_p : Y(p) \to Y(p)$ by

$$h_p = \begin{cases} \left(\frac{k'}{\nu_p(k)}\right) & \text{if } \nu_p(k) < \nu_p(120) \\ 1 & \text{if } \nu_p(k) \geq \nu_p(120). \end{cases}$$

Note that if $\nu_p(k) < \nu_p(120)$, then $\nu_p(k) = \nu_p(k')$ and $\left(\frac{k'}{\nu_p(k)}\right) \in \mathbb{Z}_{(p)}^\times$. $h_p$ is a homotopy equivalence. Put $h = \Pi h_p$, $h : Y \to Y$ is a homotopy equivalence. Since $120a_1 = 0$, we have $h \circ (k\alpha_1) \cong k'\alpha_1$ (for details see [?]). Therefore if $(120, k) = (120, k')$, then $G_k \simeq G_{k'}$. Note that $[\Sigma\mathbb{C}P^2, BSU(3)] \cong \tilde{K}^0(\Sigma\mathbb{C}P^2) = 0$. Applying the functor $[\Sigma\mathbb{C}P^2, ]$ to (1.1), we get the following exact commutative diagram:

$$\begin{array}{cccccc}
(\Omega^6)_* & [\Sigma\mathbb{C}P^2, SU(3)] & \stackrel{\alpha_k}{\longrightarrow} & [\Sigma\mathbb{C}P^2, Y] & \longrightarrow & [\Sigma\mathbb{C}P^2, BG_k] & \longrightarrow 0. \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
[X, SU(3)] & \longrightarrow & [\mathbb{C}P^2, G_k] & & & & \\
\end{array}$$

Since $\text{Im} \alpha_k \cong \text{Coker}(\Omega^6)_*$ and $\text{Im}(\Omega^6)_* = G_k$, we have $||[\mathbb{C}P^2, G_k]|| = 360/(120, k)(3, k))$. Therefore if $G_k \simeq G_{k'}$ then $(120, k) = (120, k')$.

References


