Homotopy type of gauge groups of SU(3)-bundles over S^6

by

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1 Introduction

Let G be a compact Lie group, $\pi : P \to B$ a principal G-bundle over a finite complex B. We denote by $\mathcal{G}(P)$, the group of G-equivariant self-maps covering the identity map of B. $\mathcal{G}(P)$ is called the gauge group of P.

Denote by $P_{n,k}$, the principal $\mathrm{SU}(n)$-bundle over S^4 with $c_2(P_{n,k}) = k$. In [7], the second author shows $\mathcal{G}(P_{2,k})$ is homotopy equivalent to $\mathcal{G}(P_{2,k'})$ if and only if $(12, k) = (12, k')$, where $(12, k)$ is the G.C.D. of 12 and k. Recently in [5], we show $\mathcal{G}(P_{3,k}) \simeq \mathcal{G}(P_{3,k'})$ if and only if $(24, k) = (24, k')$. On the other hand in [2] M.Crabb and W.Sutherland prove as P ranges over all principal G-bundles over B, the number of homotopy types of $\mathcal{G}(P)$ is finite if B is connected and G is a compact connected Lie group. If B is S^4 and $G = \mathrm{SU}(2)$, then there are precisely six homotopy types of $\mathcal{G}$ ([7]).

The purpose of this paper is to show the following:

Theorem 1.1. *Denote by ϵ' a generator of $\pi_6(\mathrm{BSU}(3)) \cong \mathbb{Z}$ and by $\mathcal{G}_k$, the* *gauge group of the principal* $\mathrm{SU}(3)$ *bundle over* S^6 *classified by* $k\epsilon'$. *Then* $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ *if and only if* $(120, k) = (120, k')$.

By Atiyah-Bott [1], the classifying space $B\mathcal{G}(P)$ of $\mathcal{G}(P)$ is homotopy equivalent to $\mathbf{Map}_P(B, BG)$, the connected component of maps from B to BG containing the classifying map of P. Consider the fibre sequence

$$(1.1) \quad \mathcal{G}_k \to \mathrm{SU}(3) \xrightarrow{\alpha_k} \mathbf{Map}^*_{k\epsilon'}(\mathrm{S}^6, \mathrm{BSU}(3)) \to \mathbf{Map}_{k\epsilon'}(\mathrm{S}^6, \mathrm{BSU}(3)) \xrightarrow{e_k} \mathrm{BSU}(3).$$

By Lang [8] $\mathbf{Map}^*_{k\epsilon'}(\mathrm{S}^6, \mathrm{BSU}(3))$ is homotopy equivalent to $\mathbf{Map}^*_0(\mathrm{S}^6, \mathrm{BSU}(3))$ and α_k can be identified with $\langle 1_{\mathrm{SU}(3)}, k\epsilon \rangle = k \langle 1_{\mathrm{SU}(3)}, \epsilon \rangle$ in

$$[\mathrm{SU}(3), \mathbf{Map}^*_0(\mathrm{S}^6, \mathrm{BSU}(3))] \cong [\Sigma^6\mathrm{SU}(3), \mathrm{BSU}(3)] \cong [\Sigma^5\mathrm{SU}(3), \mathrm{SU}(3)],$$

where ϵ is the adjoint of ϵ' and $\langle , \rangle$ denotes the Samelson product.

In §3 we show $\Sigma^6\mathrm{SU}(3) \simeq \Sigma^7\mathbb{C}\mathrm{P}^2 \vee \mathrm{S}^{14}$, and therefore

$$[\Sigma^6\mathrm{SU}(3), \mathrm{BSU}(3)] \cong [\Sigma^6\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \oplus \pi_{13}(\mathrm{SU}(3)).$$

In §2 we prove the unstable $\tilde{K}^1$-group $[\Sigma^6\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]$ is isomorphic to $\mathbb{Z}/120 \oplus \mathbb{Z}/3$ and $|[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]/G_k| = (120, k)(3, k)$, where $G_k = \{\alpha \in [\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \mid \langle \alpha, k\epsilon \rangle = 0\}$. Put $Y = \mathbf{Map}^*_0(\mathrm{S}^6, \mathrm{BSU}(3))$. Y is a loop space and $\pi_j(Y)$ is finite for all j. Since $\pi_{13}(\mathrm{SU}(3)) = \mathbb{Z}/6$, $120\alpha_1 = 0$. By [5], if $(120, k) = (120, k')$ then there exists a self homotopy equivalence h of Y satisfying $h \circ (k\alpha_1) \simeq k'\alpha_1$. Therefore if $(120, k) = (120, k')$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$. On the other hand applying the functor $[\Sigma\mathbb{C}\mathrm{P}^2, \ ]$ to (1.1), we get if the order of $[\Sigma\mathbb{C}\mathrm{P}^2, B\mathcal{G}_k]$ is equal to $[\Sigma\mathbb{C}\mathrm{P}^2, B\mathcal{G}_{k'}]$ then $(120, k) = (120, k')$ and prove Theorem 1.1.

2 $[\Sigma^6\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]$

First we determine $[\Sigma^6\mathbb{C}\mathrm{P}^2, \mathrm{U}(4)]$. Put $X = \Sigma^6\mathbb{C}\mathrm{P}^2 = \mathrm{S}^8 \cup_\eta e^{10}$ where η is the generator of $\pi_9(\mathrm{S}^8) \cong \mathbb{Z}/2$ and $W_4 = \mathrm{U}(\infty)/\mathrm{U}(4)$. Recall that as an algebra

$$H^*(\mathrm{BU}(\infty)) = \mathbb{Z}[c_1, c_2, \ldots]$$

where c_j is the j-th universal Chern class and

$$H^*(\mathrm{U}(\infty)) = \bigwedge(x_1, x_3, \ldots)$$

where $x_{2j-1} = \sigma(c_j)$. Consider the projection $\pi : \mathrm{U}(\infty) \to W_4$. As an algebra

$$H^*(W_4) = \bigwedge(\bar{x}_9, \bar{x}_{11}, \ldots)$$

and $\pi^*(\bar{x}_{2j+1}) = x_{2j+1}$. Put $a_{2j} = \sigma(\bar{x}_{2j+1})$. a_8 and a_{10} are generators of $H^8(\Omega W_4) \cong H^{10}(\Omega W_4) \cong \mathbb{Z}$. Note that $\mathrm{Sq}^2\rho\bar{x}_9 = 0$ where ρ is the mod 2 reduction and therefore

$$W_4 \simeq (\mathrm{S}^9 \vee \mathrm{S}^{11}) \cup e^{13} \cup \cdots,$$
$$\Omega W_4 \simeq (\mathrm{S}^8 \vee \mathrm{S}^{10}) \cup e^{12} \cup \cdots.$$

Since $\mathbf{dim}X = 10$, $[X, \Omega W_4] = [X, \mathrm{S}^8] \oplus [X, \mathrm{S}^{10}]$. Using the fact that η^2 generates $\pi_{10}(\mathrm{S}^8) \cong \mathbb{Z}/2$ we get

$$i^* : [X, \mathrm{S}^8] \to [\mathrm{S}^8, \mathrm{S}^8] \cong \mathbb{Z}$$

is monic and $\mathbf{Im}i^* = 2\mathbb{Z}$, where $i : \mathrm{S}^8 \subset X$ is the inclusion. Define a homomorphism $\lambda : [X, \Omega W_4] \to H^8(X) \oplus H^{10}(X)$ by $\lambda(\alpha) = (\alpha^* a_8, \alpha^* a_{10})$ for $\alpha \in [X, \Omega W_4]$. Then we have

Lemma 2.1. *λ is monic and $\boldsymbol{Im}\lambda = \{(n, m) | n \equiv 0 \ mod\ 2\}$.*

Consider the fibre sequence

$$\Omega \mathrm{U}(\infty) \xrightarrow{\Omega\pi} \Omega W_4 \xrightarrow{j} \mathrm{U}(4) \xrightarrow{i} \mathrm{U}(\infty).$$

Put $u = (2, 5)$ and $v = (0, 1)$. Then $u, v \in \mathbf{Im}\lambda$ and u and v generate $\mathbf{Im}\lambda$.

Lemma 2.2. *$\boldsymbol{Im}\lambda \circ (\Omega\pi)_*$ is generated by $12u$ and $120v$.*

Note that as an algebra $H^*(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}[t]/(t^3)$ for $|t| = 2$ and $K^*(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}[x]/(x^3)$ where $chx = t + \frac{t^2}{2}$. Therefore $chx^2 = t^2$. Denote by ζ_3 a generator of $\tilde{K}(\mathrm{S}^6)$. $\tilde{K}(X)$ is a free abelian group generated by $\zeta_3 \hat{\otimes} x$ and $\zeta_3 \hat{\otimes} x^2$. Since

$$(\Omega\pi)_*(\sigma(x_{2j+1})) = j! ch_j$$

(see [4]) we have

$$(\lambda \circ (\Omega\pi)_*)(\zeta_3 \hat{\otimes} x) = (24, 60),$$
$$(\lambda \circ (\Omega\pi)_*)(\zeta_3 \hat{\otimes} x^2) = (0, 120)$$

and Lemma 2.2 is obtained.

Since $\tilde{K}^1(X) = 0$, we have the following:

Theorem 2.3. *$[X, \mathrm{U}(4)] \cong \mathbb{Z}/12 \oplus \mathbb{Z}/120$.*

Denote the commutator of $\mathrm{U}(n)$ by γ and the lift of γ constructed in [4] by $\tilde{\gamma} : \mathrm{U}(n) \wedge \mathrm{U}(n) \to \Omega W_n$. In [4] using $\tau(\bar{x}_{2n+1}) = c_{n+1}$ we get

$$\tilde{\gamma}^*(a_{2n}) = \sum_{j+k=n-1} x_{2j+1} \otimes x_{2k+1}$$

where τ is the transgression with respect to the fibering

$$W_n \to \mathrm{BU}(n) \to \mathrm{BU}(\infty).$$

Using $\tau(\bar{x}_{2n+3}) \equiv c_{n+2} \bmod (c_{n+1})$ we can prove

$$\tilde{\gamma}^*(a_{2n+2}) = \sum_{j+k=n} x_{2j+1} \otimes x_{2k+1}$$

quite similarly.

Denote the inclusion $\Sigma\mathbb{C}\mathrm{P}^2 \subset \mathrm{SU}(3)$ by κ, a generator of $\pi_5(\mathrm{SU}(3))$ by ϵ, the projection $\Sigma\mathbb{C}\mathrm{P}^2 \to \mathrm{S}^5$ by q and a generator of $H^5(\mathrm{S}^5)$ by s. Put $\kappa' = \epsilon \circ q$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{SU}(3) \wedge \mathrm{SU}(3) \arrow[r, "i \wedge i"] \arrow[d, "\gamma"'] & \mathrm{U}(4) \wedge \mathrm{U}(4) \arrow[r, "\tilde{\gamma}"] \arrow[d, "\gamma"'] & \Omega W_4 \arrow[dl, "j"'] \\
\mathrm{SU}(3) \arrow[r, "i"'] & \mathrm{U}(4) &
\end{tikzcd}$$

where i is the inclusion. Note that $\kappa^*(x_3) = \sigma(t), \kappa^*(x_5) = \sigma(t^2), \kappa'^*(x_3) = 0, \kappa'^*(x_5) = 2\sigma(t^2), \epsilon^*(x_3) = 0$ and $\epsilon^*(x_5) = 2s$. Therefore we have

$$\lambda(\tilde{\gamma} \circ (i \circ \kappa \wedge i \circ \epsilon)) = (2, 2) = \alpha,$$
$$\lambda(\tilde{\gamma} \circ (i \circ \kappa' \wedge i \circ \epsilon)) = (0, 4) = \beta.$$

Since $\alpha + \beta = u + v$ and $4\alpha + 3\beta = 4u$, we have the following:

Lemma 2.4. *The subgroup of* $[X, \mathrm{U}(4)]$ *generated by* $i \circ \langle \kappa, \epsilon \rangle$ *and* $i \circ \langle \kappa', \epsilon \rangle$ *is isomorphic to* $\mathbb{Z}/120 \oplus \mathbb{Z}/3$.

On the other hand consider the exact sequence

$$(*) \quad \pi_{10}(\mathrm{SU}(3)) \to [X, \mathrm{SU}(3)] \to \pi_8(\mathrm{SU}(3)).$$

Since by [11], $\pi_{10}(\mathrm{SU}(3)) \cong \mathbb{Z}/30$ and $\pi_8(\mathrm{SU}(3)) \cong \mathbb{Z}/12$, the order of $[X, \mathrm{SU}(3)]$ is a divisor of 360. By Lemma 2.4, $(*)$ is a short exact sequence, $i_* : [X, \mathrm{SU}(3)] \to [X, \mathrm{U}(4)]$ is monic and $\mathbf{Im} i_*$ is the subgroup generated by $i \circ \langle \kappa, \epsilon \rangle$ and $i \circ \langle \kappa', \epsilon \rangle$. Therefore we have the following:

Theorem 2.5. *As a group* $[X, \mathrm{SU}(3)] \cong \mathbb{Z}/120 \oplus \mathbb{Z}/3$. $\mathbb{Z}/120$ *is generated by* $\langle \kappa + \kappa', \epsilon \rangle$ *and* $\mathbb{Z}/3$ *is generated by* $\langle 4\kappa + 3\kappa', \epsilon \rangle$.

For an integer k define

$$G_k = \left\{ a \in [\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \,|\, \langle a, k\epsilon \rangle = 0 \right\}.$$

Since $[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \cong \tilde{K}(\Sigma\mathbb{C}\mathrm{P}^2)$, $[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]$ is generated by κ and κ'. Since $\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$, $\kappa + \kappa'$ and $4\kappa + 3\kappa'$ are also generators of $[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]$. Therefore we have the following:

Lemma 2.6. $\left|[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]/G_k\right| = (120, k)(3, k)$.

3 Proof of Theorem 1.1

First we show $\Sigma^6\mathrm{SU}(3) \simeq \Sigma^7\mathbb{C}\mathrm{P}^2 \vee \mathrm{S}^{14}$. Consider the cofibering

$$\mathrm{S}^{13} \xrightarrow{\theta} \Sigma^7\mathbb{C}\mathrm{P}^2 \to \Sigma^6\mathrm{SU}(3).$$

Since $\Sigma^7\mathbb{C}\mathrm{P}^2$ is 8-connected,

$$\Sigma^\infty : [\mathrm{S}^{13}, \Sigma^7\mathbb{C}\mathrm{P}^2] \to \{\mathrm{S}^{13}, \Sigma^7\mathbb{C}\mathrm{P}^2\}$$

is isomorphic (See [11]). Note that $\Sigma^\infty(\theta) = 0$, we get $\theta = 0$. Therefore $\Sigma^6\mathrm{SU}(3) \simeq \Sigma^7\mathbb{C}\mathrm{P}^2 \vee \mathrm{S}^{14}$ and

$$\begin{aligned}
[\Sigma^6\mathrm{SU}(3), B\mathrm{SU}(3)] &\cong [\Sigma^7\mathbb{C}\mathrm{P}^2 \vee \mathrm{S}^{14}, B\mathrm{SU}(3)] \\
&\cong [\Sigma^6\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \oplus \pi_{13}(\mathrm{SU}(3)) \\
&\cong \mathbb{Z}/120 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/6
\end{aligned}$$

(See [9] and [11]). Put $Y = \Omega_0^6 B\mathrm{SU}(3)$. Y is a loop space. Since $\pi_j(Y)$ is finite for any j, $Y = \Pi Y_{(p)}$. For a non zero integer n, the exponent of n at a prime p is denoted by $\nu_p(n)$.

Let k and k' be non zero integers satisfying $(120, k) = (120, k')$. Define $h_p : Y_{(p)} \to Y_{(p)}$ by

$$h_p = \begin{cases} \left(\frac{k'}{k}\right) & \text{if } \nu_p(k) < \nu_p(120) \\ 1 & \text{if } \nu_p(k) \geq \nu_p(120). \end{cases}$$

Note that if $\nu_p(k) < \nu_p(120)$, then $\nu_p(k) = \nu_p(k')$ and $\left(\frac{k'}{k}\right) \in \mathbb{Z}_{(p)}^\times$. h_p is a homotopy equivalence. Put $h = \Pi h_p$. $h : Y \to Y$ is a homotopy equivalence. Since $120\alpha_1 = 0$, we have $h \circ (k\alpha_1) \simeq k'\alpha_1$ (for details see [5]). Therefore if $(120, k) = (120, k')$, then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$. Note that $[\Sigma\mathbb{C}\mathrm{P}^2, B\mathrm{SU}(3)] \cong \tilde{K}^0(\Sigma\mathbb{C}\mathrm{P}^2) = 0$. Applying the functor $[\Sigma\mathbb{C}\mathrm{P}^2, \ ]$ to (1.1), we get the following exact commutative diagram:

$$\begin{tikzcd}
{} \arrow[r, "(\Omega e_k)_*"] & {[\Sigma\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)]} \arrow[r, "\alpha_{k*}"] & {[\Sigma\mathbb{C}\mathrm{P}^2, Y]} \arrow[r] \arrow[d, equal] & {[\Sigma\mathbb{C}\mathrm{P}^2, B\mathcal{G}_k]} \arrow[r] \arrow[d, equal] & 0. \\
& & {[X, \mathrm{SU}(3)]} \arrow[r] & {[\mathbb{C}\mathrm{P}^2, \mathcal{G}_k]} &
\end{tikzcd}$$

Since $\mathbf{Im}\alpha_{k*} \cong \mathrm{Coker}(\Omega e_k)_*$ and $\mathbf{Im}(\Omega e_k)_* = G_k$, we have $|[\mathbb{C}\mathrm{P}^2, \mathcal{G}_k]| = 360/((120, k)(3, k))$. Therefore if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(120, k) = (120, k')$.

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