Cohomology of the classifying spaces of loop groups

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1 Introduction

Let $G$ be a compact connected Lie group and let $\mathcal{L}G$ denote the loop group $\text{Map}(S^1, G)$. In [1] it is shown that there exists a homotopy equivalence

$$BLG \simeq LBG.$$  

The purpose of this paper is to determine the cohomology of $\mathcal{L}BG$ over the Steenrod algebra for $G = U(n), SO(n)$. There are many approaches to compute the cohomology of $\mathcal{L}BG$. For example, the fibrewise homology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$ is computed in [?] Then the cohomology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$ is obtained by taking the dual in principle, but it is hard to see the algebra structure still the action of the Steenrod algebra. Since $\mathcal{L}BG$ and $EG \times_G G$ is fibrewise homotopy equivalent over $BG$, the equivariant approach can also be applied to compute the cohomology of $\mathcal{L}BG$, where $G$ acts on $G$ by the adjoint action.

Our approach is simple and different from any approaches as the above. We define the map $\hat{\sigma} : H^*(X) \to H^{*-1}(\mathcal{L}X)$ for a space $X$, which we call the inner cohomology suspension, and show that $\hat{\sigma}$ covers the cohomology suspension $\sigma : H^*(X) \to H^{*-1}(\Omega X)$. By making use of the inner cohomology suspension, we determine the map of the cohomology induced from the inclusion $\mathcal{L}BU(n) \to \mathcal{L}BU$ and $\mathcal{L}BSO(n) \to \mathcal{L}BSO$. Then we can compute the action of the Steenrod algebra by the homotopy equivalences

$$\mathcal{L}BU \simeq BU \times \Omega BU, \quad \mathcal{L}BSO \simeq BSO \times \Omega BSO.$$  

In section 1, we define the inner cohomology suspension and show the fundamental properties. In section 2, we compute the cohomology of $\mathcal{L}BU(n)$ over the Steenrod algebra by making use of the inner cohomology suspension and the similar computation is applied to the cohomology mod 2 of $\mathcal{L}BSO(n)$.

2 The inner cohomology suspension

Throughout this paper the coefficient of the cohomology is $\mathbb{Z}$ unless otherwise indicated.
Let $B$ be a topological space. In this section we define the inner cohomology suspension $\hat{\sigma}: H^*(B) \to H^{*-1}(LB)$. It is shown that $\hat{\sigma}$ covers the cohomology suspension $\tilde{\sigma}: \tilde{H}^*(B) \to H^{*-1}(\Omega B)$ and has the properties analogous to $\tilde{\sigma}$. In the special case that $B$ is an H-group, we observe that $\hat{\sigma}$ is represented by $\sigma$ and the multiplication of $B$.

At first we define the inner cohomology suspension.

**Definition 2.1.** Let $B$ be a topological space and let $\hat{e}: S^1 \times LB \to B$ be the evaluation $\hat{e}(t, l) = l(t)$ for $(t, l) \in S^1 \times LB$. The inner cohomology suspension $\hat{\sigma}: H^*(B) \to H^{*-1}(LB)$ is

$$\hat{\sigma}(x) = \hat{e}^*(x)/s,$$

where $/\!/$ denotes the slant product and $s \in H_1(S^1)$ is the Hurewicz image of $[1_{S^1}] \in \pi_1(S^1)$.

We show that $\hat{\sigma}$ covers $\tilde{\sigma}$ when $B$ is pointed. Let $B$ be a pointed space and let $\hat{e}': S^1 \times \Omega B \to B$ be the restriction of $\hat{e}$. Consider the commutative diagram below.

$$\begin{array}{ccc}
S^1 \times \Omega B & \xrightarrow{\text{proj}} & S^1 \wedge \Omega B \\
\hat{e}' \downarrow & & \downarrow \text{ad}(1_{\Omega B}) \\
B & \xrightarrow{\text{id}} & B
\end{array}$$

Since the cohomology suspension commutes with the suspension isomorphism by taking the adjoint map, we have:

**Proposition 2.1.**

$$\sigma(x) = \hat{e}'^*(x)/s,$$

where $x \in \tilde{H}^*(B)$.

**Corollary 2.1.**

$$i^* \hat{\sigma}(x) = \sigma(x),$$

where $i: \Omega B \to LB$ is the inclusion and $x \in \tilde{H}^*(B)$.

We turn to the special case such that $\hat{\sigma}$ can be represented explicitly. Let $G$ be an H-group and let $h: LG \to \Omega G \times G$ be a homotopy equivalence $h(l) = (l \cdot l(1)^{-1}, l(1))$. We denote $h^*(x \times y)$ by $xy$ for simplicity. Consider the commutative diagram

$$\begin{array}{ccc}
S^1 \times LG & \xrightarrow{h} & S^1 \times \Omega G \times G \\
\hat{e} \downarrow & & \downarrow \hat{e}' \times 1 \\
G & \xleftarrow{\mu} & G \times G,
\end{array}$$

where $\mu$ is the multiplication. Then we obtain:
Lemma 2.1. Let $x \in H^*(G)$ be $\mu^*(x) = \sum_i a_i \times b_i$, then we have

$$\hat{\sigma}(x) = \sum \sigma(a_i) b_i,$$

where we set $\sigma(y) = 0$ for $y \in H^0(G)$.

3 The cohomology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$

In this section we compute the cohomology of $\mathcal{L}BU(n)$ over the Steenrod algebra by making use of the inner cohomology suspension. We also obtain the cohomology mod 2 of $\mathcal{L}BSO(n)$ over the Steenrod algebra by the same method.

Let $c_k$ denote the $k$-th Chern class. Since $H^*(\Omega \mathcal{B}U(n)) \cong \bigwedge(\sigma(c_1), \ldots, \sigma(c_n))$, we have the following by Corollary 2.1 and the Leray-Hirsch theorem.

Proposition 3.1.

$$H^*(\mathcal{L}BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n] \otimes \bigwedge(\hat{x}_1, \ldots, \hat{x}_{2n-1}),$$

where $\hat{x}_{2k-1} = \hat{\sigma}(c_k)$.

Remark 3.1. In [Example 15.40, Part II, 3] the fibrewise homology of $\mathcal{L}BU(n)_+ \mathcal{B}U(n)$ is computed as

$$H^*_{\mathcal{B}U(n)}(BU(n) \times S^0; \mathcal{L}BU(n)_+ \mathcal{B}U(n)) \cong \mathbb{Z}[c_1, \ldots, c_n] \otimes \bigwedge(y_1, \ldots, y_{2n-1}),$$

where $|y_{2k-1}| = -2k + 1$. Then the fibrewise cohomology of $\mathcal{L}BU(n)$ is isomorphic to $H^*(\mathcal{L}BU(n))$ and obtained by taking the dual of the fibrewise homology. Since the dual of $y_{2k-1}$ is less clear geometrically, the action of the Steenrod algebra is not obtained by this method of computing $H^*(BU(n))$.

The advantage of Proposition 3.1 is that we can determine $\mathcal{L}j^*_n : H^*(\mathcal{L}BU) \to H^*(\mathcal{L}BU(n))$, where $j_n : BU(n) \to BU$ is the inclusion. By the naturality of $\hat{\sigma}$, we obtain:

Lemma 3.1. $\mathcal{L}j^*_n : H^*(\mathcal{L}BU) \to H^*(\mathcal{L}BU(n))$ is epic and

$$\text{Ker } \mathcal{L}j^*_n = (c_k, \hat{x}_{2k-1} \mid k > n)$$

Since the action of the Steenrod algebra on $\hat{x}_{2k-1}$ is less clear, we consider the other description of $H^*(\mathcal{L}BU(n))$. It is known that $BU$ is an $H$-group by the multiplication $\mu : BU \times BU \to BU$ induced from the inclusion $U(m) \times U(n) \to U(m + n)$. Then we have the homotopy equivalence $\mathcal{L}BU \simeq \Omega BU \times BU$ as in section 1. Thus we obtain that

$$H^*(\mathcal{L}BU) \cong \mathbb{Z}[c_1, c_2, \ldots] \otimes \bigwedge(y_1, y_3, \ldots),$$
where \( y_{2k-1} \) is the image of \( \sigma(c_k) \in H^*(\Omega BU) \) by the homotopy equivalence above. Consider the commutative diagram below.

\[
\begin{array}{ccc}
\Omega BU(n) & \rightarrow & LB(n) \\
\downarrow \Omega j_n & & \downarrow j_n \\
\Omega BU & \rightarrow & LB
\end{array}
\]

By the Leray-Hirsch theorem, we have:

**Proposition 3.2.**

\[ H^*(LB(n)) \cong \mathbb{Z}[c_1, \ldots, c_n] \otimes \bigwedge (x_1, \ldots, x_{2n-1}) \]

where \( x_{2k-1} \) denotes \( L_j^n(y_{2k-1}) \).

Since the action of the Steenrod algebra on \( c_i \) and \( y_i \) is known, we compute \( L_j^n(y_i) \) to determine \( H^*(LB(n)) \) over the Steenrod algebra.

By Lemma 3.1 we have:

\[ x_{2n+2k-1} = -c_1x_{2n+2k-3} - c_2x_{2n+2k-5} - \cdots - c_n x_{2k-1}. \]  

Then, by Proposition 3.2 and the degree argument, we can choose \( A_1^k, \ldots, A_n^k \in H^*(LB(n)) \) such that

\[ x_{2n+2k-1} = A_1^k x_1 + A_2^k x_3 + \cdots + A_n^k x_{2n-1}. \]  

**Proposition 3.3.**

\[ 1 + A_1^l + A_2^l + \cdots = (1 + c_1 + \cdots + c_{n-l})(1 + c_1 + \cdots + c_n)^{-1} \]

**Proof.** By substituting (2) to (1), we have:

\[ \sum_{l=1}^{n} A_{k+1}^l x_{2l-1} = \begin{cases} 
\sum_{m=1}^{n} \sum_{l=1}^{n} c_m A_{k-m+1}^l x_{2l-1} & k \geq n \\
\sum_{m=1}^{n} \sum_{l=1}^{n} c_m A_{k-m+1}^l x_{2l-1} - \sum_{m=k+1}^{n} c_m x_{2n+2k-2m+1} & k < n 
\end{cases} \]

Thus we obtain:

\[ \sum_{m=0}^{n} c_m A_{k-m+1}^l = 0 \quad k \geq n \]

\[ \sum_{m=0}^{k} c_m A_{k-m+1}^l = \begin{cases} 
-c_{n+k-l+1} & 0 \leq k < l < n \\
0 & 0 < l \leq k < n 
\end{cases} \]
Summing up the above in $k$, we have:

$$(1 + c_1 + \cdots + c_n)(1 + A_1^1 + A_2^1 + \cdots) = 1 + c_1 + \cdots + c_{n-1}$$

We determine the action of the Steenrod algebra on $H^*(ŁBG(n))$. Let $Sq, P$ be $1 + Sq^1 + Sq^2 + \cdots, 1 + P^1 + P^2 + \cdots$ and let $\pi_a$ be the modulo $a$ reduction. It is known that

$$\text{Sq}π_2y_{2k-1} = ∑_{i=0}^{∞} ∑_{l=1}^{n} \binom{k-1}{i} π_2y_{2k+2i-1},$$

$$\text{P}π_py_{2k-1} = ∑_{i=0}^{∞} ∑_{l=1}^{n} \binom{k-1}{i} π_py_{2k+2i(p-1)-1},$$

where $p$ is the odd prime. Then we obtain the following.

**Theorem 3.1.** Let $A_k^i ∈ H^{2m-2i+2k}(ŁBG(n))$ be $δ_{n+k,i}$ for $-n+1 ≤ k ≤ 0$ and as in Proposition 3.3 for $k > 0$. Then we have the following for $x_i ∈ H^*(ŁBG(n))$.

$$\text{Sq}π_2x_{2k-1} = ∑_{i=0}^{∞} ∑_{l=1}^{n} \binom{k-1}{i} π_2x_{2k+i-nx_{2l-1}}$$

$$\text{P}π_px_{2k-1} = ∑_{i=0}^{∞} ∑_{l=1}^{n} \binom{k-1}{i} π_px_{2k+i(p-1)-nx_{2l-1}}$$

We compute $H^*(ŁBSO(n))$ by the same method as $H^*(ŁBG(n))$. We denote the $k$-th Stiefel-Whitney class by $w_k$. It is well-known that

$$H^*(BSO;\mathbb{Z}/2) ≃ \bigwedge(σ(w_2), σ(w_4), σ(w_6), \ldots),$$

$$σ(w_{2n+1}) = σ(w_{n+1})^2.$$  

Then we have the following by the same way as Proposition 3.1 and Lemma 3.1.

**Lemma 3.2.** Let $j_n : BSO(n) → BSO$ be the natural inclusion. We have

$$H^*(ŁBSO(n);\mathbb{Z}/2) ≃ \mathbb{Z}/2[w_2, w_3, \ldots] ⊗ Δ(\hat{x}_1, \hat{x}_2, \ldots),$$

$Lj_n : H^*(ŁBSO;\mathbb{Z}/2) → H^*(ŁBSO(n);\mathbb{Z}/2)$ is epic and

$$\text{Ker } Lj_n = \{wk, \hat{x}_{k-1} | k > n\},$$

where $\hat{x}_k = σ(w_k + 1)$.

Let $B_k^1 ∈ H^{n−t+k}(ŁBSO(n);\mathbb{Z}/2)$ be defined as

$$1 + B_1^1 + B_2^1 + \cdots = (1 + w_2 + w_3 + \cdots + w_n)^{-1}(1 + w_2 + w_3 + \cdots + w_{n-1})$$

for $k > 0$ and $B_k^i = δ_{n+k,i}$ for $-n+1 ≤ k ≤ 0$. Since $μ^*(w_k) = ∑_{i+j=k} w_i ⊗ w_j$ and Lemma 3.2 holds, we have the following analogously to Proposition 3.2 and Proposition 3.3, where $μ : BSO × BSO → BSO$ is the multiplication induced from the inclusion $SO(n) × SO(m) → SO(n+m)$.  

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Lemma 3.3.

\[ H^\ast(LBSO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \ldots] \otimes \Delta(x_1, x_2, \ldots), \]

\[ \mathcal{L}j_n^\ast(\sigma(w_n + k)) = B_k^1 x_1 + B_k^2 x_2 + \cdots + B_k^n x_{n-1}, \]

where \( x_i = \mathcal{L}j_n^\ast(\sigma(w_{i+1})) \) for \( 1 \leq l \leq n - 1 \).

We put \( m = \lfloor n/2 \rfloor \) and \( s_k \) to be the smallest number such that \( 2^s(2k-1) \geq n \).

By Lemma 3.2 and 3.3, we obtain:

Theorem 3.2.

\[ H^\ast(LBSO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \ldots, w_n] \otimes \mathbb{Z}/2[x_1, x_3, \ldots, x_{2m-1}]/I, \]

\[ Sqx_k = \sum_{i=0}^{\infty} \sum_{l=1}^{n-1} \binom{k}{i} B_{k+1-n}^i x_l, \]

where \( I = (x_{2k} - x_k^2, x_{2k-1}^2 - \sum_{l=1}^{n-1} B_{(2k-1)x_l}^i x_l | 1 \leq k \leq m). \)

Remark 3.2. The fibrewise homology mod 2 of \( LBSO(n) \) is computed in [2]. But the method in Remark 3.1 is not applied.

Remark 3.3. The second author once pointed out that \( H^\ast(LBSO(n); \mathbb{Z}/2) \) was obtained by use of the Whitehead product.

References

