# Cohomology of the classifying spaces of loop groups

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## 1 Introduction

Let $G$ be a compact connected Lie group and let $\mathcal{L}G$ denote the loop group $\mathrm{Map}(S^1, G)$. In [1] it is shown that there exists a homotopy equivalence

$$B\mathcal{L}G \simeq \mathcal{L}BG.$$

The purpose of this paper is to determine the cohomology of $\mathcal{L}BG$ over the Steenrod algebra for $G = U(n), SO(n)$. There are many approaches to compute the cohomology of $\mathcal{L}BG$. For example, the fibrewise homology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$ is computed in [**?**]. Then the cohomology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$ is obtained by taking the dual in principle, but it is hard to see the algebra structure still the action of the Steenrod algebra. Since $\mathcal{L}BG$ and $EG \times_G G$ is fibrewise homotopy equivalent over $BG$, the equivarinant approach can also be applied to compute the cohomology of $\mathcal{L}BG$, where $G$ acts on $G$ by the adjoint action.

Our approach is simple and different from any approaches as the above. We define the map $\hat{\sigma} : H^*(X) \to H^{*-1}(\mathcal{L}X)$ for a space $X$, which we call *the inner cohomology suspension*, and show that $\hat{\sigma}$ covers the cohomology suspension $\sigma : \tilde{H}^*(X) \to H^{*-1}(\Omega X)$. By making use of the inner cohomology suspension, we determine the map of the cohomology induced from the inclusion $\mathcal{L}BU(n) \to \mathcal{L}BU$ and $\mathcal{L}BSO(n) \to \mathcal{L}BSO$. Then we can compute the action of the Steenrod algebra by the homotopy equivalences

$$\mathcal{L}BU \simeq BU \times \Omega BU, \quad \mathcal{L}BSO \simeq BSO \times \Omega BSO.$$

In section 1, we define the inner cohomology suspension and show the fundamental properties. In section 2, we compute the cohomology of $\mathcal{L}BU(n)$ over the Steenrod algebra by making use of the inner cohomology suspension and the similar computation is applied to the cohomology mod 2 of $\mathcal{L}BSO(n)$.

## 2 The inner cohomology suspension

Throughout this paper the coefficient of the cohomology is $\mathbf{Z}$ unless otherwise indicated.

Let $B$ be a topological space. In this section we define the inner cohomology suspension $\hat{\sigma} : H^*(B) \to H^{*-1}(\mathcal{L}B)$. It is shown that $\hat{\sigma}$ covers the cohomology suspension $\sigma : \tilde{H}^*(B) \to H^{*-1}(\Omega B)$ and has the properties analogous to $\sigma$. In the special case that $B$ is an H-group, we observe that $\hat{\sigma}$ is represented by $\sigma$ and the multiplication of $B$.

At first we define the inner cohomology suspension.

**Definition 2.1.** Let $B$ be a topological space and let $\hat{e} : S^1 \times \mathcal{L}B \to B$ be the evaluation $\hat{e}(t, l) = l(t)$ for $(t, l) \in S^1 \times \mathcal{L}B$. The inner cohomology suspension $\hat{\sigma} : H^*(B) \to H^{*-1}(\mathcal{L}B)$ is

$$\hat{\sigma}(x) = \hat{e}^*(x)/s,$$

where / denotes the slant product and $s \in H_1(S^1)$ is the Hurewictz image of $[1_{S^1}] \in \pi_1(S^1)$.

We show that $\hat{\sigma}$ covers $\sigma$ when $B$ is pointed. Let $B$ be a pointed space and let $\hat{e}' : S^1 \times \Omega B \to B$ be the restriction of $\hat{e}$. Consider the commutative diagram below.

$$\begin{array}{ccc} S^1 \times \Omega B & \xrightarrow{\text{proj}} & S^1 \wedge \Omega B \\ {\scriptstyle \hat{e}'}\downarrow & & \downarrow{\scriptstyle \text{ad}(1_{\Omega B})} \\ B & = & B \end{array}$$

Since the cohomology suspension commutes with the suspension isomorphism by taking the adjoint map, we have:

**Proposition 2.1.**

$$\sigma(x) = \hat{e}'^*(x)/s,$$

*where* $x \in \tilde{H}^*(B)$.

**Corollary 2.1.**

$$i^*\hat{\sigma}(x) = \sigma(x),$$

*where* $i : \Omega B \to \mathcal{L}B$ *is the inclusion and* $x \in \tilde{H}^*(B)$.

We turn to the special case such that $\hat{\sigma}$ can be represented explicitly. Let $G$ be an H-group and let $h : \mathcal{L}G \to \Omega G \times G$ be a homotopy equivalence $h(l) = (l \cdot l(1)^{-1}, l(1))$. We denote $h^*(x \times y)$ by $xy$ for simplicity. Consider the commutative diagram

$$\begin{array}{ccc} S^1 \times \mathcal{L}G & \xrightarrow{h} & S^1 \times \Omega G \times G \\ {\scriptstyle \hat{e}}\downarrow & & \downarrow{\scriptstyle \hat{e}' \times 1} \\ G & \xleftarrow{\mu} & G \times G, \end{array}$$

where $\mu$ is the multiplication. Then we obtain :

**Lemma 2.1.** *Let $x \in H^*(G)$ be $\mu^*(x) = \sum_i a_i \times b_i$, then we have*

$$\hat{\sigma}(x) = \sum_i \sigma(a_i) b_i,$$

*where we set $\sigma(y) = 0$ for $y \in H^0(G)$.*

# 3 The cohomology of $\mathcal{L}BU(n)$ and $\mathcal{L}BSO(n)$

In this section we compute the cohomology of $\mathcal{L}BU(n)$ over the Steenrod algebra by making use of the inner cohomology suspension. We also obtain the cohomology mod 2 of $\mathcal{L}BSO(n)$ over the Steenrod algebra by the same method.

Let $c_k$ denote the $k$-th Chern class. Since $H^*(\Omega BU(n)) \cong \bigwedge(\sigma(c_1), \dots, \sigma(c_n))$, we have the following by Corollary 2.1 and the Leray-Hirsch theorem.

**Proposition 3.1.**

$$H^*(\mathcal{L}BU(n)) \cong \mathbf{Z}[c_1, \dots, c_n] \otimes \bigwedge(\hat{x}_1, \dots, \hat{x}_{2n-1}),$$

*where $\hat{x}_{2k-1} = \hat{\sigma}(c_k)$.*

*Remark 3.1.* In [Example 15.40, Part II, 3] the fibrewise homology of $\mathcal{L}BU(n)_{+BU(n)}$ is computed as

$$H^*_{BU(n)}\{BU(n) \times S^0; \mathcal{L}BU(n)_{+BU(n)}\} \cong \mathbf{Z}[c_1, \dots, c_n] \otimes \bigwedge(y_1, \dots, y_{2n-1}),$$

where $|y_{2k-1}| = -2k+1$. Then the fibrewise cohomology of $\mathcal{L}BU(n)$ is isomorphic to $H^*(\mathcal{L}BU(n))$ and obtained by taking the dual of the fibrewise homology. Since the dual of $y_{2k-1}$ is less clear geometrically, the action of the Steenrod algebra is not obtained by this method of computing $H^*(\ BU(n))$.

The advantage of Proposition 3.1 is that we can determine $\mathcal{L}j_n^* : H^*(\mathcal{L}BU) \to H^*(\mathcal{L}BU(n))$, where $j_n : BU(n) \to BU$ is the inclusion. By the naturality of $\hat{\sigma}$, we obtain :

**Lemma 3.1.** *$\mathcal{L}j_n^* : H^*(\mathcal{L}BU) \to H^*(\mathcal{L}BU(n))$ is epic and*

$$\operatorname{Ker} \mathcal{L}j_n^* = (c_k, \hat{x}_{2k-1} \mid k > n)$$

Since the action of the Steenrod algebra on $\hat{x}_{2k-1}$ is less clear, we consider the other description of $H^*(\mathcal{L}BU(n))$. It is known that $BU$ is an H-group by the multiplication $\mu : BU \times BU \to BU$ induced from the inclusion $U(m) \times U(n) \to U(m+n)$. Then we have the homotopy equivalence $\mathcal{L}BU \simeq \Omega BU \times BU$ as in section 1. Thus we obtain that

$$H^*(\mathcal{L}BU) \cong \mathbf{Z}[c_1, c_2, \dots] \otimes \bigwedge(y_1, y_3, \dots),$$

where $y_{2k-1}$ is the image of $\sigma(c_k) \in H^*(\Omega BU)$ by the homotopy equivalence above. Consider the commutative diagram below.

$$\begin{array}{ccccc} \Omega BU(n) & \longrightarrow & \mathcal{L}BU(n) & \longrightarrow & BU(n) \\ \downarrow{\scriptstyle \Omega j_n} & & \downarrow{\scriptstyle \mathcal{L}j_n} & & \downarrow{\scriptstyle j_n} \\ \Omega BU & \longrightarrow & \mathcal{L}BU & \longrightarrow & BU \end{array}$$

By the Leray-Hirsch theorem, we have:

**Proposition 3.2.**

$$H^*(\mathcal{L}BU(n)) \cong \mathbf{Z}[c_1, \dots, c_n] \otimes \bigwedge(x_1, \dots, x_{2n-1}),$$

*where* $x_{2k-1}$ *denotes* $\mathcal{L}j_n^*(y_{2k-1})$.

Since the action of the Steenrod algebra on $c_i$ and $y_i$ is known, we compute $\mathcal{L}j_n^*(y_i)$ to determine $H^*(\mathcal{L}BU(n))$ over the Steenrod algebra. Since $\mu^*(c_k) = \sum_{i+j=k} c_i \otimes c_j$ for $c_k \in H^*(BU)$, we have the following by Lemma 2.1.

$$\hat{x}_{2k-1} = y_{2k-1} + c_1 y_{2k-3} + \cdots + c_{k-1} y_1 \in H^*(\mathcal{L}BU).$$

By Lemma 3.1 we have:

$$x_{2n+2k-1} = -c_1 x_{2n+2k-3} - c_2 x_{2n+2k-5} - \cdots - c_n x_{2k-1}. \tag{1}$$

Then, by Proposition 3.2 and the degree argument, we can choose $A_k^1, \dots, A_k^n \in H^*(\mathcal{L}BU(n))$ such that

$$x_{2n+2k-1} = A_k^1 x_1 + A_k^2 x_3 + \cdots + A_k^n x_{2n-1}. \tag{2}$$

**Proposition 3.3.**

$$1 + A_1^l + A_2^l + \cdots = (1 + c_1 + \cdots + c_{n-l})(1 + c_1 + \cdots + c_n)^{-1}$$

*Proof.* By substituting (2) to (1), we have:

$$\sum_{l=1}^{n} A_{k+1}^l x_{2l-1} = \begin{cases} \displaystyle\sum_{m=1}^{n}\sum_{l=1}^{n} c_m A_{k-m+1}^l x_{2l-1} & k \geq n \\ \displaystyle\sum_{m=1}^{k}\sum_{l=1}^{n} c_m A_{k-m+1}^l x_{2l-1} - \sum_{m=k+1}^{n} c_m x_{2n+2k-2m+1} & k < n \end{cases}$$

Thus we obtain:

$$\sum_{m=0}^{n} c_m A_{k-m+1}^l = 0 \qquad k \geq n$$

$$\sum_{m=0}^{k} c_m A_{k-m+1}^l = \begin{cases} -c_{n+k-l+1} & 0 \leq k < l < n \\ 0 & 0 < l \leq k < n \end{cases}$$

Summing up the above in $k$, we have:

$$(1+c_1+\cdots+c_n)(1+A_1^l+A_2^l+\cdots)=1+c_1+\cdots+c_{n-l}$$

□

We determine the action of the Steenrod algebra on $H^*(\mathcal{L}BU(n))$. Let $Sq, \mathcal{P}$ be $1+Sq^1+Sq^2+\cdots, 1+\mathcal{P}^1+\mathcal{P}^2+\cdots$ and let $\pi_a$ be the modulo $a$ reduction. It is known that

$$Sq\pi_2 y_{2k-1}=\sum_{i=0}^{\infty}\binom{k-1}{i}\pi_2 y_{2k+2i-1},$$
$$\mathcal{P}\pi_p y_{2k-1}=\sum_{i=0}^{\infty}\binom{k-1}{i}\pi_p y_{2k+2i(p-1)-1},$$

where $p$ is the odd prime. Then we obtain the following.

**Theorem 3.1.** *Let $A_k^l \in H^{2n-2l+2k}(\mathcal{L}BU(n))$ be $\delta_{n+k,l}$ for $-n+1\le k\le 0$ and as in Proposition 3.3 for $k>0$. Then we have the following for $x_i \in H^*(\mathcal{L}BU(n))$.*

$$Sq\pi_2 x_{2k-1}=\sum_{i=0}^{\infty}\sum_{l=1}^{n}\binom{k-1}{i}\pi_2 A_{k+i-n}^l x_{2l-1}$$
$$\mathcal{P}\pi_p x_{2k-1}=\sum_{i=0}^{\infty}\sum_{l=1}^{n}\binom{k-1}{i}\pi_p A_{k+i(p-1)-n}^l x_{2l-1}$$

We copmpute $H^*(\mathcal{L}BSO(n))$ by the same method as $H^*(\mathcal{L}BU(n))$.

We denote the $k$-th Stiefel-Whitney class by $w_k$. It is well-known that

$$H^*(BSO;\mathbf{Z}/2)\cong\bigwedge(\sigma(w_2),\sigma(w_4),\sigma(w_6),\dots),$$
$$\sigma(w_{2n+1})=\sigma(w_{n+1})^2.$$

Then we have the following by the same way as Proposition 3.1 and Lemma 3.1.

**Lemma 3.2.** *Let $j_n : BSO(n)\to BSO$ be the natural inclusion. We have*

$$H^*(\mathcal{L}BSO(n);\mathbf{Z}/2)\cong\mathbf{Z}/2[w_2,w_3,\dots]\otimes\Delta(\hat{x}_1,\hat{x_2},\dots),$$

*$\mathcal{L}j_n : H^*(\mathcal{L}BSO;\mathbf{Z}/2)\to H^*(\mathcal{L}BSO(n);\mathbf{Z}/2)$ is epic and*

$$\operatorname{Ker}\mathcal{L}j_n=\{w_k,\hat{x}_{k-1}|k>n\},$$

*where $\hat{x}_k=\hat{\sigma}(w_k+1)$.*

Let $B_k^l\in H^{n-l+k}(\mathcal{L}BSO(n);\mathbf{Z}/2)$ be defined as

$$1+B_1^l+B_2^l+\cdots=(1+w_2+w_3+\cdots+w_n)^{-1}(1+w_2+w_3+\cdots+w_{n-l})$$

for $k>0$ and $B_k^l=\delta_{n+k,l}$ for $-n+1\le k\le 0$. Since $\mu^*(w_k)=\sum_{i+j=k}w_i\otimes w_j$ and Lemma 3.2 holds , we have the following analogously to Proposition 3.2 and Proposition 3.3, where $\mu : BSO\times BSO\to BSO$ is the multiplication induced from the inclusion $SO(n)\times SO(m)\to SO(n+m)$.

**Lemma 3.3.**

$$H^*(\mathcal{L}BSO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_2, w_3, \dots] \otimes \Delta(x_1, x_2, \dots),$$

$$\mathcal{L}j_n^*(\sigma(w_n + k)) = B_k^1 x_1 + B_k^2 x_2 + \cdots + B_k^n x_{n-1},$$

*where* $x_l = \mathcal{L}j_n^*(\sigma(w_{l+1}))$ *for* $1 \le l \le n-1$.

We put $m = [n/2]$ and $s_k$ to be the smallest number such that $2^s(2k-1) \ge n$. By Lemma 3.2 and 3.3, we obtain:

**Theorem 3.2.**

$$H^*(\mathcal{L}BSO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_2, w_3, \dots, w_n] \otimes \mathbf{Z}/2[x_1, x_3, \dots, x_{2m-1}]/I,$$

$$Sqx_k = \sum_{i=0}^{\infty} \sum_{l=1}^{n-1} \binom{k}{i} B_{k+i-n}^l x_l,$$

*where* $I = (x_{2k} - x_k^2, x_{2k-1}^{2^{s_k}} - \sum_{l=1}^{n-1} B_{(2k-1)2^{s_k}}^l x_l | 1 \le k \le m)$.

*Remark 3.2.* The fibrewise homology mod 2 of $\mathcal{L}BSO(n)$ is computed in [2]. But the method in Remark 3.1 is not applied.

*Remark 3.3.* The second author once pointed out that $H^*(\mathcal{L}BSO(n); \mathbf{Z}/2)$ was obtained by use of the Whitehead product.

# References


[1] M.F. Atiyah,R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615.

[2] M. Crabb, Fibrewise homology, Glasg. Math. J. 43 (2001), no. 2, 199-208.

[3] M. Crabb and I. James, Fibrewise Homotopy Theory, Springer, 1998.