Cohomology of the classifying spaces of gauge groups over 3-manifolds in low dimensions

by

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Abstract

The concern in this paper is to calculate the cohomology algebra degree less than or equal to 3 of the function space $\mathrm{Map}(M, BG)$, where G is a simply connected compact Lie group and M is a closed orientable 3-manifold. This gives a simple proof and an improvement of the result [3, Theorem 1.2].

1 Introduction

Let G be a simply connected compact Lie group and let M denote a closed orientable 3-manifold. Since BG is 3-connected, any principal G-bundles over M are trivial. Then we call the gauge group of the trivial G-bundle over M the gauge group over M and denote it by $\mathcal{G}$. It is well-known that

$$B\mathcal{G} \simeq \mathrm{Map}(M, BG)$$

([2]). The cohomology of $B\mathcal{G}$ in low dimensions is computed in [3] by making use of the Eilenberg-Moore spectral sequence and is obtained as:

Theorem A ([3, Theorem 1.2]). *Suppose that* $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, R) = 0$. *Let G be a simply-connected compact Lie group such that the integral cohomology of BG is torsion free and let M be a closed orientable 3-manifold. We denote* $H^i(\mathrm{Map}(M, BG))$ *by* H^i*. Then there exists a short exact sequence*

$$0 \to H_1(M; R)^{\oplus r} \oplus R^{\oplus s} \oplus H^1 \otimes H^2 \xrightarrow{\alpha} H^3 \to (R/2R)^{\oplus r} \to 0,$$

where $\alpha|_{H^1 \otimes H^2}$ *is the cup product,* $r = \mathrm{rank} H^4(BG)$ *and* $s = \mathrm{rank} H^6(BG)$*. Moreover H^1 is a free R-module for any R, and H^2 is also free if R is a PID.*

The purpose of this paper is to refine Theorem A and to give a simple proof. Since the attention on the structure of G is not payed in [3], there are unnecessary assumptions and $H^i(\mathrm{Map}(M,BG))(i \leq 3)$ is not completely determined in Theorem A. We pay our attention on the structure of G and we determine the integral cohomology of $H^i(\mathrm{Map}(M,BG))(i \leq 3)$ with less assumptions. It is known that G is the direct product of simply connected compact simple Lie groups ([5]). Then we reduce Theorem A to the case that G is a simply connected compact simple Lie group and obtain

Theorem 1.1. *Let G be a simply connected compact simple Lie group and Let M be a closed orientable 3-manifold. We denote $H^i(\mathrm{Map}(M,BG))$ by H^i. Then we have:*

$$H^i \cong \begin{cases} \mathbb{Z} & i=0 \\ H^1(\Omega^2 G) & i=1 \\ H_2(M) & i=2 \\ H^1(\Omega^2 G)\otimes H_2(M)\oplus H_1(M)\oplus H^3(\Omega^2 G) & i=3 \end{cases}$$

Moreover, the cup product $H^1 \otimes H^2 \to H^3$ maps $H^1 \otimes H^2$ isomorphically onto the direct summand $H^1(\Omega^2 G)\otimes H_2(M) \subset H^3$.

Remark 1.2. *$H^*(\Omega^2 G)$ is as follows for a simply connected compact simple Lie group G.*

	type of G		
$H^i(\Omega^2 G)$	$A_l(l \geq 3)$	$C_l(l \geq 1)$	otherwise
$i=1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$i=2$	0	0	0
$i=3$	$\mathbb{Z}$	$\mathbb{Z}_2$	0

Remark 1.3. *Since $\bigvee^{g-1} S^1 \hookrightarrow \overbrace{\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2}^{g}$ induces an isomorphism on mod p cohomology for each odd prime p, Theorem 1.1 in [3] is easily shown without the assumption that $H^*(G)$ is p-torsion free by [4, Proposition 4.2].*

2 Approximation of G by infinite loop spaces

Let G be a simply connected compact Lie group. In this section we approximate G by an infinite loop space in low dimensions.

It is known that G is the direct product of simply connected compact simple Lie groups. Since simply connected compact simple Lie groups are classified by their Lie algebras as $A_l, B_l, C_l, D_l (l \geq 1), E_l (l = 6, 7, 8), F_4, G_2$, we give an approximation to each type. Note that the correspondence in low ranks is known as ([5]):

$$A_1 = B_1 = C_1,\ B_2 = C_2,\ D_2 = A_1 \times A_1,\ D_3 = A_3.$$

Proposition 2.1. *Let G be a simply connected compact simple Lie group. Then there exists an infinite loop space $\mathbf{B}$ and a 7-equivalence $BG \to \mathbf{B}$.*

Proof. For G is of type $A_l (l \geq 3)$, there exists a 7-equivalence $BG \to BSU$. For G is of type $C_l (l \geq 1)$, there exists a 7-equivalence $BG \to BSp$. For G otherwise, we have $\pi_i(G) = 0$ $(i = 1, 2, 4, 5)$ ([5]). Then a representative of a generator of $H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$ is a 7-equivalence $BG \to K(\mathbb{Z}, 4)$. $\square$

Corollary 2.2. *Let M be a 3-dimensional complex. Then we have $H^i(\mathrm{Map}(M, \mathbf{B})) \cong H^i(\mathrm{Map}(M, BG))$ $(i \leq 3)$, where $\mathbf{B}$ is as in Proposition 2.1.*

Proof. Let $\mathrm{Map}_*(X, Y)$ denotes the space of basepoint preserving maps from X to Y, where X, Y are based spaces.

We consider the following commutative diagram

$$\begin{array}{ccc} \pi_k(\mathrm{Map}_*(M, BG)) & \longrightarrow & \pi_k(\mathrm{Map}_*(M, \mathbf{B})) \\ \cong\downarrow & & \cong\downarrow \\ [S^k \wedge M, BG] & \longrightarrow & [S^k \wedge M, \mathbf{B}]. \end{array}$$

Since the second row is an isomorphism for $k \leq 3$ and a surjection for $k = 4$ by J.H.C. Whitehead theorem, we have $\mathrm{Map}_*(M, BG) \to \mathrm{Map}_*(M, \mathbf{B})$ is a 4-equivalence. Consider the following commutative diagram of evaluation fibrations.

$$\begin{array}{ccccc} \mathrm{Map}_*(M, BG) & \longrightarrow & \mathrm{Map}(M, BG) & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_*(M, \mathbf{B}) & \longrightarrow & \mathrm{Map}(M, \mathbf{B}) & \longrightarrow & \mathbf{B} \end{array}$$

Since $\mathrm{Map}_*(M, BG) \to \mathrm{Map}_*(M, \mathbf{B})$ is a 4-equivalence and $BG \to \mathbf{B}$ is a 7-equivalence, we obtain $\mathrm{Map}(M, BG) \to \mathrm{Map}(M, \mathbf{B})$ is a 4-equivalence. $\square$

3 Proof of Theorem 1.1

Let G be a simply connected compact simple Lie group. By Proposition 2.1 there exists an infinite loop space $\mathbf{B}$ and a 7-equivalence $BG \to \mathbf{B}$. Since $\mathbf{B}$ is a homotopy group, we have a homotopy equivalence

$$\begin{array}{rcl} \mathrm{Map}(M, \mathbf{B}) & \simeq & \mathrm{Map}_*(M, \mathbf{B}) \times \mathbf{B} \\ f & \mapsto & (f \cdot f(*)^{-1}, f(*)), \end{array}$$

where $*$ denotes the basepoint of M. Then we compute $H^*(\mathrm{Map}_*(M, \mathbf{B}))$ to determine $H^*(\mathrm{Map}(M, \mathbf{B}))$.

Proposition 3.1. *We have*

$$H^k(\mathrm{Map}_*(M,\mathbf{B})) \cong \bigoplus_{i+j=k} H^i(\mathrm{Map}_*(M^2,\mathbf{B})) \otimes H^j(\Omega^3\mathbf{B})$$

for $k \le 3$, where M^2 is the 2-skeleton of M.

Proof. Since a closed orientable 3-manifold is parallelizable, the top cell of M is split off stably ([1]). Actually by Freudenthal suspension theorem the top cell of M is split off after double suspension. Then we have

$$\begin{aligned}\mathrm{Map}_*(M,\mathbf{B}) &\simeq \mathrm{Map}_*(M,\Omega^2B^2\mathbf{B}))\\ &\simeq \mathrm{Map}_*(\Sigma^2 M, B^2\mathbf{B})\\ &\simeq \mathrm{Map}_*(\Sigma^2 M^2 \vee S^5, B^2\mathbf{B})\\ &\simeq \mathrm{Map}_*(M^2 \vee S^3, \mathbf{B})\\ &\cong \mathrm{Map}_*(M^2,\mathbf{B}) \times \Omega^3\mathbf{B}.\end{aligned}$$

Since $H^k(\Omega^3\mathbf{B})$ ($k < 3$) is either 0 or $\mathbb{Z}$, the proof is completed by Künneth Theorem. $\square$

To compute $H^i(\mathrm{Map}_*(M^2,\mathbf{B}))$ ($i \le 3$) we need the following technical lemma. Let X, Y, Z be based spaces and $f : X \to Y$ be a based map. We denote $\mathrm{Map}_*(f,1) : \mathrm{Map}_*(Y,Z) \to \mathrm{Map}_*(X,Z)$ by $f^\#$.

Lemma 3.2. *Let X be a based space such that there is a $(p+q+1)$-equivalence $X \to K(\mathbb{Z}, p+q)$ and let $f : \bigvee^l S^p \to \bigvee^m S^p$ be a based map. Suppose $(f^\#)^* : H^q(\Pi^l\Omega^p X) \to H^q(\Pi^m\Omega^p X)$ is represented by a matrix A for a certain basis. Then $f_* : H_p(\bigvee^l S^p) \to H_p(\bigvee^m S^p)$ is also represented by A for a suitable basis.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} H_q(\Pi^m\Omega^p X) & \xrightarrow[hur]{\cong} & \pi_q(\Pi^m\Omega^p X) & = & [\bigvee^m S^{p+q}, X] & \xrightarrow{\cong} & H^{p+q}(\bigvee^m S^{p+q}) \\ {\scriptstyle (f^\#)_*}\downarrow & & {\scriptstyle (f^\#)_*}\downarrow & & \downarrow & & {\scriptstyle (\Sigma^q f)^*}\downarrow \\ H_q(\Pi^l\Omega^p X) & \xrightarrow[hur]{\cong} & \pi_q(\Pi^l\Omega^p X) & = & [\bigvee^l S^{p+q}, X] & \xrightarrow{\cong} & H^{p+q}(\bigvee^l S^{p+q}), \end{array}$$

where *hur* is the Hurewicz homomorphism. Since $\Omega^p X \to K(\mathbb{Z}, q)$ is a $(q+1)$-equivalence, the proof is completed by taking the dual. $\square$

Proposition 3.3. $H^i(\mathrm{Map}_*(M^2,\mathbf{B})) \cong \begin{cases} 0 & i=1 \\ H_2(M) & i=2 \\ H_1(M) & i=3 \end{cases}$

Proof. Note the cofibration sequence $\bigvee^l S^1 \xrightarrow{f} \bigvee^m S^1 \to M^2 \to \bigvee^l S^2$, then we have the fibration

$$\Pi^l\Omega^2\mathbf{B} \to \mathrm{Map}_*(M^2,\mathbf{B}) \to \Pi^m\Omega\mathbf{B}.$$

We consider the Leray-Serre spectral sequence (E_r, d_r) of the fibration above. Since $E_2^{p,q} \cong H^p(\Pi^m\Omega\mathbf{B}) \otimes H^q(\Pi^l\Omega^2\mathbf{B})$, $\mathbf{B}$ is 3-connected and $H^4(\mathbf{B}) \cong \mathbb{Z}$, the non-trivial differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ $(p+q \leq 4)$ occurs only when $r = 3$ and $(p,q) = (0,2)$. Then we obtain $H^1(\mathrm{Map}_*(M^2, \mathbf{B})) = 0$. We determine $d_3 : E_3^{0,2} \to E_3^{3,0}$ to compute $H^i(\mathrm{Map}_*(M^2, \mathbf{B}))$ $(i = 2, 3)$. Consider the commutative diagram

$$\begin{array}{ccccccc}
\bigvee^l S^1 & \xrightarrow{f} & \bigvee^m S^1 & \longrightarrow & M^2 & \longrightarrow & \bigvee^l S^2 \\
{\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle \Sigma f}\downarrow \\
\bigvee^m S^1 & \xrightarrow{1} & \bigvee^m S^1 & \longrightarrow & \bigvee^m D^2 & \longrightarrow & \bigvee^m S^2,
\end{array}$$

then we have the following commutative diagram.

$$\begin{array}{ccccc}
\Pi^m\Omega\mathbf{B} & \longleftarrow & M^2 & \longleftarrow & \Pi^l\Omega^2\mathbf{B} \\
{\scriptstyle 1}\uparrow & & \uparrow & & {\scriptstyle (\Sigma f)^\#}\uparrow \\
\Pi^m\Omega\mathbf{B} & \longleftarrow & P(\Pi^m\Omega\mathbf{B}) & \longleftarrow & \Pi^m\Omega^2\mathbf{B}
\end{array}$$

Compare the Leray-Serre spectral sequence of fibrations above, we obtain $d_3 = \tau(\Sigma f)^{\#*} : E_3^{0,2} \to E_3^{3,0}$, where $\tau : H^2(\Pi^m\Omega^2\mathbf{B}) \xrightarrow{\cong} H^3(\Pi^m\Omega\mathbf{B})$ is the transgression.

Let A be a matrix which represents $((\Sigma f)^\#)^* : H^2(\Pi^l\Omega^2\mathbf{B}) \to H^2(\Pi^m\Omega^2\mathbf{B})$. By Lemma 3.2, $(\Sigma f)_* : H_2(\bigvee^m S^2) \to H_2(\bigvee^l S^2)$ is represented by A and so is $f_* : H_1(\bigvee^m S^1) \to H_1(\bigvee^l S^1)$. Then we have the exact sequence

$$0 \to H_2(M^2) \to H_1(\bigvee^m S^1) \xrightarrow{A} H_1(\bigvee^l S^1) \to H_1(M^2) \to 0.$$

Since $H_i(M^2) \cong H_i(M)$ $(i \leq 2)$, we have

$$\begin{aligned}
H^2(\mathrm{Map}_*(M^2, \mathbf{B})) &\cong \mathrm{Ker}\{d_3 : E_3^{0,2} \to E_3^{3,0}\} \cong \mathrm{Ker}A \cong H_2(M), \\
H^3(\mathrm{Map}_*(M^2, \mathbf{B})) &\cong \mathrm{Coker}\{d_3 : E_3^{0,2} \to E_3^{3,0}\} \cong \mathrm{Coker}A \cong H_1(M).
\end{aligned}$$

□

Proof of Theorem 1.1. By Corollary 2.2, Proposition 3.1 and Proposition 3.3, Theorem 1.1 is proved. □

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