Unstable K^1-group $[\Sigma^{2n-2}\mathbb{C}P^2, \mathrm{U}(n)]$
and
homotopy type of certain gauge groups

by

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1 Introduction

Let G be a compact connected Lie group and let P be a principal G-bundle over a connected finite complex B. We denote by $\mathcal{G}(P)$, the group of G-equivariant self-maps of P covering the identity map on B. $\mathcal{G}(P)$ is called the gauge group of P.

For fixed B and G, M.C.Crabb and W.A.Sutherland show as P ranges over all principal G-bundles with base space B, the number of homotopy types of $\mathcal{G}(P)$ is finite ([2]). Principal $\mathrm{SU}(n)$-bundles over S^4 are classified by their second Chern class. Denote by $P_{n,k}$ the principal $\mathrm{SU}(n)$-bundle over S^4 with $c_2(P_{n,k}) = k$. The second author shows in [6] $\mathcal{G}(P_{2,k})$ is homotopy equivalent to $\mathcal{G}(P_{2,k'})$ if and only if $(12, k) = (12, k')$. Therefore when B is S^4 and $G = \mathrm{SU}(2)$, there are precisely six homotopy types of $\mathcal{G}(P)$.

The purpose of this paper is to show the following:

Theorem 1. $\mathcal{G}(P_{3,k})$ is homotopy equivalent to $\mathcal{G}(P_{3,k'})$ if and only if $(24,k)$ is equal to $(24,k')$.

Theorem 2. If $\mathcal{G}(P_{n,k})$ is homotopy equivalent to $\mathcal{G}(P_{n,k'})$, then $(n(n^2-1),k) = (n(n^2-1),k')$.

Remark 3. In [9] W.A.Sutherland shows if $\mathcal{G}(P_{n,k})$ is homotopy equivalent to $\mathcal{G}(P_{n,k'})$ then $(n(n^2-1)/(2,n+1),k)$ is equal to $(n(n^2-1)/(2,n+1),k')$. Therefore if n is even, then Theorem 2 is the result of Sutherland. If n is odd, Theorem 2 is an improvement of [9].

Denote by x_0 and y_0, the base points of S^4 and $\mathrm{BSU}(n)$. Denote by ϵ' a generator of $\pi_4(\mathrm{BSU}(n))$. Note that $k\epsilon'$ is the classifying map of $P_{n,k}$. Put

$$M_{n,k} = \{f : \mathrm{S}^4 \to \mathrm{BSU}(n) | f \simeq k\epsilon'\}$$
$$M^*_{n,k} = \{f \in M_{n,k} | f(x_0) = y_0\}.$$

By Atiyah-Bott [1] $M_{n,k} \simeq \mathrm{B}\mathcal{G}(P_{n,k})$. Note that $M^*_{n,k} \simeq M^*_{n,0}$.

Consider the fibre sequence

$$(*) \quad \mathcal{G}(P_{n,k}) \to \mathrm{SU}(n) \xrightarrow{h_{n,k}} M^*_{n,0} \to M_{n,k} \xrightarrow{e} \mathrm{BSU}(n).$$

In $[\mathrm{SU}(n), M^*_{n,0}] \cong [\Sigma^4\mathrm{SU}(n), \mathrm{BSU(n)}] \cong [\Sigma^3\mathrm{SU}(n), \mathrm{SU}(n)]$, $h_{n,k}$ is equal to $\gamma\circ(k\epsilon\wedge 1_{\mathrm{SU}(n)})$ where γ is the commutator map and ϵ is a generator of $\pi_3(\mathrm{SU(n)})$ (see Lang [7]).

Define $\xi_k : [\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{SU}(n)] \to [\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{SU}(n)]$ by $\xi_k(\alpha) = \gamma\circ(k\epsilon\wedge\alpha)$ for $\alpha \in [\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{SU}(n)]$. In section 2, for odd n, we determine $[\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)]$ and show $\mathbf{Coker}\xi_k$ is finite and the order of $\mathbf{Coker}\xi_k$ is equal to the order of $\mathbf{Coker}\xi_{k'}$ if and only if $(n(n^2-1),k)$ is equal to $(n(n^2-1),k')$. Applying the functor $[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \;]$ to $(*)$, we get an exact sequence

$$\begin{tikzcd}
{[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{SU}(n)]} \arrow[r] \arrow[dr, "(h_{n,k})_*"'] & {[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, M^*_{n,0}]} \arrow[r] \arrow[d, "="] & {[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, M_{n,k}]} \arrow[r] & 0, \\
& {[\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{SU}(n)]} \arrow[ur] &
\end{tikzcd}$$

since $[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{BSU}(n)] = \mathrm{K}^1(\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2) = 0$.

Note that $(h_{n,k})_*$ is equal to ξ_k. If $\mathcal{G}(P_{n,k}) \simeq \mathcal{G}(P_{n,k'})$ then (as a set) $[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, M_{n,k}] \cong [\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, M_{n,k'}]$. Therefore we get Theorem 2.

Let $\hat{M}^*_{3,0}$ be the 2-connected cover of $M^*_{3,0}$. Since $\mathrm{SU}(3)$ is 2-connected, $[\mathrm{SU}(3), \hat{M}^*_{3,0}] \to [\mathrm{SU}(3), M^*_{3,0}]$ is bijective. Note that $\hat{M}^*_{3,0}$ is a loop space, $\pi_j(\hat{M}^*_{3,0})$ is finite for any j and $h_{3,k} = kh_{3,1}$. In section 3 we show $24h_{3,1} = 0$. If $(24,k) = (24,k')$, then we construct a homotopy equivalence

$$\left(\frac{k'}{k}\right)_{24} : \hat{M}^*_{n,0} \to \hat{M}^*_{n,0}$$

such that $\left(\frac{k'}{k}\right)_{24} \circ (kh_{3,1}) \simeq k'h_{3,1}$. Therefore $\Omega\hat{M}_{3,k} \simeq \Omega\hat{M}_{3,k'}$ if $(24,k) = (24,k')$. Then we show $\mathcal{G}(P_{3,k}) \simeq \mathrm{S}^1 \times \Omega\hat{M}_{3,k}$ and prove Theorem 1.

2 The group $[\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)]$

In this section we assume n is odd. Put $W_n = \mathrm{U}(\infty)/\mathrm{U}(n)$ and consider the fibre sequence

$$\Omega\mathrm{U}(\infty) \xrightarrow{\Omega\pi} \Omega W_n \to \mathrm{U}(n) \to \mathrm{U}(\infty) \xrightarrow{\pi} W_n.$$

Since $(W_n)^{(2n+3)} \simeq \mathrm{S}^{2n+1} \cup_\eta e^{2n+3}$, where η is the generator of $\pi_{2n+2}(\mathrm{S}^{2n+1})$. Denote a generator of $H^{2n+4}(K(\mathbb{Z}, 2n+1))$ by ϵ and consider the 2-stage Postnikov system

$$E \to K(\mathbb{Z}, 2n+1) \xrightarrow{\epsilon} K(\mathbb{Z}, 2n+4).$$

Let x be the generator of $H^{2n+1}(W_n) \cong \mathbb{Z}$. Since $H^{2n+4}(W_n) = 0$, x has a lift

$$\tilde{x} : W_n \to E,$$

such that $\tilde{x}$ is a $(2n+3)$-equivalence. Therefore if $\dim X \leq 2n+2$, then

$$\Omega\tilde{x}_* : [X, \Omega W_n] \to [X, \Omega E]$$

is an isomorphism. Define a homomorphism $\lambda : [X, \Omega W_n] \to H^{2n}(X) \oplus H^{2n+2}(X)$ by $\lambda(\alpha) = (\alpha^*(a_{2n}), \alpha^*(a_{2n+2}))$, where a_{2n} and a_{2n+2} are generators of $H^{2n}(\Omega W_n) \cong H^{2n+2}(\Omega W_n) \cong \mathbb{Z}$ and $\alpha \in [X, \Omega W_n]$. If $X = \Sigma^{2n-2}\mathbb{C}\mathrm{P}^2$, we have the following:

Lemma 2.1. If $X = \Sigma^{2n-2}\mathbb{C}\mathrm{P}^2$, $\lambda : [X, \Omega W_n] \to H^{2n}(X) \oplus H^{2n+2}(X)$ is monic and $\mathbf{Im}\lambda = \{(a,b) | a \equiv b \bmod 2\}$.

Lemma 2.2. If $X = \Sigma^{2n-2}\mathbb{C}\mathrm{P}^2$, $\mathbf{Im}\lambda \circ (\Omega\pi)_*$ is generated by $(n!, \frac{1}{2}(n+1)!)$ and $(0, (n+1)!)$.

Proof. First recall that $(\Omega\pi)^*(a_{2n}) = n!ch_n$ and $(\Omega\pi)^*(a_{2n+2}) = (n+1)!ch_{n+1}$. Denote by ξ_n a generator of $\tilde{K}(\mathrm{S}^{2n})$. $K(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}[x]/(x^3)$, $H^*(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}[t]/(t^3)$ and $chx = t + \frac{t^2}{2}$. Therefore

$$ch(\xi_{n-1}\hat{\otimes}x) = \sigma^{2n-2}t + (\sigma^{2n-2}t^2/2), \quad ch(\xi_{n-1}\hat{\otimes}x^2) = \sigma^{2n-2}t^2.$$

Note that $[X, \mathrm{U}(\infty)] = \tilde{K}(X)$. Then

$$\lambda \circ (\Omega\pi)_*(\xi_{n-1}\hat{\otimes}x) = (n!, \frac{1}{2}(n+1)!)$$
$$\lambda \circ (\Omega\pi)_*(\xi_{n-1}\hat{\otimes}x^2) = (0, (n+1)!).$$

□

Since $\tilde{K}^1(\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2) = 0$, we get

Lemma 2.3. $[\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)] \cong \mathbf{Coker}(\Omega\pi)_*$.

Put $u = (1,1), v = (0,2)$ and $l = \frac{1}{2}n!$. Then

$$(n!, \frac{1}{2}(n+1)!) = (2l, (n+1)l) = 2lu + \binom{n-1}{2} lv$$
$$(0, (n+1)!) = (0, 2(n+1)l) = (n+1)lv.$$

If $n \equiv 3 \bmod 4$,

$$2l(u + \frac{n-3}{4}v) = 2lu + \frac{n-3}{2}lv \equiv -lv \bmod \mathbf{Im}(\Omega\pi)_*$$

and therefore

$$2l(n+1)(u + \frac{n-3}{4}v) = -l(n+1)v \equiv 0 \bmod \mathbf{Im}(\Omega\pi)_*.$$

Note that

$$l(2n + \frac{n-1}{2}v) \equiv 2lu + \binom{n-1}{2} lv \equiv 0 \bmod \mathbf{Im}(\Omega\pi)_*.$$

Since

$$\begin{vmatrix} 1 & \frac{n-3}{4} \\ 2 & \frac{n-1}{2} \end{vmatrix} = \frac{n-1}{2} - \frac{n-3}{2} = 1,$$

$u' = (u + \frac{n-3}{4}v)$ and $v' = (2u + \frac{n-1}{2}v)$ are generators of $\mathbf{Im}\lambda$. Note that the order of $\mathbf{Coker}(\Omega\pi)_* = 2(n+1)l^2$ and we have $\mathbf{Coker}(\Omega\pi)_* = \mathbb{Z}/(n+1)! \oplus \mathbb{Z}/(\frac{1}{2}n!)$.

If $l \equiv 1 \bmod 4$

$$l(2u + \frac{n-3}{2}v) = 2lu + \binom{n-3}{2} lv \equiv lv \bmod \mathbf{Im}(\Omega\pi)_*$$

and therefore

$$(n+1)l(2u + \frac{n-3}{2}v) \equiv (n+1)lv \equiv 0 \bmod \mathbf{Im}(\Omega\pi)_*.$$

Note that

$$2l(u + \frac{n-1}{4}v) \equiv 0 \bmod \mathbf{Im}(\Omega\pi)_*.$$

Since

$$\begin{vmatrix} 2 & \frac{n-3}{2} \\ 1 & \frac{n-1}{4} \end{vmatrix} = \frac{n-1}{2} - \frac{n-3}{2} = 1,$$

$u' = 2u + \frac{n-3}{2}v$ and $v' = u + \frac{n-1}{4}v$ are generators of $\mathbf{Im}\lambda$. Note that the order of $\mathbf{Coker}(\Omega\pi)_* = 2(n+1)l^2$ and we have $\mathbf{Coker}(\Omega\pi)_* = \mathbb{Z}/\left(\frac{1}{2}(n+1)!\right) \oplus \mathbb{Z}/(n!)$.

Theorem 2.4. $[\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)] = \begin{cases} \mathbb{Z}/(n+1)! \oplus \mathbb{Z}/(\frac{1}{2}n!) & \text{if } n \equiv 3 \bmod 4 \\ \mathbb{Z}/\left(\frac{1}{2}(n+1)!\right) \oplus \mathbb{Z}/n! & \text{if } n \equiv 1 \bmod 4. \end{cases}$

Consider the Samelson product

$$\xi_k : [\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)] \to [\Sigma^{2n-2}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)]$$

given by $\xi_k(\alpha) = \gamma \circ (k\epsilon \wedge \alpha)$. Note that $[\Sigma^{2n-5}\mathbb{C}\mathrm{P}^2, \mathrm{U}(n)] \cong \tilde{K}(\Sigma^{2n-4}\mathbb{C}\mathrm{P}^2)$. The basis of $\tilde{K}(\Sigma^{2n-4}\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ is $\xi_{n-2}\hat{\otimes}x, \xi_{n-2}\hat{\otimes}x^2$. Put $L = (n-2)!$.

$$c_{n-1}(\xi_{n-2}\hat{\otimes}x) = L\sigma^{2n-4}t, \qquad c_n(\xi_{n-2}\hat{\otimes}x) = \frac{1}{2}(n-1)L\sigma^{2n-4}t^2$$
$$c_{n-1}(\xi_{n-2}\hat{\otimes}x^2) = 0, \qquad c_n(\xi_{n-2}\hat{\otimes}x^2) = (n-1)L\sigma^{2n-4}t^2$$

Let α_j be the adjoint of $\xi_{n-2}\hat{\otimes}x^j$ $(j = 1, 2)$. For $\alpha : \Sigma^{2n-5}\mathbb{C}\mathrm{P}^2 \to \mathrm{U}(n)$, put $\tilde{\alpha} = \tilde{\gamma} \circ (\epsilon \wedge \alpha)$, where $\tilde{\gamma}$ is the lift of γ constructed in [4].

Note that

$$H^*(\mathrm{U}(n)) = \bigwedge (\sigma(c_1), \cdots, \sigma(c_n)).$$

and

$$\tilde{\gamma}^*(a_{2n}) = \sum_{i+j=n-1} \sigma(c_i) \otimes \sigma(c_j).$$

Using the method in [4], we have

$$\tilde{\gamma}^*(a_{2n+2}) = \sum_{i+j=n} \sigma(c_i) \otimes \sigma(c_j).$$

Therefore $\lambda(\tilde{\alpha}_1) = (L, \frac{1}{2}(n-1)L)$ and $\lambda(\tilde{\alpha}_2) = (0, (n-1)L)$. If $n \equiv 3 \bmod 4$ then $v = v' - 2u'$. Put $\alpha_2' = \alpha_2 + (n-1)\alpha_1$ then $\tilde{\alpha}_1 = Lu'$ and $\tilde{\alpha}_2' = \frac{1}{2}(n-1)Lv'$. If $n \equiv 1 \bmod 4$ then $v = 2v' - u'$. Put $\alpha_2' = \alpha_2 + (n-1)\alpha_1$ then $\tilde{\alpha}_1 = \frac{1}{2}Lu'$ and $\tilde{\alpha}_2' = (n-1)Lv'$. Therefore we have

Lemma 2.5. The order of $\mathbf{Coker}\xi_k$ is equal to

$$\frac{1}{2}(n-2)!(n(n^2-1), k) \cdot (n-1)!(n, k)$$

Corollary 2.6. The order of $\mathbf{Coker}\xi_k$ is equal to the order of $\mathbf{Coker}\xi_{k'}$ if and only if $(n(n^2-1), k) = (n(n^2-1), k')$.

3 Proof of Theorem 1

Consider the cofibering

$$\mathrm{S}^{11} \xrightarrow{k} \Sigma^5\mathbb{C}\mathrm{P}^2 \to \Sigma^4\mathrm{SU}(3).$$

Since $\Sigma^5\mathbb{C}\mathrm{P}^2$ is 6-connected, the suspension map

$$\Sigma^\infty : [\mathrm{S}^{11}, \Sigma^5\mathbb{C}\mathrm{P}^2] \to \left\{\mathrm{S}^{11}, \Sigma^5\mathbb{C}\mathrm{P}^2\right\}$$

is an isomorphism (see [10]). Note that $\Sigma^\infty(k) = 0$ (see [5]). Therefore $k = 0, \Sigma^4\mathrm{SU}(3) \simeq \Sigma^5\mathbb{C}\mathrm{P}^2 \vee \mathrm{S}^{12}$ and

$$\begin{aligned}
[\Sigma^4\mathrm{SU}(3), \mathrm{BSU}(3)] &\cong [\Sigma^5\mathbb{C}\mathrm{P}^2, \mathrm{BSU}(3)] \oplus \pi_{12}(\mathrm{BSU}(3)) && (1)\\
&\cong [\Sigma^4\mathbb{C}\mathrm{P}^2, \mathrm{SU}(3)] \oplus \pi_{11}(\mathrm{SU}(3)) && (2)\\
&\cong \mathbb{Z}/24 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4, && (3)
\end{aligned}$$

where $\pi_{11}(\mathrm{SU}(3)) \cong \mathbb{Z}/4$ (see [8]). Therefore we have:

Lemma 3.1. $24(\gamma \circ (\epsilon \wedge 1_{\mathrm{SU}(3)})) = 0$.

Let X be a connected loop space, $*$ its base point, $\mu : X \times X \to X$ its loop multiplication and $\iota : X \to X$ its homotopy inverse. For an integer n define self map $n : X \to X$ as follows: $0 = *, 1 = 1_X, n = \mu \circ ((n-1) \times 1_X) \circ \Delta$ for a positive integer n. If $n < 0$ then $n = \iota \circ (-n)$. Let Y be a finite complex and $\alpha : Y \to X$. In the group $[Y, X]$, $n\alpha$ is represented by $n \circ \alpha$. For a prime p, consider the localization $l_{(p)} : X \to X_{(p)}$. For a map $f : X \to X$, there exists a map $f_{(p)} : X_{(p)} \to X_{(p)}$ satisfying $f_{(p)} \circ l_{(p)} \simeq l_{(p)} \circ f$. $f_{(p)}$ is unique up to homotopy. $X_{(p)}$ is a loop space and $l_{(p)}$ is a loop map. In $X_{(p)}$, $n_{(p)} \simeq n$. If $(n,p) = 1$, $n : X_{(p)} \to X_{(p)}$ is a homotopy equivalence. For $\left(\frac{a}{b}\right) \in \mathbb{Z}_{(p)}$ $((a,b) = 1, (b,p) = 1))$, $b : X_{(p)} \to X_{(p)}$ is a homotopy equivalence, and define $\left(\frac{a}{b}\right) = a \circ b^{-1}$. If $(a,p) = 1$, $\left(\frac{a}{b}\right)$ is a homotopy equivalence.

Lemma 3.2. Let k, k' and d be non zero integers satisfying $(k,d) = (k',d)$. If $\pi_j(X)$ is finite for any j and $d\alpha = 0$, then there exists a homotopy equivalence

$$\left(\frac{k'}{k}\right)_d : X \to X$$

satisfying $k' \circ \alpha \simeq \left(\frac{k'}{k}\right)_d \circ k \circ \alpha$.

Proof. For an integer $n \neq 0$ and $n = p^r q^s \cdots$ is the factorization into prime powers, we define $\nu_p(n) = r$. Define $h_p : X_{(p)} \to X_{(p)}$ by

$$h_p = \begin{cases} \left(\frac{k'}{k}\right) & \text{if } \nu_p(d) > \nu_p(k) \\ 1_{X_{(p)}} & \text{if } \nu_p(d) \leq \nu_p(k) \end{cases}$$

and $h = \prod_p h_p$. Since $\nu_p((k,d)) = \min(\nu_p(k), \nu_p(d))$, if $\nu_p(d) > \nu_p(k)$, then $\nu_p(d) > \nu_p(k')$ and $\nu_p(k') = \nu_p(k)$. The order of $l_{(p)} \circ k \circ \alpha$ is a power of p. If $\nu_p(d) \leq \nu_p(k)$ then $\nu_p(d) \leq \nu_p(k')$ and therefore $l_{(p)} \circ k \circ \alpha \simeq l_{(p)} \circ k' \circ \alpha \simeq 0$. Consider $l = (\prod l_{(p)}) \circ \Delta : X \to \Pi X_{(p)}$. l is a homotopy equivalence. Put $\left(\frac{k'}{k}\right)_d = l^{-1} \circ h \circ l$. We need only show $h \circ l \circ k \circ \alpha \simeq l \circ k' \circ \alpha$ or $h_{(p)} \circ l_{(p)} \circ k \circ \alpha \simeq l_{(p)} \circ k' \circ \alpha$.

If $\nu_p(d) > \nu_p(k)$

$$h_{(p)} \circ l_{(p)} \circ k \circ \alpha \simeq h_{(p)} \circ k \circ l_{(p)} \circ \alpha \simeq k' \circ k^{-1} \circ k \circ l_{(p)} \circ \alpha \simeq k' \circ l_{(p)} \circ \alpha \simeq l_{(p)} \circ k' \circ \alpha.$$

If $\nu_p(d) \leq \nu_p(k)$, $h_{(p)} = 1_X(p)$ and we have

$$h_{(p)} \circ l_{(p)} \circ k \circ \alpha \simeq h_{(p)} \circ l_{(p)} \circ k' \circ \alpha \simeq 0.$$

Therefore $\left(\frac{k'}{k}\right)_d \circ k \circ \alpha \simeq k' \circ \alpha$. $\left(\frac{k'}{k}\right)_d$ is clearly a homotopy equivalence. $\Box$

Now we can prove Theorem 1. Since BSU(3) is 3-connected $M^*_{3,k} \simeq M^*_{3,0} \to M_{3,k}$ is 2-equivalence. Note that $\pi_1(M^*_{3,0}) \cong \pi_5(\mathrm{BSU}(3)) = 0$ and $\pi_2(M^*_{3,0}) \cong \pi_6(\mathrm{BSU}(3)) = \mathbb{Z}$. Denote the generators of $H^2(M_{3,k}) \cong H^2(M^*_{3,0}) \cong \mathbb{Z}$ by b_2 and b'_2 respectively. The homotopy fibre of b_2 and b'_2 are denoted by $\hat{M}_{3,k}$ and $\hat{M}^*_{3,0}$ respectively. Therefore $\Omega\hat{M}_{3,k}$ is the universal covering of $\Omega M_{3,k} \simeq \mathcal{G}_{3,k}$. Since $\pi_1(\mathcal{G}_{3,k}) \cong \mathbb{Z}$, $\mathcal{G}_{3,k} \simeq \mathrm{S}^1 \times \Omega\hat{M}_{3,k}$. Note that $p_* : [\mathrm{SU}(3), \hat{M}^*_{3,0}] \to [\mathrm{SU}(3), M^*_{3,0}]$ is a bijection where $p : \hat{M}^*_{3,0} \to M^*_{3,0}$ is the projection. Therefore there exists $\hat{h}_{3,k} : \mathrm{SU}(3) \to \hat{M}^*_{3,0}$ such that $p \circ \hat{h}_{3,k} \simeq h_{3,k}$ and $\Omega\hat{M}_{3,k}$ is the homotopy fibre of $\hat{h}_{3,k}$. $\hat{h}_{3,k}$ is unique up to homotopy and $\hat{h}_{3,k} \simeq k\hat{h}_{3,1}$. Since $24h_{3,1} = 0$, $24\hat{h}_{3,1} = 0$. $\hat{M}^*_{3,0}$ is a loop space and $\pi_j(\hat{M}^*_{3,0})$ is finite for any j. Now by Lemma 3.2 we get if $(24, k) = (24, k')$ then $\Omega\hat{M}_{3,k} \simeq \Omega\hat{M}_{3,k'}$, and Theorem 1 is proven.

References

[1] M.F.Atiyah and R.Bott, The Yang-Mills equations over Riemann surfaces, Phils. Trans. Ray. Soc. London Ser A., 308 (1982), 523-615.

[2] M.C.Crabb and W.A.Sutherland, Counting homotopy types of gauge groups, Proc. London. Math. Soc., (3)81(2000), 747-768.

[3] H.Hamanaka, On $[X, \mathrm{U}(n)]$ when $\dim X = 2n + 1$, to appear in J. Math. Kyoto Univ.

[4] H.Hamanaka and A.Kono, On $[X, \mathrm{U}(n)]$ when $\dim X = 2n$, J. Math. Kyoto Univ., 42(2003).

[5] R.Kane, The homotopy of Hopf spaces, North-Holland Math. Library 40.

[6] A.Kono, A note on the homotopy type of certain gauge groups, Proc. Royal Soc. Edinburgh, 117A(1991), 295-297.

[7] G.E.Lang, The evaluation map and EHP sequence, Pacific J. Math., 44(1973), 201-210.

[8] H.Matsunaga, The homotopy groups $\pi_{2n+i}(\mathrm{U}(n))$ for $i = 3, 4$ and 5. Mem. Fac. Sci. Kyushu Univ., 15(1961), 72-81.

[9] W.A.Sutherland, Function spaces related to gauge groups, Proc. Roy. Soc. Edinburgh, 121A(1992), 185-190.

[10] H.Toda, A survey of homotopy theory, Adv. in Math. 10(1973), 417-455.