# L-S category of quasi-projective spaces

Daisuke KISHIMOTO and Akira KONO

## 1 Introduction

Let $\mathbf{k}$ be the real or the complex number or the quaternion, according as $d = 1, 2$ or 4 and $G(n, \mathbf{k}) = O(n), U(n)$ or $Sp(n)$. In [5] the quasi-projective space $Q_n$ is defined as

$$Q_n = S^{dn-1} \times S^{d-1} / \sim,$$

where $\sim$ is the equivalence relations

$$(x, \lambda) \sim (x\nu, \nu^{-1}\lambda\nu) \text{ for } \nu \in S^{d-1} \text{ and } (x, 1) \sim (y, 1).$$

Then there exists the natural map $Q_n \to G(n, \mathbf{k})$ and the cell decomposition of $G(n, \mathbf{k})$ is based on this map ([10]). It is easily seen that $Q_n = \mathbf{R}P^{n-1} \cup *$ for $\mathbf{k} = \mathbf{R}$ and $Q_n = \Sigma(\mathbf{C}P^{n-1} \cup *)$ for $\mathbf{k} = \mathbf{C}$. In the case that $\mathbf{k} = \mathbf{H}$, it was first announced to be the three fold suspension of $\mathbf{H}P^{n-1} \cup *$, but it was withdrawn ([11]).

The purpose of this paper is to investigate the L-S category of the stunted quasi-projective space

$$Q_{n,m} = Q_n / Q_m$$

for $\mathbf{k} = \mathbf{H}$. The L-S category is considered as *normalized*, which, denoted by $\operatorname{cat} X$, is the least number such that the diagonal map $\Delta : X \to X^{n+1}$ is compressed into the fat wedge $X^{[n+1]}$. Then it is easily seen that $\operatorname{cat} \Sigma X = 1$. Let $\operatorname{cup} X$ denote the cup-length of $X$ which is the greatest number such that there exists a multiplicative reduced cohomology theory $\tilde{h}^*$ and the cohomology classes $x_i \in \tilde{h}^*(X)$ satisfying $x_1 x_2 \cdots x_n \neq 0$. It is well known that

$$\operatorname{cup} X \leq \operatorname{cat} X.$$

Then we consider $\operatorname{cup} Q_{n,m}$ to investigate $\operatorname{cat} Q_{n,m}$. Since $Q_k - Q_{k-1} = e^{4k-1}$, the cell decomposition of $Q_n$ is

$$Q_n \simeq S^3 \cup e^7 \cup \ldots \cup e^{4n-1}.$$

Then $\tilde{H}^*(Q_{n,m}; \mathbf{Z})$ is concentrated in the odd dimensions and the product of any elements in $\tilde{H}^*(Q_{n,m}; \mathbf{Z})$ vanishes. Then it is natural to expect $\operatorname{cup} Q_{n,m} = 0$. But it is shown in section 5 that the connective $KO$-theory of $Q_{n,m}$ has non-trivial products for suitable $n, m$. This implies that $Q_{n,m}$ is not even the co-H-space, still the suspension type for suitable $n, m$.

## 2 Atiyah-Hirzebruch spectral sequence

Let $h^*$ be a cohomology theory and $X$ be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $h^*(X)$ is the spectral sequence with

$$E_1^{p,q} \cong H^*(X; h^*(\text{point}))$$

converging to $h^*(X)$. In [2] the differential of the second term of the Atiyah-Hirzebruch spectral sequence for $KO$-theory is given by

$$d_1^{p,q} = \begin{cases} Sq^2\pi_2 & q \equiv 0 \ (8) \\ Sq^2 & q \equiv -1 \ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_2$ is the modulo 2 reduction. Then we have that the differential of the second term of the Atiyah-Hirzebruch spectral sequence for $ko$-theory is

$$d_1^{p,q} = \begin{cases} Sq^2\pi_2 & q \leq 0, \ q \equiv 0 \ (8) \\ Sq^2 & q \leq 0, \ q \equiv -1 \ (8) \\ 0 & \text{otherwise,} \end{cases} \tag{$*$}$$

where $ko$ denotes the connective $KO$-theory.

We denote $ko^*(\text{point})$ by $ko^*$. Recall that

$$ko^* \cong \mathbf{Z}[\eta, x, \beta]/(2\eta, \eta^3, \eta x, x^2 - 4\beta),$$

where $|\eta| = -1, |x| = -4, |\beta| = -8$. The next lemma gives tools to compute the Atiyah-Hirzebruch spectral sequence of $ko^*(X)$ for a special $X$ and is due to [3].

**Lemma 2.1.** *Let $X$ be a finite CW-complex such that $\tilde{H}^*(X; \mathbf{Z})$ is free and concentrated in even or odd dimension, and $d_r : E_r \to E_r$ be the first non-trivial differential of the Atiyah-Hirzebruch spectral sequence for $ko^*(X)$. Then we have :*

1. $r \equiv 2 \ (8)$.
2. *There exists $x \in E_r^{p,0}$ such that $\eta x \neq 0$ and $\eta d_r x \neq 0$.*

## 3 $ko$-groups of symplectic Stiefel manifolds

In this section we determine the $ko^*$-group of the symplectic Stiefel manifold

$$V_{n,m} = Sp(n)/Sp(m)$$

by making use of the Atiyah-Hirzebruch spectral sequence.

Let $G_{k,l}$ be the symplectic Grassmaniann $Sp(k)/Sp(l) \times Sp(k-l)$ and $X^E$ be the Thom complex of a vector bundle $E \to X$. The stable splitting in [8]

$$V_{n,m} \underset{s}{\simeq} \bigvee_{q=1}^{n-m} G_{n-m,q}^{E_q}$$

implies that the Atiyah-Hirzebruch spectral sequence of $ko^*(V_{n,m})$ splits into those of $ko^*(G_{n-m,q}^{E_q})$. Then we consider the Atiyah-Hirzebruch spectral sequence of $ko^*(G_{k,l}^E)$ for a vector bundle $E \to G_{k,l}$ to obtain $ko^*(V_{n,m})$.

It is well known that

$$H^*(G_{k,l}); \mathbf{Z} \cong \mathbf{Z}[x_1, \ldots, x_l, y_1, \ldots, y_{k-l}]/(z_1, \ldots, z_k),$$

where $z_i = \sum_j x_i y_{i-j}$ and $|x_i| = |y_i| = 4i$. Let $E \to G_{k,l}$ be a vector bundle. By the Thom isomorphism we have

$$H^*(G_{k,l}^E; \mathbf{Z}) \cong \phi_E H^*(G_{k,l}; \mathbf{Z}),$$

where $\phi_E$ is the Thom class of $E$. Then we have

$$|a| - |b| \equiv 0 \ \ (4)$$

for any $a, b \in H^*(G_{k,l}^E; \mathbf{Z})$. Therefore we have the following by Lemma 2.1 and $(*)$.

**Proposition 3.1.** *The Atiyah-Hirzebruch spectral sequence of $ko^*(G_{k,l}^E)$ collapses at the second term.*

**Corollary 3.1.** *The Atiyah-Hirzebruch spectral sequence of $ko^*(V_{n,m})$ collapses at the second term.*

Since

$$H^*(V_{n,m}; \mathbf{Z}) \cong \bigwedge\nolimits_{\mathbf{Z}}(e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}), \ \ |e_i| = i,$$

we have

$$\mathrm{Gr}\, ko^*(V_{n,m}) \cong \bigwedge\nolimits_{ko^*}(e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}), \ \ |e_i| = i,$$

where $\mathrm{Gr}\, ko^*(V_{n,m})$ denotes the associated graded $ko^*$-algebra of $ko^*(V_{n,m})$. Since $\mathrm{Gr}\, ko^*(V_{n,m})$ is a free $ko^*$-module, the extension of $\mathrm{Gr}\, ko^*(V_{n,m})$ to $ko^*(V_{n,m})$ as $ko^*$-modules is trivial. Let $R$ be a ring and $\Delta_R(x_1, \ldots, x_k)$ be the $R$-algebra whose $R$-module base is $\{x_{i_1} \cdots x_{i_l} | 0 \leq l \leq k, i_1 < \ldots < i_l\}$, then we have :

**Proposition 3.2.**

$$ko^*(V_{n,m}) \cong \Delta_{ko^*}(e_{4m+3}, w_{4m+7}, \ldots, e_{4n-1}), \ \ |e_i| = i.$$

*Remark* 3.1. We can set that $e_i \in ko^*(V_{n,m})$ and $e_i \in ko^*(Sp(n))$ has the correspondence by the homomorphism induced from the projection $Sp(n) \to V_{n,m}$.

*Remark* 3.2. The same argument as Proposition 3.2 holds for $KO^*(V_{n,m})$. Then the natural transformation $T : ko^*(V_{n,m}) \to KO^*(V_{n,m})$ is monic.

For the last of this section we see the co-algebra structure of $ko^*(Sp(n))/(\beta)$.

**Proposition 3.3.** *$ko^*(Sp(n))/(\beta)$ is primitively generated.*

*Proof.* Since $H^*(Sp(n); \mathbf{Z})$ is primitively generated, so is $\mathrm{Gr}\, ko^*(Sp(n))$. Then the degree argument completes the proof. □

# 4 Multiplicative structure of $ko^*(V_{n,m})$

Let $\mathbf{c} : KO^*(\,\cdot\,) \to K^*(\,\cdot\,)$ and $\mathbf{r} : K^*(\,\cdot\,) \to KO^*(\,\cdot\,)$ be the complexification and the realization map.

**Proposition 4.1.** *$x^2 \in \eta ko^*(V_{n,m})$ for any $x \in ko^*(V_{n,m})$.*

*Proof.* By [4] $K^*(Sp(n))$ is the exterior algebra. Then we have

$$\mathbf{c}(x^2) = 0 \text{ for any } x \in KO^*(Sp(n)).$$

Since $\mathbf{rc} = 2$, we have

$$2x^2 = 0 \text{ for any } x \in KO^*(Sp(n)).$$

By Remark 3.2 we have

$$2x^2 = 0 \text{ for any } x \in ko^*(Sp(n)).$$

By Proposition 3.2 $ko^*(Sp(n))$ is a free $ko^*$-algebra. Then the proof is completed by the argument in Remark 3.1. □

*Remark* 4.1. Theorem 5.4 of [9] yields that the similar argument of Proposition 3.2 and 4.1 holds for $KO$-theory and $m = 0$.

We investigate further multiplicative structure of $ko^*(V_{n,m})$ by making use of the projective plane of $\Omega V_{n,m}$. Then we consider $ko^*(\Omega V_{n,m})$.

In [6] it is shown that

$$H_*(\Omega Sp(n); \mathbf{Z}) \cong \mathbf{Z}[z_2, z_6, \ldots . z_{4n-2}], \quad |z_i| = i,$$
$$Sq^2_* z_{4i+2} = z_{4i},$$

where $z_{4i}$ is inductively defined as $z_{4i} = z_{2i}^2$. Then it is easily seen that

$$H_*(\Omega V_{n,m}; \mathbf{Z}) \cong \mathbf{Z}[z_{4m+2}, z_{4m+6}, \ldots . z_{4n-2}], \quad |z_i| = i,$$
$$Sq^2_* z_{4i+2} = z_{4i}.$$

Then we have

$$Sq^2(z_{4m+2}^2)^* = z_{8m+6}^*$$

for $(z_{4m+2}^2)^*, z_{8m+6}^* \in H^*(\Omega V_{n,m}; \mathbf{Z})$ and $n \geq 2m+2$, where $z^* \in H^*(\Omega V_{n,m}; \mathbf{Z})$ is the Kronecker dual of $z \in H_*(\Omega V_{n,m}; \mathbf{Z})$. Consider the Atiyah-Hirzebruch spectral sequence of $ko^*(\Omega V_{n,m})$, then we can assume that $z \in E_2^{*,0}$ for $z \in H^*(\Omega V_{n,m}; \mathbf{Z})$. Therefore we have the following by $(*)$.

**Proposition 4.2.** *$\eta(z_{4m+2}^2)^* = 0$ in $ko^*(\Omega V_{n,m})$ for $n \geq 2m+2$.*

Consider the cofibration

$$\Sigma(\Omega V_{n,m} \wedge \Omega V_{n,m}) \to \Sigma\Omega V_{n,m} \to P_2(\Omega V_{n,m}),$$

where $P_2(\Omega V_{n,m})$ is the projective plane of $\Omega V_{n,m}$. Then we have the long exact sequence

$$\to \tilde{ko}^{*-1}(\Omega V_{n,m}) \xrightarrow{\triangle} \tilde{ko}^{*-1}(\Omega V_{n,m} \wedge \Omega V_{n,m}) \xrightarrow{\lambda} ko^*(P_2(\Omega V_{n,m})) \xrightarrow{i} \tilde{ko}^*(\Omega V_{n,m}) \to,$$

where $\triangle(x) = \mu^*(x) - 1 \times x - x \times 1$ and $\mu$ is the loop multiplication of $\Omega V_{n,m}$. We see that $i$ and $\lambda$ satisfy

$$\text{if } i(a) = x, i(b) = y, \text{ then } \lambda(x \times y) = ab \ \ ([7]).$$

Consider the inclusion of the bottom cell of $V_{n,m}$

$$j : S^{4m+3} \hookrightarrow \Sigma\Omega V_{n,m} \hookrightarrow P_2(\Omega V_{n,m}) \xrightarrow{j'} B(\Omega V_{n,m}) \simeq V_{n,m}.$$

By Proposition 3.2 we see that $j^*(e_{4m+3})$ generates $ko^*(S^{4m+3})$. Then we can assume that

$$j'^*(e_{4m+3}) = i(z_{4m+2}^*).$$

If $\triangle(x) = \eta z_{4m+2} \times z_{4m+2}$, then we have

$$x = \eta(z_{4m+2}^2)^*.$$

Therefore we have the following by Proposition 4.2.

**Theorem 4.1.**

$$\eta e_{4m+3}^2 \neq 0 \text{ for } e_{4m+3} \in ko^*(V_{n,m}) \text{ and } n \geq 2m+2.$$

**Corollary 4.1.** *$e_{4m+3}^2 = \eta e_{8m+7}$ in $ko^*(V_{n,m})/(\beta)$ for $n \geq 2m+2$.*

*Proof.* It is easily seen that $\eta e_{4m+3}^2$ is primitive in $ko^*(Sp(n))/(\beta)$ by Proposition 3.3. Then we have

$$e_{4m+3}^2 = \eta e_{8m+7} \text{ in } ko^*(Sp(n))/(\beta)$$

by Theorem 4.1 and Proposition 3.3. Then the proof is completed by Remark 3.1. □

# 5 L-S category of quasi-projective spaces

In this section we consider the L-S category of $Q_{n,m}$ by applying the results in the previous sections.

By the cell decomposition of $Q_{n,m}$ and $V_{n,m}$ we have

$$H^*(Q_{n,m};\mathbf{Z}) \cong \mathbf{Z}[e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}]/(e_i e_j),$$

and

$$j^*(e_i) = e_i$$

for the natural map $j : Q_{n,m} \to V_{n,m}$ ([11]). Then the Atiyah-Hirzebruch spectral sequence of $ko^*(Q_{n,m})$ collapses at the second term by Lemma 2.1 and the degree reason. Therefore we have the following by Proposition 3.1 and 3.2.

**Proposition 5.1.**

$$ko^*(Q_{n,m}) \cong < e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1} >$$

*as $ko^*$-modules and*

$$j^*(e_i) = e_i.$$

**Lemma 5.1.**

$$\operatorname{cup} Q_{n,m} \geq \begin{cases} 2 & n \geq 2m+2 \\ 3 & n \geq 4m+4. \end{cases}$$

*Proof.* For $n \geq 2m+2$ we have $\eta e_{8m+7} \neq 0 \ \ (\beta)$ by Proposition 5.1. Then, by Corollary 4.1, we have

$$e_{4m+3}^2 = \eta e_{8m+7} \neq 0 \ \ (\beta)$$

and this proves the first case. For $n \geq 4m+4$ we have $\eta^2 e_{16m+15} \neq 0$ by Proposition 5.1. Consider the natural projection $Q_{n,m} \to Q_{n,2m}$, then we have $e_{8m+7}^2 = \eta e_{16m+15}$ in $ko^*(Q_{n,m})/(\beta)$ by Corollary 4.1. Therefore we have

$$e_{4m+3}^2 e_{8m+7} = \eta e_{8m+7}^2 = \eta^2 e_{16m+15} \neq 0 \ \ (\beta)$$

and this proves the second case. $\square$

Since $\operatorname{cup} X \leq \operatorname{cat} X$, it is also shown the following in the proof of Lemma 5.1 ([10]).

**Corollary 5.1.**

$$\operatorname{cat} Sp(n) \geq \begin{cases} n+1, & n = 2, 3, \\ n+2, & n \leq 4. \end{cases}$$

Since $\operatorname{cup} X \leq \operatorname{cat} X$, we obtain the lower bounds for $\operatorname{cat} Q_{n,m}$ by Lemma 5.1. We obtain the upper bounds by the next lemma ([1]).

**Lemma 5.2.** *Let $X$ be a $(n-1)$-connected CW-complex. Then we have*

$$\operatorname{cat} X \leq \frac{\dim X}{n}.$$

*Proof.* We can assume that the $(n-1)$-skeleton of $X$ is a point. Then the $kn$-skeleta of $X^{k+1}$ and $X^{[k+1]}$ are the same. For $\dim X \leq kn$ the diagonal map $\Delta : X \to X^{k+1}$ can be compressed into $X^{[k+1]}$ by the cellular approximation theorem. Therefore the proof is completed. $\square$

Finally we obtain $\operatorname{cat} Q_{n,m}$, for the several cases by Lemma 5.1 and Lemma 5.2.

**Theorem 5.1.**

$$\operatorname{cat} Q_{n,m} = \begin{cases} 1 & m+1 \leq n \leq 2m+1 \\ 2 & 2m+2 \leq n \leq 3m+2 \end{cases}$$

*and*

$$\operatorname{cat} Q_{n,m} \geq 3 \text{ for } n \geq 4m+4.$$

**Corollary 5.2.** *$Q_n$ is not the suspension type for $n \geq 2$.*

# References


[1] O. Cornea, G. Lupton, J. Oprea, D. Tanré, Lusternik-Schnirelmann category, Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003.

[2] M. Fujii, $KO$-groups of projective spaces, Osaka J. Math., 4 (1967), 141-49.

[3] S. Hara and A. Kono, $KO$-theory of complex Grassmannians, J. Math. Kyoto Univ., 31 (1991), 827-33.

[4] L. H. Hodgkin, On the $K$-theory of Lie groups, Topology 6 (1966). 1-36.

[5] I. M. James, Spaces associated with Stiefel manifolds, Proc. London Math. Soc. (3), 9 (1959), 115-40.

[6] A. Kono and K. Kozima, The space of loops on a symplectic group, Japan J. Math. 4 (1978), no. 2, 461-486.

[7] J.P. Lin, H-spaces with finiteness conditions, Handbook of Algebraic Topology, North-Holland, 1095-141, Amsterdam, 1995.

[8] H. Miller, Stable splitting of Stiefel manifolds, Topology, 24 (1985), 411-19.

[9] R. M. Seymour, The real $K$-theory of Lie groups and Homogeneous spaces, Quart. J. Math. Oxford Ser. (2), 24 (1973), 7-30.

[10] W. Singhof, On the Lusternik-Schnirelmann category of Lie groups: I, II, Math. Z. 145 (1975), 111-16; 51 (1976), 143-48.

[11] I. Yokota, On the cells of symplectic groups, Proc. Japan Acad., 32 (1956), 399-400.