L-S category of quasi-projective spaces

Daisuke KISHIMOTO and Akira KONO

1 Introduction

Let $k$ be the real or the complex number or the quaternion, according as $d = 1, 2$ or 4 and $G(n, k) = O(n), U(n)$ or $Sp(n)$. In [5] the quasi-projective space $Q_n$ is defined as

$$Q_n = S^{d n - 1} \times S^{d - 1} / \sim,$$

where $\sim$ is the equivalence relations

$$(x, \lambda) \sim (x \nu, \nu^{-1} \lambda \nu) \text{ for } \nu \in S^{d - 1} \text{ and } (x, 1) \sim (y, 1).$$

Then there exists the natural map $Q_n \to G(n, k)$ and the cell decomposition of $G(n, k)$ is based on this map ([10]). It is easily seen that $Q_n = \mathbb{R}P^{n-1} \cup \ast$ for $k = \mathbb{R}$ and $Q_n = \Sigma(\mathbb{C}P^{n-1} \cup \ast)$ for $k = \mathbb{C}$. In the case that $k = \mathbb{H}$, it was first announced to be the three fold suspension of $\mathbb{H}P^{n-1} \cup \ast$, but it was withdrawn ([11]).

The purpose of this paper is to investigate the L-S category of the stunted quasi-projective space $Q_{n,m} = Q_n/Q_m$ for $k = \mathbb{H}$. The L-S category is considered as normalized, which, denoted by $\text{cat} X$, is the least number such that the diagonal map $\Delta : X \to X^{n+1}$ is compressed into the fat wedge $X^{[n+1]}$. Then it is easily seen that $\text{cat} \Sigma X = 1$.

Let $\text{cup} X$ denote the cup-length of $X$ which is the greatest number such that there exists a multiplicative reduced cohomology theory $\tilde{h}^\ast$ and the cohomology classes $x_i \in \tilde{h}^\ast(X)$ satisfying $x_1x_2\cdots x_n \neq 0$. It is well known that

$$\text{cup} X \leq \text{cat} X.$$

Then we consider $\text{cup} Q_{n,m}$ to investigate $\text{cat} Q_{n,m}$. Since $Q_n = Q_{k-1} = e^{4k-1}$, the cell decomposition of $Q_n$ is

$$Q_n \simeq S^3 \cup e^7 \cup \ldots \cup e^{4n-1}.$$
2 Atiyah-Hirzebruch spectral sequence

Let $h^*$ be a cohomology theory and $X$ be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $h^*(X)$ is the spectral sequence with

$$E_1^{p,q} \cong H^*(X; h^*(\text{point}))$$

converging to $h^*(X)$. In [2] the differential of the second term of the Atiyah-Hirzebruch spectral sequence for $KO$-theory is given by

$$d_1^{p,q} = \begin{cases} Sq^2 \pi_2 & q \equiv 0 \ (8) \\ Sq^2 & q \equiv -1 \ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_2$ is the modulo 2 reduction. Then we have that the differential of the second term of the Atiyah-Hirzebruch spectral sequence for $ko$-theory is

$$d_1^{p,q} = \begin{cases} Sq^2 \pi_2 & q \leq 0, \ q \equiv 0 \ (8) \\ Sq^2 & q \leq 0, \ q \equiv -1 \ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where $ko$ denotes the connective $KO$-theory.

We denote $ko^*(\text{point})$ by $ko^*$. Recall that

$$ko^* \cong \mathbb{Z}[\eta, x, \beta]/(2\eta, \eta^3, \eta x, x^2 - 4\beta),$$

where $|\eta| = -1, |x| = -4, |\beta| = -8$. The next lemma gives tools to compute the Atiyah-Hirzebruch spectral sequence of $ko^*(X)$ for a special $X$ and is due to [3].

Lemma 2.1. Let $X$ be a finite CW-complex such that $\tilde{H}^*(X; \mathbb{Z})$ is free and concentrated in even or odd dimension, and $d_r : E_r \to E_r$ be the first non-trivial differential of the Atiyah-Hirzebruch spectral sequence for $ko^*(X)$. Then we have:

1. $r \equiv 2 \ (8)$.

2. There exists $x \in E_r^{p,0}$ such that $\eta x \neq 0$ and $\eta d_r x \neq 0$.

3 $ko$-groups of symplectic Stiefel manifolds

In this section we determine the $ko^*$-group of the symplectic Stiefel manifold

$$V_{n,m} = Sp(n)/Sp(m)$$

by making use of the Atiyah-Hirzebruch spectral sequence.
Let $G_{k,l}$ be the symplectic Grassmannian $Sp(k)/Sp(l) \times Sp(k-l)$ and $X^E$ be the Thom complex of a vector bundle $E \to X$. The stable splitting in [8]

$$V_{n,m} \cong \bigvee_{q=1}^{m-n} G_{n-m,q}^E$$

implies that the Atiyah-Hirzebruch spectral sequence of $ko^*(V_{n,m})$ splits into those of $ko^*(G_{n-m,q}^E)$. Then we consider the Atiyah-Hirzebruch spectral sequence of $ko^*(G_{k,l}^E)$ for a vector bundle $E \to G_{k,l}$ to obtain $ko^*(V_{n,m})$.

It is well known that

$$H^*(G_{k,l}; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_l, y_1, \ldots, y_{k-l}]/(z_1, \ldots, z_k),$$

where $z_i = \sum_j x_i y_{i-j}$ and $|x_i| = |y_i| = 4i$. Let $E \to G_{k,l}$ be a vector bundle. By the Thom isomorphism we have

$$H^*(G_{k,l}^E; \mathbb{Z}) \cong \phi_E H^*(G_{k,l}; \mathbb{Z}),$$

where $\phi_E$ is the Thom class of $E$. Then we have

$$|a| - |b| \equiv 0 \quad (4)$$

for any $a, b \in H^*(G_{k,l}^E; \mathbb{Z})$. Therefore we have the following by Lemma 2.1 and $(*)$.

**Proposition 3.1.** The Atiyah-Hirzebruch spectral sequence of $ko^*(G_{k,l}^E)$ collapses at the second term.

**Corollary 3.1.** The Atiyah-Hirzebruch spectral sequence of $ko^*(V_{n,m})$ collapses at the second term.

Since

$$H^*(V_{n,m}; \mathbb{Z}) \cong \bigwedge \mathbb{Z}(e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}), \quad |e_i| = i,$$

we have

$$\text{Gr} ko^*(V_{n,m}) \cong \bigwedge ko^*(e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}), \quad |e_i| = i,$$

where $\text{Gr} ko^*(V_{n,m})$ denotes the associated graded $ko^*$-algebra of $ko^*(V_{n,m})$. Since $\text{Gr} ko^*(V_{n,m})$ is a free $ko^*$-module, the extension of $\text{Gr} ko^*(V_{n,m})$ to $ko^*(V_{n,m})$ as $ko^*$-modules is trivial. Let $R$ be a ring and $\Delta_R(x_1, \ldots, x_k)$ be the $R$-algebra whose $R$-module base is $\{x_{i_1}, \ldots, x_{i_l} | 0 \leq l \leq k, i_1 < \ldots < i_l\}$, then we have :

**Proposition 3.2.**

$$ko^*(V_{n,m}) \cong \Delta_{ko^*}(e_{4m+3}, w_{4m+7}, \ldots, e_{4n-1}), \quad |e_i| = i.$$
Remark 3.1. We can set that $e_i \in ko^*(V_{n,m})$ and $e_i \in ko^*(Sp(n))$ has the correspondence by the homomorphism induced from the projection $Sp(n) \to V_{n,m}$.

Remark 3.2. The same argument as Proposition 3.2 holds for $KO^*(V_{n,m})$. Then the natural transformation $T : ko^*(V_{n,m}) \to KO^*(V_{n,m})$ is monic.

For the last of this section we see the co-algebra structure of $ko^*(Sp(n))/\langle \beta \rangle$.

Proposition 3.3. $ko^*(Sp(n))/\langle \beta \rangle$ is primitively generated.

Proof. Since $H^*(Sp(n); \mathbb{Z})$ is primitively generated, so is $Gr^* ko^*(Sp(n))$. Then the degree argument completes the proof. \qed

4 Multiplcative structure of $ko^*(V_{n,m})$

Let $c : KO^*(\cdot) \to K^*(\cdot)$ and $r : K^*(\cdot) \to KO^*(\cdot)$ be the complexification and the realization map.

Proposition 4.1. $x^2 \in \eta ko^*(V_{n,m})$ for any $x \in ko^*(V_{n,m})$.

Proof. By [4] $K^*(Sp(n))$ is the exterior algebra. Then we have $$c(x^2) = 0 \text{ for any } x \in KO^*(Sp(n)).$$

Since $rc = 2$, we have

$$2x^2 = 0 \text{ for any } x \in KO^*(Sp(n)).$$

By Remark 3.2 we have

$$2x^2 = 0 \text{ for any } x \in ko^*(Sp(n)).$$

By Proposition 3.2 $ko^*(Sp(n))$ is a free $ko^*$-algebra. Then the proof is completed by the argument in Remark 3.1. \qed

Remark 4.1. Theorem 5.4 of [9] yields that the similar argument of Proposition 3.2 and 4.1 holds for $KO$-theory and $m = 0$.

We investigate further multiplicative structure of $ko^*(V_{n,m})$ by making use of the projective plane of $\Omega V_{n,m}$. Then we consider $ko^*(\Omega V_{n,m})$.

In [6] it is shown that

$$H_*(\Omega Sp(n); \mathbb{Z}) \cong \mathbb{Z}[z_2, z_6, \ldots, z_{4n-2}], \ |z_i| = i,$$

$$S q_i^2 z_{4i+2} = z_{4i},$$

where $z_{4i}$ is inductively defined as $z_{4i} = z_{2i}^2$. Then it is easily seen that

$$H_*(\Omega V_{n,m}; \mathbb{Z}) \cong \mathbb{Z}[z_{4m+2}, z_{4m+6}, \ldots, z_{4n-2}], \ |z_i| = i,$$

$$S q_i^2 z_{4i+2} = z_{4i}.$$
Then we have
\[ Sq^2(z_{4m+2})^* = z_{8m+6}^* \]
for \((z_{4m+2})^*, z_{8m+6}^* \in H^*(\Omega V_{n,m}; \mathbb{Z})\) and \(n \geq 2m + 2\), where \(z^* \in H^*(\Omega V_{n,m}; \mathbb{Z})\) is the Kronecker dual of \(z \in H_*(\Omega V_{n,m}; \mathbb{Z})\). Consider the Atiyah-Hirzebruch spectral sequence of \(ko^*(\Omega V_{n,m})\), then we can assume that \(z \in E_2^{s,t} \) for \(z \in H^*(\Omega V_{n,m}; \mathbb{Z})\). Therefore we have the following by (*).

**Proposition 4.2.** \(\eta(z_{4m+2})^* = 0\) in \(ko^*(\Omega V_{n,m})\) for \(n \geq 2m + 2\).

Consider the cofibration
\[
\Sigma(\Omega V_{n,m} \wedge \Omega V_{n,m}) \rightarrow \Sigma \Omega V_{n,m} \rightarrow P_2(\Omega V_{n,m}),
\]
where \(P_2(\Omega V_{n,m})\) is the projective plane of \(\Omega V_{n,m}\). Then we have the long exact sequence
\[
\rightarrow ko^{s-1}(\Omega V_{n,m}) \xrightarrow{\Delta} ko^{s-1}(\Omega V_{n,m} \wedge \Omega V_{n,m}) \xrightarrow{\lambda} ko^*(P_2(\Omega V_{n,m})) \xrightarrow{i} ko^*(\Omega V_{n,m}) \rightarrow,
\]
where \(\Delta(x) = \mu^*(x) - 1 \times x - x \times 1\) and \(\mu\) is the loop multiplication of \(\Omega V_{n,m}\).

We see that \(i\) and \(\lambda\) satisfy
\[
\text{if } i(a) = x, i(b) = y, \text{ then } \lambda(xy) = ab \quad ([7]).
\]

Consider the inclusion of the bottom cell of \(V_{n,m}\)
\[
j : S^{4m+3} \hookrightarrow \Sigma \Omega V_{n,m} \hookrightarrow P_2(\Omega V_{n,m}) \xrightarrow{\eta} B(\Omega V_{n,m}) \simeq V_{n,m}.
\]
By Proposition 3.2 we see that \(j^*(e_{4m+3})\) generates \(ko^*(S^{4m+3})\). Then we can assume that
\[
j^*(e_{4m+3}) = i(z_{4m+2}^*).
\]
If \(\Delta(x) = \eta z_{4m+2} \times z_{4m+2}\), then we have
\[
x = \eta(z_{4m+2}^*).
\]
Therefore we have the following by Proposition 4.2.

**Theorem 4.1.**
\[
\eta e_{4m+3}^2 \neq 0 \text{ for } e_{4m+3} \in ko^*(V_{n,m}) \text{ and } n \geq 2m + 2.
\]

**Corollary 4.1.** \(e_{4m+3}^2 = \eta e_{8m+7}\) in \(ko^*(V_{n,m})/(\beta)\) for \(n \geq 2m + 2\).

**Proof.** It is easily seen that \(\eta e_{4m+3}^2\) is primitive in \(ko^*(Sp(n))/(\beta)\) by Proposition 3.3. Then we have
\[
e_{4m+3}^2 = \eta e_{8m+7} \text{ in } ko^*(Sp(n))/(\beta)
\]
by Theorem 4.1 and Proposition 3.3. Then the proof is completed by Remark 3.1.
5 L-S category of quasi-projective spaces

In this section we consider the L-S category of \( Q_{n,m} \) by applying the results in the previous sections.

By the cell decomposition of \( Q_{n,m} \) and \( V_{n,m} \) we have

\[
H^*(Q_{n,m}; \mathbb{Z}) \cong \mathbb{Z}[e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1}] / (e_iei_j),
\]

and

\[
j^*(e_i) = e_i
\]

for the natural map \( j : Q_{n,m} \to V_{n,m} \) ([11]). Then the Atiyah-Hirzebruch spectral sequence of \( ko^*(Q_{n,m}) \) collapses at the second term by Lemma 2.1 and the degree reason. Therefore we have the following by Proposition 3.1 and 3.2.

**Proposition 5.1.**

\[
ko^*(Q_{n,m}) \cong < e_{4m+3}, e_{4m+7}, \ldots, e_{4n-1} >
\]
as \( ko^* \)-modules and

\[
j^*(e_i) = e_i.
\]

**Lemma 5.1.**

\[
cup_{Q_{n,m}} \geq \begin{cases} 
2 & n \geq 2m + 2 \\
3 & n \geq 4m + 4.
\end{cases}
\]

**Proof.** For \( n \geq 2m + 2 \) we have \( \eta e_{8m+7} \neq 0 \) (\( \beta \)) by Proposition 5.1. Then, by Corollary 4.1, we have

\[
e_{4m+3}^2 = \eta e_{8m+7} \neq 0 \ (\beta)
\]

and this proves the first case. For \( n \geq 4m + 4 \) we have \( \eta^2 e_{16m+15} \neq 0 \) by Proposition 5.1. Consider the natural projection \( Q_{n,m} \to Q_{n,2m} \), then we have

\[
e_{8m+7}^2 = \eta e_{16m+15} \in ko^*(Q_{n,m}) / (\beta)
\]

by Corollary 4.1. Therefore we have

\[
e_{4m+3}^2 e_{8m+7} = \eta e_{8m+7}^2 = \eta^2 e_{16m+15} \neq 0 \ (\beta)
\]

and this proves the second case.

Since \( \cup X \leq \text{cat} X \), it is also shown the following in the proof of Lemma 5.1 ([10]).

**Corollary 5.1.**

\[
\text{cat} Sp(n) \geq \begin{cases} 
n + 1, & n = 2, 3, \\
n + 2, & n \leq 4.
\end{cases}
\]

Since \( \cup X \leq \text{cat} X \), we obtain the lower bounds for \( \text{cat} Q_{n,m} \) by Lemma 5.1. We obtain the upper bounds by the next lemma ([1]).

**Lemma 5.2.** Let \( X \) be a \((n-1)\)-connected CW-complex. Then we have

\[
\text{cat} X \leq \frac{\dim X}{n}.
\]
Proof. We can assume that the \((n-1)\)-skeleton of \(X\) is a point. Then the \(kn\)-skeleta of \(X^{k+1}\) and \(X^{[k+1]}\) are the same. For \(\dim X \leq kn\) the diagonal map \(\Delta : X \to X^{k+1}\) can be compressed into \(X^{[k+1]}\) by the cellular approximation theorem. Therefore the proof is completed.

Finally we obtain \(\text{cat} \ Q_{n,m}\), for the several cases by Lemma 5.1 and Lemma 5.2.

**Theorem 5.1.**

\[
\text{cat} \ Q_{n,m} = \begin{cases} 
1 & m + 1 \leq n \leq 2m + 1 \\
2 & 2m + 2 \leq n \leq 3m + 2
\end{cases}
\]

and

\[
\text{cat} \ Q_{n,m} \geq 3 \text{ for } n \geq 4m + 4.
\]

**Corollary 5.2.** \(Q_n\) is not the suspension type for \(n \geq 2\).

**References**


