THE NUMBER OF ALGEBRAIC CYCLES WITH BOUNDED DEGREE

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Dedicated to Professor Masaki Maruyama on his 60th birthday

ABSTRACT. Let X be a projective scheme over a finite field. In this paper, we consider the asymptotic behavior of the number of effective cycles on X with bounded degree as it goes to the infinity. By this estimate, we can define a certain kind of zeta functions associated with groups of cycles. We also consider an analogue in Arakelov geometry.

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INTRODUCTION

Let X be a projective scheme over a finite field $\mathbb{F}_q$. Counting $\mathbb{F}_{q^r}$-valued points of X is one of classical problems in algebraic geometry. This is equivalent to decide the number n_k of 0-cycles of degree k. Actually, the generating function

$$Z(t) = \sum_{k=0}^{\infty} n_k t^k$$

of the sequence $\{n_k\}_{k=0}^{\infty}$ is nothing more than the zeta function of X. As we know, studies on this zeta function gave great influence on the development of algebraic geometry. Accordingly, it is very natural to expect a certain kind of generalization by considering a counting problem of higher dimensional algebraic cycles.

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To proceed with our problem, let us fix an ample line bundle H on X. For a non-negative integer k, we denote by $n_k(X, H, l)$ the number of all effective l-dimensional cycles V on X with $\deg_H(V) = k$, where $\deg_H(V)$ is the degree of V with respect to H, which is given by

$$\deg_H(V) = \deg\left(H^{\cdot l} \cdot V\right).$$

In the case where $l = 0$, since the above zeta function is rational, the asymptotic behavior of $\log_q n_k(X, H, 0)$ is, roughly speaking, linear with respect to k. However, if we consider the divisor case, we can easily see that this doesn't hold in general, so that the first natural question concerning $n_k(X, H, l)$ is to give an estimate of $n_k(X, H, l)$ as k goes to the infinity. Once we know it, we might give a convergent generating function of $\{n_k(X, H, l)\}_{k=0}^{\infty}$. The following theorem (cf. Corollary 2.2.5 and Proposition 2.2.6), which is one of the main results of this paper, is our answer for the above question.

Theorem A (Geometric version).

(1) *If H is very ample, then there is a constant C depending only on l and $\dim_{\mathbb{F}_q} H^0(X, H)$ such that*

$$\log_q n_k(X, H, l) \leq C \cdot k^{l+1}$$

for all $k \geq 0$.

(2) *If $l \neq \dim X$, then* $\displaystyle\limsup_{k\to\infty} \frac{\log_q n_k(X, H, l)}{k^{l+1}} > 0.$

As an analogue of Weil's zeta function, if we define a zeta function $Z(X, H, l)$ of l-dimensional cycles on a polarized scheme (X, H) over $\mathbb{F}_q$ to be

$$Z(X, H, l)(T) = \sum_{k=0}^{\infty} n_k(X, H, l) T^{k^{l+1}},$$

then, by the above theorem, we can see that $Z(X, H, l)(T)$ is a convergent power series at the origin.

Further, using the same techniques, we can estimate the number of rational points defined over a function field. Let C be a projective smooth curve over $\mathbb{F}_q$ and F the function field of C. Let $f : X \to C$ be a morphism of projective varieties over $\mathbb{F}_q$ and L an f-ample line bundle on X. Let X_η be the generic fiber of f. For $x \in X_\eta(\overline{F})$, we define the height of x with respect to L to be

$$h_L(x) = \frac{(L \cdot \Delta_x)}{\deg(\Delta_x \to C)},$$

where Δ_x is the Zariski closure of the image of $\operatorname{Spec}(\overline{F}) \to X_\eta \hookrightarrow X$. Then, we can see that, for a fixed k, there is a constant C such that

$$\#\{x \in X_\eta(\overline{F}) \mid [F(x) : F] \leq k \text{ and } h_L(x) \leq h\} \leq q^{C\cdot h}$$

for all $h \geq 1$. Thus, a series

$$\sum_{\substack{x\in X_\eta(\overline{F}),\\ [F(x):F]\leq k}} q^{-s h_L(x)}$$

converges for all $s \in \mathbb{C}$ with $\Re(s) \gg 0$. This is a local analogue of Batyrev-Manin-Tschinkel's height zeta functions.

Moreover, let $\mathcal{X} \to \operatorname{Spec}(O_K)$ be a flat and projective scheme over the ring O_K of integers of a number field K and let $\mathcal{H}$ be an ample line bundle on $\mathcal{X}$. For

$P \in \mathrm{Spec}(O_K)\setminus\{0\}$, we denote the modulo P reductions of $\mathcal{X}$ and $\mathcal{H}$ by $\mathcal{X}_P$ and $\mathcal{H}_P$ respectively. Then, as a corollary of our estimates, we can see an infinite product

$$L(\mathcal{X}, \mathcal{H}, l)(s) = \prod_{P \in \mathrm{Spec}(O_K)\setminus\{0\}} Z(\mathcal{X}_P, \mathcal{H}_P, l)(\#\kappa(P)^{-s})$$

converges for all $s \in \mathbb{C}$ with $\Re(s) \gg 0$, which looks like a generalization of the usual L-functions.

The next purpose of this paper is to give an analogue in Arakelov geometry. Let X be a projective arithmetic variety, i.e, a flat and projective integral scheme over $\mathbb{Z}$. Let $\overline{H}$ be an ample C^∞-hermitian $\mathbb{Q}$-line bundle on X. For a cycle V of dimension l on X, the arithmetic degree of V is defined by

$$\widehat{\deg}_{\overline{H}}(V) = \widehat{\deg}\left(\widehat{c}_1(\overline{H})^{\cdot l} \mid V\right).$$

For a real number h, we denote by $\hat{n}_{\leq h}(X, \overline{H}, l)$ (resp. $\hat{n}^{\mathrm{hor}}_{\leq h}(X, \overline{H}, l)$) the number of effective cycles (resp. horizontal effective cycles) V of dimension l on X with $\widehat{\deg}_{\overline{H}}(V) \leq h$. Then, we have the following analogue (cf. Corollary 5.3.2 and Theorem 5.4.1).

Theorem B (Arithmetic version)**.** (1) *There is a constant C such that*

$$\log \hat{n}_{\leq h}(X, \overline{H}, l) \leq C \cdot h^{l+1}$$

for all $h \geq 0$.

(2) *If $l \neq \dim X$, then* $\displaystyle\limsup_{h\to\infty} \frac{\log \hat{n}^{\mathrm{hor}}_{\leq h}(X, \overline{H}, l)}{h^{l+1}} > 0.$

Techniques involving the proof of Theorem B are much harder than the geometric case, but the outline for the proof is similar to the geometric one. We have also an estimate of rational points defined over a finitely generated field over $\mathbb{Q}$ (cf. Theorem 6.3.1). Let $Z^{\mathrm{eff}}_l(X)$ be the set of effective l-dimensional cycles on X. Then, as a consequence of Theorem B, we can see that a Dirichlet series

$$\zeta(X, \overline{H}, l)(s) = \sum_{V \in Z^{\mathrm{eff}}_l(X)} \exp(-s \cdot \widehat{\deg}_{\overline{H}}(V)^{l+1})$$

converges for $\Re(s) \gg 0$ (cf. Theorem 7.3.1).

Here let us give a sketch of the proof of Theorem A. A lower estimate of $n_k(X, H, l)$ is not difficult. To keep arguments simple, we only consider its upper estimate in the case where $l = 1$. First of all, we may clearly assume that X is the n-dimensional projective space. Note that the n-dimensional projective space is birationally equivalent to the n-times products $X_n = (\mathbb{P}^1_{\mathbb{F}_q})^n$ of the projective line $\mathbb{P}^1_{\mathbb{F}_q}$, so that once we get an upper estimate of the number of one cycles on X_n, then we can expect our desired result on the projective space, which can be actually done by using a comparison lemma (cf. Lemma 2.2.2). Why is X_n better than the projective space for our consideration? In order to use induction on n for the proof, it is very convenient that there are a lot of morphisms to lower dimensional cases. In this sense, X_n is a better choice.

Let $p_{n,i} : X_n \to \mathbb{P}^1_{\mathbb{F}_q}$ be the projection to the i-th factor. Let H_n be a natural ample line bundle on X_n, i.e., $H_n = \bigotimes_{i=1}^n p^*_{n,i}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_q}}(1))$. Let $Z^{\mathrm{eff}}_1(X_n \xrightarrow{p_{n,i}} \mathbb{P}^1_{\mathbb{F}_q})$ be the set of effective one cycles on X_n generated by irreducible curves which are flat over $\mathbb{P}^1_{\mathbb{F}_q}$ via $p_{n,i}$. Then, we can see that

$$\log n_k(X_n, H_n, 1) \leq \sum_{i=1}^{n} \log_q \#\{V \in Z_1^{\mathrm{eff}}(X_n \xrightarrow{p_{n,i}} \mathbb{P}^1_{\mathbb{F}_q}) \mid \deg_{H_n}(V) \leq k\}.$$

Thus, using symmetry, it is sufficient to find a constant C_n with

$$\log_q \#\{V \in Z_1^{\mathrm{eff}}(X_n \xrightarrow{p_{n,1}} \mathbb{P}^1_{\mathbb{F}_q}) \mid \deg_{H_n}(V) \leq k\} \leq C_n \cdot k^2 \quad (k \gg 1).$$

For simplicity, we set $T_n = Z_1^{\mathrm{eff}}(X_n \xrightarrow{p_{n,i}} \mathbb{P}^1_{\mathbb{F}_q})$. In order to complete the proof, we need to see the following properties (1) – (4) for a sequence $\{T_2, T_3, \ldots, T_n, \ldots\}$

(1) For each $n \geq 2$, there is a function $h_n : T_n \to \mathbb{R}_{\geq 0}$ satisfying the below (2), (3) and (4).

(2) For each $n \geq 3$, there are maps $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_2$ such that

$$h_{n-1}(\alpha_n(x)) \leq h_n(x) \quad \text{and} \quad h_2(\beta_n(x)) \leq h_n(x)$$

for all $x \in T_n$.

(3) There is a function $A : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $A(s,t) \leq A(s',t')$ for all $0 \leq s \leq s'$ and $0 \leq t \leq t'$ and that, for $y \in T_{n-1}$ and $z \in T_2$,

$$\#\{x \in T_n \mid \alpha_n(x) = y \text{ and } \beta_n(x) = z\} \leq A(h_{n-1}(y), h_2(z)).$$

(4) There is a function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$\#\{x \in T_2 \mid h_2(x) \leq h\} \leq B(h)$$

for all $h \geq 1$.

Actually, h_n is given by $h_n(V) = \deg_{H_n}(V)$. Let $a_n : X_n \to X_{n-1}$ and $b_n : X_n \to X_2$ be the morphisms given by $a_n(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ and $b_n(x_1, \ldots, x_n) = (x_1, x_n)$ respectively. Moreover, let

$$\alpha_n = (a_n)_* : T_n \to T_{n-1} \quad \text{and} \quad \beta_n = (b_n)_* : T_n \to T_2$$

be the push-forwards of cycles by a_n and b_n respectively. Then, the property (2) is almost obvious. Since $X_2 = \mathbb{P}^1_{\mathbb{F}_q} \times \mathbb{P}^1_{\mathbb{F}_q}$, it is easy to see that if we set $B(h) = (1+h)^2 q^{(1+h)^2}$, then the property (4) is satisfied (cf. Proposition 2.1.2). Technically, it is not easy to see the property (3). For this purpose, we use Lemma 2.1.3. Consequently if we set $A(s,t) = q^{s \cdot t}$, then we have (3). By using properties (1) — (4), we can conclude

$$\#\{x \in T_n \mid h_n(x) \leq k\} \leq B(k)^{k-1} A(k,k)^{k-2} \leq q^{7(n-1)k^2}.$$

The sequence $\{T_n\}$ satisfying (1) — (4) is called a counting system (for details, see §1.2). This is a very important tool for this paper because we will make several kinds of counting systems in different contexts. From viewpoint of induction, the properties (2) and (3) are the inductive steps and the property (4) is the initial step. In the arithmetic context, the inductive steps are very similar to the geometric case. However, the initial step is much harder than the geometric case.

The outline of this paper is as follows: This paper consists of the geometric part (§2 and §3) and the arithmetic part (§4, §5 and §6). As we said before, the arithmetic part is much harder than the geometric part, so that we strongly recommend reading the geometric part first. §1 contains notation and conventions of this paper, the definition of a counting system and key lemmas for counting cycles.

In §2, we prove Theorem A. In §3, we consider a refined Northcott's theorem in the geometric case. §4 contains preliminaries for the proof of Theorem B. In §5, we consider a counting problem of cycles in the arithmetic case, that is, Theorem B. §6 contains a refined Northcott's theorem in the arithmetic case. In §7, we prove the convergence of several kinds of zeta functions arising from counting cycles. Appendix A and Appendix B contain Bogomolov plus Lang in terms of a fine polarization and a weak geometric Northcott's theorem respectively. Note that §3, §6, Appendix A and Appendix B are secondary contents of this paper.

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1. General preliminaries

1.1. **Notation and Conventions.** Here, we introduce notation and conventions used in this paper.

(1.1.1). For a point x of a scheme X, the residue field at x is denoted by $\kappa(x)$.

(1.1.2). Let X be a Noetherian scheme. For a non-negative integer l, we denote by $\mathcal{C}_l(X)$ the set of all l-dimensional integral closed subschemes on X. We set

$$Z_l(X) = \bigoplus_{V \in \mathcal{C}_l(X)} \mathbb{Z}V, \quad \text{and} \quad Z_l^{\text{eff}}(X) = \bigoplus_{V \in \mathcal{C}_l(X)} \mathbb{Z}_{\geq 0}V,$$

where $\mathbb{Z}_{\geq 0} = \{z \in \mathbb{Z} \mid z \geq 0\}$. An element of $Z_l(X)$ (resp. $Z_l^{\text{eff}}(X)$) is called an *l-dimensional cycle* (resp. *l-dimensional effective cycle*) on X.

For a subset $\mathcal{C}$ of $\mathcal{C}_l(X)$, we denote $\bigoplus_{V\in\mathcal{C}} \mathbb{Z}V$ and $\bigoplus_{V\in\mathcal{C}} \mathbb{Z}_{\geq 0}V$ by $Z_l(X;\mathcal{C})$ and $Z_l^{\text{eff}}(X;\mathcal{C})$ respectively. In this paper, we consider the following $\mathcal{C}(U)$ and $\mathcal{C}(X/Y)$ as a subset of $\mathcal{C}_l(X)$; For a Zariski open set U of X, we set

$$\mathcal{C}(U) = \{V \in \mathcal{C}_l(X) \mid V \cap U \neq \emptyset\}.$$

For a morphism $f : X \to Y$ of Noetherian schemes with Y irreducible, we set

$$\mathcal{C}(X/Y) = \{V \in \mathcal{C}_l(X) \mid \overline{f(V)} = Y\}.$$

For simplicity, we denote

$$Z_l(X;\mathcal{C}(U)),\ Z_l^{\text{eff}}(X;\mathcal{C}(U)),\ Z_l(X;\mathcal{C}(X/Y)) \text{ and } Z_l^{\text{eff}}(X;\mathcal{C}(X/Y))$$

by

$$Z_l(X;U),\quad Z_l^{\text{eff}}(X;U),\quad Z_l(X/Y) \quad \text{and} \quad Z_l^{\text{eff}}(X/Y)$$

respectively. In order to show the fixed morphism $f : X \to Y$, $Z_l(X/Y)$ and $Z_l^{\text{eff}}(X/Y)$ are sometimes denoted by $Z_l(X \xrightarrow{f} Y)$ and $Z_l^{\text{eff}}(X \xrightarrow{f} Y)$ respectively.

(1.1.3). Let R be a commutative ring with the unity. Let X be the products of projective spaces $\mathbb{P}_R^{n_1}, \ldots, \mathbb{P}_R^{n_r}$ over R, that is,

$$X = \mathbb{P}_R^{n_1} \times_R \cdots \times_R \mathbb{P}_R^{n_r}.$$

Let $p_i : X \to \mathbb{P}_R^{n_i}$ be the projection to the i-th factor. For an n-sequence $(k_1, \ldots, k_n)$ of integers, the line bundle of type $(k_1, \ldots, k_n)$ is given by

$$\bigotimes_{i=1}^{n} p_i^*(\mathcal{O}_{\mathbb{P}_R^{n_i}}(k_i))$$

and it is denoted by $\mathcal{O}_X(k_1, \ldots, k_n)$. If R is UFD, then, for a divisor D on X, there is the unique sequence $(k_1, \ldots, k_n)$ of non-negative integers and the unique section $s \in H^0(X, \mathcal{O}_X(k_1, \ldots, k_n))$ module $R^\times$ with $\operatorname{div}(s) = D$. We denote k_i by $\deg_i(D)$ and call it the *i-th degree of D*. Moreover, for simplicity, we denote

$$\underbrace{\mathbb{P}^n_R \times_R \cdots \times_R \mathbb{P}^n_R}_{r\text{-times}}$$

by $(\mathbb{P}^n_R)^r$. Note that $(\mathbb{P}^n_R)^0 = \operatorname{Spec}(R)$.

(1.1.4). For a non-negative integer n, we set

$$[n] = \begin{cases} \{1, 2, \ldots, n\} & \text{if } n \geq 1 \\ \emptyset & \text{if } n = 0. \end{cases}$$

We assume $n \geq 1$. Let us consider the scheme $(\mathbb{P}^1_R)^n$ over R, where R is a commutative ring. Let $p_i : (\mathbb{P}^1_R)^n \to \mathbb{P}^1_R$ be the projection to the i-th factor. For a subset I of $[n]$, we define $p_I : (\mathbb{P}^1_R)^n \to (\mathbb{P}^1_R)^{\#(I)}$ as follows: If $I = \emptyset$, then p_I is the canonical morphism $(\mathbb{P}^1_R)^n \to \operatorname{Spec}(R)$. Otherwise, we set $I = \{i_1, \ldots, i_{\#(I)}\}$ with $1 \leq i_1 < \cdots < i_{\#(I)} \leq n$. Then, $p_I = p_{i_1} \times \cdots \times p_{i_{\#(I)}}$, i.e., $p_I(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_{\#(I)}})$. Note that $p_{\{i\}} = p_i$.

(1.1.5). Let us fix a basis $\{X_0, \ldots, X_n\}$ of $H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1))$. The Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$ of $\mathcal{O}(1)$ with respect to the basis $\{X_0, \ldots, X_n\}$ is given by

$$\|X_i\|_{\mathrm{FS}} = \frac{|X_i|}{\sqrt{|X_0|^2 + \cdots + |X_n|^2}}.$$

For a real number λ, the metric $\exp(-\lambda)\|\cdot\|_{\mathrm{FS}}$ is denoted by $\|\cdot\|_{\mathrm{FS}_\lambda}$. Moreover, the hermitian line bundle $(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}_\lambda})$ is denoted by $\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)$.

If $X = (\mathbb{P}^1_{\mathbb{C}})^n$, then the hermitian line bundle $\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1, \ldots, 1)$ of type $(1, \ldots, 1)$ on X is given by

$$\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1, \ldots, 1) = \bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)),$$

where $p_i : X \to \mathbb{P}^1_{\mathbb{C}}$ is the projection to the i-th factor,

(1.1.6). Let f and g be real valued functions on a set S. We use the notation '$f \asymp g$' if there are positive real numbers a, a' and real numbers b, b' such that $g(s) \leq af(s) + b$ and $f(s) \leq a'g(s) + b'$ for all $s \in S$.

1.2. **Counting system.** Here let us introduce a counting system. See the introduction to understand how a counting system works for counting cycles.

Let $\{T_n\}_{n=n_0}^\infty = \{T_{n_0}, T_{n_0+1}, \ldots, T_n, \ldots\}$ be a sequence of sets. If it satisfies the following properties (1) – (4), then it is called a *counting system*.

(1) (the existence of height functions) For each $n \geq n_0$, there is a function $h_n : T_n \to \mathbb{R}_{\geq 0}$ satisfying the below (2), (3) and (4).
(2) (inductive step) For each $n \geq n_0 + 1$, there are maps $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_{n_0}$ such that

$$h_{n-1}(\alpha_n(x)) \leq h_n(x) \quad \text{and} \quad h_{n_0}(\beta_n(x)) \leq h_n(x)$$

for all $x \in T_n$.

(3) (inductive step) There is a function $A : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $A(s,t) \leq A(s',t')$ for all $0 \leq s \leq s'$ and $0 \leq t \leq t'$ and that, for $y \in T_{n-1}$ and $z \in T_{n_0}$,

$$\#\{x \in T_n \mid \alpha_n(x) = y \text{ and } \beta_n(x) = z\} \leq A(h_{n-1}(y), h_{n_0}(z)).$$

(4) (initial step) There is a function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a non-negative constant t_0 such that

$$\#\{x \in T_{n_0} \mid h_{n_0}(x) \leq h\} \leq B(h)$$

for all $h \geq t_0$.

Lemma 1.2.1. *If $\{T_n\}_{n=n_0}^{\infty}$ is a counting system as above, then*

$$\#\{x \in T_n \mid h_n(x) \leq h\} \leq B(h)^{n-n_0+1} A(h,h)^{n-n_0}$$

for all $h \geq t_0$.

Proof. For $x \in T_n$ with $h_n(x) \leq h$, by the property (2), we have $h_{n-1}(\alpha_n(x)) \leq h$ and $h_{n_0}(\beta_n(x)) \leq h$. Thus, by using (3) and (4),

$$\begin{aligned} \#\{x \in T_n \mid h_n(x) \leq h\} &\leq \#\{y \in T_{n-1} \mid h_{n-1}(y) \leq h\} \cdot \#\{z \in T_{n_0} \mid h_{n_0}(z) \leq h\} \cdot A(h,h) \\ &\leq \#\{y \in T_{n-1} \mid h_{n-1}(y) \leq h\} \cdot B(h) \cdot A(h,h). \end{aligned}$$

Therefore, we get our lemma by using induction on n. □

1.3. **Key lemmas for counting cycles.** Here we consider two key lemmas for counting cycles. The first lemma will be used to see the property (3) in a counting system.

Lemma 1.3.1. *Let X and Y be projective schemes over a field K. Let $p : X \times_K Y \to X$ and $q : X \times_K Y \to Y$ be the projection to the first factor and the projection to the second factor respectively. Let $x_1, \ldots, x_s$ (resp. $y_1, \ldots, y_t$) be closed points of X (resp. Y). Let us fix an effective 0-cycle $x = \sum_{i=1}^{s} a_i x_i$ and an effective 0-cycle $y = \sum_{j=1}^{t} b_j y_j$. Then, the number of effective 0-cycles z on $X \times_K Y$ with $p_*(z) = x$ and $q_*(z) = y$ is less than or equal to $2^{\alpha_X(x)\alpha_Y(y)}$, where $\alpha_X(x) = \sum_{i=1}^{s} \sqrt{a_i[\kappa(x_i) : K]}$ and $\alpha_Y(y) = \sum_{j=1}^{t} \sqrt{b_j[\kappa(y_j) : K]}$.*

Proof. Let z_{ijk}'s ($k = 1, \ldots, l_{ij}$) be all closed points of $\operatorname{Spec}(\kappa(x_i) \otimes_K \kappa(y_j))$. Then, an effective 0-cycle z on $X \times_K Y$ with $p_*(z) = x$ and $q_*(z) = y$ can be written by the form $\sum_{ijk} c_{ijk} z_{ijk}$. Hence,

$$p_*(z) = \sum_i \left(\sum_{j,k} [\kappa(z_{ijk}) : \kappa(x_i)] c_{ijk} \right) x_i$$

and

$$q_*(z) = \sum_j \left(\sum_{i,k} [\kappa(z_{ijk}) : \kappa(y_j)] c_{ijk} \right) y_j.$$

Thus,

$$c_{ijk} \leq \min\{a_i, b_j\} \leq \sqrt{a_i b_j}.$$

Therefore, the number $N(x,y)$ of effective 0-cycles z on $X \times_K Y$ with $p_*(z) = x$ and $q_*(z) = y$ is less than or equal to $\prod_{ij}(1+\sqrt{a_i b_j})^{l_{ij}}$. Considering the following commutative diagram:

$$\begin{tikzcd}[column sep=tiny]
& \operatorname{Spec}(\kappa(x_i) \otimes_K \kappa(y_j)) \arrow[dl] \arrow[dr] & \\
\operatorname{Spec}(\kappa(x_i)) \arrow[dr] & & \operatorname{Spec}(\kappa(y_j)) \arrow[dl] \\
& \operatorname{Spec}(K) &
\end{tikzcd}$$

we note that

$$\begin{aligned}
l_{ij} &\leq \min\left\{\dim_{\kappa(y_j)}(\kappa(x_i) \otimes_K \kappa(y_j)), \dim_{\kappa(x_i)}(\kappa(x_i) \otimes_K \kappa(y_j))\right\} \\
&= \min\left\{[\kappa(x_i) : K], [\kappa(y_j) : K]\right\} \leq \sqrt{[\kappa(x_i) : K][\kappa(y_j) : K]}.
\end{aligned}$$

Moreover, $1 + u \leq 2^u$ for $u \in \{0\} \cup [1, \infty)$. Hence,

$$\begin{aligned}
N(x,y) &\leq \prod_{ij}(1+\sqrt{a_i \cdot b_j})^{\sqrt{[\kappa(x_i):K][\kappa(y_j):K]}} \\
&\leq \prod_{ij} 2^{\sqrt{a_i[\kappa(x_i):K]}\sqrt{b_j[\kappa(y_j):K]}} = 2^{\sum_{ij}\sqrt{a_i[\kappa(x_i):K]}\sqrt{b_j[\kappa(y_j):K]}}.
\end{aligned}$$

Thus, we get our lemma. □

The following lemma will be also used to count cycles.

Lemma 1.3.2. *Let $\pi : X' \to X$ be a finite morphism of normal integral schemes. Let $Z = \sum_{i=1}^n a_i Z_i$ be an effective cycle on X, where Z_i's are integral. Then the number of effective cycles Z' on X' with $\pi_*(Z') = Z$ is less than or equal to $2^{\deg(\pi)\sum_{i=1}^n a_i}$.*

Proof. We denote by $\alpha(Z)$ the number of effective cycles Z' on X' with $\pi_*(Z') = Z$. Let $Z'_{i1}, \ldots, Z'_{it_i}$ be all integral subschemes lying over Z_i. Then, $t_i \leq \deg(\pi)$. Let Z' be an effective cycle Z' on X' with $\pi_*(Z') = Z$. Then, we can set $Z' = \sum_{i=1}^n \sum_{j=1}^{t_i} a_{ij} Z_{ij}$. Since $\pi_*(Z') = Z$, the number of possible $(a_{i1}, \ldots, a_{it_i})$'s is at most $(1 + a_i)^{\deg(\pi)}$. Therefore,

$$\alpha(Z) \leq \prod_{i=1}^n (1 + a_i)^{\deg(\pi)}.$$

Here note that $1 + u \leq 2^u$ for $u \in \{0\} \cup [1, \infty)$. Hence we get our lemma. □

2. Counting cycles in the geometric case

The main purpose of this section is to find a universal upper bound of the number of effective cycles with bounded degree on the projective space over a finite field (cf. Theorem 2.2.1), namely,

> Fix non-negative integers n and l. Then, there is a constant $C(n,l)$ depending only on n and l such that the number of effective l-dimensional cycles on $\mathbb{P}^n_{\mathbb{F}_q}$ with degree k is less than or equal to $q^{C(n,k)k^{l+1}}$.

The plan for the proof of the above theorem is the following: As we described in the introduction, first we consider a similar problem on the products $(\mathbb{P}^1_{\mathbb{F}_q})^n$ of the projective line. The advantage of $(\mathbb{P}^1_{\mathbb{F}_q})^n$ is that it has a lot of morphisms, so that induction on its dimension works well. In §2.1, we estimate the number of cycles on $(\mathbb{P}^1_{\mathbb{F}_q})^n$. In §2.2, we prove the above result. Especially, we compare the number of cycles on $\mathbb{P}^n_{\mathbb{F}_q}$ with the number of cycles on $(\mathbb{P}^1_{\mathbb{F}_q})^n$ in terms of the natural birational map $\mathbb{P}^n_{\mathbb{F}_q} \dashrightarrow (\mathbb{P}^1_{\mathbb{F}_q})^n$.

2.1. **Cycles on $(\mathbb{P}^1_{\mathbb{F}_q})^n$.** Here we consider the following proposition. An idea of the proof can be found in the introduction.

Proposition 2.1.1. *There is a constant $C(n,l)$ depending only n and l such that*

$$\#\{V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg_{\mathcal{O}_{(\mathbb{P}^1)^n}(1,\ldots,1)}(V) \leq h\} \leq q^{C(n,l)\cdot h^{l+1}}$$

for all $h \geq 1$.

First, let us begin with the case of divisors, which gives the initial step in a counting system.

Proposition 2.1.2. *Let $k_1,\ldots,k_n$ be non-negative integers. Then*

$$\#\{D \in \mathrm{Div}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg_i(D) \leq k_i \ \forall i = 1,\ldots,n\} \leq \frac{q^{\prod_{i=1}^n (k_i+1)} - 1}{q-1} \cdot \prod_{i=1}^n (k_i+1).$$

Proof. In the following, the symbol D is an effective divisor on $(\mathbb{P}^1_{\mathbb{F}_q})^n$.

$$\begin{aligned}
\#\{D \mid \deg_i(D) \leq k_i \ (\forall i)\} &= \sum_{0\leq e_1\leq k_1,\ldots,0\leq e_n\leq k_n} \#\{D \mid \deg_i(D) = e_i \ (\forall i)\} \\
&= \sum_{0\leq e_1\leq k_1,\ldots,0\leq e_n\leq k_n} \frac{q^{(e_1+1)\cdots(e_n+1)} - 1}{q-1} \\
&\leq (k_1+1)\cdots(k_n+1)\frac{q^{(k_1+1)\cdots(k_n+1)} - 1}{q-1}
\end{aligned}$$

□

Let us start the proof of Proposition 2.1.1. First we assume $l = 0$. Let us see

$$\#\{V \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg(V) \leq h\} \leq q^{3nh}$$

for $h \geq 1$. We prove this by induction on n. If $n = 1$, then our assertion follows from Proposition 2.1.2, so that we assume $n > 1$. Let $q : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^{n-1}$ be the projection given by $q(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1})$. For a fixed $W \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^{n-1})$, let us estimate the number of $\{V \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid q_*(V) = W\}$. We set $W = \sum_{i=1}^e a_i y_i$. For $V \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n)$ with $q_*(V) = W$, let $V = V_1 + \cdots + V_e$ be the decomposition of effective 0-cycles with $q_*(V_i) = a_i y_i$ $(i = 1,\ldots,e)$. Then,

$V_i \in Z_0^{\mathrm{eff}}(\mathbb{P}^1_{\kappa(y_i)})$ and the degree of V_i in $\mathbb{P}^1_{\kappa(y_i)}$ is a_i. Thus, the possible number of V_i is less than or equal to $\#(\kappa(y_i))^{3a_i}$. Thus,

$$\#\{V \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid q_*(V) = W\} \leq \prod_{i=1}^{e} \#(\kappa(y_i))^{3a_i} = \prod_{i=1}^{e} q^{3[\kappa(y_i):\mathbb{F}_q]a_i} = q^{3\deg(W)}.$$

Therefore, since $\deg(V) = \deg(q_*(V))$, using the hypothesis of induction,

$$\begin{aligned} \#\{V \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg(V) \leq h\} &\leq \#\{W \in Z_0^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^{n-1}) \mid \deg(W) \leq h\} \cdot q^{3h} \\ &\leq q^{3(n-1)h} \cdot q^{3h} = q^{3nh}. \end{aligned}$$

Next we assume $l \geq 1$. For a subset I of $[n] = \{1, \ldots, n\}$ with $\#(I) = l$, let us consider the morphism $p_I : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^l$ (for the definition of p_I, see (1.1.4)). We denote by $Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \xrightarrow{p_I} (\mathbb{P}^1_{\mathbb{F}_q})^l)$ the set of effective cycles on $(\mathbb{P}^1_{\mathbb{F}_q})^n$ generated by l-dimensional subvarieties which dominates $(\mathbb{P}^1_{\mathbb{F}_q})^l$ via p_I (cf. (1.1.2)). Then, it is easy to see that

$$Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) = \sum_{I \subseteq [n], \#(I)=l} Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \xrightarrow{p_I} (\mathbb{P}^1_{\mathbb{F}_q})^l).$$

Thus, since

$$\#(\{I \mid I \subseteq [n], \#(I) = l\}) = \binom{n}{l} \leq 2^n,$$

it is sufficient to see that there is a constant $C'(n,l)$ depending only on n and l such that

$$\{V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \xrightarrow{p_I} (\mathbb{P}^1_{\mathbb{F}_q})^l) \mid \deg_H(V) \leq h\} \leq q^{C'(n,l)h^{l+1}}$$

for all $h \geq 1$. By re-ordering the coordinate of $(\mathbb{P}^1_{\mathbb{F}_q})^n$, we can find an automorphism

$$\iota : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^n$$

with $\pi_{[l]} \cdot \iota = \pi_I$ and $\iota^*(\mathcal{O}_{(\mathbb{P}^1)^n}(1, \ldots, 1)) = \mathcal{O}_{(\mathbb{P}^1)^n}(1, \ldots, 1)$. Thus, we may assume that $I = [l]$. We denote $p_{[l]}$ by p. Let $p_i : (\mathbb{P}^1_{\mathbb{F}_q})^n \to \mathbb{P}^1_{\mathbb{F}_q}$ be the projection to the i-th factor. For $n \geq l+1$, we set

$$T_n = Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \xrightarrow{p} (\mathbb{P}^1_{\mathbb{F}_q})^l)$$

and $h_n(V) = \deg_{\mathcal{O}(1,\ldots,1)}(V)$ for $V \in T_n$. We would like to see that $\{T_n\}_{n=l+1}^{\infty}$ is a counting system. Let $a_n : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^{n-1}$ and $b_n : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^{l+1}$ be morphisms given by $a_n = p_{[n-1]}$ and $b_n = p_{[l]\cup\{n\}}$, namely,

$$a_n(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}) \quad \text{and} \quad b_n(x_1, \ldots, x_n) = (x_1, \ldots, x_l, x_n).$$

Here, $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_{l+1}$ are given by

$$\alpha_n(V) = (a_n)_*(V) \quad \text{and} \quad \beta_n(V) = (b_n)_*(V).$$

Then, since

$$\mathcal{O}_{(\mathbb{P}^1)^n}(1, \ldots, 1) = (a_n)^*(\mathcal{O}_{(\mathbb{P}^1)^{n-1}}(1, \ldots, 1)) \otimes p_n^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

and

$$\mathcal{O}_{(\mathbb{P}^1)^n}(1, \ldots, 1) = (b_n)^*(\mathcal{O}_{(\mathbb{P}^1)^{l+1}}(1, \ldots, 1)) \otimes \bigotimes_{j=l+1}^{n-1} p_j^*(\mathcal{O}_{\mathbb{P}^1}(1)),$$

it is easy to see that $h_{n-1}(\alpha_n(V)) \leq h_n(V)$ and $h_{l+1}(\beta_n(V)) \leq h_n(V)$. Note that the diagram

$$\begin{tikzcd}
 & (\mathbb{P}^1_{\mathbb{F}_q})^n \arrow[dl, "a_n"'] \arrow[dr, "b_n"] & \\
(\mathbb{P}^1_{\mathbb{F}_q})^{n-1} \arrow[dr, "p_{[l]}"'] & & (\mathbb{P}^1_{\mathbb{F}_q})^{l+1} \arrow[dl, "p_{[l]}"] \\
 & (\mathbb{P}^1_{\mathbb{F}_q})^l &
\end{tikzcd}$$

is a fiber product. Thus, by the following Lemma 2.1.3, if we set $A(s,t) = q^{st}$, then, for $y \in T_{n-1}$ and $z \in T_{l+1}$,

$$\#\{x \in T_n \mid \alpha_n(x) = y,\ \beta_n(x) = z\} \leq A(h_{n-1}(y), h_{l+1}(z)).$$

Here,

$$\{D \in T_{l+1} \mid h_{l+1}(D) \leq h\} \subseteq \{D \in T_{l+1} \mid \deg_i(D) \leq h \text{ for all } i = 1, \ldots, l+1\}.$$

Thus, by Proposition 2.1.2, if we set $B(h) = (1+h)^{l+1} q^{(1+h)^{l+1}}$, then

$$\#\{D \in T_{l+1} \mid h_{l+1}(D) \leq h\} \leq B(h).$$

Hence, we obtain a counting system $\{T_n\}_{n=l+1}^{\infty}$. Therefore, by Lemma 1.2.1,

$$\begin{aligned}
\#\{x \in T_n \mid h_n(x) \leq h\} &\leq B(h)^{n-l} A(h,h)^{n-l-1} \\
&= (1+h)^{(n-l)(l+1)} q^{(n-l)(1+h)^{l+1}} q^{(n-l-1)h^2} \\
&\leq q^{(n-l)(l+1)h} q^{(n-l)(2h)^{l+1}} q^{(n-l-1)h^2} \leq q^{(n-l)(2^{l+1}+l+2)h^{l+1}}
\end{aligned}$$

for all $h \geq 1$. □

Lemma 2.1.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of projective schemes over $\mathbb{F}_q$. We assume that S is integral and of dimension l. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projections to the first factor and the second factor respectively. Fix $D \in Z_l^{\mathrm{eff}}(X/S)$ and $E \in Z_l^{\mathrm{eff}}(Y/S)$ (for the definition of $Z_l^{\mathrm{eff}}(X/S)$ and $Z_l^{\mathrm{eff}}(Y/S)$, see* (1.1.2)*).*

(1) *Assume $l \geq 1$. Let $A_1, \ldots, A_l$ be nef line bundles on X, $B_1, \ldots, B_l$ nef line bundle on Y, and $C_1, \ldots, C_l$ nef line bundles on S such that $A_i \otimes f^*(C_i)^{\otimes -1}$ and $B_i \otimes g^*(C_i)^{\otimes -1}$ are nef for all i and that $\deg(C_1 \cdots C_l) > 0$. Then,*

$$\begin{aligned}
&\log_q \left(\# \left\{ V \in Z_l^{\mathrm{eff}}(X \times_S Y/S) \mid p_*(V) = D \text{ and } q_*(V) = E \right\} \right) \\
&\qquad \leq \min \left\{ \frac{\deg(A_1 \cdots A_l \cdot D) \deg(B_1 \cdots B_l \cdot E)}{\deg{(C_1 \cdots C_l)}^2}, \right. \\
&\qquad\qquad \left. \frac{\sqrt{\theta(D)\theta(E) \deg(A_1 \cdots A_l \cdot D) \deg(B_1 \cdots B_l \cdot E)}}{\deg{(C_1 \cdots C_l)}} \right\},
\end{aligned}$$

where $\theta(D)$ (resp. $\theta(E)$) is the number of irreducible components of $\mathrm{Supp}(D)$ *(resp.* $\mathrm{Supp}(E)$*).*

(2) *Assume $l = 0$, so that $S = \mathrm{Spec}(\mathbb{F}_{q^r})$ for some positive integer r. Then,*

$$\log_q \left(\# \left\{ V \in Z_0^{\mathrm{eff}}(X \times_S Y/S) \mid p_*(V) = D \text{ and } q_*(V) = E \right\}\right) \leq \min \left\{ \frac{\deg(D)\deg(E)}{r^2}, \frac{\sqrt{\theta(D)\theta(E)\deg(D)\deg(E)}}{r} \right\}.$$

Proof. (1) We set $D = \sum_{i=1}^s a_i D_i$ and $E = \sum_{j=1}^t b_j E_j$. Then,

$$\begin{aligned} \deg(A_1 \cdots A_l \cdot D) &= \sum_{i=1}^s a_i \deg(A_1 \cdots A_l \cdot D_i) \\ &\geq \sum_{i=1}^s a_i \deg(f^*(C_1) \cdots f^*(C_l) \cdot D_i) \\ &= \sum_{i=1}^s a_i \deg(D_i \to S) \deg(C_1 \cdots C_l). \end{aligned} \tag{2.1.3.1}$$

In the same way,

$$\deg(B_1 \cdots B_l \cdot E) \geq \sum_{j=1}^t b_j \deg(E_j \to S) \deg(C_1 \cdots C_l). \tag{2.1.3.2}$$

Thus,

$$\frac{\deg(A_1 \cdots A_l \cdot D)}{\deg(C_1 \cdots C_l)} \geq \sum_{i=1}^s \sqrt{a_i \deg(D_i \to S)}$$

and

$$\frac{\deg(B_1 \cdots B_l \cdot E)}{\deg(C_1 \cdots C_l)} \geq \sum_{j=1}^t \sqrt{b_j \deg(E_j \to S)}.$$

Moreover, note that

$$\sqrt{n}\sqrt{x_1 + \cdots + x_n} \geq \sqrt{x_1} + \cdots + \sqrt{x_n}.$$

Thus, the above inequalities (2.1.3.1) and (2.1.3.2) imply

$$\sqrt{\frac{s \deg(A_1 \cdots A_l \cdot D)}{\deg(C_1 \cdots C_l)}} \geq \sum_{i=1}^s \sqrt{a_i \deg(D_i \to S)}$$

and

$$\sqrt{\frac{t \deg(B_1 \cdots B_l \cdot E)}{\deg(C_1 \cdots C_l)}} \geq \sum_{j=1}^t \sqrt{b_j \deg(E_j \to S)}.$$

Therefore, considering X, Y and $X \times_S Y$ over the generic point of S, Lemma 1.3.1 implies our assertion.

(2) We set $D = \sum_{i=1}^s a_i x_i$ and $E = \sum_{j=1}^t b_j y_j$. Then,

$$\deg(D) = \sum_{i=1}^s a_i [\kappa(x_i) : \mathbb{F}_q] = r \sum_{i=1}^s a_i [\kappa(x_i) : \mathbb{F}_{q^r}]$$

and

$$\deg(E) = \sum_{j=1}^t b_j [\kappa(y_j) : \mathbb{F}_q] = r \sum_{j=1}^t b_j [\kappa(y_j) : \mathbb{F}_{q^r}].$$

Thus, in the same way as in (1), we get our assertion. □

2.2. **Cycles on a projective variety over a finite field.** In this subsection, we consider the main problem of this section.

Theorem 2.2.1. *There is a constant $C(n,l)$ depending only on n and l such that*

$$\#\left(\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}) \mid \deg_{\mathcal{O}(1)}(V) \leq h)\}\right) \leq q^{C(n,l)\cdot h^{l+1}}$$

for all $h \geq 1$.

Let us begin with a lemma.

Lemma 2.2.2. *Let F be a field. Let $\phi : \mathbb{P}^n_F \dashrightarrow (\mathbb{P}^1_F)^n$ be the birational map given by*

$$(X_0 : \ldots : X_n) \mapsto (X_0 : X_1) \times \cdots \times (X_0 : X_n).$$

Let Σ be the boundary of $\mathbb{P}^n_F$, that is, $\Sigma = \{X_0 = 0\}$. Let $Z_l^{\mathrm{eff}}(\mathbb{P}^n_F; \mathbb{P}^n_F \setminus \Sigma)$ be the set of effective cycles generated by l-dimensional subvarieties T on $\mathbb{P}^n_F$ with $T \not\subseteq \Sigma$ (cf. (1.1.2)). For $V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_F; \mathbb{P}^n_F \setminus \Sigma)$, we denote by V' the strict transform of V by ϕ. Then,

$$n^l \deg\left(\mathcal{O}_{\mathbb{P}^n_F}(1)^{\cdot l} \cdot V\right) \geq \deg\left(\mathcal{O}_{(\mathbb{P}^1_F)^n}(1,\ldots,1)^{\cdot l} \cdot V'\right)$$

for all $V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_F; \mathbb{P}^n_F \setminus \Sigma)$.

Proof. Let Y ($\subseteq \mathbb{P}^n_F \times (\mathbb{P}^1_F)^n$) be the graph of the rational map $\phi : \mathbb{P}^n_F \dashrightarrow (\mathbb{P}^1_F)^n$. Let $\mu : Y \to \mathbb{P}^n_F$ and $\nu : Y \to (\mathbb{P}^1_F)^n$ be the morphisms induced by the projections $\mathbb{P}^n_F \times (\mathbb{P}^1_F)^n \to \mathbb{P}^n_F$ and $\mathbb{P}^n_F \times (\mathbb{P}^1_F)^n \to (\mathbb{P}^1_F)^n$ respectively. Here we claim that there is an effective Cartier divisor E on Y such that (1) $\mu(E) \subseteq \Sigma$ and (2) $\mu^*(\mathcal{O}(n)) = \nu^*(\mathcal{O}(1,\ldots,1)) \otimes \mathcal{O}_Y(E)$. Let Y_i ($\subseteq \mathbb{P}^n_F \times \mathbb{P}^1_F$) be the graph of the rational map $\mathbb{P}^n_F \dashrightarrow \mathbb{P}^1_F$ given by

$$(X_0 : \cdots : X_n) \mapsto (X_0 : X_i).$$

Let $\mu_i : Y_i \to \mathbb{P}^n_F$ and $\nu_i : Y_i \to \mathbb{P}^1_F$ be the morphisms induced by the projections $\mathbb{P}^n_F \times \mathbb{P}^1_F \to \mathbb{P}^n_F$ and $\mathbb{P}^n_F \times \mathbb{P}^1_F \to \mathbb{P}^1_F$ respectively. Let $\pi_i : (\mathbb{P}^1_F)^n \to \mathbb{P}^1_F$ be the projection to the i-th factor. Moreover, let $h_i : Y \to Y_i$ be the morphism induced by $\mathrm{id} \times \pi_i : \mathbb{P}^n_F \times (\mathbb{P}^1_F)^n \to \mathbb{P}^n_F \times \mathbb{P}^1_F$. Consequently, we have the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[d, "h_i"] \arrow[dr, "\nu"] \arrow[ddl, "\mu"'] & \\
 & Y_i \arrow[dl, "\mu_i"] \arrow[dr, "\nu_i"'] & (\mathbb{P}^1_F)^n \arrow[d, "\pi_i"] \\
\mathbb{P}^n_F & & \mathbb{P}^1_F
\end{tikzcd}$$

Note that $\mu_i : Y_i \to \mathbb{P}^n_F$ is the blowing-up by the ideal sheaf I_i generated by X_0 and X_i. Thus there is an effective Cartier divisor E_i on Y_i with $I_i\mathcal{O}_{Y_i} = \mathcal{O}_{Y_i}(-E_i)$ and $\mu_i^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{Y_i}(-E_i) = \nu_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Thus if we set $E = \sum_{i=1}^n h_i^*(E_i)$, then

$$\mu^*(\mathcal{O}(n)) = \nu^*(\mathcal{O}(1,\ldots,1)) \otimes \mathcal{O}_Y(E).$$

Hence we get our claim.

For $V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_F; \mathbb{P}^n_F \setminus \Sigma)$, let V'' be the strict transform of V by μ. Then, by using the projection formula,

$$\deg\left(\mathcal{O}(n)^{\cdot l} \cdot V\right) = \deg\left(\mu^*(\mathcal{O}(n))^{\cdot l} \cdot V''\right).$$

Moreover, by the following Sublemma 2.2.3,

$$\deg\left(\mu^*(\mathcal{O}(n))^{\cdot l} \cdot V''\right) \geq \deg\left(\nu^*(\mathcal{O}(1, \ldots, 1))^{\cdot l} \cdot V''\right)$$

Thus, using the projection formula for ν, we get our lemma because $\nu_*(V'') = V'$. □

Sublemma 2.2.3. *Let X be a projective variety over a field F and $L_1, \ldots, L_{\dim X}$, $M_1, \ldots, M_{\dim X}$ nef line bundles on X. If $L_i \otimes M_i^{\otimes -1}$ is pseudo-effective for $i = 1, \ldots, n$, then*

$$\deg(L_1 \cdots L_{\dim X}) \geq \deg(M_1 \cdots M_{\dim X}).$$

Proof. We set $E_i = L_i \otimes M_i^{\otimes -1}$. Then

$$\deg(L_1 \cdots L_{\dim X}) = \deg(M_1 \cdots M_{\dim X}) + \sum_{i=1}^{\dim X} \deg(M_1 \cdots M_{i-1} \cdot E_i \cdot L_{i+1} \cdots L_{\dim X}).$$

Thus, we get our lemma. □

Let us start the proof of Theorem 2.2.1. We prove this theorem by induction on n. Let us consider the birational map $\mathbb{P}^n_{\mathbb{F}_q} \dashrightarrow (\mathbb{P}^1_{\mathbb{F}_q})^n$ given by

$$\phi : (X_0 : \cdots : X_n) \mapsto (X_0 : X_1) \times \cdots \times (X_0 : X_n).$$

We set $U = \mathbb{P}^n_{\mathbb{F}_q} \setminus \{X_0 = 0\}$. For $V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}; U)$, we denote by V' the strict transform of V by ϕ. Then, by Lemma 2.2.2,

$$n^l \deg_{\mathcal{O}(1)}(V) \geq \deg_{\mathcal{O}(1,\ldots,1)}(V').$$

Moreover, note that if $V_1' = V_2'$ for $V_1, V_2 \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}; U)$, then $V_1 = V_2$. Therefore

$$\#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}; U) \mid \deg_{\mathcal{O}(1)}(V) \leq h\} \leq \#\{V' \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg_{\mathcal{O}(1,\ldots,1)}(V') \leq n^l h\}.$$

Here, by Proposition 2.1.1, there is a constant $C'(n,l)$ depending only n and l such that

$$\#\{V' \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n) \mid \deg_{\mathcal{O}(1,\ldots,1)}(V') \leq k\} \leq q^{C'(n,l)k^{l+1}}.$$

Hence, we have

$$\#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}; U) \mid \deg_{\mathcal{O}(1)}(V) \leq h\} \leq q^{C'(n,l)n^{l(l+1)}h^{l+1}}. \tag{2.2.4}$$

On the other hand, since $\mathbb{P}^n_{\mathbb{F}_q} \setminus U \simeq \mathbb{P}^{n-1}_{\mathbb{F}_q}$,

$$\begin{aligned}\#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}) \mid \deg_{\mathcal{O}(1)}(V) \leq h\} &\leq \#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{F}_q}; U) \mid \deg_{\mathcal{O}(1)}(V) \leq h\} \\ &\quad \cdot \#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^{n-1}_{\mathbb{F}_q}) \mid \deg_{\mathcal{O}(1)}(V) \leq h\}\end{aligned}$$

Thus, using the hypothesis of induction, if we set $C(n,l) = C(n-1,l) + n^{l(l+1)}C'(n,l)$, then we have our theorem. □

Corollary 2.2.5. *Let X be a projective variety over a finite field $\mathbb{F}_q$ and H a very ample line bundle on X. Then, for every integer l with $0 \leq l \leq \dim X$, there is a constant C depending only on l and $\dim_{\mathbb{F}_q} H^0(X, H)$ such that*

$$\#\{V \in Z_l^{\mathrm{eff}}(X) \mid \deg_H(V) \leq h\} \leq q^{Ch^{l+1}}$$

Proof. Since H is very ample, there is an embedding $\iota : X \to \mathbb{P}^n_{\mathbb{F}_q}$ with $\iota^*(\mathcal{O}(1)) = H$, where $n = \dim_{\mathbb{F}_q} H^0(X, H) - 1$. Thus, it follows from Theorem 2.2.1. □

Finally, let us consider a lower estimate of the number of effective cycles with bounded degree.

Proposition 2.2.6. *Let X be a projective variety over a finite field $\mathbb{F}_q$ and H an ample line bundle on X. Then, for every integer l with $0 \leq l < \dim X$,*

$$\limsup_{h\to\infty} \frac{\log \# \left(\{V \in Z_l^{\mathrm{eff}}(X) \mid \deg_H(V) = k\}\right)}{k^{l+1}} > 0.$$

Proof. Take $(l+1)$-dimensional subvariety Y of X. Then,

$$\# \left(\{V \in Z_l^{\mathrm{eff}}(Y) \mid \deg_{H|_Y}(V) = k\}\right) \leq \# \left(\{V \in Z_l^{\mathrm{eff}}(X) \mid \deg_H(V) = k\}\right).$$

Thus, we may assume $l = \dim X - 1$. Here, note that

$$|H^{\otimes m}| \subseteq \{D \in Z_{d-1}^{\mathrm{eff}}(X) \mid \deg_H(D) = m(H^d)\}$$

and

$$\#|H^{\otimes m}| = \frac{q^{\dim_{\mathbb{F}_q} H^0(X, H^{\otimes m})} - 1}{q-1},$$

where $d = \dim X$. Since H is ample, $\dim_{\mathbb{F}_q} H^0(X, H^{\otimes m}) = O(m^d)$. Thus we get our proposition. □

3. A refinement of Northcott's theorem in the geometric case

Let K be a function field of a projective curve over $\mathbb{F}_q$ and $\overline{K}$ the algebraic closure of K. Let X be a projective variety over K. Northcott's theorem says us that, for any k and h, the set

$$\{x \in X(\overline{K}) \mid [K(x) : K] \leq k,\ h(x) \leq h\}$$

is finite, where $h(x)$ is a height function arising from an ample line bundle on X. For a fixed k, we would like to ask the asymptotic behavior of the number of the above set as h goes to ∞. In this section, we consider it in more general contexts. Let d be an integer with $d \geq 1$ and let B be a d-dimensional projective variety over $\mathbb{F}_q$. Let H be a nef and big line bundle on B. Let X be a projective variety over $\mathbb{F}_q$ and $f : X \to B$ a surjective morphism over $\mathbb{F}_q$. Let L be a nef line bundle on X. The main result of this section is the following:

> If L_η is ample on X_η (the generic fiber of f), then, for a fixed k, there is a constant C such that the number of effective l-dimensional cycles in $Z_l^{\mathrm{eff}}(X/B)$ with $\deg(L_\eta^{\cdot l-d} \cdot V_\eta) \leq k$ and
>
> $$\deg(L^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) \leq h$$
>
> is less than or equal to $q^{C \cdot h^d}$ for all $h \geq 1$, where $\deg(L_\eta^{\cdot l-d} \cdot V_\eta)$ is the intersection number on X_η.

§3.1 contains a preliminary for the above result, where we prove a special case. In §3.2, the above theorem is proved and we treat a consequence in §3.3.

3.1. **A variant of Proposition 2.1.1.** Here we would like to consider a variant of Proposition 2.1.1, which is a special case of the main theorem of this section.

Proposition 3.1.1. *Let d, l and n be positive integers with $d \leq l \leq n$. Let $p_{[d]} : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^d$ be the morphism given in (1.1.4), i.e., $p_{[d]}(x_1, \ldots, x_n) = (x_1, \ldots, x_d)$. Let $p_i : (\mathbb{P}^1_{\mathbb{F}_q})^n \to \mathbb{P}^1_{\mathbb{F}_q}$ be the projection to the i-th factor. We set $L_n = \bigotimes_{i=1}^n p_i^*(\mathcal{O}(1))$ and $H_n = \bigotimes_{i=1}^d p_i^*(\mathcal{O}(1))$. Then, for a fixed k, there is a constant C such that*

$$\#\left\{ V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \overset{p_{[d]}}{\to} (\mathbb{P}^1_{\mathbb{F}_q})^d) \;\middle|\; \begin{array}{l} \deg(L_n^{\cdot l-d} \cdot H_n^{\cdot d} \cdot V) \leq k, \\ \deg(L_n^{\cdot l-d+1} \cdot H_n^{\cdot d-1} \cdot V) \leq h \end{array} \right\} \leq q^{C \cdot h^d}$$

for all $h \geq 1$.

Proof. We set

$$\Sigma = \{ I \mid [d] \subseteq I \subseteq [n], \#(I) = l \}.$$

Then,

$$Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \overset{p_{[d]}}{\to} (\mathbb{P}^1_{\mathbb{F}_q})^d) = \sum_{I \in \Sigma} Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{F}_q})^l).$$

Thus, it is sufficient to show that there is a constant C' such that

$$\#\left\{ V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{F}_q})^l) \;\middle|\; \begin{array}{l} \deg(L_n^{\cdot l-d} \cdot H_n^{\cdot d} \cdot V) \leq k, \\ \deg(L_n^{\cdot l-d-1} \cdot H_n^{\cdot d-1} \cdot V) \leq h \end{array} \right\} \leq q^{C' \cdot h^d}$$

for all $h \geq 1$. Re-ordering the coordinate of $(\mathbb{P}^1_{\mathbb{F}_q})^n$, we may assume that $I = [l]$. We denote p_I by p. Let $a_n : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^{n-1}$ and $b_n : (\mathbb{P}^1_{\mathbb{F}_q})^n \to (\mathbb{P}^1_{\mathbb{F}_q})^{l+1}$ be morphisms given by $a_n = p_{[n-1]}$ and $b_n = p_{[l] \cup \{n\}}$, i.e., $a_n(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ and $b_n(x_1, \ldots, x_n) = (x_1, \ldots, x_l, x_n)$. Then,

$$\begin{cases} a_n^*(L_{n-1}) \otimes p_n^*(\mathcal{O}(1)) = L_n \\ b_n^*(L_{l+1}) \otimes \bigotimes_{i=l+1}^{n-1} p_i^*(\mathcal{O}(1)) = L_n \\ a_n^*(H_{n-1}) = H_n \\ b_n^*(H_{l+1}) = H_n \end{cases} \tag{3.1.1.1}$$

Here, for $n \geq l+1$, we set

$$T_n = \{ V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{F}_q})^n \overset{p}{\to} (\mathbb{P}^1_{\mathbb{F}_q})^l) \mid \deg(L_n^{\cdot l-d} \cdot H_n^{\cdot d} \cdot V) \leq k \}.$$

Let $h_n : T_n \to \mathbb{R}$ be a map given by $h_n(V) = \deg(L_n^{\cdot l-d+1} \cdot H_n^{\cdot d-1} \cdot V)$. Then, by (3.1.1.1), we have maps $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_{l+1}$ given by $\alpha_n(V) = (a_n)_*(V)$ and $\beta_n(V) = (b_n)_*(V)$. Moreover, $h_{n-1}(\alpha_n(V)) \leq h_n(V)$ and $h_{l+1}(\beta_n(V)) \leq h_n(V)$ for all $V \in T_n$. As in Lemma 2.1.3, we denote by $\theta(V)$ the number of irreducible components of a cycle V. Then, for $V \in T_n$, it is easy to see that $\theta(V) \leq k$. Further,

$$p^*(H_l) = H_n \quad \text{and} \quad p^*(L_l) \otimes \bigotimes_{i=l+1}^{n} p_i^*(\mathcal{O}(1)) = L_n.$$

Thus, by Lemma 2.1.3, for $D \in T_{n-1}$ and $E \in T_{l+1}$,

$$\log_q \#\{V \in T_n \mid \alpha_n(V) = D, \beta_n(V) = E\} \leq \frac{k\sqrt{\deg(L_{n-1}^{\cdot l-d+1} \cdot H_{n-1}^{\cdot d-1} \cdot D)\deg(L_{l+1}^{\cdot l-d+1} \cdot H_{l+1}^{\cdot d-1} \cdot E)}}{\deg\left(L_l^{\cdot l-d+1} \cdot H_l^{\cdot d-1}\right)}.$$

Thus, if we set $A(x,y) = q^{k\sqrt{xy}}$, then

$$\#\{V \in T_n \mid \alpha_n(V) = D, \beta_n(V) = E\} \leq A(h_{n-1}(D), h_{l+1}(E)).$$

Here let us estimate $\#\{D \in T_{l+1} \mid h_{l+1}(D) \leq h\}$. In this case, D is a divisor on $(\mathbb{P}^1_{\mathbb{F}_q})^{l+1}$. Thus,

$$\deg(L_{l+1}^{\cdot l-d} \cdot H_{l+1}^{\cdot d} \cdot D) = d!(l-d)!(\deg_{d+1}(D) + \cdots + \deg_{l+1}(D))$$

and

$$\deg(L_{l+1}^{\cdot l-d+1} \cdot H_{l+1}^{\cdot d-1} \cdot D) = (d-1)!(l-d+1)!(\deg_1(D) + \cdots + \deg_{l+1}(D)).$$

Therefore, $\deg_i(D) \leq h$ for $1 = 1, \ldots d$ and $\deg_j(D) \leq k$ for $j = d+1, \ldots, l+1$. Hence, by Proposition 2.1.2, if we set $B(h) = q^{C_1 \cdot h^d}$ for some constant C_1, then

$$\#\{D \in T_{l+1} \mid h_{l+1}(D) \leq h\} \leq B(h)$$

for $h \geq 1$.

Gathering the above observations, we can see that $\{T_n\}_{n=l+1}^{\infty}$ is a counting system. Thus, by Lemma 1.2.1,

$$\#\{V \in T_n \mid h_n(V) \leq h\} \leq B(h)^{n-l} A(h,h)^{n-l-1} \leq q^{(n-l)C_1 \cdot h^d + k(n-l-1)h}.$$

for $h \geq 1$. Hence, we get our proposition. □

3.2. **Northcott's type theorem in the geometric case.** In this subsection, we prove the main theorem of this section. First, let us recall our situation.

Let B be a d-dimensional projective variety over $\mathbb{F}_q$. We assume that $d \geq 1$. Let H be a nef and big line bundle on B. Let X be a projective variety over $\mathbb{F}_q$ and $f : X \to B$ a surjective morphism over $\mathbb{F}_q$. Let L be a nef line bundle on X. In the following, the subscript η of an object on X means its restriction on the generic fiber of $f : X \to B$.

Theorem 3.2.1. *If L_η is ample, then, for a fixed k, there is a constant C such that*

$$\#\left\{ V \in Z_l^{\mathrm{eff}}(X/B) \;\middle|\; \begin{array}{l} \deg(L_\eta^{\cdot l-d} \cdot V_\eta) \leq k, \\ \deg(L^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) \leq h \end{array} \right\} \leq q^{C \cdot h^d}$$

for all $h \geq 1$.

Proof. In this proof, we consider the estimate of the number of cycles in the following cases:

(A) X, B, $f : X \to B$, L and H are given as follows: $B = (\mathbb{P}^1_{\mathbb{F}_q})^d$ and $X = (\mathbb{P}^1_{\mathbb{F}_q})^d \times (\mathbb{P}^1_{\mathbb{F}_q})^e = (\mathbb{P}^1_{\mathbb{F}_q})^{d+e}$. Let $p_i : B \to \mathbb{P}^1_{\mathbb{F}_q}$ be the projection to the i-th factor. Similarly, let $q_j : X \to \mathbb{P}^1_{\mathbb{F}_q}$ be the projection to the j-th factor. $f : X \to B$ is given the natural projection $q_1 \times \cdots \times q_d$, namely, $f(x_1, \ldots, x_{d+e}) = (x_1, \ldots, x_d)$. Moreover, $H = p_1^*(\mathcal{O}(1)) \otimes \cdots \otimes p_d^*(\mathcal{O}(1))$ and $L = q_{d+1}^*(\mathcal{O}(1)) \otimes \cdots \otimes q_{d+e}^*(\mathcal{O}(1))$.

(B) B and H are arbitrary. $X = B \times (\mathbb{P}^1_{\mathbb{F}_q})^e$, $f : X \to B$ is given by the projection to the first factor and $L = q^*(\mathcal{O}(1, \ldots, 1))$. Here $q : X \to (\mathbb{P}^1_{\mathbb{F}_q})^e$ is the natural projection.

(C) B and H are arbitrary. $X = B \times \mathbb{P}^e_{\mathbb{F}_q}$, $f : X \to B$ is given the natural projection and $L = q^*(\mathcal{O}(1))$. Here $q : X \to \mathbb{P}^e_{\mathbb{F}_q}$ is the natural projection.

(D) X, B, $f : X \to B$, L and H are arbitrary.

Let us start the estimate of the number of cycles in each case.

Step 1: First let us consider the case (A). We set $\tilde{L} = \bigotimes_{i=1}^{d+e} q_i^*(\mathcal{O}(1))$ Then, since $\tilde{L} = L \otimes f^*(H)$, we can see

$$\deg(\tilde{L}^{\cdot l-d} \cdot f^*(H)^{\cdot d} \cdot V) = \deg(L^{\cdot l-d} \cdot f^*(H)^{\cdot d} \cdot V) = d! \deg(L_\eta^{\cdot l-d} \cdot V_\eta)$$

and

$$\begin{aligned}
&\widehat{\deg}(\tilde{L}^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) \\
&\qquad = \widehat{\deg}(L^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) + (l-d+1)\deg(L^{\cdot l-d} \cdot f^*(H)^{\cdot d} \cdot V).
\end{aligned}$$

Thus, in this case, our assertion of the theorem follows from Proposition 3.1.1.

Step 2: Next let us consider the case (B). By virtue of Noether's normalization theorem, there is a dominant rational map $B \dashrightarrow (\mathbb{P}^1_{\mathbb{F}_q})^d$. Let the following diagram

$$\begin{tikzcd}
& B' \arrow[dl, "\nu"'] \arrow[dr, "\nu'"] & \\
B & & (\mathbb{P}^1_{\mathbb{F}_q})^d
\end{tikzcd}$$

be the graph of the rational map $B \dashrightarrow (\mathbb{P}^1_{\mathbb{F}_q})^d$. Here we set $X' = B' \times (\mathbb{P}^1_{\mathbb{F}_q})^e$, $B'' = (\mathbb{P}^1_{\mathbb{F}_q})^d$ and $X'' = (\mathbb{P}^1_{\mathbb{F}_q})^d \times (\mathbb{P}^1_{\mathbb{F}_q})^e$. Let $f' : X' \to B'$, $q' : X' \to (\mathbb{P}^1_{\mathbb{F}_q})^e$, $f'' : X'' \to (\mathbb{P}^1_{\mathbb{F}_q})^d$, and $q'' : X'' \to (\mathbb{P}^1_{\mathbb{F}_q})^e$ be the natural projections. Moreover, we set $L' = q'^*(\mathcal{O}(1, \ldots, 1))$ and $L'' = q''^*(\mathcal{O}(1, \ldots, 1))$.

Let $\alpha : Z_l^{\mathrm{eff}}(X/B) \to Z_l^{\mathrm{eff}}(X'/B')$ be a homomorphism given by the strict transform in terms of $\nu \times \mathrm{id} : X' \to X$. Further, let $\beta : Z_l^{\mathrm{eff}}(X'/B') \to Z_l^{\mathrm{eff}}(X''/B'')$ be the homomorphism given by the push forward $(\nu' \times \mathrm{id})_*$ of cycles. Since H is nef and big, there is a positive integer a such that

$$H^0(B', \nu^*(H)^{\otimes a} \otimes \nu'^*(\mathcal{O}(-1, \ldots, -1))) \neq 0.$$

Then, by Sublemma 2.2.3,

$$\begin{aligned}
&a^{d-1}\deg(L^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) \\
&\qquad = a^{d-1}\deg((\nu \times \mathrm{id})^*(L)^{\cdot l-d+1} \cdot (\nu \times \mathrm{id})^*(f^*(H))^{\cdot d-1} \cdot \alpha(V)) \\
&\qquad = \deg(L'^{\cdot l-d+1} \cdot f'^*(\nu^*(H)^{\otimes a})^{\cdot d-1} \cdot \alpha(V)) \\
&\qquad \geq \deg(L'^{\cdot l-d+1} \cdot f'^*(\nu'^*(\mathcal{O}(1, \ldots, 1))^{\cdot d-1} \cdot \alpha(V)) \\
&\qquad = \deg(L''^{\cdot l-d+1} \cdot f''^*(\mathcal{O}(1, \ldots, 1))^{\cdot d-1} \cdot \beta(\alpha(V))).
\end{aligned}$$

Moreover,

$$\begin{aligned}\deg(L_{\eta}^{\cdot l-d+1} \cdot V_{\eta}) &= \deg({L'}_{\eta'}^{\cdot l-d+1} \cdot \alpha(V)_{\eta'}) \\ &= \deg({L''}_{\eta''}^{\cdot l-d+1} \cdot \beta(\alpha(V))_{\eta''}),\end{aligned}$$

where the subscripts η' and η'' means the restrictions of objects to the generic fibers f' and f'' respectively.

For a fixed $V'' \in Z_l^{\mathrm{eff}}(X''/B'')$, we claim that

$$\log_q \#\{V' \in Z_l^{\mathrm{eff}}(X'/B') \mid \beta(V') = V''\} \leq \deg(\nu') \deg({L''}_{\eta''}^{l-d+1} \cdot V''_{\eta''}).$$

Let B''_0 be the maximal Zariski open set of B'' such that ν' is finite over B''_0. We set $B'_0 = {\nu'}^{-1}(B''_0)$. Then, the natural homomorphisms

$$Z_l^{\mathrm{eff}}(X'/B') \to Z_l^{\mathrm{eff}}(X'_0/B'_0) \quad \text{and} \quad Z_l^{\mathrm{eff}}(X''/B'') \to Z_l^{\mathrm{eff}}(X''_0/B''_0)$$

are bijective, where $X'_0 = B'_0 \times (\mathbb{P}^1_{\mathbb{F}_q})^e$ and $X''_0 = B''_0 \times (\mathbb{P}^1_{\mathbb{F}_q})^e$. Thus, by virtue of Lemma 1.3.2, if we set $V'' = \sum_i a_i W_i$, then

$$\log_q \#\{V' \in Z_l^{\mathrm{eff}}(X'/B') \mid \beta(V') = V''\} \leq \deg(\nu') \sum_i a_i.$$

On the other hand,

$$\sum_i a_i \leq \sum_i a_i \deg({L''}_{\eta''}^{l-d+1} \cdot W_{i\eta''}) = \deg({L''}_{\eta''}^{l-d+1} \cdot V''_{\eta''}).$$

Therefore, we get our claim.

Hence, by the above observations and Step 1, we have our case.

Step 3: Let us consider the case (C). We prove our theorem in this case by induction on e. If $e = l - d$, then our assertion is obvious. Thus we assume that $e > l - d$. Let the following diagram

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P}^e_{\mathbb{F}_q} & & (\mathbb{P}^1_{\mathbb{F}_q})^e \end{tikzcd}$$

be the graph of the rational map $\mathbb{P}^e_{\mathbb{F}_q} \dashrightarrow (\mathbb{P}^1_{\mathbb{F}_q})^e$ given by

$$(X_0 : \cdots : X_e) \mapsto (X_0 : X_1) \times \cdots \times (X_0 : X_e).$$

Then, as in Lemma 2.2.2, there is an effective Cartier divisor E on Y such that $\mu(E) \subset \{X_0 = 0\}$ and $\mu^*(\mathcal{O}(e)) = \nu^*(\mathcal{O}(1, \ldots, 1)) \otimes \mathcal{O}_Y(E)$. Here we set $X' = B \times (\mathbb{P}^1_{\mathbb{F}_q})^e$ and $L' = {q'}^*(\mathcal{O}(1, \ldots, 1))$, where $q' : X \to (\mathbb{P}^1_{\mathbb{F}_q})^e$ is the natural projection. Moreover, $f' : X' \to B$ is given by the natural projection. Then, for

$$V \in Z_l^{\mathrm{eff}}(X; X \setminus B \times \{X_0 = 0\}),$$

by Sublemma 2.2.3,

$$\begin{aligned}&e^{l-d+1} \deg(L^{\cdot l-d+1} \cdot f^*(H)^{\cdot d-1} \cdot V) \\ &\qquad = \deg((\mu \times \mathrm{id})^*(L^{\otimes e})^{\cdot l-d+1} \cdot (\mu \times \mathrm{id})^* f^*(H)^{\cdot d-1} \cdot V') \\ &\qquad \geq \deg((\nu \times \mathrm{id})^*(L')^{\cdot l-d+1} \cdot (\nu \times \mathrm{id})^* {f'}^*(H)^{\cdot d-1} \cdot V') \\ &\qquad = \deg({L'}^{\cdot l-d+1} \cdot {f'}^*(H)^{\cdot d-1} \cdot (\nu \times \mathrm{id})_*(V')),\end{aligned}$$

where V' is the strict transform of V by $\mu \times \mathrm{id}$. Further,

$$\begin{aligned} e^{l-d}\deg(L_{\eta}^{\cdot l-d}\cdot V_{\eta}) &= \deg((\mu\times\mathrm{id})^*(L^{\otimes e})_{\eta''}^{\cdot l-d}\cdot V'_{\eta''}) \\ &\geq \deg((\nu\times\mathrm{id})^*(L')_{\eta''}^{\cdot l-d}\cdot V'_{\eta''}) \\ &= \deg({L'}_{\eta'}^{\cdot l-d}\cdot(\nu\times\mathrm{id})_*(V')_{\eta'}), \end{aligned}$$

where η' and η'' means the restriction of objects on X' and $B\times Y$ to the generic fibers $X'\to B$ and $B\times Y\to B$ respectively. Here $B\times\{X_0=0\}\simeq B\times\mathbb{P}^{e-1}_{\mathbb{F}_q}$. Thus, by hypothesis of induction and Step 2, we have our case.

Step 4: Finally we consider the case (D) (general case). Clearly we may assume that L_η is very ample. Thus, there are a positive integer e and a subvariety X' of $B\times\mathbb{P}^e_{\mathbb{F}_q}$ with the following properties:

(1) Let $f': X'\to B$ (resp. $q: X'\to\mathbb{P}^e_{\mathbb{F}_q}$) be the projection to the first factor (resp. the second factor). There is a non-empty Zariski open set B_0 of B such that $f^{-1}(B_0)$ is isomorphic to ${f'}^{-1}(B_0)$ over B_0. We denote this isomorphism by ι.

(2) If we set $L'=q^*(\mathcal{O}(1))$, then $L|_{f^{-1}(B_0)}=\iota^*\left(L'|_{{f'}^{-1}(B_0)}\right)$.

Let

$$\begin{tikzcd} & X'' \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ X \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

be the graph of the rational map induced by ι. We denote $f\cdot\mu=f'\cdot\mu'$ by f''. By the property (2),

$$f''_*(\mu^*(L)\otimes{\mu'}^*({L'}^{\otimes-1}))\neq 0.$$

Thus, we can find an ample line bundle A on B such that

$$H^0(X'',\mu^*(L\otimes f^*(A))\otimes{\mu'}^*({L'}^{\otimes-1}))\neq 0.$$

Let us choose a non-zero element s of

$$H^0(X'',\mu^*(L\otimes f^*(A))\otimes{\mu'}^*({L'}^{\otimes-1})).$$

Since $f''(\mathrm{Supp}(\mathrm{div}(s))\neq B$, by Sublemma 2.2.3,

$$\begin{aligned} &\deg((L\otimes f^*(A))^{\cdot l-d+1}\cdot f^*(H)^{\cdot d-1}\cdot V) \\ &\qquad = \deg(\mu^*(L\otimes f^*(A))^{\cdot l-d+1}\cdot\mu^*f^*(H)^{\cdot d-1}\cdot V') \\ &\qquad \geq \deg({\mu'}^*(L')^{\cdot l-d+1}\cdot{\mu'}^*{f'}^*(H)^{\cdot d-1}\cdot V') \\ &\qquad = \deg({L'}^{\cdot l-d+1}\cdot{f'}^*(H)^{\cdot d-1}\cdot\mu'_*(V')), \end{aligned}$$

where V' is the strict transform of V by μ. Moreover,

$$\begin{aligned} &\deg((L\otimes f^*(A))^{\cdot l-d+1}\cdot f^*(H)^{\cdot d-1}\cdot V) = \\ &\qquad \deg(L^{\cdot l-d+1}\cdot f^*(H)^{\cdot d-1}\cdot V) \\ &\qquad\qquad +(l-d+1)\deg(A\cdot H^{\cdot d-1})\deg(L_\eta^{\cdot l-d}\cdot V_\eta). \end{aligned}$$

Therefore, by Step 3, we get our theorem. □

3.3. **Geometric height functions defined over a finitely generated field over $\mathbb{F}_q$.** Here we consider a consequence of Theorem 3.2.1. Let K be a finitely generated field over $\mathbb{F}_q$ with $d = \operatorname{tr.deg}_{\mathbb{F}_q}(K) \geq 1$. Let X be a projective variety over K and L a line bundle on X. Here we fix a projective variety B and a nef and big line bundle H on B such that the function field of B is K. We choose a pair $(\mathcal{X}, \mathcal{L})$ with the following properties:

(1) $\mathcal{X}$ is a projective variety over $\mathbb{F}_q$ and there is a morphism $f : \mathcal{X} \to B$ over $\mathbb{F}_q$ such that X is the generic fiber of f.
(2) $\mathcal{L}$ is a $\mathbb{Q}$-line bundle on $\mathcal{X}$ (i.e., $\mathcal{L} \in \operatorname{Pic}(\mathcal{X}) \otimes \mathbb{Q}$) such that $\mathcal{L}|_X$ coincides with L in $\operatorname{Pic}(X) \otimes \mathbb{Q}$.

The pair $(\mathcal{X}, \mathcal{L})$ is called a *model of* (X, L).

For $x \in X(\overline{K})$, let Δ_x be the closure of the image of $\operatorname{Spec}(\overline{K}) \xrightarrow{x} X \hookrightarrow \mathcal{X}$. Then, the height function of (X, L) with respect to (B, H) and $(\mathcal{X}, \mathcal{L})$ is defined by

$$h^{(B,H)}_{(\mathcal{X},\mathcal{L})}(x) = \frac{\deg(\mathcal{L} \cdot f^*(H)^{d-1} \cdot \Delta_x)}{[K(x) : K]}.$$

It is not difficult to see that if $(\mathcal{X}', \mathcal{L}')$ is another model of (X, L), then there is a constant C such that

$$|h^{(B,H)}_{(\mathcal{X},\mathcal{L})}(x) - h^{(B,H)}_{(\mathcal{X}',\mathcal{L}')}(x)| \leq C$$

for all $x \in X(\overline{K})$ (cf. [6, the proof of Proposition 3.3.3]). Thus, the height function is uniquely determined modulo bounded functions. In this sense, we denote the class of $h^{(B,H)}_{(\mathcal{X},\mathcal{L})}$ modulo bounded functions by $h^{(B,H)}_L(x)$. As a corollary of Theorem 3.2.1, we have the following.

Corollary 3.3.2. *Let h_L be a representative of $h^{(B,H)}_L$. If L is ample, then, for a fixed k, there is a constant C such that*

$$\{x \in X(\overline{K}) \mid h_L(x) \leq h, [K(x) : K] \leq k\} \leq q^{C \cdot h^d}$$

for all $h \geq 1$.

Proof. Since L is ample, we can find a model $(\mathcal{X}, \mathcal{L})$ of (X, L) such that $\mathcal{L}$ is nef (cf. Step 4 of Theorem 3.2.1). Thus, our assertion follows from Theorem 3.2.1. □

4. Preliminaries for the arithmetic case

In this section, we prepare notation and results for considering cycles in the arithmetic case. In this case, we use Arakelov intersection theory instead of the usual geometric intersection theory. §4.1 and §4.2 contain notation in Arakelov geometry and the proofs of miscellaneous results. In §4.3, we introduce several kinds of norms of polynomials and compare each norm with another one, which is useful to count divisors in $(\mathbb{P}^1_{\mathbb{Z}})^n$.

4.1. **Arakelov geometry.** Here we fix notation in Arakelov geometry (for details, see [6]). In this paper, a flat and quasi-projective integral scheme over $\mathbb{Z}$ is called an *arithmetic variety.* If it is smooth over $\mathbb{Q}$, then it is said to be *generically smooth.*

Let X be a generically smooth arithmetic variety. A pair (Z, g) is called an *arithmetic cycle of codimension* p if Z is a cycle of codimension p and g is a current of type $(p-1,p-1)$ on $X(\mathbb{C})$. We denote by $\widehat{Z}^p(X)$ the set of all arithmetic cycles on X. We set

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\sim,$$

where $\sim$ is the arithmetic linear equivalence.

Let $\overline{L} = (L, \|\cdot\|)$ be a C^∞-hermitian line bundle on X. Then, the homomorphism

$$\widehat{c}_1(\overline{L})\cdot : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+1}(X)$$

arising from $\overline{L}$ is define by

$$\widehat{c}_1(\overline{L}) \cdot (Z, g) = \left(\mathrm{div}(s) \text{ on } Z, [-\log(\|s\|_Z^2)] + c_1(\overline{L}) \wedge g\right),$$

where s is a rational section of $L|_Z$ and $[-\log(\|s\|_Z^2)]$ is a current given by $\phi \mapsto -\int_{Z(\mathbb{C})} \log(\|s\|_Z^2)\phi$.

Here we assume that X is projective. Then we can define the arithmetic degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{\dim X}(X) \to \mathbb{R}$$

by

$$\widehat{\deg}\left(\sum_P n_P P, g\right) = \sum_P n_P \log(\#(\kappa(P))) + \frac{1}{2}\int_{X(\mathbb{C})} g.$$

Thus, if C^∞-hermitian line bundles $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ are given, then we can get the number

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{\dim X})\right),$$

which is called the *arithmetic intersection number of* $\overline{L}_1, \ldots, \overline{L}_{\dim X}$.

Let X be a projective arithmetic variety. Note that X is not necessarily generically smooth. Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ be C^∞-hermitian line bundles on X. Choose a birational morphism $\mu : Y \to X$ such that Y is a generically smooth projective arithmetic variety. Then, we can see that the arithmetic intersection number

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{L}_1)) \cdots \widehat{c}_1(\mu^*(\overline{L}_{\dim X}))\right)$$

does not depend on the choice of the generic resolution of singularities $\mu : Y \to X$. Thus, we denote this number by

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{\dim X})\right).$$

Let $f : X \to Y$ be a morphism of projective arithmetic varieties. Let $\overline{L}_1, \ldots, \overline{L}_r$ be C^∞-hermitian line bundles on X, and $\overline{M}_1, \ldots, \overline{M}_s$ C^∞-hermitian line bundles on Y. If $r+s = \dim X$, then the following formula is called the projection formula:

$$\begin{aligned}
&(4.1.1)\quad \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_r) \cdot \widehat{c}_1(f^*(\overline{M}_1)) \cdots \widehat{c}_1(f^*(\overline{M}_s))\right) \\
&\qquad = \begin{cases} 0 & \text{if } s > \dim Y \\ \deg((L_1)_\eta \cdots (L_r)_\eta)\widehat{\deg}(\widehat{c}_1(\overline{M}_1) \cdots \widehat{c}_1(\overline{M}_s)) & \text{if } s = \dim Y \text{ and } r > 0 \\ \deg(f)\widehat{\deg}(\widehat{c}_1(\overline{M}_1) \cdots \widehat{c}_1(\overline{M}_s)) & \text{if } s = \dim Y \text{ and } r = 0, \end{cases}
\end{aligned}$$

where the subscript η means the restriction of line bundles to the generic fiber of $f : X \to Y$.

Let $\overline{L}_1, \ldots, \overline{L}_l$ be C^∞-hermitian line bundles on a projective arithmetic variety X. Let V be an l-dimensional integral closed subscheme on X. Then, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid V\right)$ is defined by

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1\big|_V) \cdots \widehat{c}_1(\overline{L}_l\big|_V)\right).$$

Note that if V is lying over a prime p with respect to $X \to \operatorname{Spec}(\mathbb{Z})$, then

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid V\right) = \log(p) \deg(L_1\big|_V \cdots L_l\big|_V).$$

Moreover, for an l-dimensional cycle $Z = \sum_i n_i V_i$ on X, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid Z\right)$ is given by

$$\sum_i n_i \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_l) \mid V_i\right).$$

Let $f : X \to Y$ be a morphism of projective arithmetic varieties. Let $\overline{M}_1, \ldots, \overline{M}_l$ be C^∞-hermitian line bundles on Y. Then, as a consequence of (4.1.1), we have

$$\widehat{\deg}\left(\widehat{c}_1(f^*(\overline{M}_1)) \cdots \widehat{c}_1(f^*(\overline{M}_l)) \mid Z\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{M}_1) \cdots \widehat{c}_1(\overline{M}_l) \mid f_*(Z)\right) \tag{4.1.2}$$

for all l-dimensional cycles Z on X.

4.2. **The positivity of C^∞-hermitian $\mathbb{Q}$-line bundles.** Here let us introduce several kinds of the positivity of a C^∞-hermitian $\mathbb{Q}$-line bundle. Let X be a projective arithmetic variety and $\widehat{\operatorname{Pic}}(X)$ the set of isometric classes of C^∞-hermitian line bundles on X. An element of $\widehat{\operatorname{Pic}}(X) \otimes \mathbb{Q}$ is called a C^∞-hermitian $\mathbb{Q}$-line bundle. For a C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{L}$ on X, we consider the following kinds of the positivity of $\overline{L}$. In the following, let d be a positive integer with $\overline{L}^{\otimes d} \in \widehat{\operatorname{Pic}}(X)$ (Note that the following definitions do not depend on the choice of d).

•**ample**: We say $\overline{L}$ is *ample* if L is ample on X, $c_1(\overline{L})$ is positive form on $X(\mathbb{C})$, and there is a positive number n such that $L^{\otimes dn}$ is generated by the set $\{s \in H^0(X, L^{\otimes dn}) \mid \|s\|_{\sup} < 1\}$.

•**nef**: We say $\overline{L}$ is *nef* if $c_1(\overline{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes Γ of X, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}) \mid \Gamma\right) \geq 0$.

•**big**: $\overline{L}$ is said to be *big* if $\operatorname{rk}_{\mathbb{Z}} H^0(X, L^{\otimes dm}) = O(m^{\dim X_{\mathbb{Q}}})$ and there is a non-zero section s of $H^0(X, L^{\otimes dn})$ with $\|s\|_{\sup} < 1$ for some positive integer n.

•**$\mathbb{Q}$-effective**: $\overline{L}$ is said to be $\mathbb{Q}$-*effective* if there is a positive integer n and a non-zero $s \in H^0(X, L^{\otimes dn})$ with $\|s\|_{\sup} \leq 1$.

•**pseudo-effective**: $\overline{L}$ is said to be *pseudo-effective* if there are (1) a sequence $\{\overline{L}_n\}_{n=1}^\infty$ of $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-line bundles, (2) C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{E}_1, \ldots, \overline{E}_r$ and (3) sequences $\{a_{1,n}\}_{n=1}^\infty, \ldots, \{a_{r,n}\}_{n=1}^\infty$ of rational numbers such that

$$\widehat{c}_1(\overline{L}) = \widehat{c}_1(\overline{L}_n) + \sum_{i=1}^r a_{i,n} \widehat{c}_1(\overline{E}_i)$$

in $\widehat{\operatorname{CH}}^1(X) \otimes \mathbb{Q}$ and $\lim_{n\to\infty} a_{i,n} = 0$ for all i, in other words, $\overline{L}$ is the limit of $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-line bundles. If $\overline{L}_1 \otimes \overline{L}_2^{\otimes -1}$ is pseudo-effective for C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \overline{L}_2$ on X, then we denote this by $\overline{L}_1 \succsim \overline{L}_2$.

•**of surface type**: $\overline{L}$ is said to be *of surface type* if there are a morphism $\phi : X \to X'$ of projective arithmetic varieties and a C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{L}'$ on X' such that $\dim X'_{\mathbb{Q}} = 1$ (i.e. X' is a projective arithmetic surface), $\overline{L}'$ is nef and big, and that $\phi^*(\overline{L}') = \overline{L}$ in $\widehat{\operatorname{Pic}}(X) \otimes \mathbb{Q}$.

In the following, we consider three lemmas which will be used later.

Lemma 4.2.3. *Let X be a projective arithmetic variety. Then, we have the following.*

(1) *Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}, \overline{M}_1, \ldots, \overline{M}_{\dim X}$ be nef C^∞-hermitian $\mathbb{Q}$-line bundles on X. If $\overline{L}_i \otimes \overline{M}_i^{\otimes -1}$ is pseudo-effective for every i, then*
$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{\dim X})\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_{\dim X})\right).$$

(2) *Let V be an effective cycle of dimension l and let $\overline{L}_1, \ldots, \overline{L}_l, \overline{M}_1, \ldots, \overline{M}_l$ be nef C^∞-hermitian $\mathbb{Q}$-line bundles on X such that, for each i, there is a non-zero global section $s_i \in H^0(X, L_i \otimes M_i^{\otimes -1})$ with $\|s_i\|_{\sup} \leq 1$. Let $V = \sum_j a_j V_j$ be the irreducible decomposition as a cycle. If $s_i|_{V_j} \neq 0$ for all i, j, then*
$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_l) \mid V\right) \geq \widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_l) \mid V\right).$$

Proof. (1) This lemma follows from [6, Proposition 2.3] and the following formula:
$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{\dim X})\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_{\dim X})\right) + \\ \sum_{i=1}^{\dim X} \widehat{\deg}\left(\widehat{c}_1(\overline{M}_1)\cdots\widehat{c}_1(\overline{M}_{i-1})\cdot\widehat{c}_1(\overline{L}_i \otimes \overline{M}_i^{\otimes -1})\cdot\widehat{c}_1(\overline{L}_{i+1})\cdots\widehat{c}_1(\overline{L}_{\dim X})\right).$$

(2) This is a consequence of (1). □

Next let us consider the following technical formula.

Lemma 4.2.4. *Let X be a projective arithmetic variety and d an integer with $1 \leq d \leq \dim X$. Let $X_1, \ldots, X_d$ be projective arithmetic surfaces (i.e. 2-dimensional projective arithmetic varieties) and $\phi_i : X \to X_i$ $(i = 1, \ldots, d)$ surjective morphisms. Let $\overline{L}_1, \ldots, \overline{L}_d$ be C^∞-hermitian $\mathbb{Q}$-line bundles on $X_1, \ldots, X_d$ respectively with $\deg((L_i)_{\mathbb{Q}}) > 0$ $(i = 1, \ldots, d)$, and let $\overline{H}_{d+1}, \ldots, \overline{H}_{\dim X}$ be C^∞-hermitian $\mathbb{Q}$-line bundles on X. We set $\overline{H}_i = \phi_i^*(\overline{L}_i)$ $(i = 1, \ldots, d)$ and $\overline{H} = \bigotimes_{i=1}^d \overline{H}_i$. Then,*
$$\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{\cdot d}\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right) \\
&\quad = d!\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right) \\
&\qquad + \frac{d!}{2}\sum_{i \neq j} \frac{\widehat{\deg}\left(\widehat{c}_1(\overline{L}_i)^{\cdot 2}\right)\deg\left(\prod_{\substack{l \neq j \\ 1 \leq l \leq d}} (H_l)_{\mathbb{Q}}\cdot(H_{d+1})_{\mathbb{Q}}\cdots(H_{\dim X})_{\mathbb{Q}}\right)}{\deg((L_i)_{\mathbb{Q}})},
\end{aligned}$$
where the subscript $\mathbb{Q}$ means the restriction to the generic fiber $X_{\mathbb{Q}}$ of $X \to \operatorname{Spec}(\mathbb{Z})$.

Proof. First of all,

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\overline{H})^{\cdot d}\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)\\
&=\sum_{\substack{a_1+\cdots+a_d=d\\ a_1\geq 0,\ldots,a_d\geq 0}}\frac{d!}{a_1!\cdots a_d!}\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)^{\cdot a_1}\cdots\widehat{c}_1(\overline{H}_d)^{\cdot a_d}\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right).
\end{aligned}
$$

Claim 4.2.4.1. *If $(a_1,\ldots,a_d)\neq(1,\ldots,1)$ and*

$$
\widehat{\deg}\left(\prod_{l=1}^{d}\widehat{c}_1(\overline{H}_l)^{\cdot a_l}\cdot\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)\neq 0,
$$

then there are $i,j\in\{1,\ldots,d\}$ such that $a_i=2$, $a_j=0$ and $a_l=1$ for all $l\neq i,j$.

Clearly, $a_l\leq 2$ for all l. Thus, there is i with $a_i=2$. Suppose that $a_j=2$ for some $j\neq i$. Then,

$$
\begin{aligned}
&\widehat{\deg}\left(\prod_{l=1}^{d}\widehat{c}_1(\overline{H}_l)^{\cdot a_l}\cdot\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)\\
&=\widehat{\deg}\left(\widehat{c}_1(\phi_i^*(\overline{L}_i))^{\cdot 2}\cdot\widehat{c}_1(\phi_j^*(\overline{L}_j))^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}\widehat{c}_1(\overline{H}_l)^{\cdot a_l}\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right).
\end{aligned}
$$

Thus, using the projection formula with respect to ϕ_i,

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\phi_i^*(\overline{L}_i))^{\cdot 2}\cdot\widehat{c}_1(\phi_j^*(\overline{L}_j))^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}\widehat{c}_1(\overline{H}_l)^{\cdot a_l}\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)\\
&\qquad=\widehat{\deg}(\widehat{c}_1(\overline{L}_i)^{\cdot 2})\deg\left(\phi_j^*(L_j)_{\eta_i}^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}(H_l)_{\eta_i}^{\cdot a_l}\cdot(H_{d+1})_{\eta_i}\cdots(H_{\dim X})_{\eta_i}\right),
\end{aligned}
$$

where η_i means the restriction of line bundles to the generic fiber of ϕ_i. Here $(X_j)_{\mathbb{Q}}$ is projective curve. Thus, we can see

$$
\deg\left(\phi_j^*(L_j)_{\eta_i}^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}(H_l)_{\eta_i}^{\cdot a_l}\cdot(H_{d+1})_{\eta_i}\cdots(H_{\dim X})_{\eta_i}\right)=0.
$$

This is a contradiction. Hence, we get our claim.

By the above claim, it is sufficient to see that

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\phi_i^*(\overline{L}_i))^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}\widehat{c}_1(\overline{H}_l)\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)\\
&\qquad=\frac{\widehat{\deg}\left(\widehat{c}_1(\overline{L}_i)^{\cdot 2}\right)\deg\left(\prod_{\substack{l\neq j\\ 1\leq l\leq d}}(H_l)_{\mathbb{Q}}\cdot(H_{d+1})_{\mathbb{Q}}\cdots(H_{\dim X})_{\mathbb{Q}}\right)}{\deg((L_i)_{\mathbb{Q}})}
\end{aligned}
$$

By the (arithmetic) projection formula with respect to ϕ_i,

$$\widehat{\deg}\left(\widehat{c}_1(\phi_i^*(\overline{L}_i))^{\cdot 2}\cdot\prod_{l=1,l\neq i,j}^{d}\widehat{c}_1(\overline{H}_l)\cdot\widehat{c}_1(\overline{H}_{d+1})\cdots\widehat{c}_1(\overline{H}_{\dim X})\right)$$
$$=\widehat{\deg}\left(\widehat{c}_1(\overline{L}_i)^{\cdot 2}\right)\deg\left(\prod_{\substack{l\neq i,j\\1\leq l\leq d}}(H_l)_{\eta_i}\cdot(H_{d+1})_{\eta_i}\cdots(H_{\dim X})_{\eta_i}\right).$$

Moreover, using the (geometric) projection formula with respect to ϕ_i again,

$$\deg\left(\prod_{\substack{l\neq j\\1\leq l\leq d}}(H_l)_{\mathbb{Q}}\cdot(H_{d+1})_{\mathbb{Q}}\cdots(H_{\dim X})_{\mathbb{Q}}\right)$$
$$=\deg((L_i)_{\mathbb{Q}})\deg\left(\prod_{\substack{l\neq i,j\\1\leq l\leq d}}(H_l)_{\eta_i}\cdot(H_{d+1})_{\eta_i}\cdots(H_{\dim X})_{\eta_i}\right).$$

Thus, we get our lemma. □

Finally let us consider the following technical lemma.

Lemma 4.2.5. *Let $\phi:\mathbb{P}^n_{\mathbb{Z}}\dashrightarrow(\mathbb{P}^1_{\mathbb{Z}})^n$ be the birational map given by*

$$(X_0:\ldots:X_{n_i})\mapsto(X_0:X_1)\times\cdots\times(X_0:X_n).$$

Let Σ be the boundary of $\mathbb{P}^n_{\mathbb{Z}}$, that is, $\Sigma=\{X_0=0\}$. Let B be a projective arithmetic variety and $\overline{H}_1,\ldots,\overline{H}_d$ nef C^∞-hermitian line bundles on B, where $d=\dim B_{\mathbb{Q}}$. Let $Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}}\times B;(\mathbb{P}^n_{\mathbb{Z}}\setminus\Sigma)\times B)$ be the set of effective cycles generated by l-dimensional integral closed subschemes T with $T\cap((\mathbb{P}^n_{\mathbb{Z}}\setminus\Sigma)\times B)\neq\emptyset$ (cf. (1.1.2)). For $V\in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}}\times B;(\mathbb{P}^n_{\mathbb{Z}}\setminus\Sigma)\times B)$, we denote by V' the strict transform of V by $\phi\times\mathrm{id}:\mathbb{P}^n_{\mathbb{Z}}\times B\dashrightarrow(\mathbb{P}^1_{\mathbb{Z}})^n\times B$. Let us fix a non-negative real number λ. Then,

$$n^{l-d}\widehat{\deg}\left(\widehat{c}_1(p^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))^{\cdot l-d}\cdot\widehat{c}_1(q^*(\overline{H}_1))\cdots\widehat{c}_1(q^*(\overline{H}_d))\,|\,V\right)$$
$$\geq\widehat{\deg}\left(\widehat{c}_1(p'^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1,\ldots,1)))^{\cdot l-d}\cdot\widehat{c}_1(q'^*(\overline{H}_1))\cdots\widehat{c}_1(q'^*(\overline{H}_d))\,|\,V'\right)$$

for all $V\in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}}\times B;(\mathbb{P}^n_{\mathbb{Z}}\setminus\Sigma)\times B)$, where $p:\mathbb{P}^n_{\mathbb{Z}}\times B\to\mathbb{P}^n_{\mathbb{Z}}$ and $p':(\mathbb{P}^1_{\mathbb{Z}})^n\times B\to(\mathbb{P}^1_{\mathbb{Z}})^n$ (resp. $q:\mathbb{P}^n_{\mathbb{Z}}\times B\to B$ and $q':(\mathbb{P}^1_{\mathbb{Z}})^n\times B\to B$) are the projections to the first factor (the second factor). Note that in the case $d=0$, we do not use the nef C^∞-hermitian line bundles $\overline{H}_1,\ldots,\overline{H}_d$.

Proof. Let Y ($\subseteq\mathbb{P}^n_{\mathbb{Z}}\times(\mathbb{P}^1_{\mathbb{Z}})^n$) be the graph of the rational map $\phi:\mathbb{P}^n_{\mathbb{Z}}\dashrightarrow(\mathbb{P}^1_{\mathbb{Z}})^n$. Let $\mu:Y\to\mathbb{P}^n_{\mathbb{Z}}$ and $\nu:Y\to(\mathbb{P}^1_{\mathbb{Z}})^n$ be the morphisms induced by the projections. Here we claim the following:

Claim 4.2.5.1. *There are an effective Cartier divisor E on Y, a non-zero section $s\in H^0(Y,\mathcal{O}_Y(E))$ and a C^∞-metric $\|\cdot\|_E$ of $\mathcal{O}_Y(E)$ such that*

(1) $\mathrm{div}(s)=E$, $\mu(E)\subseteq\Sigma$,

(2) $\mu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))^{\otimes n}=\nu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1,\ldots,1))\otimes(\mathcal{O}_Y(E),\|\cdot\|_E)$, *and that*

(3) $\|s\|_E(x) \leq 1$ *for all* $x \in Y(\mathbb{C})$.

Let Y_i $(\subseteq \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}})$ be the graph of the rational map $\mathbb{P}^n_{\mathbb{Z}} \dashrightarrow \mathbb{P}^1_{\mathbb{Z}}$ given by

$$(X_0 : \cdots : X_n) \mapsto (X_0 : X_i).$$

Let $\mu_i : Y_i \to \mathbb{P}^n_{\mathbb{Z}}$ and $\nu_i : Y_i \to \mathbb{P}^1_{\mathbb{Z}}$ be the morphisms induced by the projections $\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ respectively. Let $\pi_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ be the projection to the i-th factor. Moreover, let $h_i : Y \to Y_i$ be the morphism induced by $\mathrm{id} \times \pi_i : \mathbb{P}^n_{\mathbb{Z}} \times (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$. Consequently, we have the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[d, "h_i"] \arrow[dr, "\nu"] \arrow[ddl, "\mu"'] & \\
 & Y_i \arrow[dl, "\mu_i"] \arrow[dr, "\nu_i"'] & (\mathbb{P}^1_{\mathbb{Z}})^n \arrow[d, "\pi_i"] \\
\mathbb{P}^n_{\mathbb{Z}} & & \mathbb{P}^1_{\mathbb{Z}}
\end{tikzcd}$$

Note that Y_i is the blowing-up by the ideal sheaf I_i generated by X_0 and X_i. Thus there is an effective Cartier divisor E_i on Y_i with $I_i\mathcal{O}_{Y_i} = \mathcal{O}_{Y_i}(-E_i)$ and $\mu_i^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{Y_i}(-E_i) = \nu_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Let s_i be the canonical section of $\mathcal{O}_{Y_i}(E_i)$. We choose C^∞-metric $\|\cdot\|_i$ of $\mathcal{O}_{Y_i}(E_i)$ with

$$\mu_i^*(\mathcal{O}_{\mathbb{P}^n}(1), \|\cdot\|_{\mathrm{FS}_\lambda}) = \nu_i^*(\mathcal{O}_{\mathbb{P}^1}(1), \|\cdot\|_{\mathrm{FS}_\lambda}) \otimes (\mathcal{O}_{Y_i}(E_i), \|\cdot\|_i).$$

Let $(T_0 : T_1)$ be a coordinate of $\mathbb{P}^1_{\mathbb{Z}}$. Then, $\mu_i^*(X_0) = \nu_i^*(T_0) \otimes s_i$. Thus,

$$\frac{\exp(-\lambda)|X_0|}{\sqrt{|X_0|^2 + \cdots + |X_n|^2}} = \frac{\exp(-\lambda)|T_0|}{\sqrt{|T_0|^2 + |T_1|^2}} \|s_i\|_i,$$

which implies

$$\|s_i\|_i = \frac{\sqrt{|X_0|^2 + |X_i|^2}}{\sqrt{|X_0|^2 + \cdots + |X_n|^2}}$$

because $X_0 T_1 = X_i T_0$. Therefore, $\|s_i\|_i(x_i) \leq 1$ for all $x_i \in Y_i(\mathbb{C})$. We set $E = \sum_{i=1}^n h_i^*(E_i)$ and give a C^∞-metric $\|\cdot\|_E$ to $\mathcal{O}_Y(E)$ with

$$(\mathcal{O}_Y(E), \|\cdot\|_E) = \bigotimes_{i=1}^n h_i^*(\mathcal{O}_{Y_i}(E_i), \|\cdot\|_i).$$

Thus, if we set $s = h_1^*(s_1) \otimes \cdots \otimes h_n^*(s_n)$, then $s \in H^0(Y, \mathcal{O}_Y(E))$, $\mathrm{div}(s) = E$ and $\|s\|_E(x) \leq 1$ for all $x \in Y(\mathbb{C})$. Moreover, we have

$$\mu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))^{\otimes n} = \nu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1, \ldots, 1)) \otimes (\mathcal{O}_Y(E), \|\cdot\|_E).$$

Hence we get our claim.

For $V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}} \times B; (\mathbb{P}^n_{\mathbb{Z}} \setminus \Sigma) \times B)$, let V'' be the strict transform of V by $\mu \times \mathrm{id} : Y \times B \to \mathbb{P}^n_{\mathbb{Z}} \times B$. Let $p'' : Y \times B \to Y$ and $q'' : Y \times B \to B$ be the projections to the first factor and the second factors respectively. Then, by using the projection formula,

$$\widehat{\deg}\left(\widehat{c}_1(p^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(n)))^{\cdot l-d}\cdot\widehat{c}_1(q^*(\overline{H}_1))\cdots\widehat{c}_1(q^*(\overline{H}_d))\mid V\right)$$
$$=\widehat{\deg}\left(\widehat{c}_1(p''^*(\mu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(n))))^{\cdot l-d}\cdot\widehat{c}_1(q''^*(\overline{H}_1))\cdots\widehat{c}_1(q''^*(\overline{H}_d))\mid V''\right).$$

Moreover, by virtue of (2) of Lemma 4.2.3,

$$\widehat{\deg}\left(\widehat{c}_1(p''^*(\mu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(n))))^{\cdot l-d}\cdot\widehat{c}_1(q''^*(\overline{H}_1))\cdots\widehat{c}_1(q''^*(\overline{H}_d))\mid V''\right)$$
$$\geq\widehat{\deg}\left(\widehat{c}_1(p''^*(\nu^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1,\ldots,1))))^{\cdot l-d}\cdot\widehat{c}_1(q''^*(\overline{H}_1))\cdots\widehat{c}_1(q''^*(\overline{H}_d))\mid V''\right)$$

Thus, using the projection formula for $\nu\times\mathrm{id}: Y\times B\to(\mathbb{P}^1_{\mathbb{Z}})^n\times B$, we get our lemma because $(\nu\times\mathrm{id})_*(V'')=V'$. □

4.3. **Comparisons of norms of polynomials.** In this subsection, we introduce several kinds of norms of polynomials and compare each norm with another one.

Let $S_n=\mathbb{C}[z_1,\ldots,z_n]$ be the ring of n-variable polynomials over $\mathbb{C}$. We define norms $|f|_\infty$ and $|f|_2$ of $f=\sum_{i_1,\ldots,i_n}a_{i_1,\ldots,i_n}z_1^{i_1}\cdots z_n^{i_n}\in S_n$ as follows:

$$|f|_\infty=\max_{i_1,\ldots,i_n}\{|a_{i_1,\ldots,i_n}|\}\quad\text{and}\quad|f|_2=\sqrt{\sum_{i_1,\ldots,i_n}|a_{i_1,\ldots,i_n}|^2}.$$

Moreover, the degree of f with respect to the variable z_i is denoted by $\deg_i(f)$.

First of all, we have obvious inequalities:

$$|f|_\infty\leq|f|_2\leq\sqrt{(\deg_1(f)+1)\cdots(\deg_n(f)+1)}|f|_\infty. \tag{4.3.1}$$

We set

$$S_n^{(d_1,\ldots,d_n)}=\{f\in S_n\mid\deg_i(f)\leq d_i\quad(\forall i=1,\ldots,n)\}.$$

Note that

$$\dim_{\mathbb{C}}S_n^{(d_1,\ldots,d_n)}=(d_1+1)\cdots(d_n+1). \tag{4.3.2}$$

For $f_1,\ldots,f_l\in S_n$, we set

$$v(f_1,\ldots,f_l)=\exp\left(\int_{\mathbb{C}^n}\log\left(\max_i\{|f_i|\}\right)\omega_1\wedge\cdots\wedge\omega_n\right), \tag{4.3.3}$$

where ω_i's are the $(1,1)$-forms on $\mathbb{C}^n$ given by

$$\omega_i=\frac{\sqrt{-1}dz_i\wedge d\bar{z}_i}{2\pi(1+|z_i|^2)^2}.$$

Let us begin with the following proposition.

Proposition 4.3.4. *For $f_1,\ldots,f_l\in S_n^{(d_1,\ldots,d_n)}$, we have the following.*

(1) $\max_i\{|f_i|_\infty\}\leq 2^{d_1+\cdots+d_n}v(f_1,\ldots,f_l)$.
(2) $v(f_1,\ldots,f_l)\leq\sqrt{2}^{d_1+\cdots+d_n}\sqrt{(|f_1|_2)^2+\cdots+(|f_l|_2)^2}$.

Proof. (1) Since

$$\max_i\left\{\int_{\mathbb{C}^n}\log\left(|f_i|\right)\omega_1\wedge\cdots\wedge\omega_n\right\}\leq\int_{\mathbb{C}^n}\log\left(\max_i\{|f_i|\}\right)\omega_1\wedge\cdots\wedge\omega_n,$$

(1) is a consequence of [6, Lemma 4.1].

For the proof of (2), we set

$$\mathbb{D}_0 = \{z \in \mathbb{C} \mid 0 < |z| < 1\} \quad \text{and} \quad \mathbb{D}_1 = \{z \in \mathbb{C} \mid 1 < |z|\}.$$

Then,

$$\begin{aligned}\int_{\mathbb{C}^n} & \log\left(\max_i\{|f_i|\}\right)\omega_1 \wedge \cdots \wedge \omega_n \\ &= \sum_{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n} \int_{\mathbb{D}_{\epsilon_1}\times\cdots\times\mathbb{D}_{\epsilon_n}} \log\left(\max_i\{|f_i|\}\right)\omega_1 \wedge \cdots \wedge \omega_n.\end{aligned}$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, let us consider a holomorphic map

$$\varphi_\epsilon : \mathbb{D}_0 \times \cdots \times \mathbb{D}_0 \to \mathbb{D}_{\epsilon_1} \times \cdots \times \mathbb{D}_{\epsilon_n}$$

given by $\varphi_\epsilon(z_1, \ldots, z_n) = (z_1^{\iota(\epsilon_1)}, \ldots, z_n^{\iota(\epsilon_n)})$, where $\iota : \{0,1\} \to \{-1,1\}$ is a map given by $\iota(0) = 1$ and $\iota(1) = -1$. Then, since $\varphi_\epsilon^*(\omega_1 \wedge \cdots \wedge \omega_n) = \omega_1 \wedge \cdots \wedge \omega_n$,

$$\begin{aligned}\int_{\mathbb{D}_{\epsilon_1}\times\cdots\times\mathbb{D}_{\epsilon_n}} & \log\left(\max_i\{|f_i|\}\right)\omega_1 \wedge \cdots \wedge \omega_n \\ &= \int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_i(z_1^{\iota(\epsilon_1)}, \ldots, z_n^{\iota(\epsilon_n)})|\}\right)\omega_1 \wedge \cdots \wedge \omega_n.\end{aligned}$$

Here we can find $f_{i,\epsilon} \in S_n^{(d_1,\ldots,d_n)}$ such that

$$f_i(z_1^{\iota(\epsilon_1)}, \ldots, z_n^{\iota(\epsilon_n)}) = \frac{f_{i,\epsilon}(z_1, \ldots, z_n)}{z_1^{\epsilon_1 d_1} \cdots z_n^{\epsilon_n d_n}}$$

and $|f_i|_2 = |f_{i,\epsilon}|_2$. Note that

$$\int_{\mathbb{D}_0^n} \log(|z_i|)\omega_1 \wedge \cdots \wedge \omega_n = -\frac{\log(2)}{2^n}$$

for all i. Therefore,

$$\begin{aligned}\int_{\mathbb{D}_{\epsilon_1}\times\cdots\times\mathbb{D}_{\epsilon_n}} & \log\left(|\max_i\{|f_i|\}|\right)\omega_1 \wedge \cdots \wedge \omega_n \\ &= \int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_{i,\epsilon}|\}\right)\omega_1 \wedge \cdots \wedge \omega_n - \sum_{i=1}^n \epsilon_i d_i \int_{\mathbb{D}_0^n} \log(|z_i|)\omega_1 \wedge \cdots \wedge \omega_n \\ &= \int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_{i,\epsilon}|\}\right)\omega_1 \wedge \cdots \wedge \omega_n + \frac{\log(2)}{2^n}\sum_{i=1}^n \epsilon_i d_i.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_{\mathbb{C}^n} & \log\left(\max_i\{|f_i|\}\right)\omega_1 \wedge \cdots \wedge \omega_n \\ &= \sum_{\epsilon\in\{0,1\}^n} \int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_{i,\epsilon}|\}\right)\omega_1 \wedge \cdots \wedge \omega_n + \log(\sqrt{2})(d_1 + \cdots + d_n).\end{aligned}$$

Hence, by the lemma below (Lemma 4.3.5), we can conclude

$$\int_{\mathbb{C}^n} \log\left(\max_i\{|f_i|\}\right) \omega_1 \wedge \cdots \wedge \omega_n$$
$$\leq \sum_{\epsilon \in \{0,1\}^n} \frac{\log\left(\sqrt{(|f_1|_2)^2 + \cdots + (|f_l|_2)^2}\right)}{2^n} + \log(\sqrt{2})(d_1 + \cdots + d_n)$$
$$= \log\left(\sqrt{(|f_1|_2)^2 + \cdots + (|f_l|_2)^2}\right) + \log(\sqrt{2})(d_1 + \cdots + d_n).$$

□

Lemma 4.3.5. *For all $f_1, \ldots, f_l \in S_n$,*

$$\exp\left(\int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_i|\}\right) (2\omega_1) \wedge \cdots \wedge (2\omega_n)\right) \leq \sqrt{(|f_1|_2)^2 + \cdots + (|f_l|_2)^2}.$$

Proof. Let us begin with the following sublemma:

Sublemma 4.3.6. *Let M be a differential manifold and Ω a volume form on M with $\int_M \Omega = 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function with $\varphi'' \geq 0$. Let u be a real valued function on M. If u and $\varphi(u)$ are integrable on M, then $\varphi\left(\int_M u\Omega\right) \leq \int_M \varphi(u)\Omega$.*

Proof. We set $c = \int_M u\Omega$. Since the second derivative of φ is non-negative, we can see

$$(x - c)\varphi'(c) \leq \varphi(x) - \varphi(c)$$

for all $x \in \mathbb{R}$. Therefore, we get

$$\int_M (u - c)\varphi'(c)\Omega \leq \int_M (\varphi(u) - \varphi(c))\Omega.$$

On the other hand, the left hand side of the above inequality is zero, and the right hand side is $\int_M \varphi(u)\Omega - \varphi(c)$. Thus, we have our desired inequality. □

Let us go back to the proof of Lemma 4.3.5. Applying the above lemma to the case $\varphi = \exp$,

$$\exp\left(\int_{\mathbb{D}_0^n} \log\left(\max_i\{|f_i|^2\}\right) (2\omega_1) \wedge \cdots \wedge (2\omega_n)\right)$$
$$\leq \int_{\mathbb{D}_0^n} \max_i\left\{|f_i|^2\right\} (2\omega_1) \wedge \cdots \wedge (2\omega_n)$$
$$\leq \int_{\mathbb{D}_0^n} \sum_i |f_i|^2 (2\omega_1) \wedge \cdots \wedge (2\omega_n)$$

We set $f_i = \sum_{e_1,\ldots,e_n} a^{(i)}_{e_1,\ldots,e_n} z_1^{e_1} \cdots z_n^{e_n}$ for all i. Then

$$\sum_i \int_{\mathbb{D}_0^n} |f_i|^2 (2\omega_1) \wedge \cdots \wedge (2\omega_n) =$$
$$\sum_i \sum_{\substack{e_1,\ldots,e_n,\\ e'_1,\ldots,e'_n}} a^{(i)}_{e_1,\ldots,e_n} \overline{a^{(i)}_{e'_1,\ldots,e'_n}} \int_{\mathbb{D}_0^n} z_1^{e_1} \bar{z_1}^{e'_1} \cdots z_n^{e_n} \bar{z_n}^{e'_n} (2\omega_1) \wedge \cdots \wedge (2\omega_n).$$

It is easy to see that
$$\int_{\mathbb{D}_0^n} z_1^{e_1}\bar{z_1}^{e'_1}\cdots z_n^{e_n}\bar{z_n}^{e'_n}(2\omega_1)\wedge\cdots\wedge(2\omega_n)=0$$
if $(e_1,\ldots,e_n)\neq(e'_1,\ldots,e'_n)$. Moreover,
$$\int_{\mathbb{D}_0^n}|z_1|^{2e_1}\cdots|z_n|^{2e_n}(2\omega_1)\wedge\cdots\wedge(2\omega_n)=\left(\int_{\mathbb{D}_0}|z_1|^{2e_1}2\omega_1\right)\cdots\left(\int_{\mathbb{D}_0}|z_n|^{2e_n}2\omega_n\right).$$
Thus, it is sufficient to see that
$$\int_{\mathbb{D}_0}|z|^{2e}\frac{\sqrt{-1}dz\wedge d\bar{z}}{\pi(1+|z|^2)^2}\leq 1$$
for all $e\geq 0$. We set $z=r\exp(2\pi\sqrt{-1}\theta)$, then
$$\int_{\mathbb{D}_0}|z|^{2e}\frac{\sqrt{-1}dz\wedge d\bar{z}}{\pi(1+|z|^2)^2}=\int_0^1\frac{4r^{2e+1}}{(1+r^2)^2}dr=\int_0^1\frac{2t^e}{(t+1)^2}dt$$
If $e=0$, then the above integral is 1. Further if $e\geq 1$, then
$$\int_0^1\frac{2t^e}{(t+1)^2}dt\leq\int_0^1 2t^e dt=\frac{2}{e+1}\leq 1.$$
□

Next let us consider the following proposition, which tells us the behavior of the norm $|\cdot|_\infty$ by the product of two polynomials.

Proposition 4.3.7. *For $f,g\in\mathbb{C}[z_1,\ldots,z_n]$,*
$$|f\cdot g|_\infty\leq|f|_\infty\cdot|g|_\infty\cdot\prod_i^n(1+\min\{\deg_i(f),\deg_i(g)\}).$$

Proof. For $I\in(\mathbb{Z}_{\geq 0})^n$, the i-th entry of I is denoted by $I(i)$. A partial order '$\leq$' on $(\mathbb{Z}_{\geq 0})^n$ is defined as follows:
$$I\leq J\quad\stackrel{\text{def}}{\Longleftrightarrow}\quad I(i)\leq J(i)\text{ for all }i=1,\ldots,n$$
Moreover, for $I\in\mathbb{Z}_{\geq 0}^n$, the monomial $z_1^{I(1)}\cdots z_n^{I(n)}$ is denoted by z^I.

Let us fix two non-zero polynomials
$$f=\sum_{I\in(\mathbb{Z}_{\geq 0})^n}a_I z^I\quad\text{and}\quad g=\sum_{I\in(\mathbb{Z}_{\geq 0})^n}b_I z^I.$$
We set $I_1=(\deg_1(f),\ldots,\deg_n(f))$, $I_2=(\deg_1(g),\ldots,\deg_n(g))$ and
$$d=\prod_i^n(1+\min\{\deg_i(f),\deg_i(g)\}).$$
First, we note that, for a fixed $I\in\mathbb{Z}_{\geq 0}^n$,
$$\#\{(J,J')\in\mathbb{Z}_{\geq 0}^n\times\mathbb{Z}_{\geq 0}^n\mid J+J'=I,\ J\leq I_1\text{ and }J'\leq I_2\}\leq d.$$
On the other hand,
$$f\cdot g=\sum_I\left(\sum_{\substack{J+J'=I\\ J\leq I_1,J'\leq I_2}}a_J b_{J'}\right)z^I.$$

Thus,

$$|f \cdot g|_\infty \le \max_I \left\{ \sum_{\substack{J+J'=I \\ J \le I_1, J' \le I_2}} |a_J b_{J'}| \right\} \le \max_I \left\{ \sum_{\substack{J+J'=I \\ J \le I_1, J' \le I_2}} |f|_\infty |g|_\infty \right\} \le d|f|_\infty |g|_\infty$$

□

For $f \in \mathbb{C}[z_1, \ldots, z_n]$, we denote by $\mathrm{lc}_i(f)$ the coefficient of the highest terms of f as a polynomial of z_i, that is, if we set

$$f = a_n z_i^n + \cdots + a_0 \quad (a_i \in \mathbb{C}[z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n],\ a_n \neq 0),$$

then $\mathrm{lc}_i(f) = a_n$. Note that $\mathrm{lc}_i(0) = 0$ and

$$\mathrm{lc}_i : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n].$$

For an element σ of the n-th symmetric group $\mathfrak{S}_n$, we set

$$\mathrm{lc}_\sigma(f) = \mathrm{lc}_{\sigma(n)} \circ \cdots \circ \mathrm{lc}_{\sigma(1)}(f).$$

Then we have the following proposition, which gives a lower bound of $v(f)$.

Proposition 4.3.8. *For a non-zero $f \in \mathbb{C}[z_1, \ldots, z_n]$,*

$$\int_{\mathbb{C}^n} \log(|f|)\omega_1 \wedge \cdots \wedge \omega_n \ge \max_{\sigma \in \mathfrak{S}_n} \{\log(|\mathrm{lc}_\sigma(f)|)\}.$$

In particular, if $f \in \mathbb{Z}[z_1, \ldots, z_n]$, then

$$\int_{\mathbb{C}^n} \log(|f|)\omega_1 \wedge \cdots \wedge \omega_n \ge 0.$$

Proof. Changing the order of variables, it is sufficient to see that

$$\int_{\mathbb{C}^n} \log(|f|)\omega_1 \wedge \cdots \wedge \omega_n \ge \log(|\mathrm{lc}_n \circ \cdots \circ \mathrm{lc}_1(f)|).$$

We prove this by induction on n. First we assume $n = 1$. Then, for $f = \alpha(z - c_1) \cdots (z - c_l)$,

$$\int_{\mathbb{C}} \log(|f|)\omega_1 = \log|\alpha| + \frac{1}{2}\sum_{i=1}^{l} \log(1 + |c_i|^2) \ge \log|\alpha|.$$

Next we consider a general n. By the hypothesis of induction, we can see that

$$\int_{\mathbb{C}^{n-1}} \log(|f|)\omega_1 \wedge \cdots \wedge \omega_{n-1} \ge \log|\mathrm{lc}_{n-1} \circ \cdots \circ \mathrm{lc}_1(f)|$$

as a function with respect to z_n. Thus,

$$\int_{\mathbb{C}^n} \log(|f|)\omega_1 \wedge \cdots \wedge \omega_n \ge \int_{\mathbb{C}} \log|\mathrm{lc}_{n-1} \circ \cdots \circ \mathrm{lc}_1(f)|\omega_n \ge \log|\mathrm{lc}_n \circ \cdots \circ \mathrm{lc}_1(f)|.$$

□

5. Counting cycles in the arithmetic case

In this section, we consider an arithmetic analogue of §2. Actually, we prove the following:

> Let X be a projective arithmetic variety and $\overline{H}$ an ample C^∞-hermitian line bundle on X. Then, there is a constant C such that the number of l-dimensional cycles V with $\widehat{\deg}(\widehat{c}_1(\overline{H}) \mid V) \leq h$ is less than or equal to $\exp(C \cdot h^{l+1})$ for all $h \geq 1$.

A scheme for the proof of the above theorem is similar to the geometric case (cf. §2). However, some parts are much harder than the geometric case. Especially, upper and lower estimates of the number of divisors on $(\mathbb{P}^1_{\mathbb{Z}})^n$ are difficult, so that we treat them in the separate subsection §5.1. This upper estimate gives rise to the initial step of a counting system for cycles on $(\mathbb{P}^1_{\mathbb{Z}})^n$. Once we get this, we can obtain the upper estimate of cycles on $(\mathbb{P}^1_{\mathbb{Z}})^n$ similarly, which is treated in §5.2. As in §2.2, the results on $(\mathbb{P}^1_{\mathbb{Z}})^n$ can be generalized to estimates on an arbitrary arithmetic variety, which is considered in §5.3 and §5.4.

5.1. **Counting arithmetic divisors.** Here let us consider several problems concerning the number of arithmetic divisors with bounded arithmetic degree. Let us begin with the following proposition.

Proposition 5.1.1. *Let us fix a positive real number λ, a subset I of $\{1, \ldots, n\}$ and a function $\alpha : I \to \mathbb{Z}_{\geq 0}$. For a divisor D on $(\mathbb{P}^1_{\mathbb{Z}})^n$, we set*

$$\delta_\lambda(D) = \widehat{\deg}(\widehat{c}_1(p_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \cdots \widehat{c}_1(p_n^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \mid D),$$

where $p_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Then, there is a constant $C(\lambda, \alpha)$ depending only on λ and $\alpha : I \to \mathbb{Z}$ such that

$$\# \left\{ D \in Z_n^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \delta_\lambda(D) \leq h \text{ and } \deg_i(D) \leq \alpha(i) \text{ for all } i \in I \right\} \leq \exp\left(C(\lambda, \alpha) \cdot h^{n+1-\#(I)} \right)$$

for $h \geq 1$. (Note that in the case where $I = \emptyset$, no condition on $\deg_1(D), \ldots, \deg_n(D)$ is posed.)

Proof. Fix a basis $\{X_i, Y_i\}$ of $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1))$ of the i-th factor of $\left(\mathbb{P}^1_{\mathbb{Z}}\right)^n$. We denote by $\mathbb{Z}[X_1, Y_1, \ldots, X_n, Y_n]^{(k_1, \ldots, k_n)}$ the set of homogeneous polynomials of multidegree $(k_1, \ldots, k_n)$. Then,

$$H^0\left((\mathbb{P}^1_{\mathbb{Z}})^n, \bigotimes_{i=1}^n p_i^*(\mathcal{O}(k_i)) \right) = \mathbb{Z}[X_1, Y_1, \ldots, X_n, Y_n]^{(k_1, \ldots, k_n)}.$$

Let D be an effective divisor on $(\mathbb{P}^1_{\mathbb{Z}})^n$ with $\deg_i(D) = k_i$ $(i = 1, \ldots, n)$. Then there is

$$P \in \mathbb{Z}[X_1, Y_1, \ldots, X_n, Y_n]^{(k_1, \ldots, k_n)} \setminus \{0\}$$

with $\operatorname{div}(P) = D$. Let us evaluate

$$\widehat{\deg}\left(\widehat{c}_1(p_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \cdots \widehat{c}_1(p_n^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \cdot \widehat{c}_1\left(\bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_0}(k_i)) \right) \right)$$

in terms of P. Since $\widehat{\deg}\left(\widehat{c}_1\left(\overline{\mathcal{O}}^{\mathrm{FS}_0}_{\mathbb{P}^1_{\mathbb{Z}}}(1)\right)^2\right) = 1/2$, we can see

$$(\lambda + 1/2)(k_1 + \cdots + k_n) = \delta_\lambda(D) - \int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log \|P\|_{\mathrm{FS}_0} \omega_1 \wedge \cdots \wedge \omega_n,$$

where $\omega_i = p_i^*(c_1(\overline{\mathcal{O}}^{\mathrm{FS}}(1)))$ $(i = 1, \ldots, n)$. We set $p(x_1, \ldots, x_n) = P(x_1, 1, \ldots, x_n, 1)$. Then,

$$\|P\|_{\mathrm{FS}_0} = \frac{|p|}{(1 + x_1^2)^{k_1/2} \cdots (1 + x_n^2)^{k_n/2}}.$$

Note that

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log\left((1 + x_1^2)^{k_1/2} \cdots (1 + x_n^2)^{k_n/2}\right) \omega_1 \wedge \cdots \wedge \omega_n = \sum_{i=1}^n \frac{k_i}{2}.$$

Thus,

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log |p| \omega_1 \wedge \cdots \wedge \omega_n = \delta_\lambda(D) - \lambda(k_1 + \cdots + k_n). \tag{5.1.1.1}$$

On the other hand, by Proposition 4.3.4,

$$\begin{aligned}\log |P|_\infty = \log |p|_\infty &\leq \log(2)(\deg_1(p) + \cdots + \deg_n(p)) + \int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log |p| \omega_1 \wedge \cdots \wedge \omega_n \\ &\leq \log(2)(k_1 + \cdots + k_n) + \int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log |p| \omega_1 \wedge \cdots \wedge \omega_n,\end{aligned}$$

where $|P|_\infty$ is the maximal of the absolute values of coefficients of P. Thus,

$$\log |P|_\infty \leq \delta_\lambda(D) + (k_1 + \cdots + k_n)(\log 2 - \lambda). \tag{5.1.1.2}$$

We assume that $\delta_\lambda(D) \leq h$. Then, since it follows from Proposition 4.3.8 that

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log |p| \omega_1 \wedge \cdots \wedge \omega_n \geq 0,$$

(5.1.1.1) implies

$$k_1 + \cdots + k_n \leq h/\lambda. \tag{5.1.1.3}$$

Moreover, using (5.1.1.2), if $\lambda \leq \log 2$, then

$$\log |P|_\infty \leq h + (k_1 + \cdots + k_n)(\log 2 - \lambda) \leq h + \frac{h}{\lambda}(\log 2 - \lambda) = \frac{h \log 2}{\lambda}.$$

Thus, if we set

$$g(h, \lambda) = \begin{cases} \exp(h \log 2/\lambda) & \text{if } 0 < \lambda \leq \log 2 \\ \exp(h) & \text{if } \lambda > \log 2, \end{cases}$$

then

$$|P|_\infty \leq g(h, \lambda). \tag{5.1.1.4}$$

Therefore,

$$\begin{aligned}&\#\{P \in \mathbb{Z}[X_1, Y_1, \ldots, X_n, Y_n]^{(k_1, \ldots, k_n)} \setminus \{0\} \mid \delta_\lambda(\mathrm{div}(P)) \leq h\} \\ &\qquad\qquad \leq (2g(h, \lambda) + 1)^{(k_1+1)\cdots(k_n+1)}\end{aligned}$$

Hence if we set

$$N_\alpha(h) = \#\{D \in Z_n^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \delta_\lambda(D) \le h \text{ and } \deg_i(D) \le \alpha(i) \text{ for all } i \in I\},$$

then we can see

$$\begin{aligned} N_\alpha(h) &\le \sum_{\substack{k_1+\cdots+k_n \le h/\lambda \\ k_i \le \alpha(i) (\forall i \in I)}} (2g(h,\lambda)+1)^{(k_1+1)\cdots(k_n+1)} \\ &\le (h/\lambda+1)^{l-\#(I)} \left(\prod_{i\in I}(\alpha(i)+1)\right) (2g(h,\lambda)+1)^{(h/\lambda+1)^{n-\#(I)} \prod_{i\in I}(\alpha(i)+1)}. \end{aligned}$$

Note that in the case where $I=\emptyset$, the number $\prod_{i\in I}(\alpha(i)+1)$ in the above inequality is treated as 1. Thus, we get our lemma. □

Next we consider a lower estimate of the number of divisors, which is not easy because no member of a linear system has the same arithmetic degree.

Proposition 5.1.2. *Let us fix a positive real number λ. For a divisor D on $(\mathbb{P}^1_{\mathbb{Z}})^n$, we set*

$$\delta_\lambda(D) = \widehat{\deg}(\widehat{c}_1(p_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))\cdots \widehat{c}_1(p_n^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \mid D),$$

where $p_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Let $x_1,\ldots,x_s$ be closed points of $(\mathbb{P}^1_{\mathbb{Z}})^n$. Then, we have

$$\limsup_{h\to\infty} \frac{\log \#\{D \in \mathrm{Div}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \delta_\lambda(D) \le h \text{ and } x_i \notin \mathrm{Supp}(D) \text{ for all } i\}}{h^{n+1}} > 0.$$

Moreover, if $n \ge 1$, then

$$\limsup_{h\to\infty} \frac{\log \#\{D \in \mathrm{Div}^{\mathrm{eff}}_{\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \delta_\lambda(D) \le h \text{ and } x_i \notin \mathrm{Supp}(D) \text{ for all } i\}}{h^{n+1}} > 0,$$

where $\mathrm{Div}^{\mathrm{eff}}_{\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n)$ is the set of all effective divisors on $(\mathbb{P}^1_{\mathbb{Z}})^n$ generated by prime divisors flat over $\mathbb{Z}$.

Proof. Let us fix a coordinate $\{X_i, Y_i\}$ of the i-th factor of $(\mathbb{P}^1_{\mathbb{Z}})^n$. Then, note that

$$\bigoplus_{k_1\ge 0,\ldots,k_n\ge 0} H^0\left((\mathbb{P}^1_{\mathbb{Z}})^n, \bigotimes_{i=1}^n p_i^*(\mathcal{O}(k_i))\right) = \mathbb{Z}[X_1,Y_1,\ldots,X_n,Y_n].$$

We set $l = 4\prod_i \#(\kappa(x_i))$. Then, $l=0$ in $\kappa(x_i)$ for all i. Since $H = \bigotimes_{i=1}^n p_i^*(\mathcal{O}(1))$ is ample, there is a positive integer k_0 with $H^1((\mathbb{P}^1_{\mathbb{Z}})^n, H^{\otimes k_0} \otimes m_{x_1} \otimes \cdots \otimes m_{x_s}) = 0$, where m_{x_i} is the maximal ideal at x_i. Thus, the homomorphism

$$H^0((\mathbb{P}^1_{\mathbb{Z}})^n, H^{\otimes k_0}) \to \bigoplus_{i=1}^s H^{\otimes k_0} \otimes \kappa(x_i)$$

is surjective. Hence, there is $P_0 \in H^0((\mathbb{P}^1_{\mathbb{Z}})^n, H^{\otimes k_0})$ with $P_0(x_i) \ne 0$ for all i. Clearly, we may assume that P_0 is primitive as a polynomial in $\mathbb{Z}[X_1,Y_1,\ldots,X_n,Y_n]$.

For $m \ge 1$ and $Q \in H^0((\mathbb{P}^1_{\mathbb{Z}})^n, H^{\otimes mk_0})$, we set $\alpha_m(Q) = P_0^m + lQ$. Note that $\alpha_m(Q)(x_i) \ne 0$ for all i. Thus, we get a map

$$\phi_m : H^0((\mathbb{P}^1_{\mathbb{Z}})^n, H^{\otimes mk_0}) \to \{D \in \mathrm{Div}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid x_i \notin \mathrm{Supp}(D) \text{ for all } i\}$$

given by $\phi_m(Q) = \mathrm{div}(\alpha_m(Q))$. Here we claim that ϕ_m is injective. Indeed, if $\phi_m(Q) = \phi_m(Q')$, then $\alpha_m(Q) = \alpha_m(Q')$ or $\alpha_m(Q) = -\alpha_m(Q')$. Clearly, if

$\alpha_m(Q) = \alpha_m(Q')$, then $Q = Q'$, so that we assume $\alpha_m(Q) = -\alpha_m(Q')$. Then $P_0^m = -2\prod_i \#(\kappa(x_i))(Q+Q')$. Since P_0 is primitive, so is P_0^m. This is a contradiction.

We set $d = (1+k_0)^n$. Let us choose a positive number c with

$$c \geq \max\{\log(2d|P_0|_\infty), (\lambda+1)k_0 n\}.$$

Claim 5.1.2.1. *If* $|Q|_\infty \leq \dfrac{\exp(cm)}{2l}$, *then* $\delta_\lambda(\phi_m(Q)) \leq 2cm$.

We set $p_0 = P_0(x_1, 1, \ldots, x_n, 1)$ and $q = Q(x_1, 1, \ldots, x_n, 1)$. Then

$$\alpha_m(Q)(x_1, 1, \ldots, x_n, 1) = p_0^m + lq.$$

By (5.1.1.1) in the proof of Proposition 5.1.1,

$$\delta_\lambda(\phi_m(Q)) = \lambda k_0 mn + \int_{(\mathbb{P}^1_{\mathbb{C}})^n} \log|p_0^m + lq|\omega_1 \wedge \cdots \wedge \omega_n.$$

Thus, using (2) of Proposition 4.3.4 and (4.3.1),

$$\begin{aligned}
\delta_\lambda(\phi_m(Q)) &\leq \lambda k_0 mn + \frac{k_0 mn}{2}\log 2 + \frac{n\log(1+k_0 m)}{2} + \log|p_0^m + lq|_\infty \\
&\leq (\lambda+1)k_0 nm + \log|p_0^m + lq|_\infty.
\end{aligned}$$

On the other hand, using Lemma 4.3.7 and $\exp(c) \geq 2d|p_0|_\infty$,

$$\begin{aligned}
|p_0^m + lq|_\infty &\leq d^{m-1}|p_0|_\infty^m + l|q|_\infty \leq (d|p_0|_\infty)^m + l|Q|_\infty \leq (d|p_0|_\infty)^m + \frac{\exp(cm)}{2} \\
&\leq \frac{\exp(cm)}{2^m} + \frac{\exp(cm)}{2} \leq \exp(cm).
\end{aligned}$$

Therefore, since $c \geq (\lambda+1)k_0 n$, we have

$$\delta_\lambda(\phi_m(Q)) \leq (\lambda+1)k_0 nm + cm \leq cm + cm = 2cm.$$

Let us go back to the proof of our proposition. Since $H^0\left((\mathbb{P}^1_{\mathbb{Z}})^n, \bigotimes_{i=1}^n p_i^*(\mathcal{O}(k_0 m)\right)$ is a free abelian group of rank $(1+k_0 m)^n$,

$$\begin{aligned}
\#\left\{Q \in H^0\left((\mathbb{P}^1_{\mathbb{Z}})^n, \bigotimes_{i=1}^n p_i^*(\mathcal{O}(k_0 m))\right) \,\middle|\, |Q|_\infty \leq \frac{\exp(cm)}{2l}\right\} \\
&= \left(1 + 2\left[\frac{\exp(cm)}{2l}\right]\right)^{(1+k_0 m)^n} \\
&\geq \left(1 + \left[\frac{\exp(cm)}{2l}\right]\right)^{(1+k_0 m)^n} \\
&\geq \left(\frac{\exp(cm)}{2l}\right)^{(1+k_0 m)^n}.
\end{aligned}$$

Therefore, by the above claim,

$$\begin{aligned}
\log\#\{D \in \operatorname{Div}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \ &\delta_\lambda(D) \leq 2cm \text{ and } x_i \notin \operatorname{Supp}(D) \text{ for all } i\} \\
&\geq (1+k_0 m)^n(cm - \log(2l)).
\end{aligned}$$

Thus, we get the first assertion.

From now, we assume $n > 0$. We denote by $\mathcal{D}(h)$ (resp. $\mathcal{D}_{\mathrm{hor}}(h)$) the set

$$\{D \in \mathrm{Div}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \ \delta_\lambda(D) \leq h \text{ and } x_i \not\in \mathrm{Supp}(D) \text{ for all } i\}$$

$$\left(\text{resp.} \quad \{D \in \mathrm{Div}^{\mathrm{eff}}_{\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \ \delta_\lambda(D) \leq h \text{ and } x_i \not\in \mathrm{Supp}(D) \text{ for all } i\}\right).$$

For $D \in \mathcal{D}(h)$, let $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$ be the unique decomposition such that D_{hor} is horizontal over $\mathbb{Z}$ and D_{ver} is vertical over $\mathbb{Z}$. Note that $\delta_\lambda(D) = \delta_\lambda(D_{\mathrm{hor}}) + \delta_\lambda(D_{\mathrm{ver}})$, $\delta_\lambda(D_{\mathrm{hor}}) \geq 0$ and $\delta_\lambda(D_{\mathrm{ver}}) \geq 0$. Thus, $\delta_\lambda(D_{\mathrm{hor}}) \leq h$ and $\delta_\lambda(D_{\mathrm{ver}}) \leq h$. Therefore, we have a map

$$\beta_h : \mathcal{D}(h) \to \mathcal{D}_{\mathrm{hor}}(h)$$

given by $\beta_h(D) = D_{\mathrm{hor}}$. Since $\beta_h(D) = D$ for $D \in \mathcal{D}_{\mathrm{hor}}(h)$, β_h is surjective. Here let us consider a fiber $\beta_h^{-1}(D)$ for $D \in \mathcal{D}_{\mathrm{hor}}(h)$. First of all, an element $D' \in \beta_h^{-1}(D)$ has a form

$$D' = D + \mathrm{div}(n) \quad (n \in \mathbb{Z} \setminus \{0\}).$$

Since $\delta_\lambda(\mathrm{div}(n)) = \log|n| \leq h$, we can see that $\#\beta_h^{-1}(D) \leq \exp(h)$. Thus,

$$\#\mathcal{D}(h) = \sum_{D \in \mathcal{D}_{\mathrm{hor}}(h)} \#\beta_h^{-1}(D) \leq \sum_{D \in \mathcal{D}_{\mathrm{hor}}(h)} \exp(h) = \exp(h) \cdot \#\mathcal{D}_{\mathrm{hor}}(h).$$

Hence, we get

$$\limsup_{h \to \infty} \frac{\log \#\mathcal{D}_{\mathrm{hor}}(h)}{h^{n+1}} > 0.$$

□

Remark 5.1.3. In Proposition 5.1.2, we set $\overline{H}^\lambda = \bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))$. Then, using Lemma 4.2.4, we can see

$$\widehat{\deg}_{\overline{H}^\lambda}(D) = n! \left(\delta_\lambda(D) + \frac{1 + 4\lambda}{4} \sum_{i=1}^n \deg_i(D) \right).$$

Moreover, using Lemma 4.3.8 and (5.1.1.1),

$$\lambda \sum_{i=1}^n \deg_i(D) \leq \delta_\lambda(D).$$

Thus,

$$\widehat{\deg}_{\overline{H}^\lambda}(D) \leq \frac{(8\lambda + 1)n!}{4\lambda} \delta_\lambda(D).$$

Hence, Proposition 5.1.2 implies that if $n \geq 1$, then

$$\limsup_{h \to \infty} \frac{\log \#\{D \in \mathrm{Div}^{\mathrm{eff}}_{\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{H}^\lambda}(D) \leq h,\ x_i \not\in \mathrm{Supp}(D)\ (\forall i)\}}{h^{n+1}} > 0.$$

5.2. **Arithmetic cycles on the products of $\mathbb{P}^1_{\mathbb{Z}}$.** In this subsection, we consider the number of cycles on $(\mathbb{P}^1_{\mathbb{Z}})^n$. First, let us consider horizontal cycles.

Proposition 5.2.1. *Let us fix a positive real number λ. Let $p_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ be the projection to the i-th factor. We set $\overline{H}^\lambda = \bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))$. For $1 \leq l \leq n$, we denote by $Z^{\mathrm{eff}}_{l,\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n)$ the set of all effective cycles on $(\mathbb{P}^1_{\mathbb{Z}})^n$ generated by l-dimensional integral closed subschemes of $(\mathbb{P}^1_{\mathbb{Z}})^n$ which dominate* $\mathrm{Spec}(\mathbb{Z})$ *by the canonical morphism $(\mathbb{P}^1_{\mathbb{Z}})^n \to \mathrm{Spec}(\mathbb{Z})$. Then, there is a constant C such that*

$$\#\{V \in Z^{\mathrm{eff}}_{l,\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{H}^\lambda}(V) \leq h\} \leq \exp(C \cdot h^{l+1})$$

for all $h \geq 1$.

Proof. We set $\Sigma = \{I \mid I \subseteq [n], \#(I) = l-1\}$. Then, it is easy to see that

$$Z^{\mathrm{eff}}_{l,\mathrm{hor}}((\mathbb{P}^1_{\mathbb{Z}})^n) = \sum_{I \in \Sigma} Z^{\mathrm{eff}}_l((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}),$$

where p_I is the morphism given in (1.1.4). Thus, it is sufficient to show that there is a constant C' such that

$$\#\{V \in Z^{\mathrm{eff}}_l((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}) \mid \widehat{\deg}_{\overline{H}^\lambda}(V) \leq h\} \leq \exp(C' \cdot h^{l+1})$$

for all $h \geq 1$. By re-ordering the coordinate of $(\mathbb{P}^1_{\mathbb{Z}})^n$, we may assume that $I = [l-1]$. We denote $p_{[l-1]}$ by p. We set

$$T_n = Z^{\mathrm{eff}}_l((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}).$$

Let us see that $\{T_n\}_{n=l}^{\infty}$ is a counting system. First we define $h_n : T_n \to \mathbb{R}_{\geq 0}$ to be

$$h_n(V) = \widehat{\deg}_{\overline{H}^\lambda}(V).$$

Let $a_n : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^{n-1}$ and $b_n : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^l$ be the morphisms given by $a_n = p_{[n-1]}$ and $b_n = p_{[l-1]\cup\{n\}}$. Then, we have maps $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_l$ defined by $\alpha_n(V) = (a_n)_*(V)$ and $\beta_n(V) = (b_n)_*(V)$. Here, it is easy to see that

$$h_{n-1}(\alpha_n(V)) \leq h_n(V) \quad \text{and} \quad h_l(\beta_n(V)) \leq h_n(V)$$

for all $V \in T_n$. Moreover, by the following Lemma 5.2.2, if we set

$$e_l = \begin{cases} \widehat{\deg}\left(\widehat{c}_1(\otimes_{i=1}^{l-1} p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))\right) \text{ on } (\mathbb{P}^1_{\mathbb{Z}})^{l-1} & \text{if } l \geq 2 \\ \widehat{\deg}(\widehat{c}_1(\mathbb{Z}, \exp(-\lambda)|\cdot|)) \text{ on } \mathrm{Spec}(\mathbb{Z}) & \text{if } l = 1 \end{cases}$$

and

$$A(s,t) = \exp\left(\frac{s \cdot t}{e_l}\right),$$

then

$$\#\{x \in T_n \mid \alpha_n(x) = y, \beta_n(x) = z\} \leq A(h_{n-1}(y), h_l(z))$$

for all $y \in T_{n-1}$ and $z \in T_l$. Further, by Proposition 5.1.1, if we set

$$B(h) = \exp(C'' \cdot h^{l+1})$$

for some constant C'', then

$$\{x \in T_1 \mid h_1(x) \leq h\} \leq B(h)$$

for all $h \geq 1$. Thus, we can see that $\{T_n\}_{n=l}^{\infty}$ is a counting system. Therefore, by virtue of Lemma 1.2.1, we get our proposition. □

Lemma 5.2.2. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of projective arithmetic varieties. We assume that S is of dimension $l \geq 1$. Let $\overline{A}_1, \ldots, \overline{A}_l$ be nef C^∞-hermitian line bundles on X, $\overline{B}_1, \ldots, \overline{B}_l$ nef C^∞-hermitian line bundles on Y, and $\overline{C}_1, \ldots, \overline{C}_l$ nef C^∞-hermitian line bundles on S such that $\overline{A}_i \otimes f^*(\overline{C}_i)^{\otimes -1}$ and $\overline{B}_i \otimes g^*(\overline{C}_i)^{\otimes -1}$ are nef for all i and that $\widehat{\deg}\left(\widehat{c}_1(\overline{C}_1) \cdots \widehat{c}_1(\overline{C}_l)\right) > 0$. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projections to the first factor and* *the second factor respectively. Fix* $D \in Z_l^{\mathrm{eff}}(X/S)$ *and* $E \in Z_l^{\mathrm{eff}}(Y/S)$ *(for the definition of* $Z_l^{\mathrm{eff}}(X/S)$ *and* $Z_l^{\mathrm{eff}}(Y/S)$*, see* (1.1.2)*). Then,*

$$
\begin{aligned}
&\log\left(\#\left\{V \in Z_l^{\mathrm{eff}}(X \times_S Y/S) \mid p_*(V) = D \text{ and } q_*(V) = E\right\}\right)\\
&\qquad \leq \min\left\{ \frac{\widehat{\deg}(\widehat{c}_1(\overline{A}_1)\cdots\widehat{c}_1(\overline{A}_l) \mid D)\widehat{\deg}(\widehat{c}_1(\overline{B}_1)\cdots\widehat{c}_1(\overline{B}_l) \mid E)}{\widehat{\deg}\left(\widehat{c}_1(\overline{C}_1)\cdots\widehat{c}_1(\overline{C}_l)\right)^2},\right.\\
&\qquad\qquad \left.\frac{\sqrt{\theta(D)\theta(E)\widehat{\deg}(\widehat{c}_1(\overline{A}_1)\cdots\widehat{c}_1(\overline{A}_l) \mid D)\widehat{\deg}(\widehat{c}_1(\overline{B}_1)\cdots\widehat{c}_1(\overline{B}_l) \mid E)}}{\widehat{\deg}\left(\widehat{c}_1(\overline{C}_1)\cdots\widehat{c}_1(\overline{C}_l)\right)}\right\},
\end{aligned}
$$

where $\theta(D)$ *(resp.* $\theta(E)$*) is the number of irreducible components of* $\mathrm{Supp}(D)$ *(resp.* $\mathrm{Supp}(E)$*).*

Proof. We set $D = \sum_{i=1}^s a_i D_i$ and $E = \sum_{j=1}^t b_j E_j$. Then,

$$
\begin{aligned}
\text{(5.2.2.1)}\quad \widehat{\deg}(\widehat{c}_1(\overline{A}_1)\cdots\widehat{c}_1(\overline{A}_l) \mid D) &= \sum_i a_i \widehat{\deg}(\widehat{c}_1(\overline{A}_1)\cdots\widehat{c}_1(\overline{A}_l) \mid D_i)\\
&\geq \sum_{i=1}^s a_i \widehat{\deg}(\widehat{c}_1(f^*(\overline{C}_1))\cdots\widehat{c}_1(f^*(\overline{C}_l)) \mid D_i)\\
&= \sum_{i=1}^s a_i \deg(D_i \to S)\widehat{\deg}(\widehat{c}_1(\overline{C}_1)\cdots\widehat{c}_1(\overline{C}_l))).
\end{aligned}
$$

In the same way,

$$
\text{(5.2.2.2)}\quad \widehat{\deg}(\widehat{c}_1(\overline{B}_1)\cdots\widehat{c}_1(\overline{B}_l) \mid E) \geq \sum_{j=1}^t b_j \deg(E_j \to S)\widehat{\deg}(\widehat{c}_1(\overline{C}_1)\cdots\widehat{c}_1(\overline{C}_l)).
$$

Thus, in the same way as in Lemma 2.1.3, we have our assertion. □

Next we consider vertical cycles.

Proposition 5.2.3. *Let* $Z_{l,\mathrm{ver}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n)$ *be the set of effective cycles on* $(\mathbb{P}^1_{\mathbb{Z}})^n$ *generated by* l*-dimensional integral subschemes which are not flat over* $\mathbb{Z}$*. Then, there is a constant* $B(n,l)$ *depending only on* n *and* l *such that*

$$
\#\{V \in Z_{l,\mathrm{ver}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{\mathcal{O}}(1,\ldots,1)}(V) \leq h\} \leq \exp\left(B(n,l)\cdot h^{l+1}\right)
$$

for $h \geq 1$*.*

Proof. For simplicity, we denote $(\mathbb{P}^1_{\mathbb{Z}})^n$ and $\overline{\mathcal{O}}(1,\ldots,1)$ by X and $\overline{H}$ respectively. Let $\pi : X \to \mathrm{Spec}(\mathbb{Z})$ be the canonical morphism. Let k be a positive integer and $k = \prod_i p_i^{a_i}$ the prime decomposition of k. We set $X_{p_i} = \pi^{-1}([p_i])$. Then,

$$
\begin{aligned}
&\#\{V \in Z_{l,\mathrm{ver}}^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(V) = \log(k)\}\\
&\qquad\qquad = \prod_i \#\{V_i \in Z_{l,\mathrm{ver}}^{\mathrm{eff}}(X) \mid \pi(V_i) = [p_i] \text{ and } \deg_{H|_{X_{p_i}}}(V_i) = a_i\}.
\end{aligned}
$$

Let $C'(n,l)$ be a constant as in Proposition 2.1.1. We set

$$
C''(n,l) = \max\{C'(n,l), 1/\log(2)\}.
$$

Note that $C''(n,l)\log(p) \geq 1$ for all primes p. Thus,

$$\begin{aligned}\log \#\{V \in Z^{\mathrm{eff}}_{l,\mathrm{ver}}(X) \mid \widehat{\deg}_{\overline{H}}(V) = \log(k)\} &\leq \sum_i C''(n,l)\log(p_i)a_i^{l+1} \\ &\leq \left(\sum_i C''(n,l)\log(p_i)a_i\right)^{l+1} \\ &= C''(n,l)^{l+1}\log(k)^{l+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\#\{V \in Z^{\mathrm{eff}}_{l,\mathrm{ver}}(X) \mid \widehat{\deg}_{\overline{H}}(V) \leq h\} &\leq \sum_{k=1}^{[\exp(h)]} \#\{V \in Z^{\mathrm{eff}}_{l,\mathrm{ver}}(X) \mid \widehat{\deg}_{\overline{H}}(V) = \log(k)\} \\ &\leq \sum_{k=1}^{[\exp(h)]} \exp\left(C''(n,l)^{l+1}\log(k)^{l+1}\right) \\ &\leq \exp(h)\cdot\exp\left(C''(n,l)^{l+1}h^{l+1}\right) \\ &= \exp\left(C''(n,l)^{l+1}h^{l+1}+h\right)\end{aligned}$$

Thus, we get the proposition. □

By using Proposition 5.2.1 and Proposition 5.2.3, we have the following:

Theorem 5.2.4. *For all non-negative integers l and n with $0 \leq l \leq n$, there is a constant C such that*

$$\#\{V \in Z^{\mathrm{eff}}_l((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{H}^\lambda}(V) \leq h\} \leq \exp(C\cdot h^{l+1})$$

for all $h \geq 1$.

5.3. **A upper estimate of cycles with bounded arithmetic degree.** Here let us consider the following theorem, which is one of the main results of this paper.

Theorem 5.3.1. *Let us fix a positive real number λ. For all non-negative integers l with $0 \leq l \leq \dim X$, there is a constant C such that*

$$\#\{V \in Z^{\mathrm{eff}}_l(\mathbb{P}^n_{\mathbb{Z}}) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \leq h\} \leq \exp(C\cdot h^{l+1})$$

for all $h \geq 1$.

Proof. Let us consider the birational map $\phi : \mathbb{P}^n_{\mathbb{Z}} \dashrightarrow (\mathbb{P}^1_{\mathbb{Z}})^n$ given by

$$(X_0 : \cdots : X_n) \mapsto (X_0 : X_1)\times\cdots\times(X_0 : X_n).$$

We set $U = \mathbb{P}^n_{\mathbb{Z}} \setminus \{X_0 = 0\}$. Let $Z^{\mathrm{eff}}_l(\mathbb{P}^n_{\mathbb{Z}}; U)$ be the set of effective cycles generated by l-dimensional closed integral subschemes T with $T \cap U \neq \emptyset$ (cf. (1.1.2)). For $V \in Z^{\mathrm{eff}}_l(\mathbb{P}^n_{\mathbb{Z}}; U)$, we denote by V' the strict transform of V by ϕ. Then, by applying Lemma 4.2.5 in the case $B = \mathrm{Spec}(\mathbb{Z})$,

$$n^l\widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \geq \deg_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1,\ldots,1)}(V').$$

Moreover, if $V'_1 = V'_2$ for $V_1, V_2 \in Z^{\mathrm{eff}}_l(\mathbb{P}^n_{\mathbb{F}_q}; U)$, then $V_1 = V_2$. Therefore,

$$\begin{aligned}\text{(5.3.1.1)}\quad &\#\{V \in Z^{\mathrm{eff}}_l(\mathbb{P}^n_{\mathbb{Z}}; U) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \leq h\} \\ &\qquad\leq \#\{V' \in Z^{\mathrm{eff}}_l((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1,\ldots,1)}(V') \leq n^l h\}.\end{aligned}$$

On the other hand, since $\mathbb{P}^n_{\mathbb{Z}} \setminus U \simeq \mathbb{P}^{n-1}_{\mathbb{Z}}$,

$$\begin{aligned}(5.3.1.2)\quad \#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}}) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \leq h\} \\ \leq \#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^n_{\mathbb{Z}}; U) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \leq h\} \\ \cdot \#\{V \in Z_l^{\mathrm{eff}}(\mathbb{P}^{n-1}_{\mathbb{Z}}) \mid \widehat{\deg}_{\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)}(V) \leq h\}\end{aligned}$$

Thus, using (5.3.1.1), (5.3.1.2), Theorem 5.2.4 and the hypothesis of induction, we have our theorem. □

Corollary 5.3.2. *Let X be a projective arithmetic variety and $\overline{H}$ an ample C^∞-hermitian line bundle on X. For all non-negative integers l with $0 \leq l \leq \dim X$, there is a constant C such that*

$$\#\{V \in Z_l^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(V) \leq h\} \leq \exp(C \cdot h^{l+1})$$

for all $h \geq 1$.

Proof. Since X is projective over $\mathbb{Z}$, there is an embedding $\iota : X \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$ over $\mathbb{Z}$. We fix a positive real number λ. Then, there is a positive integer a such that $\overline{H}^{\otimes a} \otimes \iota^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(-1))$ is ample. Thus,

$$a^l \widehat{\deg}_{\overline{H}}(V) \geq \widehat{\deg}_{\iota^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))}(V)$$

for all $V \in Z_l^{\mathrm{eff}}(X)$. Hence, our assertion follows from Theorem 5.3.1. □

5.4. **A lower estimate of cycles with bounded arithmetic degree.** Here we consider the lower bound of the number of cycles.

Theorem 5.4.1. *Let X be a projective arithmetic variety and $\overline{H}$ an ample C^∞-hermitian line bundle on X. Then, for $0 \leq l < \dim X$,*

$$\limsup_{h\to\infty} \frac{\log \#\{V \in Z_l^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(V) \leq h\}}{h^{l+1}} > 0.$$

Moreover, if $0 < l < \dim X$, then

$$\limsup_{h\to\infty} \frac{\log \#\{V \in Z_{l,\mathrm{hor}}^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(V) \leq h\}}{h^{l+1}} > 0.$$

Proof. Choose a closed integral subscheme Y of X such that $\dim Y = l + 1$ and Y is flat over $\mathbb{Z}$. First, we assume that $l = 0$. Then, the canonical morphism $\pi : Y \to \mathrm{Spec}(\mathbb{Z})$ is finite. For $n \in \mathbb{Z} \setminus \{0\}$,

$$\widehat{\deg}(\pi^*(\mathrm{div}(n))) = \deg(\pi)\widehat{\deg}(\mathrm{div}(n)) = \deg(\pi)\log|n|.$$

Thus,

$$\#\{V \in Z_0^{\mathrm{eff}}(Y) \mid \widehat{\deg}(V) \leq h\} \geq \#\{\pi^*(\mathrm{div}(n)) \mid n \in \mathbb{Z} \setminus \{0\} \text{ and } \deg(\pi)\log|n| \leq h\}.$$

Note that

$$\pi^*(\mathrm{div}(n)) = \pi^*(\mathrm{div}(n')) \implies \mathrm{div}(n) = \mathrm{div}(n') \implies n = \pm n'.$$

Thus,

$$\#\{\pi^*(\mathrm{div}(n)) \mid n \in \mathbb{Z} \setminus \{0\} \text{ and } \deg(\pi)\log|n| \leq h\} = [\exp(h/\deg(\pi))].$$

Therefore,

$$\limsup_{h\to\infty} \frac{\log \#\{V \in Z_0^{\mathrm{eff}}(X) \mid \widehat{\deg}(V) \leq h\}}{h} > 0.$$

From now on, we assume that $l > 0$. Since

$$\#\{D \in \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}(Y) \mid \widehat{\deg}_{\overline{H}}(D) \leq h\} \subseteq \#\{V \in Z_{l,\mathrm{hor}}^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(V) \leq h\},$$

we may assume that $\dim X = l + 1$.

Let us take a birational morphism $\mu : X' \to X$ of projective arithmetic varieties such that there is a generically finite morphism $\nu : X' \to (\mathbb{P}^1_{\mathbb{Z}})^n$, where $n = \dim X_{\mathbb{Q}}$. We set $\overline{A} = \bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))$ on $(\mathbb{P}^1_{\mathbb{Z}})^n$ for some positive real number λ. Let us choose a positive rational number a such that there is a non-zero section $s \in H^0(X', \nu^*(A)^{\otimes a} \otimes \mu^*(H)^{\otimes -1})$ with $\|s\|_{\sup} \leq 1$ (cf. [6, Proposition 2.2]). Let X_0 be a Zariski open set of X such that μ is an isomorphism over X_0. Moreover, let B be the non-flat locus of ν. Let

$$(X' \setminus \mu^{-1}(X_0)) \cup \mathrm{Supp}(\mathrm{div}(s)) \cup B = Z_1 \cup \cdots \cup Z_r$$

be the irreducible decomposition. Choose a closed point z_i of $Z_i \setminus \bigcup_{j\neq i} Z_j$ for each i. Then, by Proposition 5.1.2 and Remark 5.1.3,

$$\limsup_{h\to\infty} \frac{\log \#\{D' \in \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{A}}(D') \leq h,\ \nu(z_i) \not\in \mathrm{Supp}(D')\ (\forall i)\}}{h^{n+1}} > 0.$$

Let D' be an element of $\mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n)$ with $\nu(z_i) \not\in \mathrm{Supp}(D')$ for all i. First, we claim that $\nu^*(D')$ is horizontal over $\mathbb{Z}$. Assume the contrary, that is, $\nu^*(D')$ contains a vertical irreducible component Γ. Then, $\nu_*(\Gamma) = 0$, which implies $\Gamma \subseteq B$. Thus, there is z_i with $z_i \in \Gamma$. Hence,

$$z_i \in \mathrm{Supp}(\nu^*(D')) = \nu^{-1}(\mathrm{Supp}(D')),$$

which contradicts the assumption $\nu(z_i) \not\in \mathrm{Supp}(D')$.

By the above claim, we can consider a map

$$\phi : \{D' \in \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \nu(z_i) \not\in \mathrm{Supp}(D) \text{ for all } i\} \to \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}(X)$$

given by $\phi(D') = \mu_*(\nu^*(D'))$. Here we claim that ϕ is injective. We assume that $\phi(D'_1) = \phi(D'_2)$. Since $z_i \not\in \mathrm{Supp}(\nu^*(D'_\epsilon))$ for $\epsilon = 1, 2$ and all i, no component of $\nu^*(D'_\epsilon)$ is contained in $X' \setminus \mu^{-1}(X_0)$. Thus, we have $\nu^*(D'_1) = \nu^*(D'_2)$. Hence

$$\deg(\nu) D'_1 = \nu_*(\nu^*(D'_1)) = \nu_*(\nu^*(D'_2)) = \deg(\nu) D'_2.$$

Therefore, $D'_1 = D'_2$.

Let D' be an element of $\mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n)$ with $\nu(z_i) \not\in \mathrm{Supp}(D')$ for all i. Since no component of $\nu^*(D')$ is contained in $\mathrm{Supp}(\mathrm{div}(s))$, we can see

$$\begin{aligned}\widehat{\deg}_{\overline{H}}(\phi(D')) &= \widehat{\deg}_{\mu^*(\overline{H})}(\nu^*(D')) \leq \widehat{\deg}_{\nu^*(\overline{A})^{\otimes a}}(\nu^*(D')) \\ &= a^n \widehat{\deg}_{\nu^*(\overline{A})}(\nu^*(D')) = a^n \deg(\nu) \widehat{\deg}_{\overline{A}}(D').\end{aligned}$$

Thus

$$\begin{aligned}&\#\{D' \in \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n) \mid \widehat{\deg}_{\overline{A}}(D') \leq h \text{ and } \nu(z_i) \not\in \mathrm{Supp}(D) \text{ for all } i\} \\ &\qquad\qquad \leq \#\{D \in \mathrm{Div}_{\mathrm{hor}}^{\mathrm{eff}}(X) \mid \widehat{\deg}_{\overline{H}}(D) \leq \deg(\nu) a^n h\}.\end{aligned}$$

Therefore, we get our theorem. $\square$

6. A refinement of Northcott's theorem in the arithmetic case

In this section, we consider an arithmetic analogue of §3. In some sense, this is an original case because Northcott's theorem was proved first over a number field. Here we treat a more general case, namely, diophantine geometry over a field K of finite type over $\mathbb{Q}$. For this purpose, we need a polarization of K, as we introduced in [6]. In §6.1, we explain it and also introduce a fine polarization of K, which guarantees the richness of the height function induced from this polarization (cf. §6.2, especially, Proposition 6.2.2.1). In §6.3, we prove the following main result of this section:

> Let $f : X \to B$ be a morphism of projective arithmetic varieties. Let X_η be the generic fiber of $f : X \to B$. Let $\overline{H}_1, \ldots, \overline{H}_d$ be a fine polarization of B, where $d = \dim B_{\mathbb{Q}}$. Let $\overline{L}$ be a nef C^∞-hermitian line bundle on X such that L is ample on the generic fiber X_η of $f : X \to B$. For an integer l with $d+1 \le l \le \dim X$, let $Z_l^{\mathrm{eff}}(X/B)$ be the set of effective cycles on X generated by integral closed l-dimensional subschemes Γ on X with $f(\Gamma) = B$. Then, for a fixed k, there is a constant C such that the number of elements V of $Z_l^{\mathrm{eff}}(X/B)$ with $\deg(L_{X_\eta}^{\cdot l-d-1} \cdot V_{X_\eta}) \le k$ and
> $$\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d} \cdot \widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \cdot V\right) \le h$$
> is less than or equal to $\exp(C \cdot h^{d+1})$ for all $h \ge 1$.

6.1. A polarization of a finitely generated field over $\mathbb{Q}$.

Some details of this subsection can be found in [6]. Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and let B be a projective arithmetic variety such that K is the function field of B. Here we fix notation.

•**polarization**: A collection $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ of B and nef C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{H}_1, \ldots, \overline{H}_d$ on B is called a *polarization of K*.

•**big polarization**: A polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ is said to be *big* if $\overline{H}_1, \ldots, \overline{H}_d$ are nef and big.

•**fine polarization**: A polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ is said to be *fine* if there are a generically finite morphism $\mu : B' \to B$ of projective arithmetic varieties, and C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \ldots, \overline{L}_d$ on B' such that $\overline{L}_1, \ldots, \overline{L}_d$ are of surface type (see §4.2 for its definition), $\mu^*(\overline{H}_i) \succsim \overline{L}_i$ for all i, and that $\overline{L}_1 \otimes \cdots \otimes \overline{L}_d$ is nef and big.

The concept of a fine polarization seems to be technical and complicated, but it gives rise to a good arithmetic height function (cf. Proposition 6.2.2.1).

Let us consider the following proposition.

Proposition 6.1.1. *If a polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ is fine, then there are generically finite morphisms $\mu : B' \to B$ and $\nu : B' \to (\mathbb{P}^1_{\mathbb{Z}})^d$ with the following property: for any real number λ, there are positive rational numbers $a_1, \ldots, a_d$ such that*
$$\mu^*(\overline{H}_i) \succsim \nu^*(q_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))^{\otimes a_i}$$
for all $i = 1, \ldots, d$, where $q_i : (\mathbb{P}^1_{\mathbb{Z}})^d \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor.

Proof. By the definition of fineness, there are a generically finite morphism $\mu : B' \to B$ of projective arithmetic varieties, morphisms $\phi_i : B' \to B_i$ ($i = 1, \ldots, d$)

of projective arithmetic varieties, and nef and big C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{Q}_i$ on B_i $(i = 1, \ldots, d)$ such that B_i's are arithmetic surfaces, $\mu^*(\overline{H}_i) \gtrsim \phi_i^*(\overline{Q}_i)$ for all i, and that $\phi_1^*(\overline{Q}_1) \otimes \cdots \otimes \phi_d^*(\overline{Q}_d)$ is nef and big. Here, there are dominant rational maps $\psi_i : B_i \dashrightarrow \mathbb{P}^1_{\mathbb{Z}}$ for $i = 1, \ldots, d$. Replacing B' and B_i's by their suitable birational models, we may assume ψ_i's are morphisms. Let $\nu : B' \to (\mathbb{P}^1_{\mathbb{Z}})^d$ be a morphism given by $\nu(x) = (\psi_1(\phi_1(x)), \ldots, \psi_d(\phi_d(x)))$. Let us fix a real number λ. Then, since $\overline{Q}_i$ is nef and big, there is a positive rational number a_i with $\overline{Q}_i \gtrsim \psi_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))^{\otimes a_i}$. Thus,

$$\mu^*(\overline{H}_i) \gtrsim \phi_i^*(\overline{Q}_i) \gtrsim \phi_i^*(\psi_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))^{\otimes a_i} = \nu^*(q_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))^{\otimes a_i}.$$

Finally, we need to see that ν is generically finite. For this purpose, it is sufficient to see that $\nu^*(q_1^*(\mathcal{O}(1)) \otimes \cdots \otimes q_d^*(\mathcal{O}(1)))$ is nef and big on $B'_{\mathbb{Q}}$. Indeed, we can find a positive rational number a such that $\psi_i^*(\mathcal{O}(1)) \otimes Q_i^{\otimes -a}$ is ample over $(B_i)_{\mathbb{Q}}$ for all i. Thus,

$$\bigotimes_{i=1}^{d} \phi_i^*(\psi_i^*(\mathcal{O}(1)) \otimes Q_i^{\otimes -a}) = \nu^* \left(\bigotimes_{i=1}^{d} q_i^*(\mathcal{O}(1)) \right) \otimes \left(\bigotimes_{i=1}^{d} \phi_i^*(Q_i) \right)^{\otimes -a}$$

is semiample on $B'_{\mathbb{Q}}$. Hence, $\nu^*(q_1^*(\mathcal{O}(1)) \otimes \cdots \otimes q_d^*(\mathcal{O}(1)))$ is nef and big because $\phi_1^*(Q_1) \otimes \cdots \otimes \phi_d^*(Q_d)$ is nef and big. □

Finally we would like to give a simple and sufficient condition for the fineness of a polarization. Let k be a number field, and O_k the ring of integer in k. Let $B_1, \ldots, B_l$ be projective and flat integral schemes over O_k whose generic fibers over O_k are geometrically irreducible. Let K_i be the function field of B_i and d_i the transcendence degree of K_i over k. We set $B = B_1 \times_{O_k} \cdots \times_{O_k} B_l$ and $d = d_1 + \cdots + d_l$. Then, the function field of B is the quotient field of $K_1 \otimes_k K_2 \otimes_k \cdots \otimes_k K_l$, which is denoted by K, and the transcendence degree of K over k is d. For each i $(i = 1, \ldots, l)$, let $\overline{H}_{i,1}, \ldots, \overline{H}_{i,d_i}$ be nef and big C^∞-hermitian $\mathbb{Q}$-line bundles on B_i. We denote by q_i the projection $B \to B_i$ to the i-th factor. Then, we have the following.

Proposition 6.1.2. *A polarization $\overline{B}$ of K given by*

$$\overline{B} = \left(B; q_1^*(\overline{H}_{1,1}), \ldots, q_1^*(\overline{H}_{1,d_1}), \ldots, q_l^*(\overline{H}_{l,1}), \ldots, q_l^*(\overline{H}_{l,d_l})\right)$$

is fine. In particular, a big polarization is fine.

Proof. Since there is a dominant rational map $B_i \dashrightarrow \left(\mathbb{P}^1_{\mathbb{Z}}\right)^{d_i}$ by virtue of Noether's normalization theorem, we can find a birational morphism $\mu_i : B'_i \to B_i$ of projective integral schemes over O_k and a generically finite morphism $\nu_i : B'_i \to \left(\mathbb{P}^1_{\mathbb{Z}}\right)^{d_i}$. We set $B' = B'_1 \times_{O_k} \cdots \times_{O_k} B'_l$, $\mu = \mu_1 \times \cdots \times \mu_l$ and $\nu = \nu_1 \times \cdots \times \nu_l$. Note that $\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)$ is ample on $\mathbb{P}^1_{\mathbb{Z}}$ for a $\lambda > 0$. Then, since $\mu_i^*(\overline{H}_{i,j})$ is big, there is a positive integer $a_{i,j}$ with $\mu_i^*(\overline{H}_{i,j})^{\otimes a_{i,j}} \gtrsim \nu_i^*\left(p_j^*\left(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)\right)\right)$ (cf. [6, Proposition 2.2]), that is, $\mu_i^*(\overline{H}_{i,j}) \gtrsim \nu_i^*\left(p_j^*\left(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)^{\otimes 1/a_{i,j}}\right)\right)$. Thus, we get our proposition. □

6.2. **Height functions over a finitely generated field.** First, we give the definition of height functions (for details, see [6]).

6.2.1. *The definition of height functions.* Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a polarization of K. Let X be a geometrically irreducible projective variety over K and L an ample line bundle on X. Let us take a projective integral scheme $\mathcal{X}$ over B and a C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ such that X is the generic fiber of $\mathcal{X} \to B$ and L is equal to $\mathcal{L}_K$ in $\operatorname{Pic}(X) \otimes \mathbb{Q}$. The pair $(\mathcal{X}, \overline{\mathcal{L}})$ is called a model of (X, L). Then, for $x \in X(\overline{K})$, we define $h^{\overline{B}}_{(\mathcal{X},\mathcal{L})}(x)$ to be

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = \frac{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}) \cdot \prod_{j=1}^d \widehat{c}_1(f^*(\overline{H}_j)) \mid \Delta_x\right)}{[K(x) : K]},$$

where Δ_x is the Zariski closure in $\mathcal{X}$ of the image of $\operatorname{Spec}(\overline{K}) \to X \hookrightarrow \mathcal{X}$, and $f : \mathcal{X} \to B$ is the canonical morphism. By virtue of [6, Corollary 3.3.5], if $(\mathcal{X}', \mathcal{L}')$ is another model of (X, L) over B, then there is a constant C with

$$|h^{\overline{B}}_{(\mathcal{X},\mathcal{L})}(x) - h^{\overline{B}}_{(\mathcal{X}',\mathcal{L}')}(x)| \le C$$

for all $x \in X(\overline{K})$. Hence, we have the unique height function $h^{\overline{B}}_L$ modulo the set of bounded functions. In the case where $X = \mathbb{P}^n_K$, if we set

$$h^{\overline{B}}_{nv}(x) = \sum_{\substack{\Gamma \text{ is a prime} \\ \text{divisor on } B}} \max_i\{-\operatorname{ord}_\Gamma(\phi_i)\} \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid \Gamma\right) + \int_{B(\mathbb{C})} \log\left(\max_i\{|\phi_i|\}\right) c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d)$$

for $x = (\phi_0 : \cdots : \phi_n) \in \mathbb{P}^n(K)$, then $h^{\overline{B}}_{\mathcal{O}(1)} = h^{\overline{B}}_{nv} + O(1)$ on $\mathbb{P}^n(K)$.

6.2.2. *The similarity of height functions.* Here we consider the following proposition, which tells us that height functions arising from fine polarizations are similar.

Proposition 6.2.2.1. *Let X be a geometrically irreducible projective variety over K, and L an ample line bundle on X. Let $\overline{B}$ and $\overline{B}'$ be fine polarizations of K. Then $h^{\overline{B}'}_L \asymp h^{\overline{B}'}_L$ on $X(\overline{K})$ (For the notation $\asymp$, see* (1.1.6)*).*

Proof. First we do a general observation. Let B be a projective arithmetic variety with $d = \dim B_{\mathbb{Q}}$. Let $\overline{H}_1, \ldots, \overline{H}_d$ be C^∞-hermitian $\mathbb{Q}$-line bundles of surface type on B. By its definition, for each i, there are a morphism $\phi_i : B \to B_i$ of flat and projective integral schemes over $\mathbb{Z}$ and a C^∞-hermitian $\mathbb{Q}$-line bundle $\overline{L}_i$ on B_i such that $\dim(B_i)_{\mathbb{Q}} = 1$, $\overline{L}_i$ is nef and big, and that $\phi_i^*(\overline{L}_i) = \overline{H}_i$ in $\widehat{\operatorname{Pic}}(B) \otimes \mathbb{Q}$. We set $\overline{H} = \bigotimes_{i=1}^d \overline{H}_i$ and

$$\lambda_i = \exp\left(-\frac{\widehat{\deg}(\widehat{c}_1(\overline{L}_i)^2)}{\deg((L_i)_{\mathbb{Q}})}\right).$$

Let K be the function field of B. Here we consider several kinds of polarizations of K as follows:

$$\begin{cases} \overline{B}_0 = (B; \overline{H}, \ldots, \overline{H}), \\ \overline{B}_1 = (B; \overline{H}_1, \ldots, \overline{H}_d), \\ \overline{B}_{i,j} = (B; \overline{H}_1, \ldots, \overline{H}_{j-1}, (\mathcal{O}_B, \lambda_i |\cdot|_{can}), \overline{H}_{j+1}, \ldots, \overline{H}_d) \quad \text{for } i \neq j. \end{cases}$$

Let X be a geometrically irreducible projective variety over K, and L an ample line bundle on X. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a model of (X, L) over B. Then, for all $x \in X(\overline{K})$,

$$h^{\overline{B}_0}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = d! h^{\overline{B}_1}_{(\mathcal{X},\overline{\mathcal{L}})}(x) + \frac{d!}{2} \sum_{i \neq j} h^{\overline{B}_{i,j}}_{(\mathcal{X},\overline{\mathcal{L}})}(x). \tag{6.2.2.2}$$

Indeed, by Lemma 4.2.4,

$$h^{\overline{B}_0}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = d! h^{\overline{B}_1}_{(\mathcal{X},\overline{\mathcal{L}})}(x) + \frac{d!}{2} \sum_{i \neq j} \frac{\widehat{\deg}(\widehat{c}_1(\overline{L}_i)^2) \deg\left(\mathcal{L}_{\mathbb{Q}} \cdot \prod_{l=1, l\neq j}^{d} f^* \phi_l^*(\overline{L}_l)_{\mathbb{Q}} \cdot (\Delta_x)_{\mathbb{Q}}\right)}{\deg((L_i)_{\mathbb{Q}})[K(x):K]},$$

where $f : \mathcal{X} \to B$ is the canonical morphism. Moreover,

$$h^{\overline{B}_{i,j}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = \frac{-\log(\lambda_i) \displaystyle\int_{\Delta_x(\mathbb{C})} c_1(\overline{\mathcal{L}}) \wedge \bigwedge_{l=1, l\neq j}^{d} c_1(f^* \phi_l^*(\overline{L}_l))}{[K(x):K]}.$$

On the other hand,

$$\int_{\Delta_x(\mathbb{C})} c_1(\overline{\mathcal{L}}) \wedge \bigwedge_{l=1, l\neq j} c_1(\pi^* \phi_l^*(\overline{L}_l)) = \deg\left(\mathcal{L}_{\mathbb{Q}} \cdot \prod_{l=1, l\neq j}^{d} f^* \phi_l^*(\overline{L}_l)_{\mathbb{Q}} \cdot (\Delta_x)_{\mathbb{Q}}\right).$$

Thus, we obtain

$$h^{\overline{B}_{i,j}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L}_i)^2) \deg\left(\mathcal{L}_{\mathbb{Q}} \cdot \prod_{l=1, l\neq j}^{d} f^* \phi_l^*(\overline{L}_l)_{\mathbb{Q}} \cdot (\Delta_x)_{\mathbb{Q}}\right)}{\deg((L_i)_{\mathbb{Q}})[K(x):K]}.$$

Therefore, we get (6.2.2.2).

Using (6.2.2.2), we can find a constant C such that

$$h^{\overline{B}_0}_L(x) \leq C h^{\overline{B}_1}_L(x) + O(1) \tag{6.2.2.3}$$

for all $x \in X(\overline{K})$ because there is a positive integer m such that

$$\overline{H}_j^{\otimes m} \gtrsim (\mathcal{O}_B, \lambda_i |\cdot|_{can})$$

for every i, j.

Let us start the proof of Proposition 6.2.2.1. It is sufficient to see that there are a positive real number a and a real number b such that $h^{\overline{B}}_L \leq a h^{\overline{B}'}_L + b$. We set $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ and $\overline{B}' = (B'; \overline{H}'_1, \ldots, \overline{H}'_d)$. Since $\overline{B}'$ is fine, by Proposition 6.1.1, there are generically finite morphisms $\mu' : B'' \to B'$ and $\nu : B'' \to \left(\mathbb{P}^1_{\mathbb{Z}}\right)^d$ of flat and projective integral schemes over $\mathbb{Z}$, and nef and big C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \ldots, \overline{L}_d$ on $\mathbb{P}^1_{\mathbb{Z}}$ such that $\mu'^*(\overline{H}'_i) \gtrsim \nu^*(p_i^*(\overline{L}_i))$ for all i, where $p_i : \left(\mathbb{P}^1_{\mathbb{Z}}\right)^d \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Changing B'' if necessarily, we may assume that there is a generically finite morphism $\mu : B'' \to B$.

Let us consider polarizations

$$\overline{B}_1 = (B''; \mu^*(\overline{H}_1), \ldots, \mu^*(\overline{H}_d)) \quad \text{and} \quad \overline{B}'_1 = (B''; \mu'^*(\overline{H}'_1), \ldots, \mu'^*(\overline{H}'_d))$$

and compare $h^{\overline{B}}$ with $h^{\overline{B}_1}$ (resp. $h^{\overline{B}'}$ with $h^{\overline{B}'_1}$). By virtue of the projection formula, we may assume that $B = B' = B''$ and $\mu = \mu' = \text{id}$.

We set $\overline{H} = \nu^*\left(\bigotimes_{l=1}^d p_l^*(\overline{L}_l)\right)$. Then, $(B;\overline{H},\ldots,\overline{H})$ is a big polarization. Thus, by [6, (5) of Proposition 3.3.7], there is a positive integer b_1 such that

$$h_L^{\overline{B}} \le b_1 h_L^{(B;\overline{H},\ldots,\overline{H})} + O(1).$$

Moreover, by (6.2.2.3), we can find a positive constant b_2 with

$$h_L^{(B;\overline{H},\ldots,\overline{H})} \le b_2 h_L^{(B;\nu^*p_1^*(\overline{L}_1),\ldots,\nu^*p_d^*(\overline{L}_d))} + O(1).$$

On the other hand, since $\overline{H}'_i \succsim \nu^*(p_i^*(\overline{L}_i))$ for all i,

$$h_L^{(B;\nu^*p_1^*(\overline{L}_1),\ldots,\nu^*p_d^*(\overline{L}_d))} \le h_L^{(B;\overline{H}_1,\ldots,\overline{H}_d)} + O(1).$$

Hence, we get our proposition. □

6.3. **Northcott's type theorem in the arithmetic case.** The purpose of this subsection is to prove the following theorem, which is a kind of refined Northcott's theorem.

Theorem 6.3.1. *Let $f : X \to B$ be a morphism of projective arithmetic varieties. Let K be the function field of B. Let $\overline{H}_1,\ldots,\overline{H}_d$ be a fine polarization of B, where $d = \dim B_{\mathbb{Q}}$. Let $\overline{L}$ be a nef C^∞-hermitian line bundle on X such that L_K is ample. For an integer l with $d+1 \le l \le \dim X$, as in* (1.1.2)*, let $Z_l^{\mathrm{eff}}(X/B)$ be the set of effective cycles on X generated by integral closed l-dimensional subschemes Γ on X with $f(\Gamma) = B$. We denote by $Z_l^{\mathrm{eff}}(X/B,k,h)$ the set of effective cycle $V \in Z_l^{\mathrm{eff}}(X/B)$ with $\deg(L_K^{\cdot l-d-1}\cdot V_K) \le k$ and*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d}\cdot \widehat{c}_1(f^*(\overline{H}_1))\cdots\widehat{c}_1(f^*(\overline{H}_d)) \,|\, V\right) \le h.$$

Then, for a fixed k, there is a constant C such that

$$\#Z_l^{\mathrm{eff}}(X/B,k,h) \le \exp(C\cdot h^{d+1})$$

for all $h \ge 1$.

Let us begin with a variant of Proposition 5.2.1.

Proposition 6.3.2. *Let us fix a positive real number λ. Let n and d be non-negative integers with $n \ge d+1$. Let $p_{[d]} : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^d$ be the morphism as in* (1.1.4)*. Let $p_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ be the projection to the i-th factor. For an integer l with $d+1 \le l \le n$, we denote by $Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n/(\mathbb{P}^1_{\mathbb{Z}})^d)$ the set of all effective cycles on $(\mathbb{P}^1_{\mathbb{Z}})^n$ generated by l-dimensional integral closed subschemes of $(\mathbb{P}^1_{\mathbb{Z}})^n$ which dominate $(\mathbb{P}^1_{\mathbb{Z}})^d$ by $p_{[d]}$. We set*

$$\widehat{\deg}_{[d]}(V) = \widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d}\cdot\prod_{j=1}^d \widehat{c}_1(p_j^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))) \,\middle|\, V\right)$$

for $V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n/(\mathbb{P}^1_{\mathbb{Z}})^d)$, where $\overline{L} = \bigotimes_{i=1}^n p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))$. (Note that $\widehat{\deg}_{[0]}(V)$ is given by $\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l} \,|\, V\right)$.) Let K be the function field of $(\mathbb{P}^1_{\mathbb{Z}})^d$. For $V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n/(\mathbb{P}^1_{\mathbb{Z}})^d)$, we denote by $\deg_K(V)$ the degree of V in the generic fiber of $\pi : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^d$ with respect to $\mathcal{O}_{(\mathbb{P}^1_K)^{n-d}}(1,\ldots,1)$. Then, for a fixed k, there is a constant C such that

$$\#\{V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n/(\mathbb{P}^1_{\mathbb{Z}})^d) \mid \widehat{\deg}_{[d]}(V) \le h \text{ and } \deg_K(V) \le k\} \le \exp(C\cdot h^{d+1})$$

for all $h \ge 1$.

Proof. We set
$$\Sigma = \{I \mid [d] \subseteq I \subseteq [n], \#(I) = l-1\}.$$
Then, it is easy to see that
$$Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n/(\mathbb{P}^1_{\mathbb{Z}})^d) = \sum_{I \in \Sigma} Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}).$$
Thus, it is sufficient to see that, for each I, there is a constant C' such that
$$\#\{V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p_I}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}) \mid \widehat{\deg}_{[d]}(V) \le h \text{ and } \deg_K(V) \le k\} \le \exp(C' \cdot h^{d+1})$$
for all $h \geq 1$. By changing the coordinate, we may assume that $I = [l-1]$. Here we denote p_I by p. For $n \geq l$, we set
$$T_n = \{V \in Z_l^{\mathrm{eff}}((\mathbb{P}^1_{\mathbb{Z}})^n \overset{p}{\to} (\mathbb{P}^1_{\mathbb{Z}})^{l-1}) \mid \deg_K(V) \le k\}.$$
Let $a_n : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^{n-1}$ and $b_n : (\mathbb{P}^1_{\mathbb{Z}})^n \to (\mathbb{P}^1_{\mathbb{Z}})^l$ be morphisms given by $a_n = p_{[n-1]}$ and $b_n = p_{[l-1]\cup\{n\}}$. Then, since
$$(a_n)_K^*(\mathcal{O}_{(\mathbb{P}^1_K)^{n-d-1}}(1,\ldots,1)) \otimes p_n^*(\mathcal{O}_{\mathbb{P}^1_K}(1)) = \mathcal{O}_{(\mathbb{P}^1_K)^{n-d}}(1,\ldots,1)$$
and
$$(b_n)_K^*(\mathcal{O}_{(\mathbb{P}^1_K)^{l-d}}(1,\ldots,1)) \otimes \bigotimes_{i=l}^{n-1} p_i^*(\mathcal{O}_{\mathbb{P}^1_K}(1)) = \mathcal{O}_{(\mathbb{P}^1_K)^{n-d}}(1,\ldots,1),$$
we have maps $\alpha_n : T_n \to T_{n-1}$ and $\beta_n : T_n \to T_l$ given by $\alpha_n(V) = (a_n)_*(V)$ and $\beta_n(V) = (b_n)_*(V)$. Moreover, we set
$$h_n(V) = \widehat{\deg}_{[d]}(V)$$
for $V \in T_n$. Then, it is easy to see that
$$h_{n-1}(\alpha_n(V)) \le h_n(V) \quad \text{and} \quad h_l(\beta_n(V)) \le h_n(V)$$
for all $V \in T_n$. Note that
$$k \geq \deg_K(V) \geq \theta(V) = \text{the number of irreducible components of } V.$$
Here we set e_l as follows: if $l \geq 2$ and $d \geq 1$, then
$$e_l = \widehat{\deg}\left(\widehat{c}_1\left(\bigotimes_{i=1}^{l-1} p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))\right)^{\cdot l-d} \prod_{i=1}^{d} \widehat{c}_1(p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1)))\right) \text{ on } (\mathbb{P}^1_{\mathbb{Z}})^{l-1};$$
if $l \geq 2$ and $d = 0$, then
$$e_l = \widehat{\deg}\left(\widehat{c}_1\left(\bigotimes_{i=1}^{l-1} p_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_\lambda}(1))\right)^{\cdot l}\right) \text{ on } (\mathbb{P}^1_{\mathbb{Z}})^{l-1};$$
if $l = 1$, then
$$e_l = \widehat{\deg}(\widehat{c}_1(\mathbb{Z}, \exp(-\lambda)|\cdot|)) \text{ on } \mathrm{Spec}(\mathbb{Z}).$$
Moreover, we set
$$A(s,t) = \frac{k\sqrt{s \cdot t}}{e_l}.$$
Then, by Lemma 5.2.2,
$$\#\{x \in T_n \mid \alpha_n(x) = y,\ \beta_n(x) = z\} \le A(h_{n-1}(y), h_l(z))$$

for all $x \in T_{n-1}$ and $y \in T_l$. Further, in the case where $n = l$,

$$\deg_K(V) = \deg_{d+1}(V) + \cdots + \deg_n(V).$$

Therefore, by Proposition 5.1.1, there is a constant C'' such that

$$\#\{x \in T_l \mid h_l(x) \leq h\} \leq \exp(C'' \cdot h^{d+1})$$

for all $h \geq 1$. Hence, by Lemma 1.2.1, there is a constant C' such that

$$\#\{x \in T_n \mid h_n(V) \leq h\} \leq \exp(C' \cdot h^{d+1})$$

for all $h \geq 1$. Thus, we get our assertion. $\square$

Let us start the proof of Theorem 6.3.1. First, we claim the following:

Claim 6.3.2.1. *Let $\overline{H}'_1, \ldots, \overline{H}'_d$ be nef C^∞-hermitian $\mathbb{Q}$-line bundles on B with $\overline{H}'_i \succsim \overline{H}_i$ for all i. If the assertion of the theorem holds for $\overline{H}_1, \ldots, \overline{H}_d$, then so does for $\overline{H}'_1, \ldots, \overline{H}'_d$.*

By virtue of Lemma 4.2.3,

$$\begin{aligned}\widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d} \cdot \widehat{c}_1(f^*(\overline{H}'_1)) \cdots \widehat{c}_1(f^*(\overline{H}'_d)) \mid V\right)& \\ \geq \widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d} \cdot \widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid V\right)&\end{aligned}$$

for all $V \in Z_l^{\mathrm{eff}}(X/B)$. Thus, we get our claim.

Next we claim the following:

Claim 6.3.2.2. *We assume that the generic fiber of $f : X \to B$ is geometrically irreducible. Let $\mu : B' \to B$ be a generically finite morphism of projective arithmetic varieties. Let X' be the main part of $X \times_B B'$, i.e., X' is the Zariski closure of the generic fiber $X \times_B B' \to B'$ in $X \times_B B'$. Let $f' : X' \to B'$ and $\mu' : X' \to X$ be the induced morphisms by the projections $X \times_B B' \to B'$ and $X \times_B B' \to X$ respectively. Then, the assertion of the theorem holds for $f : X \to B$, $\overline{L}$ and $\overline{H}_1, \ldots, \overline{H}_d$ if and only if so does for $f' : X' \to B'$, $\mu'^*(\overline{L})$ and $\mu^*(\overline{H}_1), \ldots, \mu^*(\overline{H}_d)$.*

First of all, note that the following diagram is commutative.

$$\begin{array}{ccc} X' & \xrightarrow{\mu'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow[\mu]{} & B \end{array}$$

Thus, for $V' \in Z_l^{\mathrm{eff}}(X'/B')$, we have

$$\begin{aligned}\widehat{\deg}\left(\widehat{c}_1(\mu'^*(\overline{L}))^{\cdot l-d} \cdot \widehat{c}_1(f'^*(\mu^*(\overline{H}_1))) \cdots \widehat{c}_1(f'^*(\mu^*(\overline{H}_d))) \mid V'\right)& \\ = \widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l-d} \cdot \widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \mu'_*(V)\right)&\end{aligned} \tag{6.3.2.3}$$

and

$$\deg(\mu'^*(L)_{K'}^{\cdot l-d-1} \cdot V'_{K'}) = \deg(L_K^{\cdot l-d-1} \cdot \mu'_*(V')_K), \tag{6.3.2.4}$$

where K' is the function field of B'.

First we assume that the assertion of the theorem holds for $f : X \to B$, $\overline{L}$ and $\overline{H}_1, \ldots, \overline{H}_d$. Then, by using (6.3.2.3) and (6.3.2.4), it is sufficient to see that, for a fixed $V \in Z_l^{\mathrm{eff}}(X/B)$, the number of effective cycles V' on X' with $\mu'_*(V') = V$ is less than or equal to

$$\exp\left(\deg(\mu')\deg\left(L_K^{\cdot l-d-1}\cdot V_K\right)\right).$$

For, let $V = \sum e_i V_i$ be the irreducible decomposition. Then by Lemma 1.3.2, the above number is less than or equal to $\exp(\deg(\mu')\sum e_i)$. On the other hand, we can see

$$\sum e_i \leq \sum_i e_i \deg\left(L_K^{\cdot l-d-1}\cdot (V_i)_K\right) = \deg\left(L_K^{\cdot l-d-1}\cdot V_K\right).$$

Next we assume that the assertion of the theorem holds for $f' : X' \to B'$, $\mu'^*(\overline{L})$ and $\mu^*(\overline{H}_1),\ldots,\mu^*(\overline{H}_d)$. In this case, by using (6.3.2.3) and (6.3.2.4), it is sufficient to construct a homomorphism

$$\mu'^\star : Z_l(X/B) \to Z_l(X'/B')$$

with $\mu'_*(\mu'^\star(V)) = \deg(\mu')V$. Let B_0 be the locus of points of B over which $\mu : B' \to B$ and $f : X \to B$ are flat. Here we set $X_0 = f^{-1}(B_0)$, $B'_0 = \mu^{-1}(B_0)$ and $X'_0 = f'^{-1}(B'_0)$. Then, we can see that $X'_0 = X_0 \times_{B_0} B'_0$ by [5, Lemma 4.2]. Thus, $\mu'_0 = \mu'|_{X'_0}$ is flat. For $V \in Z_l^{\mathrm{eff}}(X/B)$, no component of $\mathrm{Supp}(V)$ is contained in $X \setminus X_0$. Hence, $\mu'^\star(V)$ is defined by the Zariski closure of ${\mu'_0}^*(V|_{X_0})$.

Here, let us consider a case where $X = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} B$, f is the natural projection $X \to B$ and $\overline{L} = p^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^n_{\mathbb{Z}}}(1))$, where $p : X \to \mathbb{P}^n_{\mathbb{Z}}$ is the natural projection. Since the polarization $\overline{H}_1,\ldots,\overline{H}_d$ is fine, by Proposition 6.1.1, there are generically finite morphisms $\mu : B' \to B$ and $\nu : B' \to \left(\mathbb{P}^1_{\mathbb{Z}}\right)^d$ of projective arithmetic varieties, and positive rational numbers $a_1,\ldots,a_d$ such that $\mu^*(\overline{H}_i) \succsim \nu^*(r_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^1_{\mathbb{Z}}}(a_i)))$ for all i, where $r_i : \left(\mathbb{P}^1_{\mathbb{Z}}\right)^d \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor.

$$B \xleftarrow{\ \mu\ } B' \xrightarrow{\ \nu\ } (\mathbb{P}^1_{\mathbb{Z}})^d \xrightarrow{\ r_i\ } \mathbb{P}^1_{\mathbb{Z}}$$

We set $X' = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} B'$, $B'' = (\mathbb{P}^1_{\mathbb{Z}})^d$ and $X'' = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} B''$. Let $p' : X' \to \mathbb{P}^n_{\mathbb{Z}}$ and $p'' : X'' \to \mathbb{P}^n_{\mathbb{Z}}$ be the projections to the first factor and $f' : X' \to B'$ and $f'' : X'' \to B''$ the projections to the last factor. Here we claim the following.

Claim 6.3.2.5. *The assertion of the theorem holds for $f'' : X'' \to B''$, $\overline{L}''$ and $r_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^1_{\mathbb{Z}}}(1)),\ldots,r_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^1_{\mathbb{Z}}}(1))$, where $\overline{L}'' = p''^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^n_{\mathbb{Z}}}(1))$.*

Fixing l, we prove this lemma by induction on n. If $n = l-d-1$, then the assertion is trivial, so that we assume $n > l-d-1$. Let $\psi : \mathbb{P}^n_{\mathbb{Z}} \dashrightarrow (\mathbb{P}^1_{\mathbb{Z}})^n$ be the rational map given by

$$(X_0 : \cdots : X_n) \mapsto (X_0 : X_1) \times \cdots \times (X_0 : X_n).$$

We set $\phi = \psi \times \mathrm{id} : \mathbb{P}^n_{\mathbb{Z}} \times B'' \dashrightarrow (\mathbb{P}^1_{\mathbb{Z}})^n \times B''$ and $U = (\mathbb{P}^n_{\mathbb{Z}} \setminus \{X_0 = 0\}) \times B''$. Moreover, let $g : Y = (\mathbb{P}^1_{\mathbb{Z}})^n \times B'' \to B''$ be the natural projection, and $s_i : Y = (\mathbb{P}^1_{\mathbb{Z}})^{n+d} \to \mathbb{P}^1_{\mathbb{Z}}$ the projection to the i-th factor. We set

$$\overline{M} = \bigotimes_{i=1}^{n} s_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1)).$$

For $V \in Z_l^{\mathrm{eff}}(X''/B''; U)$ (i.e. $V \in Z_l^{\mathrm{eff}}(X''/B'')$ and any component of $\mathrm{Supp}(V)$ is not contained in $X'' \setminus U$), let V' be the strict transform of V via ϕ. Then, Lemma 4.2.5 and Lemma 2.2.2,

$$
\begin{aligned}
n^{(l-d)}\widehat{\deg}&\left(\widehat{c}_1(\overline{L}'')^{\cdot l-d}\cdot \widehat{c}_1(f''^*(r_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\cdots \widehat{c}_1(f''^*(r_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\,|\,V\right)\\
&\geq \widehat{\deg}\left(\widehat{c}_1(\overline{M})^{\cdot l-d}\cdot \widehat{c}_1(g^*(r_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\cdots \widehat{c}_1(g^*(r_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\,|\,V'\right)
\end{aligned}
$$

and

$$
n^{(l-d-1)}\deg(L''^{\,l-d-1}_{K''}\cdot V_{K''}) \geq \deg(M_{K''}^{l-d-1}\cdot V'_{K''}),
$$

where K'' is the function field of B''. Moreover, if we set

$$
\overline{M}' = \bigotimes_{i=1}^{n+d} s_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1)) = \overline{M}\otimes \bigotimes_{i=n+1}^{n+d} s_i^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1)),
$$

then

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\overline{M}')^{\cdot l-d}\cdot \widehat{c}_1(g^*(r_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\cdots \widehat{c}_1(g^*(r_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\,|\,V'\right)\\
&\quad= \widehat{\deg}\left(\widehat{c}_1(\overline{M})^{\cdot l-d}\cdot \widehat{c}_1(g^*(r_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\cdots \widehat{c}_1(g^*(r_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))))\,|\,V'\right)\\
&\qquad\qquad + (l-d)d\deg(M_{K''}^{l-d-1}\cdot V'_{K''})\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))^2\right).
\end{aligned}
$$

On the other hand, $\{X_0 = 0\}\times B = \mathbb{P}^{n-1}_{\mathbb{Z}}\times B$. Thus, by the hypothesis of induction and Proposition 6.3.2, we have our claim.

Therefore, gathering Claim 6.3.2.1, Claim 6.3.2.2 and Claim 6.3.2.5, we have our assertion in the case where $X = \mathbb{P}^n_{\mathbb{Z}}\times_{\mathbb{Z}} B$.

Finally let us consider the proof of the theorem in a general case. Replacing $\overline{L}$ by a positive multiple of it, we may assume that L_K is very ample. Thus we have an embedding $\phi : X_K \hookrightarrow \mathbb{P}^n_K$ with $\phi^*(\mathcal{O}(1)) = L_K$. Let X' be the Zariski closure of X_K in $\mathbb{P}^n_{\mathbb{Z}}\times_{\mathbb{Z}} B$ and $f' : X' \to B$ the induced morphism. Let $p : \mathbb{P}^n_{\mathbb{Z}}\times_{\mathbb{Z}} B \to \mathbb{P}^n_{\mathbb{Z}}$ be the projection to the first factor and $\overline{L}' = p^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))\Big|_{X'}$. Then there are birational morphisms $\mu : Z \to X$ and $\nu : Z \to X'$ of projective arithmetic varieties. We set $g = f\cdot\mu = f'\cdot\nu$. Let A be an ample line bundle on B such that $g_*(\mu^*(L)\otimes\nu^*(L')^{\otimes -1})\otimes A$ is generated by global sections. Thus there is a non-zero global section $s \in H^0(Z, \mu^*(L)\otimes\nu^*(L')^{\otimes -1}\otimes g^*(A))$. Since $(\mu^*(L)\otimes\nu^*(L')^{\otimes -1})_K = \mathcal{O}_{X_K}$, we can see that $f(\mathrm{div}(s)) \subsetneq B$. We choose a metric of A with $\|s\| \leq 1$. For $V \in Z_l^{\mathrm{eff}}(X/B)$, let V_1 be the strict transform of V by μ and $V' = \nu_*(V_1)$. Then,

$$
\begin{aligned}
&\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{L}\otimes f^*(\overline{A}))^{\cdot l+1}\cdot \widehat{c}_1(g^*(\overline{H}_1))\cdots \widehat{c}_1(g^*(\overline{H}_d))\,|\,V_1\right)\\
&\qquad= \widehat{\deg}\left(\widehat{c}_1(\overline{L})^{\cdot l+1}\cdot \widehat{c}_1(f^*(\overline{H}_1))\cdots \widehat{c}_1(f^*(\overline{H}_d))\,|\,V\right)\\
&\qquad\qquad + (l+1)\deg(L_K^l\cdot V_K)\widehat{\deg}\left(\widehat{c}_1(\overline{A})\cdot \widehat{c}_1(\overline{H}_1)\cdots \widehat{c}_1(\overline{H}_d)\right)
\end{aligned}
$$

Moreover,

$$\begin{aligned}\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{L}\otimes f^*(\overline{A}))^{\cdot l+1}\cdot\widehat{c}_1(g^*(\overline{H}_1))\cdots\widehat{c}_1(g^*(\overline{H}_d))\mid V_1\right)\\ \geq \widehat{\deg}\left(\widehat{c}_1(\nu^*\overline{L}')^{\cdot l+1}\cdot\widehat{c}_1(g^*(\overline{H}_1))\cdots\widehat{c}_1(g^*(\overline{H}_d))\mid V_1\right)\\ = \widehat{\deg}\left(\widehat{c}_1(\overline{L}')^{\cdot l+1}\cdot\widehat{c}_1(f'^*(\overline{H}_1))\cdots\widehat{c}_1(f'^*(\overline{H}_d))\mid V'\right)\end{aligned}$$

Thus, we may assume that there is an embedding $X\hookrightarrow\mathbb{P}^n_{\mathbb{Z}}\times_{\mathbb{Z}}B$ and $\overline{L}=p^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}(1))$. Therefore we get our Theorem. □

6.4. **The number of rational points over a finitely generated field.** In this subsection, we would like to prove the following two theorems.

Theorem 6.4.1. *Let K be a finitely generated field over $\mathbb{Q}$, and $\overline{B}$ a fine polarization of K. Let X be a projective variety over K and L an ample line bundle on X. Then*

$$\#\{x\in X(K)\mid h^{\overline{B}}_{L}(x)\leq h\}\leq\exp(C\cdot h^{d+1})$$

for all $h\geq 0$, where $d=\operatorname{tr.deg}_{\mathbb{Q}}(K)$.

Theorem 6.4.2. *Let K be a finitely generated field over $\mathbb{Q}$ and $\overline{B}$ a fine polarization of K. Then,*

$$\limsup_{h\to\infty}\frac{\log\#\{x\in\mathbb{P}^n(K)\mid h^{\overline{B}}_{\mathcal{O}(1)}(x)\leq h\}}{h^{d+1}}>0,$$

where $d=\operatorname{tr.deg}_{\mathbb{Q}}(K)$.

Theorem 6.4.1 is a consequence of Theorem 6.3.1. Let us consider the proof of Theorem 6.4.2. If we set $d=\operatorname{tr.deg}_{\mathbb{Q}}(K)$, then there is a subfield $\mathbb{Q}(z_1,\ldots,z_d)$ such that K is finite over $\mathbb{Q}(z_1,\ldots,z_d)$. Let $\overline{B}_1$ be the standard polarization of $\mathbb{Q}(z_1,\ldots,z_d)$ as in the following Lemma 6.4.3. Then, by Lemma 6.4.3,

$$\limsup_{h\to\infty}\frac{\log\#\{x\in\mathbb{P}^n(\mathbb{Q}(z_1,\ldots,z_d))\mid h^{\overline{B}_1}_{\mathcal{O}(1)}(x)\leq h\}}{h^{d+1}}>0.$$

Let $\overline{B}_1^K$ be the polarization of K induced by $\overline{B}_1$. Then,

$$[K:\mathbb{Q}(z_1,\ldots,z_d)]h^{\overline{B}_1}_{\mathcal{O}(1)}(x)=h^{\overline{B}_1^K}_{\mathcal{O}(1)}(x)+O(1)$$

for all $x\in\mathbb{P}^n(K)$. Moreover, by Proposition 6.2.2.1, $h^{\overline{B}_1^K}_{\mathcal{O}(1)}\asymp h^{\overline{B}}_{\mathcal{O}(1)}$. Thus, we get our theorem. □

Lemma 6.4.3. *Let us consider the polarization*

$$\overline{B}_1=\left((\mathbb{P}^1_{\mathbb{Z}})^d;p_1^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^1_{\mathbb{Z}}}(1)),\ldots,p_d^*(\overline{\mathcal{O}}^{\mathrm{FS}_1}_{\mathbb{P}^1_{\mathbb{Z}}}(1))\right)$$

of $\mathbb{Q}(z_1,\ldots,z_d)$, where $p_i:(\mathbb{P}^1_{\mathbb{Z}})^d\to\mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. The polarization $\overline{B}_1$ is called the standard polarization of $\mathbb{Q}(z_1,\ldots,z_d)$. Then, we have the following:

$$\limsup_{h\to\infty}\frac{\log\#\{x\in\mathbb{P}^n(\mathbb{Q}(z_1,\ldots,z_d))\mid h^{\overline{B}_1}_{nv}(x)\leq h\}}{h^{d+1}}>0.$$

(See §6.2 for the definition of $h_{nv}^{\overline{B}_1}$*.)*

Proof. We set $\overline{H}_i = p_i^*(\overline{\mathcal{O}}_{\mathbb{P}^1_{\mathbb{Z}}}^{\mathrm{FS}_1}(1))$ for $i = 1, \ldots, d$. Clearly, it is sufficient to consider the case $n = 1$, that is, rational points of $\mathbb{P}^1$. Let Δ_∞ be the closure of $\infty \in \mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^1_{\mathbb{Z}}$. We set $\Delta_\infty^{(i)} = p_i^*(\Delta_\infty)$. Then

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d) \mid \Delta_\infty^{(i)}\right) = 1 \tag{6.4.3.1}$$

Let P be a $\mathbb{Q}(z_1, \ldots, z_d)$-valued point of $\mathbb{P}^1$. Then, there are $f_0, f_1 \in \mathbb{Z}[z_1, \cdots, z_d]$ such that f_0 and f_1 are relatively prime and $P = (f_0 : f_1)$. Thus, by (6.4.3.1)

$$\begin{aligned} h_{nv}^{\overline{B}_1}(P) = \sum_i \max\{\deg_i(f_0), \deg_i(f_1)\} \\ + \int_{(\mathbb{P}^1)^d} \log\left(\max\{|f_0|, |f_1|\}\right) c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d). \end{aligned}$$

Let a be a positive number with $1 - 2da > 0$. We set

$$\mathcal{S}(h) = \{f \in \mathbb{Z}[z_1, \ldots, z_d] \mid v(1, f) \le \exp((1 - da)h) \text{ and } \deg_i(f) \le [ah] \text{ for all } i\}.$$

(See (4.3.3) for the definition v.)

First we claim that $h_{nv}^{\overline{B}_1}((1 : f)) \le h$ for all $f \in \mathcal{S}(h)$. If $f = 0$, then the assertion is obvious. We assume that $f \neq 0$. Then,

$$h_{nv}^{\overline{B}_1}((1 : f)) = \sum_{i=1}^d \deg_i(f) + \log(v(1, f)) \le d[ah] + (1 - da)h \le h.$$

Next we claim that

$$|f|_\infty \le \frac{\exp((1 - 2ad)h)}{\sqrt{2}} \quad \Longrightarrow \quad v(1, f) \le \exp((1 - ad)h).$$

For this purpose, we may assume that $f \neq 0$. Moreover, note that $\sqrt{1 + x^2} \le \sqrt{2}x$ for $x \ge 1$. Thus, using (4.3.1) and Proposition 4.3.4,

$$\begin{aligned} v(1, f) &\le \sqrt{2}^{d[ah]}\sqrt{1 + |f|_2^2} \le \sqrt{2}\sqrt{2}^{dah}|f|_2 \le \sqrt{2}\sqrt{2}^{dah}(1 + [ah])^{d/2}|f|_\infty \\ &\le \sqrt{2}\exp(dah/2)\exp(dah/2)|f|_\infty \le \sqrt{2}\exp(dah)\frac{\exp((1 - 2ad)h)}{\sqrt{2}} \\ &= \exp((1 - ad)h). \end{aligned}$$

By the second claim,

$$\begin{aligned} \#\mathcal{S}(h) &\ge \left(1 + 2\left[\frac{\exp((1 - 2ad)h)}{\sqrt{2}}\right]\right)^{([ah]+1)^d} \ge \left(\frac{\exp((1 - 2ad)h)}{\sqrt{2}}\right)^{(ah)^d} \\ &\ge \exp((1 - 2ad)h - 1)^{(ah)^d} = \exp(a^d(1 - 2ad)h^{d+1} - a^d h^d). \end{aligned}$$

Thus, we get our lemma by the first claim. □

7. The convergence of Zeta functions of algebraic cycles

In this section, we would like to propose a kind of zeta functions arising from the number of algebraic cycles. First let us consider a local case, i.e., the case over a finite field.

7.1. **The local case.** Let X be a projective variety over a finite field $\mathbb{F}_q$ and H an ample line bundle on X. For a non-negative integer k, we denote by $n_k(X, H, l)$ the number of all effective l-dimensional cycles V on X with $\deg_H(V) = k$. We define a zeta function $Z(X, H, l)$ of l-dimensional cycles on a polarized scheme (X, H) over $\mathbb{F}_q$ to be

$$Z(X, H, l)(T) = \sum_{k=0}^{\infty} n_k(X, H, l) T^{k^{l+1}}.$$

Then, we have the following:

Theorem 7.1.1. *$Z(X, H, l)(T)$ is a convergent power series at the origin.*

Proof. First note that $n_{m^l k}(X, H^{\otimes m}, l) = n_k(X, H, l)$. Moreover, if we choose $m > 0$ with $H^{\otimes m}$ very ample, then, by Corollary 2.2.5, there is a constant C with $n_k(X, H^{\otimes m}, l) \le q^{Ck^{l+1}}$. Thus,

$$n_k(X, H, l) = n_{m^l k}(X, H^{\otimes m}, l) \le q^{C' k^{l+1}},$$

where $C' = Cm^{l(l+1)}$. Therefore, if $|q^{C'} T| < 1$, then

$$\sum_{k=0}^{\infty} n_k(X, H, l) |T^{k^{l+1}}| \le \sum_{k=0}^{\infty} |q^{C'} T|^{k^{l+1}} \le \sum_{k=0}^{\infty} |q^{C'} T|^k = \frac{1}{1 - |q^{C'} T|}.$$

Thus, we get our theorem. □

See Remark 7.4.2 for Wan's zeta functions. Next, let us consider height zeta functions in the local case, which is a local analogue of Batyrev-Manin-Tschinkel's height zeta functions (cf. [1]).

Theorem 7.1.2. *Let K be a finitely generated field over a finite field $\mathbb{F}_q$ with $d = \operatorname{tr.deg}_{\mathbb{F}_q}(K) \ge 1$. Let X be a projective variety over K and L a ample line bundle on X. Let h_L be a representative of the class of height functions associated with (X, L) as in 3.3. Then, for a fixed k, a series*

$$\sum_{\substack{x \in X(\overline{K}),\\ [K(x):K] \le k}} q^{-s(h_L(x))^d}$$

converges absolutely and uniformly on the compact set in $\{s \in \mathbb{C} \mid \Re(s) > C\}$ for some C.

Proof. We set

$$X_n = \{x \in X(\overline{K}) \mid n - 1 < h_L(x) \le n \text{ and } [K(x) : K] \le k\}$$

for $n > 1$ and

$$X_1 = \{x \in X(\overline{K}) \mid h_L(x) \le 1 \text{ and } [K(x) : K] \le k\}.$$

Then, by Corollary 3.3.2, there is a constant C such that $\#(X_n) \le q^{Cn^d}$ for all $n \ge 1$. Hence,

$$\begin{aligned}\sum_{\substack{x \in X(\overline{K}),\\ [K(x):K] \le k}} |q^{-s(h_L(x))^d}| &= \sum_{n=1}^{\infty} \sum_{x \in X_n} q^{-\Re(s)(h_L(x))^d} \\ &\le \sum_{n=1}^{\infty} q^{Cn^d} q^{-\Re(s) n^d} = \sum_{n=1}^{\infty} \left(q^{-(\Re(s) - C)}\right)^{n^d}.\end{aligned}$$

Thus, we have our assertion. □

7.2. **The global case.** Let K be a number field and O_K the ring of integers in K. Let $f : X \to \operatorname{Spec}(O_K)$ be a flat and projective scheme over O_K and H an f-ample line bundle on X. For $P \in \operatorname{Spec}(O_K) \setminus \{0\}$, we denote by X_P the fiber of f at P. Here let us consider an infinite product

$$L(X, H, l)(s) = \prod_{P \in \operatorname{Spec}(O_K) \setminus \{0\}} Z(X_P, H_P, l)(\#(\kappa(P))^{-s})$$

for $s \in \mathbb{C}$. Then, we have the following:

Theorem 7.2.1. *There is a constant C such that the infinite product $L(X, H, l)(s)$ converges absolutely and uniformly on the compact set in $\{s \in \mathbb{C} \mid \Re(s) > C\}$.*

Proof. Since $Z(X_P, H_P^{\otimes n}, l)(q^{-s}) = Z(X_P, H_P, l)(q^{-sn^{l(l+1)}})$, replacing H by $H^{\otimes n}$ for some positive number n, we may assume that H is very f-ample. For non-negative integer k, we set

$$n_k(X_P, H_P, l) = \#\{V \in Z_l^{\mathrm{eff}}(X_P) \mid \deg_{H_P}(V) = k\}.$$

Then,

$$Z(X_P, H_P, l)(\#(\kappa(P))^{-s}) = 1 + \sum_{k=1}^{\infty} n_k(X_P, H_P, l)\#(\kappa(P))^{-sk^{l+1}}.$$

We denote $\sum_{k=1}^{\infty} n_k(X_P, H_P, l)\#(\kappa(P))^{-sk^{l+1}}$ by $u_P(s)$. We set $N = \operatorname{rk}(f_*(H)) - 1$. Then, for each P, we have an embedding $\iota_P : X_P \hookrightarrow \mathbb{P}^N_{\kappa(P)}$ with $\iota_P^*(\mathcal{O}(1)) = H_P$. Thus, by Theorem 2.2.1, there is a constant C depending only on l and N with $n_k(X_P, H_P, l) \le \kappa(P)^{Ck^{l+1}}$ for all $k \ge 1$. Thus, for $s \in \mathbb{C}$ with $\Re(s) > C + 1$,

$$\begin{aligned}
|u_P(s)| &\le \sum_{k=1}^{\infty} \#(\kappa(P))^{Ck^{l+1}} \#(\kappa(P))^{-\Re(s)k^{l+1}} = \sum_{k=1}^{\infty} \#(\kappa(P))^{-(\Re(s)-C)k^{l+1}} \\
&\le \sum_{l=1}^{\infty} \#(\kappa(P))^{-(\Re(s)-C)l} = \frac{\#(\kappa(P))^{-(\Re(s)-C)}}{1 - \#(\kappa(P))^{-(\Re(s)-C)}} \le \#(\kappa(P))^{-(\Re(s)-C)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{P \in \operatorname{Spec}(O_K) \setminus \{0\}} |u_P(s)| &\le \sum_{P \in \operatorname{Spec}(O_K) \setminus \{0\}} \#(\kappa(P))^{-(\Re(s)-C)} \\
&= \sum_{p\,:\,\text{prime}} \sum_{\substack{P \in \operatorname{Spec}(O_K) \\ P \cap \mathbb{Z} = p\mathbb{Z}}} \#(\kappa(P))^{-(\Re(s)-C)} \\
&\le [K : \mathbb{Q}] \sum_{p\,:\,\text{prime}} p^{-(\Re(s)-C)} \le [K : \mathbb{Q}]\zeta(\Re(s) - C).
\end{aligned}$$

Hence, we get our theorem by the criterion of the convergence of infinite products. □

7.3. **The arithmetic case.** Next let us consider an analogue in Arakelov geometry. Let $\mathcal{X}$ be a projective arithmetic variety and $\overline{\mathcal{H}}$ an ample C^{∞}-hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$. For an effective cycle V of l-dimension, the norm of V is defined by

$$N_{\overline{\mathcal{H}}}(V) = \exp\left(\widehat{\deg}_{\overline{\mathcal{H}}}(V)^{l+1}\right).$$

Then, the zeta function of $(\mathcal{X}, \overline{\mathcal{H}})$ for cycles of dimension l is defined by

$$\zeta(\mathcal{X}, \overline{\mathcal{H}}, l)(s) = \sum_{V \in Z_l^{\mathrm{eff}}(\mathcal{X})} N_{\overline{\mathcal{H}}}(V)^{-s}$$

Theorem 7.3.1. *There is a constant C such that the above $\zeta(\mathcal{X}, \overline{\mathcal{H}}, l)(s)$ converges absolutely and uniformly on the compact set in $\{s \in \mathbb{C} \mid \Re(s) > C\}$.*

Proof. We denote $Z_l^{\mathrm{eff}}(\mathcal{X}, \overline{\mathcal{H}}, h)$ the set of all l-dimensional effective cycles on $\mathcal{X}$ with $\widehat{\deg}_{\overline{\mathcal{H}}}(V) \leq h$. By Corollary 5.3.2, there is a constant C such that

$$\#\left(Z_l^{\mathrm{eff}}(\mathcal{X}, \overline{\mathcal{H}}, h)\right) \leq \exp(C \cdot h^{l+1})$$

for all $h \geq 1$. We choose a positive constant C' with

$$\exp(C \cdot (h+1)^{l+1}) \leq \exp(C' \cdot h^{l+1})$$

for all $h \geq 1$. Moreover, for a real number x, we set $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$. Note that if $k = [\widehat{\deg}_{\overline{\mathcal{H}}}(V)]$, then $k \leq \widehat{\deg}_{\overline{\mathcal{H}}}(V) < k+1$. Thus, for $s \in \mathbb{C}$ with $\Re(s) > C'$,

$$\begin{aligned}
\sum_{V \in Z_l^{\mathrm{eff}}(\mathcal{X})} \left|N_{\overline{\mathcal{H}}}(V)^{-s}\right| &= \sum_{k=0}^{\infty} \sum_{\substack{V \in Z_l^{\mathrm{eff}}(\mathcal{X}) \\ [\widehat{\deg}_{\overline{\mathcal{H}}}(V)] = k}} |N_{\overline{\mathcal{H}}}(V)|^{-\Re(s)} \\
&\leq \sum_{k=0}^{\infty} \#(Z_l^{\mathrm{eff}}(\mathcal{X}, \overline{\mathcal{H}}, k+1)) \exp(k^{l+1})^{-\Re(s)} \\
&\leq \sum_{k=0}^{\infty} \exp(C \cdot (k+1)^{l+1}) \exp(k^{l+1})^{-\Re(s)} \\
&\leq \exp(C) + \sum_{k=1}^{\infty} \exp(-(\Re(s) - C'))^{k^{l+1}} \\
&\leq \exp(C) + \frac{\exp(-(\Re(s) - C'))}{1 - \exp(-(\Re(s) - C'))}.
\end{aligned}$$

Thus, we get our theorem. □

7.4. **Remarks.** Here let us discuss remarks of the previous zeta functions. The first one is the abscissa of convergence of zeta functions.

Remark 7.4.1. Let I be an index set and $\{\lambda_i\}_{i \in I}$ a sequence of real numbers such that the set $I(t) = \{i \in I \mid \lambda_i \leq t\}$ is finite for every real number t. Then the abscissa σ_0 of convergence of the Dirichlet series

$$\sum_{i \in I} \exp(-\lambda_i s) = \lim_{t \to \infty} \sum_{i \in I(t)} \exp(-\lambda_i s)$$

is given by

$$\sigma_0 = \limsup_{t\to\infty} \frac{\log\left(\#(I(t))\right)}{t}.$$

Let X be a projective scheme over $\mathbb{F}_q$ and H an ample line bundle on X. Moreover, let $\mathcal{X}$ be a projective arithmetic variety and $\overline{\mathcal{H}}$ an C^∞-hermitian line bundle on $\mathcal{X}$. We denote by $\sigma_0(X,H,l)$ (resp. $\sigma_0(\mathcal{X},\overline{\mathcal{H}},l)$) the abscissa of convergence of $Z(X,H,l)(q^{-s})$ (resp. $\zeta(\mathcal{X},\overline{\mathcal{H}},l)(s)$). Then, $\sigma_0(X,H,l)$ and $\sigma_0(\mathcal{X},\overline{\mathcal{H}},l)$ are given by

$$\sigma_0(X,H,l) = \limsup_{h\to\infty} \frac{\log_q \#\left(\{V \in Z_l^{\mathrm{eff}}(X) \mid \deg_H(V) \le h\}\right)}{h^{l+1}}$$

and

$$\sigma_0(\mathcal{X},\overline{\mathcal{H}},l) = \limsup_{h\to\infty} \frac{\log \#\left(\{V \in Z_l^{\mathrm{eff}}(\mathcal{X}) \mid \widehat{\deg}_{\overline{\mathcal{H}}}(V) \le h\}\right)}{h^{l+1}}$$

respectively.

For example, let X be an n-dimensional projective scheme over $\mathbb{F}_q$ with $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$, where H is ample. Then,

$$\sigma_0(X,H,n-1) = \frac{1}{\deg(H^n)^{n-1} n!}.$$

Remark 7.4.2. Let X be a projective variety over a finite field $\mathbb{F}_q$ and H an ample line bundle on X. As before, the number of all effective l-dimensional cycles V on X with $\deg_H(V) = k$ is denoted by $n_k(X,H,l)$. In [10], Wan defined a zeta function $\tilde{Z}(X,H,l)$ by

$$\tilde{Z}(X,H,l)(T) = \sum_{k=0}^{\infty} n_k(X,H,l) T^k.$$

He proved $\tilde{Z}(X,H,l)(T)$ is p-adically analytic and proposed several kinds of conjectures. Of course, $\tilde{Z}(X,H,l)(T)$ is never analytic as $\mathbb{C}$-valued functions if $0 < l < \dim X$. In order to get classical analytic functions, we need to replace T^k by $T^{k^{l+1}}$.

Remark 7.4.3. Let $\theta(T)$ be a theta function given by

$$\theta(T) = \sum_{k\in\mathbb{Z}} T^{k^2} = 1 + 2\sum_{k=1}^{\infty} T^{k^2}.$$

Let p be a prime number. Virtually, the typical p-local zeta function for 1-dimensional cycles might be

$$Z_p(T) = \theta(pT) = \sum_{k\in\mathbb{Z}} p^{k^2} T^{k^2}.$$

Here let us consider

$$l(s) = \prod_{p\,:\,\text{prime}} Z_p(p^{-s}).$$

Then, we can see

$$l(s)^{-1} = \prod_{m=1}^{\infty} \frac{\zeta(2m(s-1))\zeta(2(2m-1)(s-1))^2}{\zeta((2m-1)(s-1))^2},$$

where $\zeta(s)$ is the Riemann-zeta function. This formula follows from Jacobi's triple product formula:

$$\theta(T) = \prod_{m=1}^{\infty} (1 - T^{2m})(1 + T^{2m-1})^2$$

Appendix A. Bogomolov plus Lang in terms of a fine polarization

In this appendix, we show that the main results in [6] and [7] hold even if a polarization is fine.

Theorem A.1 ([6, Theorem 4.3])**.** *We assume that the polarization $\overline{B}$ is fine. Let X be a geometrically irreducible projective variety over K, and L an ample line bundle on X. Then, for any number M and any positive integer e, the set*

$$\{x \in X(\overline{K}) \mid h_L^{\overline{B}}(x) \le M, \quad [K(x) : K] \le e\}$$

is finite.

Theorem A.2 ([7, Theorem A])**.** *We assume that the polarization $\overline{B}$ is fine. Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Let*

$$\langle\ ,\ \rangle_L^{\overline{B}} : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$$

be a paring given by

$$\langle x, y\rangle_L^{\overline{B}} = \frac{1}{2}\left(\hat{h}_L^{\overline{B}}(x+y) - \hat{h}_L^{\overline{B}}(x) - \hat{h}_L^{\overline{B}}(x)\right).$$

For $x_1, \ldots, x_l \in A(\overline{K})$, we denote $\det\left(\langle x_i, x_j\rangle_L^{\overline{B}}\right)$ *by $\delta_L^{\overline{B}}(x_1, \ldots, x_l)$.*

Let Γ be a subgroup of finite rank in $A(\overline{K})$ (i.e., $\Gamma \otimes \mathbb{Q}$ is finite-dimensional), and X a subvariety of $A_{\overline{K}}$. Fix a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\Gamma \otimes \mathbb{Q}$. If the set

$$\{x \in X(\overline{K}) \mid \delta_L^{\overline{B}}(\gamma_1, \ldots, \gamma_n, x) \le \epsilon\}$$

is Zariski dense in X for every positive number ϵ, then X is a translation of an abelian subvariety of $A_{\overline{K}}$ by an element of

$$\Gamma_{div} = \{x \in A(\overline{K}) \mid nx \in \Gamma \textit{ for some positive integer } n\}.$$

The proof of Theorem A.1 and Theorem A.2: Here, let us give the proof of Theorem A.1, Theorem A.2. Theorem A.1 is obvious by [6, Theorem 4.3] and Proposition 6.2.2.1, or Theorem 6.3.1. Theorem A.2 is a consequence of [7], Proposition 6.2.2.1 and the following lemma.

Lemma A.3. *Let V be a vector space over $\mathbb{R}$, and $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ be two inner products on V. If $\langle x, x\rangle \le \langle x, x\rangle'$ for all $x \in V$, then* $\det(\langle x_i, x_j\rangle) \le \det(\langle x_i, x_j\rangle')$ *for all $x_1, \ldots, x_n \in V$.*

Proof. If $x_1, \ldots, x_n$ are linearly dependent, then our assertion is trivial. Otherwise, it is nothing more than [4, Lemma 3.4]. □

Remark A.4. In order to guarantee Northcott's theorem, the fineness of a polarization is crucial. The following example shows us that even if the polarization is ample in the geometric sense, Northcott's theorem does not hold in general.

Let $k = \mathbb{Q}(\sqrt{29})$, $\epsilon = (5 + \sqrt{29})/2$, and $O_k = \mathbb{Z}[\epsilon]$. We set

$$E = \operatorname{Proj}\left(O_k[X, Y, Z]/(Y^2Z + XYZ + \epsilon^2 YZ^2 - X^3)\right).$$

Then, E is an abelian scheme over O_k. Thus, as in the proof of [6, Proposition 3.1.1], we can construct a nef C^∞-hermitian line bundle $\overline{H}$ on E such that $[2]^*(\overline{H}) = \overline{H}^{\otimes 4}$ and H_k is ample on E_k, $c_1(\overline{H})$ is positive on $E(\mathbb{C})$, and that $\widehat{\deg}\left(\widehat{c}_1(\overline{H})^2\right) = 0$. Let K be the function field of E. Then, $\overline{B} = (E; \overline{H})$ is a polarization of K. Here we claim that Northcott's theorem dose not hold for the polarization $(E, \overline{H})$ of K.

Let $p_i : E \times_{O_k} E \to E$ be the projection to the i-th factor. Then, considering $p_2 : E \times_{O_k} E \to E$, $(E \times_{O_k} E, p_1^*(\overline{H}))$ gives rise to a model of (E_K, H_K). Let Γ_n be the graph of $[2]^n : E \to E$, i.e., $\Gamma_n = \{([2]^n(x), x) \mid x \in E\}$. Moreover, let x_n be a K-valued point of E_K arising from Γ_n. Then, if we denote the section $E \to \Gamma_n$ by s_n, then

$$
\begin{aligned}
h^{\overline{B}}_{H_K}(x_n) &= \widehat{\deg}\left(p_1^*(\overline{H}) \cdot p_2^*(\overline{H}) \cdot \Gamma_n\right) = \widehat{\deg}\left(s_n^*(p_1^*(\overline{H})) \cdot s_n^*(p_2^*(\overline{H}))\right) \\
&= \widehat{\deg}\left(([2]^n)^*(\overline{H}) \cdot \overline{H}\right) = \widehat{\deg}\left(\overline{H}^{\otimes 4^n} \cdot \overline{H}\right) = 4^n \widehat{\deg}\left(\overline{H} \cdot \overline{H}\right) = 0.
\end{aligned}
$$

On the other hand, x_n's are distinct points in $E_K(K)$.

Appendix B. Weak geometric Northcott's theorem

In §3, we worked over a finite field, so that we can prove Northcott's type theorem in general. However, if we consider it over an algebraically closed field of characteristic zero, Northcott's type theorem does not hold in general. Nevertheless, we have the following weak form.

Proposition B.1. *Let X be a smooth projective variety over an algebraically closed field k of characteristic zero, C a smooth projective curve over k, and $f : X \to C$ a surjective morphism whose generic fiber is geometrically irreducible. Let L be an ample line bundle on X. If $\deg(f_*(\omega^n_{X/C})) > 0$ for some $n > 0$, then, for any number A, the set*

$$\{\Delta \mid \Delta \textit{ is a section of } f : X \to C \textit{ with } (L \cdot \Delta) \le A\}$$

is not dense in X.

Proof. Let us begin with the following lemma.

Lemma B.2. *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties over an algebraically closed field k of characteristic zero. If there are a projective smooth algebraic variety T over k and a dominant rational map $\phi : T \times_k Y \dashrightarrow X$ over Y, then the double dual $f_*(\omega^n_{X/Y})^{\vee\vee}$ of $f_*(\omega^n_{X/Y})$ is a free $\mathcal{O}_Y$-sheaf for all $n \ge 0$.*

Proof. Let A be a very ample line bundle on T. If $\dim T > \dim f$ and T_1 is a general member of $|A|$, then $\phi|_{T_1 \times Y} : T_1 \times Y \dashrightarrow X$ still dominates X. Thus, considering induction on $\dim T$, we may assume that $\dim T = \dim f$.

Let $\mu : Z \to T \times Y$ be a birational morphism of smooth projective varieties such that $\psi = \phi \cdot \mu : Z \to X$ is a morphism. Then, ψ is generically finite. Thus, there is a natural injection $\psi^*(\omega_{X/Y}) \hookrightarrow \omega_{Z/Y}$. Hence, $\psi^*(\omega^n_{X/Y}) \hookrightarrow \omega^n_{Z/Y}$ for all $n > 0$. Therefore,

$$\omega^n_{X/Y} \hookrightarrow \psi_*(\psi^*(\omega^n_{X/Y})) \hookrightarrow \psi_*(\omega^n_{Z/Y}).$$

Applying f_* to the above injection, we have

$$f_*(\omega^n_{X/Y}) \hookrightarrow f_*(\psi_*(\omega^n_{Z/Y})).$$

Further, letting p be the natural projection $p : T \times Y \to Y$,

$$f_*(\psi_*(\omega^n_{Z/Y})) = p_*(\mu_*(\omega^n_{Z/Y})) = p_*(\omega^n_{T\times Y/Y}) = H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y.$$

Thus, $f_*(\omega^n_{X/Y})^{\vee\vee}$ is a subsheaf of the free sheaf $H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y$.

Here we claim

$$\left(c_1\left(f_*(\omega^n_{X/Y})^{\vee\vee}\right) \cdot H^{d-1}\right) \geq 0, \tag{B.2.1}$$

where H is an ample line bundle on Y and $d = \dim Y$. This is an immediate consequence of weak positivity of $f_*(\omega^n_{X/Y})^{\vee\vee}$ due to Viehweg [9]. We can however conclude our claim by a weaker result of Kawamata [3], namely $\deg(f_*(\omega^n_{X/Y})) \geq 0$ if $\dim Y = 1$. For, considering complete intersections by general members of $|H^m|$ ($m \gg 0$), we may assume $\dim Y = 1$.

We can find a projection $\alpha : H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus r_n}$ such that $r_n = \operatorname{rk} f_*(\omega^n_{X/Y})^{\vee\vee}$ and the composition

$$f_*(\omega^n_{X/Y})^{\vee\vee} \hookrightarrow H^0(T, \omega^n_T) \otimes_k \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{O}_Y^{\oplus r_n}$$

is injective. Therefore, since $f_*(\omega^n_{X/Y})^{\vee\vee}$ is reflexive, the above homomorphism is an isomorphism by (B.2.1). □

Let us go back to the proof of Proposition B.1. Let $\operatorname{Hom}_k(C, X)$ be a scheme consisting of morphisms from C to X. Then, there is a morphism $\alpha : \operatorname{Hom}_k(C, X) \to \operatorname{Hom}_k(C, C)$ given by $\alpha(s) = f \cdot s$. We set $\operatorname{Sec}(f) = \alpha^{-1}(\mathrm{id}_C)$. Then, there is a natural morphism $\beta : \operatorname{Sec}(f) \times C \to X$ given by $\beta(s, y) = s(y)$. Since L is ample,

$$\{\Delta \mid \Delta \text{ is a section of } f : X \to C \text{ with } (L \cdot \Delta) \leq A\}$$

is a bounded family, so that there are finitely many connected components

$$\operatorname{Sec}(f)_1, \ldots, \operatorname{Sec}(f)_r$$

of $\operatorname{Sec}(f)$ such that, for all sections Δ with $(L \cdot \Delta) \leq A$, there is $s \in \operatorname{Sec}(f)_i$ for some i with $\Delta = s(C)$. On the other hand, by Lemma B.2, $\operatorname{Sec}(f)_i \times C \to X$ is not a dominant morphism for every i. Thus, we get our proposition. □

References

[1] V. V. Batyrev and Yu. Manin, Sur le nombre de points rationnels de hauteur bornée des variétés algébriques, Math. Ann. 286 (1990), 27–43.

[2] S. Kawaguchi and A. Moriwaki, Inequalities for semistable families for arithmetic varieties, J. Math. Kyoto Univ. 36, 97–182 (2001).

[3] Y. Kawamata, Kodaira dimension of algebraic fiber spaces over curves, Invent. Math., 66 (1982), 57–71.

[4] A. Moriwaki, Inequality of Bogomolov-Gieseker type on arithmetic surfaces, Duke Math. J. 74 (1994), 713–761.

[5] A. Moriwaki, The continuity of Deligne's pair, IMRN, 19, (1999), 1057–1066.

[6] A. Moriwaki, Arithmetic height functions over finitely generated fields, Invent. math., 140, (2000), 101–142.

[7] A. Moriwaki, A generalization of conjectures of Bogomolov and Lang over finitely generated fields, Duke Math. J. 107, (2001), 85–102.

[8] A. Moriwaki, The canonical arithmetic height of subvarieties of an abelian variety over a finitely generated field, J. reine angew. Math. 530 (2001), 33-54.

[9] E. Viehweg, Die additivität der Kodaira dimension für projektive faserraüme über varietäten des allgemeinen typs, J. Reine Angew. Math., 330 (1982), 132–142.

[10] D. Wan, Zeta functions of algebraic cycles over finite fields, manuscripta math. 74 (1992), 413–444.
[11] D. B. Zagier, Zetafunktionen und quadratische Körper, Springer Verlag (1981).

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